Simulations of gravitational collapse in null coordinates. III. Hyperbolicity

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We investigate the well-posedness of the characteristic initial-boundary value problem for the Einstein equations in Bondi-like coordinates (including Bondi, double-null and affine). We propose a definition of strong hyperbolicity of a system of partial differential equations of any order, and show that the Einstein equations in Bondi-like coordinates in their second-order form used in numerical relativity do not meet it, in agreement with results of Giannakopoulos *et al.* for specific first-order reductions. In the principal part, frozen coefficient approximation that one uses to examine hyperbolicity, we explicitly construct the general solution to identify the solutions that obstruct strong hyperbolicity. Independently, we present a first-order symmetric hyperbolic formulation of the Einstein equations in Bondi gauge, linearized about Schwarzschild, thus completing work by Frittelli. This establishes an energy norm (L^2 in the metric perturbations and selected first and second derivatives), in which the initial-boundary value problem, with initial data on an outgoing null cone and boundary data on a timelike cylinder or an ingoing null cone, is well posed, thus verifying a conjecture by Giannakopoulos *et al.* Unfortunately, our method does not extend to the pure initial-value problem on a null cone with regular vertex.

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I. INTRODUCTION

Well-posedness of a system of partial differential equations (from now on, PDEs) is defined as the existence and uniqueness of solutions and their continuous dependence on the initial and boundary data. For formulations of the Einstein equations where surfaces of the time coordinate u are outgoing null cones, three PDE problems, among others, are of interest: (1) the double null initial value problem, with free data posed on an outgoing null cone $u=0$ and ingoing null cone $v=0$ that intersect in a spacelike 2-sphere; (2) the initial-boundary value problem, with free initial data posed on $u=0$ and boundary data on a timelike cylinder $r=r_0$, again intersecting in a spacelike 2-sphere; and (3) the initial value problem on a null cone $u=0$ with regular vertex $r=0$.

The mathematical literature has focused on the proof of well-posedness of such problems in *some* coordinate system and formulation that is convenient for the proof. This might be called geometric well-posedness. By contrast, in numerical relativity, well-posedness of the continuum problem is necessary for the existence of a stable discretization, but other considerations are equally important for the choice of formulation of the Einstein equations, and so one wants a proof of well-posedness in the formulation of choice.

We give two examples of this difference in emphasis. A famous proof of geometric well-posedness of the Cauchy problem was given by Fourès-Bruhat in harmonic coordinates [1], but these became useful for numerical relativity only through the breakthrough work of Pretorius [2], which incorporated two key modifications of lower-order terms: specific choices of the "gauge source functions" of [3] that avoid coordinate singularities, and the addition of "constraint damping" terms to suppress violations of the harmonic and Einstein constraints arising from numerical error [4]. In a second example, much of the mathematical relativity literature uses double-null coordinates on ingoing and outgoing null cones, but these are not expected to be useful in numerical relativity beyond spherical symmetry because the ingoing null cones form caustics. (It is one purpose of the present series of papers to establish if outgoing null cones also form caustics in specific strong-field applications.)

Geometric well-posedness for the double initial value problem was established by Rendall [5] for smooth data, covering a small neighborhood of the intersection 2-sphere. The proof used harmonic coordinates, made to coincide with u and v on the initial surfaces $u=0$ and $v=0$. This result was improved by Luk [6] for a larger region (small finite distances to the future of the two initial null surfaces) and rougher data. The proof used double-null coordinates, and estimates in H^1 of a null tetrad and the Ricci rotation coefficients and curvature components in this tetrad. The initial value problem on a regular null cone is even harder because of the need to characterize data near the tip that will give a regular solution. Existence was proved by Chruściel [7], see also [8].

In the present series of papers, we focus on class of coordinates (u, x, θ, φ) introduced in [9,10], and called

"Bondi-like" in [11], where the surfaces of constant (retarded) time u are outgoing null cones and the lines of constant (u, θ, φ) are their generators. (To avoid confusion about terminology, we mention already that "Bondi coordinates" are further defined by the radial coordinate x being the area radius.)

In addition, we are interested in formulations of the Einstein equations where a maximum number of equations can be solved as ordinary differential equations (from now on, ODEs) along the generators of the coordinate null cones, and only a minimum number, namely two or three, contain u-derivatives: intuitively, the latter are evolution equations for the two polarizations of gravitational waves, plus, in double-null coordinates only, a third one for the area radius R.

Such formulations are attractive for numerical relativity because they are "maximally constrained," meaning that one solves the same constraint equations on each time slice as on the initial time slice (so that only free data are evolved), and because these constraints are not elliptic equations but inhomogeneous first-order linear ODEs that can be solved by integration. (An exception to this statement are affine null coordinates, where one of the constraints is a homogeneous linear second-order ODE.)

Time evolutions beyond spherical symmetry on null cones emanating from a regular center have been carried out, for example, for supernova core collapse [12] and for scalar field critical collapse [13], and vacuum evolutions using Cauchy-characteristic matching along a timelike cylinder, for example in [14].

However, we are not aware of any previous well-posedness result specifically in Bondi-like coordinates. In an incomplete attempt, Frittelli [15] constructed a first-order symmetric hyperbolic form of the vacuum Einstein equations in Bondi coordinates, linearized about Minkowski space. An "energy" estimate, in L^2 of the reduction variables, then follows. However, the perturbation $\tilde{V}$ of the metric coefficient V was omitted from the system. This is possible in the linearization about Schwarzschild because $\tilde{V}$ couples to the other perturbations, but not vice versa.

The present paper is an attempt to reconcile the results of [5–7], and the suggestive incomplete result of [15], with recent work of Giannakopoulos and collaborators. Their paper [11] found that a first-order reduction of the null cone formulation is weakly, but not strongly, hyperbolic (and hence not symmetric hyperbolic), that the lack of strong hyperbolicity is essentially a gauge problem in any Bondi-like gauge [16], and that it appears to indeed break the convergence of numerical solutions with resolution, both in toy models and in an open-source code for the null initial-boundary value problem [17].

The structure of the paper is as follows. As in the previous papers in this series [13,18], in Sec. II we restrict to twist-free axisymmetry, with a minimally coupled massless scalar field ψ as matter. We briefly restate the metric, the mathematical structure of the field equations and the gauge choices we consider. We then linearize the equations, first around an arbitrary background and then around Minkowski spacetime, drop lower-order terms, and "freeze" the background coefficients by treating them as constants. We now have a homogeneous system of linear PDEs with constant coefficients that we shall call the "toy model" of the original system. The symbol of the toy model is also the principal symbol of the original system, and strong hyperbolicity is an algebraic property of this principal symbol.

In contrast to [11,15], we do not reduce the PDEs to first order, but leave them in the form in which they are solved numerically. We generalize the textbook definition of strong hyperbolicity of first-order systems of PDEs to systems of PDEs of arbitrary order. We show that, by this criterion, all known Bondi-like null gauges are only weakly hyperbolic. We also find the general solution of the toy model itself in closed form, and hence identify the polynomial solutions that obstruct strong hyperbolicity. (Appendix A reminds the reader of a textbook example of this phenomenon).

In Sec. III, we use a completely different approach. We relax the restriction to twist-free axisymmetry, but linearize about the Schwarzschild solution, and restrict to Bondi gauge. We present a first-order symmetric hyperbolic form of the full linearized equations, thus completing the result of [15]. (We correct some minor errors in Appendix B.) To include all 6 metric coefficients in the estimate, we need to include also 10 first and 7 second derivatives of the metric as variables in the system and hence in the estimate. These are far from all first and second derivatives, and their choice is crucial for symmetric hyperbolicity. To connect with Sec. II with Sec. III, we give the equivalent symmetric hyperbolic form of the toy model in Appendix C.

We then present well-posedness estimates for the initial value problem on two intersecting null cones and the initial-boundary value problem on a null cone intersecting a timelike world tube, closely following [19,20]. Unfortunately, these methods do not allow us to derive an estimate for the pure Cauchy problem on an outgoing null cone with regular vertex, see Appendix D for the technical obstructions.

We summarize and conclude in Sec. IV.

II. OBSTRUCTIONS TO STRONG HYPERBOLICITY OF THE SECOND-ORDER FORM OF THE EINSTEIN EQUATIONS IN BONDI-LIKE GAUGES

A. Metric and field equations in twist-free axisymmetry

We begin with a brief review of our setup in this ection, see [18] for full details. We can write the metric of any twist-free axisymmetric spacetime in the form

$$ds^2 = -2Gdudx - Hdu^2 + R^2\left[e^{2Sf}S^{-1}(dy + Sbdu)^2 + e^{-2Sf}Sd\varphi^2\right]. \quad (1)$$

We use the angular coordinate $y \coloneqq -\cos\theta$, so that the range $0 \le \theta \le \pi$ corresponds to $-1 \le y \le 1$, and the shorthand $S \coloneqq 1 - y^2$. The other angular coordinate φ has the usual range $0 \le \varphi < 2\pi$. The Killing vector generating the axisymmetry is ∂_φ. (We use the convention of equating vector fields with derivative operators.) (G, H, R, b, f) and the scalar field ψ depend on (u, x, y). In [18] we assumed that $R = 0$ occurs at $x = 0$ and is a regular center, but we do not make this assumption here.

Each surface $\mathcal{N}_u^+$ of constant u is an outgoing null cone, and is ruled by the null geodesics $\mathcal{L}^+_{u,y,\varphi}$, which are also coordinate lines. Each surface $\mathcal{S}_{u,x}$ of constant u and x is assumed to be spacelike, and has topology S^2 and area $4\pi R^2$. The outgoing future-directed null vector field normal to $\mathcal{S}_{u,x}$ is $U \coloneqq G^{-1}\partial_x$, and is also tangent to the affinely parametrized generators of $\mathcal{N}_u^+$. The ingoing null normal on $\mathcal{S}_{u,x}$ is

$$\Xi = \partial_u - \frac{H}{2G}\partial_x - bS\partial_y. \quad (2)$$

It is normalized to $\Xi^a U_a = -1$.

The field equations we want to solve are the Einstein equations

$$E_{ab} \coloneqq R_{ab} - 8\pi\nabla_a\psi\nabla_b\psi = 0 \quad (3)$$

and the massless, minimally coupled wave equation

$$\nabla^a\nabla_a\psi = 0. \quad (4)$$

(We use units where $c = G = 1$.) A subset of the Einstein equations, plus the wave equation, take the form

$$\left(\ln\frac{G}{R_{,x}}\right)_{,x} = S_G[R, f, \psi], \quad (5)$$

$$\left(\frac{R^4 e^{2Sf} b_{,x}}{G}\right)_{,x} = S_b[R, f, \psi, G], \quad (6)$$

$$(R\Xi R)_{,x} = S_R[R, f, \psi, G, b], \quad (7)$$

$$(R\Xi f)_{,x} = S_f[R, f, \psi, G, b] - (\Xi R)f_{,x}, \quad (8)$$

$$(R\Xi\psi)_{,x} = S_\psi[R, f, \psi, G, b] - (\Xi R)\psi_{,x}, \quad (9)$$

where Ξ is the derivative operator defined in (2). We call these the "hierarchy equations." H and ∂_u appear only in the combination Ξ. The right-hand sides $S[f, \ldots]$ are given in full in [18]. They contain the derivatives $f_{,x}$, $f_{,y}$, $f_{,xy}$ and $f_{,yy}$ (but not $f_{,xx}$), and similarly for R, G, b and ψ, with the exception that $\psi_{,xy}$ and $b_{,yy}$ do not appear.

B. Gauges and solution algorithm

With H given (for example $H = 0$ in double-null gauge), the hierarchy equations take the form of first-order linear ODEs in x for b, G, ΞR, Ξf, and $\Xi\psi$ that can be solved by integration. With R given (for example $R = x$ in Bondi gauge), (7) is solved for H, given $R_{,u}$. In a third group of gauges, where G is given (for example $G = 1$ in affine gauge), (5) becomes a second-order linear ODE in x for R that is solved first, the other equations are again solved by integration for b, ΞR and Ξf, and $(7)_{,x}$ and $(5)_{,u}$ are combined to find an equation that can be solved for H by integration.

With $f_{,u}$, $R_{,u}$ and $\psi_{,u}$ now known, f, ψ, and in double-null gauge also R, are now advanced in u, and the hierarchy equations are then solved again. Note that the algorithm is "maximally constrained" in the sense that the hypersurface equations are solved at each time step, as they are from the free initial data.

Consider now the characteristic initial-boundary value problem, with outgoing null boundary $(u = 0, x > 0)$ and timelike or null inner boundary $(x = 0, u > 0)$. (Geometrically, one can think of the problem on two intersecting null cones as a pure initial-value problem, but in Bondi-like coordinates it is more natural to think of data on $u = 0$ as initial data and data on $v = 0$ as boundary data, as we solve the constraints by integration in v.)

In double null gauge we specify R, f and ψ on $u = 0$, and ΞR, Ξf, $\Xi\psi$, b, $b_{,x}$ and G on $x = 0$, or nine functions of two variables. In Bondi gauge we specify f and ψ on $u = 0$, and Ξf, $\Xi\psi$, b, $b_{,x}$, G and H on $x = 0$, or eight functions of two variables. Finally, in affine gauge, we specify f and ψ on $u = 0$, and R, $R_{,x}$, Ξf, $\Xi\psi$, b, $b_{,x}$, H and $H_{,x}$ on $x = 0$, or ten functions of two variables.

In each case, the data on $u = 0$ can be specified freely, while the data on $x = 0$ are constrained by some of the remaining Einstein equations, which do not contain x-derivatives. We do not discuss these constraints here, but evaluate the hyperbolicity of the evolution equations with the constraints relaxed.

C. Definition of strong hyperbolicity

As already noted, a necessary condition for well-posedness of a nonlinear PDE is well-posedness of its linearization, in the principal part, frozen coefficient approximation. We denote the field equations linearized around a background solution ϕ by

$$L(\mathbf{x}, \phi, \nabla)\delta\phi = 0, \quad (10)$$

We denote the principal part of L by L^p. The principal symbol is the matrix-valued function $L^p(\mathbf{x}, \phi, i\mathbf{k})$ of the wave number covector $\mathbf{k}$. For a quasilinear system (such as the Einstein equations), the principal symbol of the linearization is the same as the principal part of the full equations.

We can use the principal symbol to find plane-wave solutions to the linearized field equations in the frozen coefficient approximation: substituting $\delta\phi(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}\widehat{\delta\phi}(\mathbf{k})$ into $L(\nabla)\delta\phi(\mathbf{x}) = 0$ gives $L(i\mathbf{k})\widehat{\delta\phi}(\mathbf{k}) = 0$, which is now a system of linear algebraic equations. Nontrivial solutions exist only for $\mathbf{k}$ such that $\det L(i\mathbf{k}) = 0$. These $\mathbf{k}$ are called characteristic covectors, and vectors $\widehat{\delta\phi(\mathbf{k})}$ in the corresponding null space of $L(i\mathbf{k})$ are called characteristic variables. The linearized problem with frozen coefficients is well-posed in L^2 if the plane-wave solutions with real $\mathbf{k}$ are complete in the sense that we can map them smoothly and one-to-one to the boundary and initial data.

Consider now the Cauchy problem for a system of linear PDEs with constant coefficients. Initial data are imposed on a spacelike surface $t = 0$, with normal covector $\mathbf{n} \coloneqq dt$, and we assume that the initial data fall off as $|\mathbf{x}| \to 0$, so that we can Fourier-transform the initial data and the solution in $\mathbf{x}$. Then a sufficient criterion for well-posedness is given by strong hyperbolicity. We offer the following definitions:

A system of PDE is defined to be *strictly hyperbolic* in the time direction $\mathbf{n}$, in a neighborhood of the solution ϕ, if, first, $\det L^p(\phi, i\mathbf{n}) \neq 0$, and second, all roots ω of $\det L^p(\phi, i\omega\mathbf{n} + i\bar{\mathbf{k}}) = 0$ are real and distinct, for all real $\bar{\mathbf{k}}$ that are not zero or proportional to $\mathbf{n}$ (see, for example, [21]). (We shall use the notation $\mathbf{k}$ for arbitrary covectors, $\bar{\mathbf{k}}$ for what intuitively are vectors in "space", and $\mathbf{k}_\omega \coloneqq \omega\mathbf{n} + \bar{\mathbf{k}}$.)

If the roots are not all distinct, then the system is *strongly hyperbolic* if the null spaces of $L^p(\phi, i\omega\mathbf{n} + i\bar{\mathbf{k}})$ have dimension corresponding to the multiplicity of each root ω, and these null spaces depend smoothly on ω. This generalizes the textbook definition for first-order systems (see, for example, [22]).

If the roots are all real, but the system is not strongly hyperbolic, it is called *weakly hyperbolic*.

Strict hyperbolicity implies strong hyperbolicity, and for the linearized equations with frozen coefficients both mean that after Fourier-transforming in $\mathbf{x}$, initial data on a surface with normal covector $\mathbf{n}$ can be decomposed into plane waves $\exp(i\mathbf{k}\cdot\mathbf{x})\delta\phi(\mathbf{k})$, with $\delta\phi(\mathbf{k})$ in the null spaces mentioned above, each of which propagates with a speed ω in the direction $-\mathbf{k}$. This in turn implies local well-posedness of the pure initial-value problem.

If we reduce a strongly hyperbolic system of arbitrary higher order to first order by introducing suitable derivatives of the original variables as reduction variables, the general solution translated into these variables still consists of purely oscillating plane waves. Hence any such first-order reduction is strongly hyperbolic. However, the converse is not true: a higher order system that is only weakly hyperbolic according to our definition may or may not admit a strongly hyperbolic first-order reduction.

D. The principal symbol

To fix notation, we write our variables in the order $\phi^\dagger \coloneqq (G, b, R, f, H, \psi)$, the coordinates in the order $\mathbf{x} \coloneqq (u, x, y)$, so that $\nabla \coloneqq (\partial_u, \partial_x, \partial_y)$, and we use the notation $\mathbf{k} \coloneqq (\mu, \xi, \eta)$ for the wave number covector.

For our field equations, the principal terms clearly include the second derivatives of G, b, R, f and ψ. The only derivative of H that appears is $H_{,x}$, and this should therefore be considered principal. Finally, because we want to think of (5) as determining G, we include the first derivative $G_{,x}$ as a principal term in this equation, but not in the other equations, which also contain second derivatives of G. The equations are quasilinear in this sense.

Note that until we have fixed a specific null gauge, the principal symbol will have five rows, corresponding to the linearizations of the field equations (5)–(9), but six columns, corresponding to six metric perturbations. The preliminary principal symbol and the corresponding vector of perturbation variables are

$$L^p(\mathbf{x},\phi,i\mathbf{k}) = \begin{pmatrix} \frac{i}{G}\xi & 0 & \frac{1}{R_{,x}}\xi^2 & 0 & 0 & 0 \\ -\frac{R^2}{G}\eta\xi & -\frac{e^{2Sf}R^4}{G}\xi^2 & -2R\eta\xi & 2R^2S\eta\xi & 0 & 0 \\ \frac{R^2}{4}Y & \frac{R^2S}{4}\eta\xi & X+\frac{GR}{2}Y & -\frac{GR^2S}{2}Y & -i\frac{RR_{,x}}{2G}\xi & 0 \\ \frac{R}{4S}Y & \frac{R}{4}\eta\xi & 0 & X & -i\frac{Rf_{,x}}{2G}\xi & 0 \\ 0 & 0 & 0 & 0 & -i\frac{R\psi_{,x}}{2G}\xi & X+\frac{GR}{2}Y \end{pmatrix}, \qquad \delta\phi(\mathbf{x}) \coloneqq \begin{pmatrix} \delta G \\ \delta b \\ \delta R \\ \delta f \\ \delta H \\ \delta\psi \end{pmatrix}. \tag{11}$$

We have defined the shorthands

$$X \coloneqq R\xi\left(-\mu + bS\eta + \frac{H}{2G}\xi\right), \tag{12}$$

$$Y := \frac{e^{-2Sf}S}{R^2}\eta^2. \tag{13}$$

X is the principal symbol of the derivative operator $\partial_x R\Xi$ that appears on the left-hand side of each of the hierarchy equations. Y is the symbol of the principal angular derivative $-g^{yy}\partial_y\partial_y$.

Each of the three classes of Bondi-like gauges that we have already discussed simply eliminates one of the columns of (11), giving us a square 5×5 symbol: this is the δH column in generalized double-null gauges, where H is given and $\delta H = 0$; the δR column in Bondi and related gauges, where R is given and $\delta R = 0$; and the δG column in affine and related gauges, where G is given and $\delta G = 0$. We do not write the 5×5 principal symbols for these gauges out explicitly, as they can be read off trivially from (11).

In any null gauge, the "time" direction $\mathbf{n} = du = (1,0,0)$, that is, using u as the time coordinate, fails the first criterion for strong hyperbolicity. That is of course expected, as the surfaces of constant u are characteristic. Instead we follow [11] and analyse hyperbolicity on spacelike time slices with normal covector

$$\mathbf{n} := \left(1, \frac{1}{A}, 0\right), \tag{14}$$

with $A > 0$. For now we leave A and $\bar{\mathbf{k}} = (\mu, \xi, \eta)$ general, so that

$$\mathbf{k}_\omega = \left(\omega + \mu, \frac{\omega}{A} + \xi, \eta\right). \tag{15}$$

E. Double-null and related gauges

In double-null gauge $H = 0$ and $\delta H = 0$. In related gauges [18], δH is given in terms of δG, and so the resulting $\delta H_{,x}$ is not principal in the equations where it appears, because these contain second derivatives of G. In either case we can delete the δH column in the temporary principal symbol to obtain the 5×5 principal symbol for these gauges, and we remover δH from $\delta\phi$.

The first row of $L^p(\phi, i\mathbf{k})$ now contains first and second order in $\mathbf{k}$ terms, and the other rows are homogeneous in $\mathbf{k}$ of second order. However, the cofactor of the entry $\xi^2/R_{,x}$ in the first row, third column, of $L^p(\phi, i\mathbf{k})$ is zero, and so $\det L^p(\phi, i\mathbf{k})$ is homogeneous of order nine, rather than ten, in $\mathbf{k}$.

The nine roots ω of $L^p(\phi, i\omega\mathbf{n} + i\mathbf{k}) = 0$ are

$$\omega_0 := -A\xi \tag{16}$$

with multiplicity four,

$$\omega_\pm := \omega_c \pm \Delta\omega, \tag{17}$$

each with multiplicity two, and

$$\omega_2 := 2\omega_c + A\xi = -2Z\bar{\mu} - A\xi, \tag{18}$$

with multiplicity one. Here we have defined the shorthand

$$\omega_c := -Z\bar{\mu} - A\xi, \tag{19}$$

$$\Delta\omega := \sqrt{Z^2\bar{\mu}^2 + AGZY}, \tag{20}$$

$$\bar{\mu} := \mu - A\xi - bS\eta, \tag{21}$$

$$Z := \frac{AG}{2AG - H}. \tag{22}$$

The roots $\omega_\pm$ with multiplicity two have corresponding null spaces of dimension two, and the null space of ω_2 has dimension one, but the null space of ω_0 has only dimension two, not four, so two fewer than the multiplicity of the root.

Attention must also be given to special directions in which roots ω merge. In the special direction given by $\bar{\mu} = 0$, the roots $\omega_2 = \omega_0$ have merged and ω_0 is now a quintuple root, but the corresponding null space is still only two-dimensional, so (exceptionally) three fewer than the multiplicity of the root.

The characteristic covectors $\mathbf{k}_\omega$ themselves are independent of the parameter A of the time slicing, as long as $A > 0$, and to find all of them it is sufficiently general to give $\bar{\mathbf{k}}$ only two algebraically independent components. In particular, $\mathbf{k}_\pm$ and $\mathbf{k}_2$ can be written as

$$\mathbf{k}_\pm = \left(\frac{GY}{2\tilde{\xi}} + \frac{H}{2G}\tilde{\xi} + bS\eta, \tilde{\xi}, \eta\right), \tag{23}$$

$$\mathbf{k}_2 = \left(\frac{H}{2G}\tilde{\xi} + bS\eta, \tilde{\xi}, \eta\right), \tag{24}$$

where now $\tilde{\xi}$ and η are arbitrary, except that $\tilde{\xi} > 0$ for $\mathbf{k}_+$ and $\tilde{\xi} < 0$ for $\mathbf{k}_-$. Similarly, $\mathbf{k}_0$ can be written as

$$\mathbf{k}_0 = (\tilde{\mu}, 0, \eta), \tag{25}$$

where now $\tilde{\mu}$ and η are arbitrary. We see that $\mathbf{k}_0$ parametrizes plane waves that do not depend on x. The $\mathbf{k}_\pm$ are null covectors, while $\mathbf{k}_0$ and $\mathbf{k}_2$ are spacelike. They obey $U\cdot\mathbf{k}_0 = 0$ and $\Xi\cdot\mathbf{k}_2 = 0$.

F. Bondi and related gauges

In Bondi and related gauges, R is given and $\delta R = 0$, and so we eliminate the column corresponding to it. We now have eight roots ω: ω_0 with multiplicity four and null space of dimension two, so two fewer than the multiplicity of the root, and $\omega_\pm$ with multiplicity two and the same null spaces of dimension two as in the double null case. There is no problem of merging roots.

G. Affine and related gauges

In affine gauge, G is given and $\delta G = 0$, and so we eliminate the column corresponding to it. With $G = G(u)$, (5) and (7) take the form

$$R_{,xx} = (\ldots)R, \tag{26}$$

$$(RR_{,u})_{,x} + \frac{1}{2G}(RR_{,x}H)_{,x} = (\ldots), \tag{27}$$

where the dots stand for terms already known at the point where the equations are solved. (26) is then solved as a second-order ODE in x for R. Starting from $(27)_{,x}$, we use $(26)_{,u}$ to eliminate $R_{,uxx}$ and (27) to eliminate $R_{,ux}$, and obtain an equation of the form $H_{,xx} = (\ldots)$.

To reflect this, in affine gauge we multiply the third row of (11), which comes from (7), by $i\xi$. This will obviously multiply $\det L_p(i\mathbf{k})$ by a factor of $i\xi$, and hence will gives us an extra root ω_0, without changing the dimension of the corresponding null space.

We now have ten roots ω: ω_0 with multiplicity six and null space of dimension three, so three fewer than the multiplicity of the root and $\omega_\pm$ with multiplicity two and the same null spaces of dimension two as in the double null case. There is no problem of merging roots.

H. Linearization about Minkowski spacetime and frozen coefficient approximation

The principal symbol, and hence the characteristic covectors and variables are simpler in the linearization about Minkowski, but the multiplicities of the roots and dimensions of the corresponding null spaces are unchanged, so we now restrict to this case.

Linearizing about Minkowski spacetime, so that $b = f = \psi = 0$, $G = H = 1$ and $R = x$ in the background, we have the 5×6 principal symbol

$$L^p_{\rm Mink}(\mathbf{x}, i\mathbf{k}) = \begin{pmatrix} i\xi & 0 & \xi^2 & 0 & 0 & 0 \\ -r^2\eta\xi & -r^4\xi^2 & -2r\eta\xi & 2r^2S\eta\xi & 0 & 0 \\ \frac{S}{4}\eta^2 & \frac{r^2S}{4}\eta\xi & \frac{S}{2r}\eta^2 - r\mu\xi + \frac{r}{2}\xi^2 & -\frac{S^2}{2}\eta^2 & -i\frac{r}{2}\xi & 0 \\ \frac{1}{4r}\eta^2 & \frac{r}{4}\eta\xi & 0 & -r\mu\xi + \frac{r}{2}\xi^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{S}{2r}\eta^2 - r\mu\xi + \frac{r}{2}\xi^2 \end{pmatrix}. \tag{28}$$

In the frozen coefficient approximation, we denote the frozen value of R in the background by r, but the radial argument of the linear perturbations by x.

We can simplify (28) further, as follows. We define the coordinate $\bar{y}$ and corresponding wave number k as

$$k \coloneqq \frac{\sqrt{S}}{r}\eta, \qquad \bar{y} \coloneqq \frac{r}{\sqrt{S}}y, \quad \Rightarrow \quad k\bar{y} = \eta y, \qquad \partial_y = \frac{r}{\sqrt{S}}\partial_{\bar{y}}. \tag{29}$$

Note that in the frozen coefficient approximation both y and $\bar{y}$ are considered to have infinite range, while S appears only as a frozen coefficient, ignoring that $S = 1 - y^2 = \sin^2\theta$. Similarly, r stands for a frozen coefficient, while x is considered to have infinite range. We multiply the rows of (28) by the constants $(1, \sqrt{S}, 4, 4S, 2)$ respectively, and renormalize the perturbations by constants as

$$\overline{\delta\phi}^\dagger \coloneqq \left(\delta G, (r\sqrt{S})\delta b, (2/r)\delta R, (2S)\delta f, (2/r)\delta H, \delta\psi\right) \coloneqq \left(\delta G, \overline{\delta b}, \overline{\delta R}, \overline{\delta f}, \overline{\delta H}, \delta\psi\right), \tag{30}$$

which corresponds to multiplying the columns of (28) by constants. With this notation, the preliminary 5×6 principal symbol of the linearization about Minkowski space takes the form

$$L^p_{\rm Mink,frozen}(i\mathbf{k}) = \begin{pmatrix} i\xi & 0 & \frac{r}{2}\xi^2 & 0 & 0 & 0 \\ -k\xi & -\xi^2 & -k\xi & k\xi & 0 & 0 \\ k^2 & k\xi & k^2 - 2\mu\xi + \xi^2 & -k^2 & -i\xi & 0 \\ k^2 & k\xi & 0 & -2\mu\xi + \xi^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k^2 - 2\mu\xi + \xi^2 \end{pmatrix}. \tag{31}$$

We have been able to absorb all but one of the background-dependent coefficients into our redefinitions.

We also simplify the calculation of characteristic covectors by restricting our ansatz to $\mathbf{n} \coloneqq (1,1,0)$ and $\bar{k} = (0, \xi, k)$, so that

$$\mathbf{k}_\omega = (\omega, \omega + \xi, k). \tag{32}$$

In all three classes of Bondi-like gauges, $\det L^p(i\mathbf{k}_\omega) = 0$ has roots

$$\omega_\pm = \pm\sqrt{\xi^2 + k^2}, \tag{33}$$

corresponding to characteristic covectors

$$\mathbf{k}_\pm = \left(\pm\sqrt{\xi^2 + k^2}, \xi \pm \sqrt{\xi^2 + k^2}, k\right), \tag{34}$$

with multiplicity two and null spaces of dimension two. In addition, in double null gauge we have $\omega_0 = -\xi$, or

$$\mathbf{k}_0 = (-\xi, 0, k), \tag{35}$$

with multiplicity four but null space of dimension two, and $\omega_2 = \xi$, or

$$\mathbf{k}_2 = (\xi, 2\xi, k), \tag{36}$$

with multiplicity one. In Bondi gauge we have ω_0 with multiplicity four but null space of dimension two, and in affine gauge ω_0 with multiplicity six but null space of dimension four. So in each gauge the obstruction to strong hyperbolicity is that the multiplicity of the characteristic vector $\mathbf{k}_0$ is two more than the dimension of the corresponding null space.

In the special case $\xi = 0$, we have

$$\mathbf{k}_\pm = (\pm k, \pm k, k), \qquad \mathbf{k}_2 = \mathbf{k}_0 = (0, 0, k), \tag{37}$$

and in the special case $k = 0$, we have

$$\mathbf{k}_+ = \mathbf{k}_2 = (\xi, 2\xi, 0), \qquad \mathbf{k}_- = \mathbf{k}_0 = (-\xi, 0, 0). \tag{38}$$

For $\xi = k = 0$, there are no characteristic covectors.

I. A toy model for the obstruction to strong hyperbolicity

If a weakly hyperbolic system of homogeneous linear PDEs with constant coefficients fails to be strongly hyperbolic because there are not enough plane-wave solutions, the missing solutions must be a polynomial times a plane wave. (The reader is reminded of a well-known textbook example in Appendix A.)

We now use the principal part, frozen coefficient approximation as a toy model for the full linearized equations. The fact that it has constant coefficients allows us to find the general solution in closed form, and in particular the polynomial solutions that obstruct strong hyperbolicity.

In the linearization about Minkowski spacetime the scalar wave equation decouples from the metric perturbation in any choice of Bondi-like null gauge, and so we can consider it separately. In the frozen coefficient approximation, it takes the form

$$-2\delta\psi_{,ux} + \delta\psi_{,xx} + \delta\psi_{,\bar{y}\bar{y}} = 0. \tag{39}$$

Note this is the actual wave equation on $2+1$-dimensional Minkowski spacetime with metric

$$ds^2 = -2du dx - du^2 + dx^2 + d\bar{y}^2. \tag{40}$$

A plane-wave ansatz gives the solution

$$\delta\psi(u, x, \bar{y}) = \sum_\pm \int\int \widehat{\delta\psi}_\pm(\xi, k) e^{i\mathbf{k}_\pm\cdot\mathbf{x}} d\xi dk. \tag{41}$$

The functions $\widehat{\delta\psi}_\pm(\xi, k)$ map one-to-one to, for example, characteristic data in L^2 on null cones $u = 0$ and $v \coloneqq u + 2x = 0$, or to Cauchy data in L^2 on $t \coloneqq u + x = 0$. Hence this solution is complete by function counting, with no polynomial solutions required.

We now derive the general solution for the metric perturbations. We restrict to vacuum, $\delta\psi = 0$, without loss of generality. As the missing plane-wave solutions are for $\mathbf{k}_0 = (-\xi, 0, k)$, which parametrizes functions that are independent of x, we expect the missing solutions to be polynomial in x. The physical gravitational waves can be expressed in terms of derivatives of a solution Ψ of the *scalar* wave equation [18,23], which can be parametrized as

$$\Psi(u, x, \bar{y}) = \sum_\pm \int\int \hat{c}_\pm(\xi, k) e^{i\mathbf{k}_\pm\cdot\mathbf{x}} d\xi dk, \tag{42}$$

in complete parallel to (41). In double-null gauge, we also introduce the shorthand

$$\Psi_2(u, x, \bar{y}) = \int\int \hat{c}_2(\xi, k) e^{i\mathbf{k}_2\cdot\mathbf{x}} d\xi dk. \tag{43}$$

for the general solution in L^2 of the advection equation

$$2\Psi_{2,u} - \Psi_{2,x} = 0 \tag{44}$$

along ingoing null cones.

In *double null gauge* and its generalizations, we set $\overline{\delta H} = 0$. The equations in the principal part, frozen coefficient approximation can be read off from (31) with the $\widehat{\delta H}$ column deleted. They are

$$\delta G_{,x} - \frac{r}{2}\overline{\delta R}_{,xx} = 0, \tag{45}$$

$$\delta G_{,x\bar{y}} + \overline{\delta b}_{,xx} + \overline{\delta R}_{,x\bar{y}} - \overline{\delta f}_{,x\bar{y}} = 0, \tag{46}$$

$$\begin{aligned} -\delta G_{,\bar{y}\bar{y}} - \overline{\delta b}_{,x\bar{y}} + 2\overline{\delta R}_{,ux} \\ - \overline{\delta R}_{,xx} - \overline{\delta R}_{,\bar{y}\bar{y}} + \overline{\delta f}_{,\bar{y}\bar{y}} = 0, \end{aligned} \tag{47}$$

$$-\delta G_{,\bar{y}\bar{y}} - \overline{\delta b}_{,x\bar{y}} + 2\overline{\delta f}_{,ux} - \overline{\delta f}_{,xx} = 0. \tag{48}$$

The general solution is

$$\delta G = \delta G_{0,u} + \frac{r}{2}\Psi_{2,x}, \tag{49}$$

$$\overline{\delta b} = \overline{\delta b}_0 + \overline{\delta b}_{1,u} x - \frac{r}{2}\Psi_{2,\bar{y}} + \Psi_{,\bar{y}}, \tag{50}$$

$$\overline{\delta R} = \overline{\delta f}_0 + \frac{1}{2}(\overline{\delta b}_{1,\bar{y}} + \delta G_{0,\bar{y}\bar{y}})x + \Psi_2, \tag{51}$$

$$\overline{\delta f} = \overline{\delta f}_0 + \frac{1}{2}(\overline{\delta b}_{1,\bar{y}} + \delta G_{0,\bar{y}\bar{y}})x + \Psi_2 + \Psi_{,x}. \tag{52}$$

The five free functions (of two variables) $\hat{c}_\pm$, $\hat{c}_2$, $\overline{\delta b}_0$, $\overline{\delta f}_0$ parametrize plane waves, while the free functions δG_0 and $\overline{\delta b}_1$ parametrize linear-in-x solutions. These seven free functions correspond to seven characteristic covectors: $\mathbf{k}_\pm$ and $\mathbf{k}_2$, all with multiplicity one, and $\mathbf{k}_0$ with multiplicity four. They also correspond to the freedom to set five functions $(b, b_{,x}, G, \Xi R, \Xi f)$ of $(u, \bar{y})$ at $x = 0$ and two functions (R, f) of $(x, \bar{y})$ at $u = 0$. Hence our solution is complete by function counting.

The linear-in-x solutions are problematic not because they grow (and so are not in L^2), but because they grow arbitrarily rapidly in x for boundary data at $x = 0$ that oscillate arbitrarily rapidly in y.

In *Bondi gauge* and its generalizations, we set $\overline{\delta R} = 0$. The equations in the principal part, frozen coefficient approximation can be read off from (31) with the $\widehat{\delta R}$ column deleted. They are

$$\delta G_{,x} = 0, \tag{53}$$

$$\delta G_{,x\bar{y}} + \overline{\delta b}_{,xx} - \overline{\delta f}_{,x\bar{y}} = 0, \tag{54}$$

$$-\delta G_{,\bar{y}\bar{y}} - \overline{\delta b}_{,x\bar{y}} + \overline{\delta f}_{,\bar{y}\bar{y}} - \overline{\delta H}_{,x} = 0, \tag{55}$$

$$-\delta G_{,\bar{y}\bar{y}} - \overline{\delta b}_{,x\bar{y}} + 2\overline{\delta f}_{,ux} - \overline{\delta f}_{,xx} = 0. \tag{56}$$

The general solution is

$$\delta G = \delta G_0, \tag{57}$$

$$\overline{\delta b} = \overline{\delta b}_0 - \delta G_{0,\bar{y}} x + \Psi_{,\bar{y}}, \tag{58}$$

$$\overline{\delta f} = \overline{\delta f}_0 + \Psi_{,x}, \tag{59}$$

$$\overline{\delta H} = \overline{\delta H}_0 + \overline{\delta f}_{0,\bar{y}\bar{y}} x. \tag{60}$$

The four free functions $\hat{c}_\pm$, $\overline{\delta b}_0$ and $\overline{\delta H}_0$ parametrize plane waves, while δG_0 and $\overline{\delta f}_0$ parametrize linear-in-x solutions. These six free functions correspond to six characteristic covectors: $\mathbf{k}_\pm$ with multiplicity one and $\mathbf{k}_0$ with multiplicity four. They also correspond to the freedom to set five functions $(b, b_{,x}, G, \Xi f, H)$ at $x = 0$ and one function f at $u = 0$. Hence our solution is complete by function counting.

Finally, in *affine gauge* and its generalizations, we set $\delta G = 0$. The equations in the principal part, frozen coefficient approximation can be read off from (31) with the $\widehat{\delta G}$ column deleted. They are

$$\overline{\delta R}_{,xx} = 0, \tag{61}$$

$$\overline{\delta b}_{,xx} + \overline{\delta R}_{,x\bar{y}} - \overline{\delta f}_{,x\bar{y}} = 0, \tag{62}$$

$$-\overline{\delta b}_{,x\bar{y}} + 2\overline{\delta R}_{,ux} - \overline{\delta R}_{,xx} - \overline{\delta R}_{,\bar{y}\bar{y}} + \overline{\delta f}_{,\bar{y}\bar{y}} - \overline{\delta H}_{,x} = 0, \tag{63}$$

$$-\overline{\delta b}_{,x\bar{y}} + 2\overline{\delta f}_{,ux} - \overline{\delta f}_{,xx} = 0. \tag{64}$$

Following what is done in the full nonlinear equations, we take an x-derivative of (63), and use derivatives of the other equations to simplify it. In the principal part, frozen coefficient approximation the result is simply

$$\overline{\delta H}_{,xx} = 0. \tag{65}$$

The general solution of (61)–(65) is

$$\overline{\delta b} = \overline{\delta b}_0 + 2\delta R_{1,u} x + \Psi_{,\bar{y}}, \tag{66}$$

$$\overline{\delta R} = \overline{\delta R}_0 + \overline{\delta R}_{1,\bar{y}} x, \tag{67}$$

$$\overline{\delta f} = \overline{\delta f}_0 + \overline{\delta R}_{1,\bar{y}} x + \Psi_{,x}, \tag{68}$$

$$\overline{\delta H} = \overline{\delta H}_0 + \overline{\delta H}_1 x. \tag{69}$$

The six free functions $\hat{c}_\pm$, $\overline{\delta b}_0$, $\overline{\delta H}_0$, $\overline{\delta R}_0$ and $\overline{\delta f}_0$ parametrize plane waves, while $\overline{\delta R}_1$ and $\overline{\delta H}_1$ parametrize linear-in-x solutions. These eight free constants correspond to eight characteristic covectors: $\mathbf{k}_\pm$ with multiplicity one and $\mathbf{k}_0$ with multiplicity six. They also correspond to the freedom to set seven functions $(b, b_{,x}, G, R, R_{,x}, \Xi f, H, H_{,x})$ at $x = 0$ and one function f at $u = 0$. Hence our solution is once again complete by function counting.

If we solved (63) instead of (65) then $\overline{\delta H}_1$ would not be free, but given by $\overline{\delta H}_1 = \overline{\delta f}_{0,\bar{y}\bar{y}} - \overline{\delta R}_{0,\bar{y}\bar{y}}$. However, this is a

TABLE I. Function counting for the general solution (in the frozen coefficient approximation) versus free initial and boundary data. Here all barred functions are arbitrary functions of (u,y), while all hatted functions are arbitrary functions of (ξ,η). The mixed physical space/Fourier space notation has been chosen simply for ease of function counting, but see (41)–(43) for how to translate everything into real space. The massless scalar matter field decouples from the metric perturbations in the linearization about Minkowski, in any gauge, and is therefore listed separately. Initial data are imposed on $u=0$ and boundary data are imposed on $v=0$ or $r=r_0$.

Gauge	Initial data	Boundary data	Characteristic covectors	Free functions	Number
Double null	R,f	$b,b_{,x},G,\Xi R,\Xi f$	$\mathbf{k}_\pm,\mathbf{k}_2,\mathbf{k}_0\times 4$	$\hat c_\pm,\hat c_2,\overline{\delta G}_0,\overline{\delta b}_0,\overline{\delta b}_1,\overline{\delta f}_0$	7
Bondi	f	$b,b_{,x},G,\Xi f,H$	$\mathbf{k}_\pm,\mathbf{k}_0\times 4$	$\hat c_\pm,\overline{\delta G}_0,\overline{\delta b}_0,\overline{\delta f}_0,\overline{\delta H}_0$	6
Affine	f	$b,b_{,x},R,R_{,x},\Xi f,H,H_{,x}$	$\mathbf{k}_\pm,\mathbf{k}_0\times 6$	$\hat c_\pm,\overline{\delta b}_0,\overline{\delta R}_0,\overline{\delta R}_1,\overline{\delta f}_0,\overline{\delta H}_0,\overline{\delta H}_1$	8
Scalar field	ψ	$\Xi\psi$	$\mathbf{k}_\pm$	$\hat\psi_\pm$	2

feature of the toy model only, where are able to solve in closed form for $\overline{\delta R}$.

Table I gives an overview of free null data and normal data, the characteristic covectors, and the free functions in our explicit solution (in the frozen-coefficient approximation, in the linearization about Minkowski). In each case, if we add $\overline{\delta\psi}$ back in, the number of free function, matching the free data, becomes 9, 8 and 10 respectively, as discussed above in Sec. II E–II G. In all gauges, the boundary data at $x=0$ are subject to constraints, which we do not impose here.

III. SYMMETRIC HYPERBOLIC FIRST-ORDER REDUCTION OF THE LINEARIZED EINSTEIN EQUATIONS IN BONDI GAUGE

A. Balance laws from symmetric hyperbolic first-order systems

This subsection reviews relevant parts of [20] for completeness and to establish notation. A system of linear first-order PDEs

$$C^\mu\phi_{,\mu}=D\phi, \tag{70}$$

where $\phi\in\mathbb{R}^N$ is a vector of dependent variables, C^μ and D are real $N\times N$ matrices that depend smoothly on $x^\mu\in\mathbb{R}^n$, and the C^μ are symmetric, implies the balance law

$$j^\mu{}_{,\mu}=S, \tag{71}$$

$$j^\mu:=\phi^\dagger C^\mu\phi, \tag{72}$$

$$S:=\phi^\dagger\mathcal{D}\phi, \tag{73}$$

$$\mathcal{D}:=D+D^\dagger+C^\mu{}_{,\mu}. \tag{74}$$

The matrix $\mathcal{D}$ is real and symmetric, and so can be diagonalized with real eigenvalues.

If furthermore $C^\mu k_\mu$ depends smoothly on the covector $k_\mu\neq 0$, Gauss' law gives

$$\int_{\partial V}j^\mu d\Sigma_\mu=\int_V S\,dV \tag{75}$$

for any "control" spacetime volume V with boundary ∂V.

If furthermore there exists a time coordinate t (a smooth function with $(dt)_\mu$ everywhere future pointing), such that

$$j^t:=j^\mu(dt)_\mu \tag{76}$$

is positive definite in ϕ, the system is called symmetric hyperbolic with respect to t.

We now consider two elementary generalizations. If we redefine the dependent variables as

$$\phi:=A\tilde\phi, \tag{77}$$

where A is invertible and depends smoothly on the x^μ, (70) becomes

$$\tilde C^\mu\tilde\phi_{,\mu}=\tilde D\,\tilde\phi, \tag{78}$$

where

$$\tilde C^\mu:=A^\dagger C^\mu A, \tag{79}$$

$$\tilde D:=A^\dagger(DA-C^\mu A_{,\mu}). \tag{80}$$

The resulting balance law is the same as before, namely

$$\tilde j^\mu:=\tilde\phi^\dagger\tilde C^\mu\tilde\phi=j^\mu, \tag{81}$$

$$\tilde S:=\tilde\phi^\dagger\tilde{\mathcal{D}}\tilde\phi=S, \tag{82}$$

$$\tilde{\mathcal{D}}:=A^\dagger\mathcal{D}A, \tag{83}$$

In particular, the $A_{,\mu}$ term in $\tilde D$ in (80) cancels out of $\tilde{\mathcal{D}}$ in (83). However, we can change the eigenvalues of $\tilde{\mathcal{D}}$ and $\tilde C^\mu$ by choosing an invertible but nonorthogonal A. To see this, note that because $\mathcal{D}$ is symmetric, we have

$$\mathcal{D}=R\Lambda R^\dagger,\qquad R^\dagger R=I, \tag{84}$$

so

$$\tilde{\mathcal{D}} = A^\dagger R \Lambda R^\dagger A, \tag{85}$$

Hence $\tilde{\mathcal{D}}$ has the same eigenvalues as $\mathcal{D}$ if and only if A is orthogonal. The same is true for the matrices the $\tilde{C}^\mu$.

As a second modification, we consider a change of integration weight dV in (88). The balance law (71) is equivalent to

$$(\omega j^\mu)_{,\mu} = \omega S_\omega \tag{86}$$

with

$$S_\omega \coloneqq S + \frac{\omega_{,\mu}}{\omega} j^\mu = \phi^\dagger \left(\mathcal{D} + \frac{\omega_{,\mu}}{\omega} C^\mu \right) \phi, \tag{87}$$

and so we have

$$\int_{\partial V} j^\mu \omega d\Sigma_\mu = \int_V S_\omega \omega dV \tag{88}$$

for any smooth function $\omega > 0$.

B. Estimate on an arbitrary control volume

We now review how a symmetric hyperbolic system of homogeneous first-order PDEs gives rise to "energy" estimates that demonstrate well-posedness in the corresponding "energy norm." We follow the basic idea of [20], but give an alternative derivation.

Let $n_\mu \neq 0$ denote an outward-pointing covector field on ∂V. As the C^μ are real and symmetric, at any point in ∂V the matrix $C^\mu n_\mu$ can be diagonalized with real eigenvalues, and so the space $\mathbb{R}^N$ of dependent variables ϕ can be written as the sum of the positive, negative and zero eigenspaces of $C^\mu n_\mu$, or $\mathbb{R}^N = V_+ \oplus V_- \oplus V_0$. Hence the outward-pointing flux $j^\mu n_\mu = \phi^\dagger C^\mu n_\mu \phi$ can be written as a term that is positive definite on V_+ plus one that is negative definite on V_-. Integrating over ∂V, we can then write (88) schematically as

$$\int_V S\, dV \eqqcolon \|\mathrm{out}\|^2 - \|\mathrm{in}\|^2. \tag{89}$$

In applications, we will split $\|\mathrm{out}\|^2$ into

$$\|\mathrm{out}\|^2 = \|\mathrm{out}'\|^2 + \|\mathrm{out}''\|^2, \tag{90}$$

where $\|\mathrm{out}''\|^2$ denotes any part of the outgoing flux that we do not want to include in our estimate. We trivially obtain

$$\|\mathrm{out}'\|^2 \leq \|\mathrm{in}\|^2 + \int_V S\, dV. \tag{91}$$

Now assume there is a slicing of V by hypersurfaces of constant t such that the system is symmetric hyperbolic with respect to t. Let $c > 0$ be the smallest eigenvalue of $C^t \coloneqq C^\mu (dt)_\mu$ anywhere in V, and let d be the largest positive eigenvalue of $\mathcal{D}$ anywhere in V, or zero if $\mathcal{D}$ is negative definite everywhere in V. We have $S \leq d\phi^\dagger\phi$ and $j^t \geq c\phi^\dagger\phi$, and so we can bound the source term S of the balance law as

$$S \leq \frac{d}{c} j^t. \tag{92}$$

We slice V into surfaces of constant t,

$$\int_V S\, dV = \int_{t_0}^{t_1} \mathcal{S}_t\, dt, \qquad \mathcal{S}_t \coloneqq \int_{V\cap\Sigma_t} S\, d\Sigma_t, \tag{93}$$

where $t_1 \coloneqq \sup_V t$ and $t_0 \coloneqq \inf_V t$, and use the bound (92) to obtain

$$\mathcal{S}_t \leq \frac{d}{c} \mathcal{E}_t, \qquad \mathcal{E}_t \coloneqq \int_{V\cap\Sigma_t} j^t d\Sigma_t. \tag{94}$$

We now evaluate (91) on the control volume $V_t \coloneqq V \cap \{t' < t\}$ (the part of V to the past of $t' = t$) to obtain

$$\mathcal{E}_t = \|\mathrm{in}\|^2_{(\partial V)_t} - \|\mathrm{out}\|^2_{(\partial V)_t} + \int_{t_0}^t \mathcal{S}_{t'}\, dt' \tag{95}$$

$$\leq \|\mathrm{in}\|^2 + \frac{d}{c}\int_{t_0}^t \mathcal{E}_{t'} dt', \tag{96}$$

where we have defined $(\partial V)_t \coloneqq \partial V \cap \{t' < t\} \subseteq \partial V$. In (96) we have used $\|\mathrm{in}\|^2_{\partial V_t} \leq \|\mathrm{in}\|^2$, which follows from $\partial V_t \subseteq \partial V$.

We now differentiate to turn the integral inequality (96) into a differential one,

$$\frac{d}{dt}\mathcal{E}_t \leq \frac{d}{c}\mathcal{E}_t, \qquad \mathcal{E}_{t_0} \leq \|\mathrm{in}\|^2, \tag{97}$$

and solve this to obtain

$$\mathcal{E}_t \leq e^{\frac{d}{c}(t-t_0)} \|\mathrm{in}\|^2. \tag{98}$$

We then have

$$\int_V S\, dV \leq \int_{t_0}^{t_1} \frac{d}{c}\mathcal{E}_t dt \leq \left(e^{\frac{d}{c}(t_1-t_0)} - 1 \right) \|\mathrm{in}\|^2, \tag{99}$$

and substituting (99) into (91) we obtain the desired estimate

$$\|\mathrm{out}'\|^2 \leq e^{\frac{d}{c}(t_1-t_0)} \|\mathrm{in}\|^2, \tag{100}$$

as derived in [20], but here for arbitrary $t_1 - t_0$ and V.

C. The scalar wave equation on Schwarzschild

In spherical polar coordinates, symmetry of the matrices C^μ is less obvious than it is in Cartesian coordinates. This is best illustrated if we consider the scalar wave equation on the Schwarzschild background, which decouples to linear order from the metric perturbations, as already noted. It is

$$-2\psi_{,ur} - \frac{2}{r}\psi_{,u} + \mathcal{A}\left(\psi_{,rr} + \frac{2}{r}\psi_{,r}\right) + \frac{2m}{r^2}\psi_{,r} + \frac{1}{r^2}\nabla^a\nabla_a\psi = 0. \quad (101)$$

Here and in the following, we write all equations in covariant form with respect to the coordinates x^a on S^2. Following [19], we denote by q_{ab} the abstract round unit metric on S^2, by q^{ab} its inverse, and by ∇_a the covariant derivative with respect to q_{ab}, so $\nabla_a q_{bc} = 0$. Note that ∇_a commutes with partial ∂_u and ∂_x.

Following [19], we define the reduction variables

$$P := (r\psi)_{,r}, \qquad Q_a := \nabla_a\psi, \quad (102)$$

and obtain the first-order system

$$2P_{,u} - \mathcal{A}P_{,r} - \frac{1}{r}\nabla^a Q_a = \frac{2m}{r^2}(P-\psi), \quad (103)$$

$$Q_{a,r} - \frac{1}{r}\nabla_a P = -\frac{1}{r}Q_a, \quad (104)$$

$$\psi_{,r} = \frac{1}{r}(P-\psi). \quad (105)$$

Let $x^a = (\theta, \varphi)$ be the usual coordinates on S^2, in terms of which $q_{ab} = \mathrm{diag}(1, \sin^2\theta)$. Then

$$\nabla^a\nabla_a\psi = \psi_{,\theta\theta} + \cot\theta\psi_{,\theta} + \frac{1}{\sin^2\theta}\psi_{,\varphi\varphi}. \quad (106)$$

To make the nondiagonal matrices C^θ and C^φ symmetric, we need to expand the covector Q_a in components with respect to the noncoordinate, orthonormal basis

$$\partial_\theta, \qquad \partial_{\hat\varphi} := \frac{1}{\sin\theta}\partial_\varphi. \quad (107)$$

The system becomes

$$2P_{,u} - \mathcal{A}P_{,r} - \frac{1}{r}\left(Q_{\theta,\theta} + \frac{1}{\sin\theta}Q_{\hat\varphi,\varphi}\right) = \frac{2m}{r^2}(P-\psi) + \frac{1}{r}\cot\theta Q_\theta, \quad (108)$$

$$Q_{\theta,r} - \frac{1}{r}P_{,\theta} = -\frac{1}{r}Q_\theta, \quad (109)$$

$$Q_{\hat\varphi,r} - \frac{1}{r\sin\theta}P_{,\varphi} = -\frac{1}{r}Q_{\hat\varphi}, \quad (110)$$

$$\psi_{,r} = \frac{1}{r}(P-\psi). \quad (111)$$

The matrices C^u and C^r are diagonal and C^θ and C^φ are now symmetric. Replacing the coordinate θ by $y := -\cos\theta$ gets rid of the $\cot\theta$ term in D, see [19] (s there is $-y$ here).

However, a more elegant approach to both establishing the symmetry of the C^a and keeping track of Christoffel terms from covariant derivatives is to keep the equations covariant on S^2. From (103)–(105) we read off

$$j^u = 2P^2, \quad (112)$$

$$j^r = -\mathcal{A}P^2 + Q^aQ_a + \psi^2, \quad (113)$$

$$j^a = -\frac{2}{r}PQ^a, \quad (114)$$

$$S = \frac{4m}{r^2}P^2 - \frac{2}{r}(Q_aQ^a + \psi^2) + \frac{A}{r}P\psi. \quad (115)$$

It is easy to check that

$$j^u{}_{,u} + j^r{}_{,r} + \nabla_a j^a = S \quad (116)$$

holds if and only if (103)–(105) hold, where $\nabla_a j^a$ is the covariant divergence. This means that our integration measure dV must contain the covariant measure $d\Omega$ on the round two-sphere, or $d\Omega = \sin\theta d\theta d\varphi$ in the standard coordinates.

D. The vacuum Einstein equations, linearized in Bondi gauge about Schwarzschild

In [19], the metric is written as

$$ds^2 = -\frac{e^{2\beta}V}{r}du^2 - 2e^{2\beta}du\,dr + r^2h_{ab}(dx^a - U^a du)(dx^b - U^b du). \quad (117)$$

Keeping in mind that $y = -\cos\theta$ and $S := 1 - y^2 = \sin^2\theta$, we read off the identifications of the metric components of Secs. II and III given in Table II.

We denote the perturbations of V, β, h_{ab} and U^a about the vacuum Schwarzschild solution by $\tilde V, \tilde\beta, \tilde h_{ab}$ and $\tilde U^a$. (In contrast to [15], we have added the tildes on the perturbations $\tilde\beta$ and $\tilde U^a$ to distinguish them from the full variables.) We also replace $\tilde U_a$ and $\tilde V$ by

$$\tilde u_a := r\tilde U_a, \quad (118)$$

$$\tilde v := \frac{\tilde V}{r}. \quad (119)$$

TABLE II. Comparison of our notation for the nonlinear variables in Secs. II and III.

Sec. III	Sec. II	Schwarzschild
r	$R = x = r$	r
β	$\frac{1}{2}\ln G$	0
V	$\frac{rH}{G}$	$r - 2m$
$h_{\theta\theta}$	e^{2Sf}	1
$h_{\varphi\varphi}$	Se^{-2Sf}	S
$h_{\theta\varphi}$	0	0
U^θ	$-\sqrt{S}b$	0
U^φ	0	0

TABLE III. Comparison of our notation for the linear perturbations in Secs. II and III. We only list the independent perturbations present in twist-free axisymmetry, see (124), (125) for $\tilde{h}_{\varphi\varphi}$ and $P_{\varphi\varphi}$.

Sec. III	Sec. II	Toy model
$\tilde{\beta}$	$\frac{1}{2}\delta G$	$\frac{1}{2}\delta G$
$\tilde{v}$	$\delta H - \delta G$	$\frac{r^2}{2}\overline{\delta H} - r\delta G$
$\tilde{h}_{\theta\theta}$	$2S\delta f$	$\overline{\delta f}$
$\tilde{u}_\theta$	$-r\sqrt{S}\delta b$	$-\overline{\delta b}$
$P_{\theta\theta}$	$2S(r\delta f)_{,r}$	$r\overline{\delta f}_{,x}$
Q_θ	$-\sqrt{S}(r^2\delta b)_{,r} - \sqrt{S}\delta G_{,y}$	$-rQ$
T_θ	$\frac{1}{2}\sqrt{S}\delta G_{,y}$	$\frac{r}{2}T$
J_θ	$\sqrt{S}(2S\delta f)_{,y} - \sqrt{S}(r^2\delta b)_{,r}$	rJ

This makes all variables dimensionless, and therefore all lower-order terms become proportional to $2m/r^2$ or $1/r$. Our perturbation variables are summarised in Table III.

We introduce the reduction variables of [15], which are

$$P_{ab} := (r\tilde{h}_{ab})_{,r}, \tag{120}$$

$$Q_a := (r\tilde{u}_a)_{,r} - 2\tilde{\beta}_{,a}, \tag{121}$$

$$J_a := \nabla^b\tilde{h}_{ab} + (r\tilde{u}_a)_{,r}, \tag{122}$$

$$T_a := \tilde{\beta}_{,a}. \tag{123}$$

These comprise 8 first derivatives of the 6 metric perturbations.

Note that the linearization of the Bondi gauge condition $\det h_{ab} = \det q_{ab}$ is $q^{ab}\tilde{h}_{ab} = 0$, or in coordinates,

$$\tilde{h}_{\varphi\varphi} = -\sin^2\theta\tilde{h}_{\theta\theta}. \tag{124}$$

By definition, P_{ab} is also trace-free, and so

$$P_{\varphi\varphi} = -\sin^2\theta P_{\theta\theta}. \tag{125}$$

To complete the system with an evolution equation for $\tilde{v}$ while maintaining the symmetric hyperbolic form, we introduce the further variables

$$\mathcal{P}_a := \nabla^b P_{ab}, \tag{126}$$

$$\mathcal{Q} := \nabla^a Q_a, \tag{127}$$

$$\hat{\mathcal{Q}} := \epsilon^{ab}\nabla_a Q_b, \tag{128}$$

$$\mathcal{J} := \nabla^a J_a, \tag{129}$$

$$\hat{\mathcal{J}} := \epsilon^{ab}\nabla_a J_b, \tag{130}$$

$$\mathcal{U} := \nabla^a\tilde{u}_a, \tag{131}$$

$$\hat{\mathcal{U}} := \epsilon^{ab}\nabla_a\tilde{u}_b, \tag{132}$$

$$\mathcal{T} := \nabla^a T_a. \tag{133}$$

These comprise 2 additional first derivatives of the metric (namely $\mathcal{U}$ and $\hat{\mathcal{U}}$, for a total of 10) and 7 second derivatives of the metric. We move tensor indices $a, b, \ldots$ on S^2 with q_{ab} and q^{ab}, and note that this commutes with taking derivatives in u and r. ϵ_{ab} is the volume form on the unit 2-sphere, with defining properties

$$\nabla_a\epsilon_{bc} = 0, \qquad \epsilon_{ac}\epsilon^b{}_c = q_{ab}. \tag{134}$$

We can decompose the vector field Q_a in terms of potentials Q and $\hat{Q}$ that are determined (up to a constant) as solutions of the Poisson equation on the unit 2-sphere, as follows:

$$Q_a = \nabla_a Q - \epsilon_a{}^b\nabla_b\hat{Q}, \tag{135}$$

$$\nabla^a\nabla_a Q = \mathcal{Q}, \tag{136}$$

$$\nabla^a\nabla_a\hat{Q} = \hat{\mathcal{Q}}. \tag{137}$$

We can use (135) to show that

$$\begin{aligned} \nabla^b(\nabla_a Q_b + \nabla_b Q_a - q_{ab}\nabla^c Q_c) &= \nabla^b\nabla_b Q_a + Q_a \\ &= \nabla_a\mathcal{Q} - \epsilon_a{}^b\hat{\mathcal{Q}} + 2Q_a, \end{aligned} \tag{138}$$

where in both lines we have used

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)Q^b = -R_{ab}Q^b = -Q_a, \tag{139}$$

with $R_{ab} = q_{ab}$ the Ricci tensor on the unit round 2-sphere. The potentials Q and $\hat{Q}$ were introduced only to derive (138), and are not part of our system.

The evolution equations for the reduction variables already introduced in [15] are

$$2P_{ab,u} - \mathcal{A}P_{ab,r} + \frac{1}{r}\left(2\nabla_{(a}Q_{b)} - q_{ab}\nabla^c Q_c\right) = \frac{2m}{r^2}\left(P_{ab} - \tilde{h}_{ab}\right), \tag{140}$$

$$Q_{a,r} + \frac{1}{r}\nabla^b P_{ab} = \frac{1}{r}\left(J_a - Q_a - 2T_a + 2\tilde{u}_a\right), \tag{141}$$

$$J_{a,r} = \frac{1}{r}(2\tilde{u}_a - 4T_a), \tag{142}$$

$$\tilde{u}_{a,r} = \frac{1}{r}(Q_a + 2T_a - u_a). \tag{143}$$

$$T_{a,r} = 0, \tag{144}$$

$$\tilde{h}_{ab,r} = \frac{1}{r}(P_{ab} - \tilde{h}_{ab}), \tag{145}$$

$$\tilde{\beta}_{,r} = 0. \tag{146}$$

Equations (140)–(142) incorporate minor corrections of their counterparts in [15], see Appendix B. The evolution equations for our additional variables are

$$2\mathcal{P}_{a,u} - \mathcal{A}\mathcal{P}_{a,r} + \frac{1}{r}(\nabla_a \mathcal{Q} - \epsilon_a{}^b\nabla_b\hat{\mathcal{Q}}) = -\frac{2}{r}Q_a + \frac{2m}{r^2}(\mathcal{P}_a - J_a + Q_a + 2T_a), \tag{147}$$

$$\mathcal{Q}_{,r} + \frac{1}{r}\nabla^a\mathcal{P}_a = \frac{1}{r}(\mathcal{J} - \mathcal{Q} - 2\mathcal{T} + 2\mathcal{U}), \tag{148}$$

$$\hat{\mathcal{Q}}_{,r} + \frac{1}{r}\epsilon^{ab}\nabla_a\mathcal{P}_b = \frac{1}{r}(\hat{\mathcal{J}} - \hat{\mathcal{Q}} + 2\hat{\mathcal{U}}), \tag{149}$$

$$\mathcal{J}_{,r} = \frac{1}{r}(2\mathcal{U} - 4\mathcal{T}), \tag{150}$$

$$\hat{\mathcal{J}}_{,r} = \frac{2}{r}\hat{\mathcal{U}}, \tag{151}$$

$$\mathcal{U}_{,r} = \frac{1}{r}(\mathcal{Q} + 2\mathcal{T} - \mathcal{U}), \tag{152}$$

$$\hat{\mathcal{U}}_{,r} = \frac{1}{r}(\hat{\mathcal{Q}} - \hat{\mathcal{U}}), \tag{153}$$

$$\mathcal{T}_{,r} = 0, \tag{154}$$

$$\tilde{v}_{,r} = \frac{1}{r}\left(\frac{1}{2}\mathcal{J} - \mathcal{T} + \mathcal{U} + 2\tilde{\beta} - \tilde{v}\right). \tag{155}$$

The explicit matrices C^μ are given in Appendix B. For the more elegant covariant-on-S^2 approach to symmetric hyperbolicity, we define

$$X_{ab}{}^{cd} \coloneqq \frac{1}{2}\left(q_a{}^c q_b{}^d + q_b{}^c q_a{}^d - q_{ab}q^{cd}\right), \tag{156}$$

the projection operator into the space of symmetric trace-free 2-tensors on S^2. We can then write (140) and (141) as

$$2P_{ab,u} - \mathcal{A}P_{ab,r} + \frac{2}{r}X_{ab}{}^{cd}\nabla_d Q_c = \text{l.o.}, \tag{157}$$

$$Q_{c,r} + \frac{1}{r}X_{abc}{}^d\nabla_d P_{ab} = \text{l.o.}, \tag{158}$$

and the conserved current as

$$j^u \coloneqq 2\mathbf{P}^\dagger\mathbf{P} \tag{159}$$

$$j^r \coloneqq -\mathcal{A}\mathbf{P}^\dagger\mathbf{P} + \mathbf{Q}^\dagger\mathbf{Q} \tag{160}$$

$$j^d \coloneqq \frac{2}{r}X^{abcd}P_{ab}Q_c + \frac{1}{r}(q^{da}\mathcal{Q} + \epsilon^{da}\hat{\mathcal{Q}})P_a, \tag{161}$$

$$S = \frac{1}{r}(\ldots) + \frac{2m}{r^2}(\ldots), \tag{162}$$

where we have defined the shorthand

$$\mathbf{P}^\dagger\mathbf{P} \coloneqq \frac{1}{2}P_{ab}P^{ab} + P_aP^a \tag{163}$$

$$= P^2_{\theta\theta} + P^2_{\theta\hat{\varphi}} + P^2_\theta + P^2_{\hat{\varphi}}, \tag{164}$$

$$\mathbf{Q}^\dagger\mathbf{Q} \coloneqq Q_aQ^a + \cdots = Q^2_\theta + Q^2_{\hat{\varphi}} + \cdots, \tag{165}$$

The factor of $1/2$ in front of $P_{ab}P^{ab}$ compensates for double-counting of its algebraically independent components, see (125). The balance law(116) holds if and only if our first-order reduction of the linearized Einstein equations holds.

For brevity, we have not written out S in full. Our introduction of $\tilde{u}_a$ and $\tilde{v}$ has the advantage, relative to [15], that in the Minkowski case $m = 0$ $\mathcal{D}$ becomes $1/r$ times a matrix of integers, so all its eigenvalues take the form $\lambda_i(r) = \bar{\lambda}_i/r$, and $d = \bar{d}/r_0$, where $\bar{d}$ is the largest λ_i.

Looking at the evolution equations, in order to add $\tilde{v}$ to our system and then close it as first-order symmetric hyperbolic system we have effectively introduced a system of variables and equations that duplicates Frittelli's original system, but at one derivative higher. Geometrically, this is the level of curvature, rather than of the connection. However, our estimate includes, beside the 6 metric perturbations, only 10 first and 7 second derivatives of the metric,

TABLE IV. List of the quantities involved in our L^2 estimates, written out in terms of metric perturbations and their derivatives, and ordered by derivative of the metric and by left-moving variables **P** and right-moving variables **Q**. The quantities in the first line before the semicolon, the second line, and the third line before the semicolon were already introduced as variables in [15]. We have introduced the remaining quantities as variables to obtain a symmetric hyperbolic first-order system including all metric perturbations.

	P	**Q**
g	—	$\tilde\beta, \tilde h_{\theta\theta}, \tilde h_{\theta\hat\varphi}, \tilde u_a; \tilde v$
$\partial_r g$	$(r\tilde h_{ab})_{,r}$	$(r\tilde u_a)_{,r}$
∇g	—	$\tilde\beta_{,a}, \nabla^b\tilde h_{ab}; \nabla^a\tilde u_a, \epsilon^{ab}\nabla_a\tilde u_b$
$\partial_r\nabla g$	$\nabla^a(r\tilde h_{ab})_{,r}$,	$\nabla^a(r\tilde u_a)_{,r}, \epsilon^{ab}\nabla_a(r\tilde u_b)_{,r}$
$\nabla\nabla g$	—	$\nabla^a\nabla_a\tilde\beta, \nabla^a\nabla^b\tilde h_{ab}, \epsilon_c{}^b\nabla^a\nabla^c\tilde h_{ab}$

far short of the full set of Ricci rotation coefficients and null curvature components. These are listed in Table IV.

At both the level of the connection and the level of curvature, we have in effect two pairs of wave equations, but in different objects, (P_{ab}, Q_a), with $P_a{}^a = 0$, and $\mathcal{P}_a$, $\mathcal{Q}$, $\hat{\mathcal{Q}}$. Using (138) in (147) was essential for bringing the pair (147), (148) into a form similar to (140), (141), with the same matrices C^μ.

As the background solution is spherically symmetric, all perturbations can be split into polar and axial parts, where the axial perturbations change sign under a reflection of S^2, or reflection in space. All genuine scalars on S^2, in our case ψ, $\tilde V$ and $\tilde\beta$ are polar. Vectors on S^2 can be split into polar and axial parts as in (135) for the example of Q_a. Tracefree symmetric tensors can be split as, for example,

$$P_{ab} = \left(2\nabla_{(a}\nabla_{b)} - q_{ab}\nabla^c\nabla_c\right)P + 2\epsilon_{(a}{}^c\nabla_{b)}\nabla_c\hat P, \qquad (166)$$

in terms of a scalar P and pseudoscalar $\hat P$. Axial parts are hatted. Then axial and polar perturbations decouple from each other.

In twist-free axisymmetry, only the polar perturbations are present. We note in passing that substituting the solution $\tilde\beta = T_a = 0$ of $\tilde\beta_{,r} = 0$, and restricting the background solution to the Minkowski spacetime by setting $m = 0$, makes Eqs. (140) and (141) equivalent to Eqs. (D25), (D26) of Paper I.

We could also write the linear perturbation equations in terms of scalars and pseudoscalars only, in order to explicitly decouple polar and axial perturbations. This form of the system would not be symmetric hyperbolic, as all angular derivatives would appear in the form $\nabla^c\nabla_c$, but it could be made symmetric hyperbolic again by reintroducing first angular derivatives as reduction variables, as for the scalar wave equation. However, this would mean duplicating all vectors, for example defining $Q_a = \nabla_a Q$ to be a true vector (polar) and adding $\hat Q_a = \epsilon_{ab}\nabla^b\hat Q$ as a pseudovector (axial).

Looking back, to merely write the second-order Einstein equations first-order form with a minimum number of variables, one already has to introduce all the reduction variables (first derivatives of the metric) (120)–(123). This is true if $\tilde v$ is included or not. There is one reduction constraint

$$\nabla^b P_{ab} = \left(r\nabla^b\tilde h_{ab}\right)_{,r} = \left(r(J_a - Q_a - 2T_a)\right)_{,r}. \qquad (167)$$

The second derivative $(r\nabla^b\tilde h_{ab})_{,r}$ that can be thus written in two ways appears in only one place, and so there is a one-parameter family of first-order reductions with inequivalent principal parts. It turns out this contains the symmetric hyperbolic first-order form of the equations with the equation for $\tilde v$ excluded found by Frittelli. To bring the full system, with $\tilde v$ included, into a first-order symmetric hyperbolic form, we had to further add all the variables (126)–(133).

E. Estimates for characteristic initial value and initial-boundary value problems

We now use the symmetric hyperbolic form of the linearized Einstein equations in Bondi gauge to obtain energy estimates on control volumes of interest. We introduce the Schwarzschild time coordinate

$$t \coloneqq u + r_* \Rightarrow dt = du + dr_* = du + \mathcal{A}^{-1}dr, \qquad (168)$$

where r_* is the usual tortoise radius, and where we have defined the shorthand

$$\mathcal{A} \coloneqq 1 - \frac{2m}{r}. \qquad (169)$$

Following [15], we observe that

$$C^t \coloneqq C^\mu(dt)_{,\mu} = C^u + \mathcal{A}^{-1}C^r, \qquad (170)$$

where C^u and C^r are explicitly given in (B4) and (B5), is positive definite on $\mathbb{R}^N$ with smallest eigenvalue 1 for our system, independently of r, so for the smallest eigenvalue anywhere in V we have $c = 1$. Equivalently, with j^u and j^r given by (159) and (160), we have

$$j^t \coloneqq j^u + \mathcal{A}^{-1}j^r = \mathbf{P}^\dagger\mathbf{P} + \mathcal{A}^{-1}\mathbf{Q}^\dagger\mathbf{Q}, \qquad (171)$$

where $\mathbf{P}^\dagger\mathbf{P}$ and $\mathbf{Q}^\dagger\mathbf{Q}$ were defined in (163)–(165).

The estimates in [15,19,20] are on the control volume V_1 shown in Fig. 1: the product of S^2 with the triangle bounded by $u = u_0$, $r = r_0$, and $t = t_1$. We have

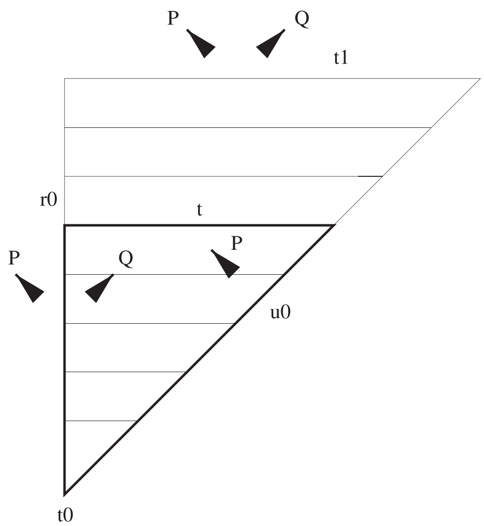

FIG. 1. Spacetime diagram of the control volume V_1, reduced by spherical symmetry, so that every point in the plot corresponds to a spacelike 2-sphere. Ingoing and outgoing spherical null surfaces are lines at 45 degrees. Horizontal lines represent surfaces of constant t, by which we slice V_1. V_1 is bounded by $r = r_0$ (left), $u = u_0$ (bottom right) and $t = t_1$ (top). The thicker line shows the volume V_t bounded by $t = t' < t_1$, which is needed for the estimation of $\mathcal{E}_t$ in (96). Arrows labeled $\mathbf{P}$ and $\mathbf{Q}$ symbolize energy fluxes. The estimate on V_1 is (176).

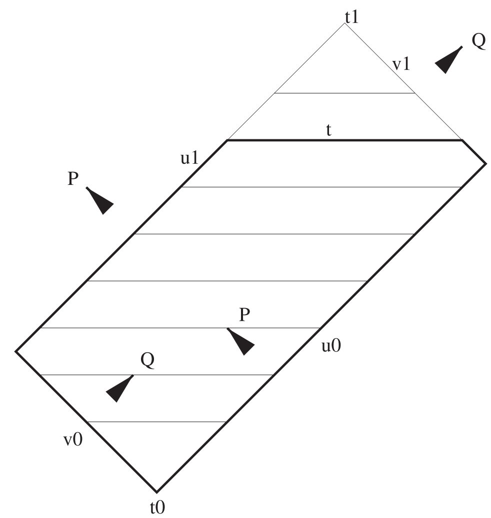

FIG. 2. Spacetime diagram of the control volume V_2, bounded by $v = v_0$ (bottom left), $u = u_0$ (bottom right), $u = u_1$ (top left) and $v = v_1$ (top right). Otherwise as in Fig. 1. The estimate on V_2 is (179).

$$-\int_{u=u_0} j^u - \int_{r=r_0} j^r + \int_{t=t_1} j^t = \int_{V_1} S. \tag{172}$$

The signs come from du at $u = u_0$ and dr at $r = r_0$ pointing into V, and dt at $t = t_1$ pointing out. We will ignore the flux out of $r = r_0$ (as is standard practice for initial-boundary value problems), and so

$$\|\mathrm{in}\|^2 = \int_{u=u_0} 2\mathbf{P}^\dagger\mathbf{P} + \int_{r=r_0} \mathbf{Q}^\dagger\mathbf{Q}, \tag{173}$$

$$\|\mathrm{out}'\|^2 = \int_{t=t_1} \mathbf{P}^\dagger\mathbf{P} + \mathcal{A}^{-1}\mathbf{Q}^\dagger\mathbf{Q} \tag{174}$$

$$\|\mathrm{out}''\|^2 = \int_{r=r_0} \mathcal{A}\mathbf{P}^\dagger\mathbf{P}. \tag{175}$$

Hence, from the general formula (100), and with $c = 1$, our estimate is

$$\int_{t=t_1} \left(\mathbf{P}^\dagger\mathbf{P} + \mathcal{A}^{-1}\mathbf{Q}^\dagger\mathbf{Q}\right) \le e^{d(r_0)(t_1-t_0)}\left(\int_{u=u_0} 2\mathbf{P}^\dagger\mathbf{P} + \int_{r=r_0} \mathbf{Q}^\dagger\mathbf{Q}\right). \tag{176}$$

Recall that d is defined as the largest positive eigenvalue of $\mathcal{D}$ in V. All elements of $\mathcal{D}$ are constants times $1/r$ or $2m/r^2$, so d depends only on r_0, the smallest value of r in V, and we have written $d = d(r_0)$ to emphasize this. On the Minkowski background, the $2m/r^2$ terms are absent, and so $d = \bar{d}/r_0$, where $\bar{d}$ is the largest eigenvalue of $r\mathcal{D}$ (with $m = 0$).

A second control volume of interest is shown in Fig. 2: the product of S^2 with the null rectangle triangle bounded by $u = u_0$ and $v = v_0$, and $u = u_1$ and $v = v_1$. The null coordinate v on the Schwarzschild background is defined by

$$v \coloneqq u + 2r_* \Rightarrow dv = du + 2\mathcal{A}^{-1}dr, \tag{177}$$

and hence

$$j^v \coloneqq j^u + 2\mathcal{A}^{-1}j^r = 2\mathcal{A}^{-1}\mathbf{Q}^\dagger\mathbf{Q}. \tag{178}$$

The corresponding estimate is

$$\int_{u=u_1} 2\mathbf{P}^\dagger\mathbf{P} + \int_{v=v_1} 2\mathcal{A}^{-1}\mathbf{Q}^\dagger\mathbf{Q} \le e^{d(r_0)(t_1-t_0)}\left(\int_{u=u_0} 2\mathbf{P}^\dagger\mathbf{P} + \int_{v=v_0} 2\mathcal{A}^{-1}\mathbf{Q}^\dagger\mathbf{Q}\right). \tag{179}$$

A third control volume of interest is shown in Fig. 3: the product of S^2 with the null right trapezoid bounded by $u = u_0$, $r = r_0$, and $u = u_1$ and $v = v_1$. Again we will ignore the flux out of $r = r_0$. The corresponding estimate is

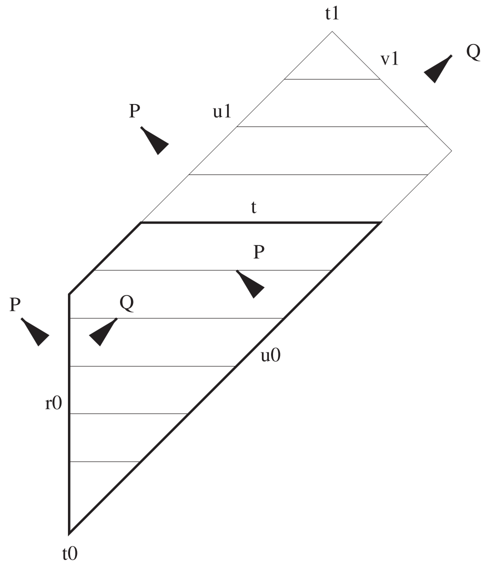

FIG. 3. Spacetime diagram of the control volume V_3, bounded by $r = r_0$ (left), $u = u_0$ (bottom right), $u = u_1$ (top left) and $v = v_1$ (top right). Otherwise as in Fig. 1. The estimate on V_3 is (180).

$$\int_{u=u_1} 2\mathbf{P}^\dagger\mathbf{P} + \int_{v=v_1} 2\mathcal{A}^{-1}\mathbf{Q}^\dagger\mathbf{Q} \leq e^{d(r_0)(t_1-t_0)}\left(\int_{u=u_0} 2\mathbf{P}^\dagger\mathbf{P} + \int_{r=r_0} \mathbf{Q}^\dagger\mathbf{Q}\right). \quad (180)$$

A hypothetical fourth estimate would be (180) with $m = 0$ and $r_0 = 0$, that is the pure null initial-value problem on an outgoing cone with regular vertex at $r = 0$. (We need to set $m = 0$ so the background has a regular center). This is the PDE problem we have been considering in Papers I and II in this series. Regularity on the central worldline $r = 0$ requires that all metric variables take their Minkowski values. Hence their perturbations, and in particular the right-moving perturbations $\mathbf{Q}$ vanish at $r = 0$. Independently, $r = 0$ is no longer a boundary. Hence the last term in (180) would be absent.

However, we recall that on Minkowski spacetime $d(r_0) = \bar{d}/r_0$, with $\bar{d} \simeq 5.0 > 0$. Hence $\exp d(r_0)(t_1 - t_0)$ grows arbitrarily rapidly with t as $r_0 \to 0$, so the limit $r_0 \to 0$ of (180) as written does not exist. This problem already arises for the scalar wave equation on Minkowski. In Appendix D we attempt some simple ways around this problem, and show that one of them works for the scalar wave equation on Minkowski, but none work for the metric perturbations. In Appendix E we carry out some numerical tests of the hypothetical estimate on a regular null cones.

IV. CONCLUSIONS

In the Introduction, we motivated the desirability of a well-posedness proof for formulations of the Einstein equations on null cones, not just geometrically, but for specific formulations of the Einstein equations that are used in numerical relativity. These formulations use "Bondi-like" coordinates, where coordinate lines of constant (u, θ, φ) are outgoing null rays, and where we evolve only free data and solve ODEs along the null rays to reconstruct the full metric on each time slice.

One of these formulations, using Bondi coordinates, has been used successfully for Cauchy-characteristic matching [24] in full generality, see for example [14]. This and the incomplete well-posedness result of [15] appeared to be in tension with recent results [11,16] that showed that minimal first-order reductions of this and similar formulations are not strongly hyperbolic, and which we have verified in Sec. II.

Our tentative resolution is that the characteristic initial-boundary value problem is well-posed in L^2 of the metric plus *selected* first and second derivatives. This is the "skewed" norm whose existence was conjectured in [11,16,17]. In Sec. III, we have proved this for the linearization of Bondi gauge about Schwarzschild, for the null initial-boundary value problem and the double null initial value problem.

Based on this proof, we conjecture that (1) the initial-boundary value problem is well-posed for the linearization about an arbitrary background; (2) this holds for any Bondi-like gauge; (3) this holds also for the initial value-problem on a null cone with regular vertex.

We expect that generalizing the symmetric hyperbolic form of the linearized Einstein equations to an arbitrary background solution can be done because we expect all additional terms to be of lower order. The existence of a symmetric hyperbolic form of the linearized Einstein equations in other gauges seems plausible because they can be written as two wave equations coupled to transport equations along the outgoing null cones, coupled only through lower-order terms.

The third conjecture appears to be the most challenging. Unfortunately, the methods of [15,19] that we have used for the estimates above do not allow an estimate for the initial value problem with initial data on a regular null cone. Chrusciel has given an existence proof for solutions of the Einstein equations with initial data specified on a null cone with regular vertex [7]. This uses harmonic coordinates, relying on results of Dossa [25] for quasilinear wave equations. Those proofs suggest that we may need to split off the lowest powers of r and corresponding lowest spherical harmonics Y_{lm} as an approximate solution (resulting in a polynomial in x and y), and control only the remainder in an energy norm.

For a proof of well-posedness of Cauchy-characteristic *matching* along the lines of the present paper to succeed, the set of variables that are passed between the Cauchy code and the null code probably has to coincide with the set of variables that appear at the matching surface in the separate well-posedness estimates on both sides.

Setting aside the difficulties with proving well-posedness, one may wonder if the linearized Einstein equations in Bondi coordinates on null cones emanating from a regular center are actually already well-posed in the norm $\int \mathbf{P}^\dagger \mathbf{P} dS$ on those null cones, and only our estimates for the lower-order terms are not sharp enough to see this. Numerical experiments with the code of [13,18] are described in Appendix E. While these cannot of course prove stability, they seem to be compatible with it.

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APPENDIX A: A TEXTBOOK EXAMPLE OF A WEAKLY HYPERBOLIC SYSTEM

A textbook example of a PDE system that is weakly but not strongly hyperbolic is given in [26], pp. 29–30:

$$\phi_{,t} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \phi_{,x}, \qquad \phi \coloneqq \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{A1}$$

It has the general solution

$$\phi(t,x) = \int_{-\infty}^{\infty} \left[a_{(k)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b_{(k)} \begin{pmatrix} ikt \\ 1 \end{pmatrix} \right] e^{ik(x+\lambda t)}\, dk. \tag{A2}$$

Clearly this is not well-posed in the L^2 norm $\int (\phi_1^2 + \phi_2^2) dx$ because the ikt term grows arbitrarily rapidly in t for k arbitrarily large. The very simplest system with this property occurs for $\lambda = 0$, so that $\phi_{2,x} = 0$ and $\phi_{1,t} = \phi_{2,x}$.

Such systems are called "weakly well-posed," which means they are not well-posed in L^2 but can be made well-posed if a higher derivative norm is used for the initial data than for the solutions, see [26], p. 39. In this example, we would include $\phi_{2,x}^2$ in the norm at $t = 0$ only, or we could introduce an additional variable $\phi_3 \coloneqq \phi_{2,x}$. They can become completely ill posed if lower-order terms are included, see [26], p. 40 for an example.

We give this example to make two points: (1) when a plane-wave ansatz does not give all the solutions of a linear PDE system with constant coefficients, we should look for additional solutions that are polynomial; (2) the problem with polynomial solutions is not that they grow but that they can grow arbitrarily rapidly in time for initial data that oscillate arbitrarily rapidly in space. (In our null toy problem, read x for "time" and $\bar{y}$ for "space.")

APPENDIX B: NOTES ON [15]

Equation (140) corrects the right-hand side of (15a) of [15] by a factor of $-1/2$. This error occurs already between the nonlinear field equation (8) of [15] and its linearization (11a).

Equation (141) corrects the right-hand side of (15b) of [15] by the addition of $Q_a + 2T_a$. Equation (142) corrects the right-hand side of (15e) of [15], by the subtraction of $Q_a + 2T_a$. These errors occurs between (11c) and (15b) and (15e), respectively. Note there is no error between (6b) and its linearization (11c).

The last two errors are related because the definitions (120) and (122) give rise to the integrability condition

$$\nabla^b P_{ab} = (r\nabla^b \tilde{h}_{ab})_{,r} = \big(r(J_a - Q_a - 2T_a)\big)_{,r}, \tag{B1}$$

or, using (144),

$$rQ_{a,r} + \nabla_b P_{ab} = rJ_{a,r} + J_a - Q_a - 2T_a. \tag{B2}$$

Using the correct expression (142) for $rJ_{a,r}$ then gives the correct equation (141) for $Q_{a,r}$.

(155) is the covariant form of Eq. (15c) of [15]. A factor of 2 is missing from the last two terms of Eq. (12) of [15], but is restored in Eq. (15c).

A second set of minor corrections concerns the explicit matrices C^μ. With $\phi^\dagger \coloneqq (P_{ab}, Q_a, \ldots)$, Frittelli states that $C^u = \mathrm{diag}(1, 1, 0\ldots, 0)$ and $C^r = \mathrm{diag}(-\mathcal{A}, 1, 1, \ldots, 1)$, but this cannot be true as stated for any choice of normalization of the equations.

The matrices C^a are not given explicitly, but become symmetric, without terms $\sin^2\theta$ appearing in C^u and C^r, only if one introduces frame components on the 2-sphere as in (107). With the variables in the order

$$\phi^\dagger = \big(P_{\theta\theta}, P_{\theta\hat{\varphi}}, Q_\theta, Q_{\hat{\varphi}}, \mathcal{P}_\theta, \mathcal{P}_{\hat{\varphi}}, \mathcal{Q}, \hat{\mathcal{Q}}, \ldots\big), \tag{B3}$$

where the dots stand for all other variables, in any order, then with our normalization of the equations the matrices are

$$C^u = \mathrm{diag}\big(C_4^u, C_4^u, 0, \ldots, 0\big), \tag{B4}$$

$$C^r = \mathrm{diag}\big(C_4^r, C_4^r, 1, \ldots, 1\big), \tag{B5}$$

$$C^\theta = \mathrm{diag}\big(C_4^\theta, C_4^\theta, 0, \ldots, 0\big), \tag{B6}$$

$$C^\varphi = \frac{1}{\sin\theta}\mathrm{diag}\big(C_4^{\hat{\varphi}}, C_4^{\hat{\varphi}}, 0, \ldots, 0\big), \tag{B7}$$

where we have defined

$$C_4^u = \mathrm{diag}(2, 2, 0, 0), \tag{B8}$$

$$C_4^r = \mathrm{diag}(-\mathcal{A}, -\mathcal{A}, 1, 1), \tag{B9}$$

$$C_4^\theta := \frac{1}{r}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{B10}$$

$$C^{\hat\varphi} := \frac{1}{r}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{B11}$$

APPENDIX C: SYMMETRIC HYPERBOLIC FORM OF THE TOY MODEL

To link the considerations of Sec. II and Sec. III, in this Appendix we construct a symmetric hyperbolic reduction of the toy model of Sec. II in Bondi gauge, using the methods of Sec. III.

We introduce the reduction variables

$$P := \overline{\delta f}_{,x}, \tag{C1}$$

$$Q := \overline{\delta b}_{,x} + \delta G_{,\bar y}, \tag{C2}$$

$$J := \overline{\delta f}_{,\bar y} - \overline{\delta b}_{,x}, \tag{C3}$$

$$T := \delta G_{,\bar y}, \tag{C4}$$

$$L := J_{,y} - T_{,y} \tag{C5}$$

Here P, Q, J and T are closely related to P_{ab}, Q_a, J_a and T_a of Sec. III. They do not carry indices because of the restriction to twist-free axisymmmetry.

A symmetric hyperbolic first-order reduction of the system (53)–(56) in terms of these variables is

$$2P_{,u} - P_{,x} - Q_{,\bar y} = 0, \tag{C6}$$

$$Q_{,x} - P_{,\bar y} = 0, \tag{C7}$$

$$\overline{\delta b}_{,x} = Q - T, \tag{C8}$$

$$J_{,x} = 0, \tag{C9}$$

$$T_{,x} = 0, \tag{C10}$$

$$\overline{\delta f}_{,x} = P, \tag{C11}$$

$$\delta G_{,x} = 0, \tag{C12}$$

$$L_{,x} = 0, \tag{C13}$$

$$\overline{\delta H}_{,x} = L, \tag{C14}$$

in the variables

$$\phi^\dagger := \left(P, Q, \overline{\delta b}, J, T, \overline{\delta f}, \delta G, L, \overline{\delta H}\right), \tag{C15}$$

The three matrices C^μ are symmetric and $C^t := C^u + C^x$ is positive definite.

The bad solution $\overline{\delta b} = -\delta G_{0,\bar y} x$ of the toy model becomes $\overline{\delta b} = -T_0 x$ in the symmetric-hyperbolic form, and the bad solution $\overline{\delta H} = \delta f_{0,\bar y\bar y} x$ becomes $\overline{\delta H} = L_0 x$. To see that the linear growth in x is by itself in conflict with our estimates, consider a toy model of the toy model, namely

$$T_{,x} = 0, \tag{C16}$$

$$b_{,x} = -T, \tag{C17}$$

where for simplicity T and b depend only on u and x. The general solution is

$$T = T_0(u), \tag{C18}$$

$$b = b_0(u) - T_0(u)(x - x_0). \tag{C19}$$

Comparing to our general framework for estimates, we see that $c = 1$, $d = 1$, $\mathbf{Q} := (T, b)$ and $\mathbf{P}$ is absent. Hence the estimate (176) reduces to

$$\int_{t=t_1} \mathbf{Q}^\dagger \mathbf{Q} \le e^{t_1 - t_0} \int_{x=x_0} \mathbf{Q}^\dagger \mathbf{Q}. \tag{C20}$$

We can use $u = t - x$, $u_1 = t_1 - x_0$, $u_0 = t_0 - x_0$ to write both sides as integrals over u, namely

$$\int_{t=t_1} \mathbf{Q}^\dagger \mathbf{Q} = \int_{u_0}^{u_1} \left[T_0(u)^2 + b_0(u)^2\right] du, \tag{C21}$$

$$\int_{x=x_0} \mathbf{Q}^\dagger \mathbf{Q} = \int_{u_0}^{u_1} \left[T_0(u)^2 + \left(b_0(u) - T_0(u)(u_1 - u)\right)^2\right] du. \tag{C22}$$

One can show that

$$\frac{\int_{t=t_1} \mathbf{Q}^\dagger \mathbf{Q}}{\int_{x=x_0} \mathbf{Q}^\dagger \mathbf{Q}} \le 1 + \frac{\Delta^2}{2}\left(1 + \sqrt{1 + \frac{\Delta^2}{4}}\right), \tag{C23}$$

$$\le e^{t_1 - t_0} \quad \text{for } t_1 \ge t_0, \tag{C24}$$

where $\Delta := u_1 - u_0 = t_1 - t_0$, and the first inequality is sharp. We should stress again that this is only a toy model, similar to the one considered in Sec. 6.4 of [27].

APPENDIX D: ATTEMPTS TO EXTEND THE ESTIMATES TO $r_0 = 0$

We attempt to overcome the problem that on the Minkowski background $d = \bar{d}/r$ with $\bar{d} > 0$, which means that we cannot take $r_0 \to 0$ in any of our estimates. In this Appendix we try three simple ways of overcoming this, first by a change of integration measure dV, as in Eq. (86). The most general weight compatible with the time translation and rotation symmetries of the Schwarzschild or Minkowski background is

$$dV = du\,\omega(r)dr\,d\Omega, \tag{D1}$$

where $d\Omega = \sin\theta d\theta d\varphi$ is the integration weight on S^2 induced by the round unit metric q_{ab}. The choice $\omega(r) = r^2$ gives dV induced by the background Schwarzschild metric $g_{\mu\nu}$.

From (86) we see that both d and c acquire a factor of ω, so ω cancels out of the ratio d/c that appears in our estimates. This leaves us with the addition of C^r to play with. For simplicity, we make the ansatz $\omega(r) = r^p$, so that (87) becomes

$$S_\omega = S + \frac{p}{r} j^r = \phi^\dagger \left(\mathcal{D} + \frac{p}{r} C^r\right)\phi. \tag{D2}$$

For the wave equation on Minkowski, we find that $\bar{d} = 0$ for $p = 1$ only, while it is positive for all other p. It is also positive on Schwarzschild for all $r < \infty$, for all p. So for the wave equation on Minkowski only, the unique choice $\omega = r$ gives us $d = 0$.

For the linearized Einstein equations on the Minkowski background, we find that the largest eigenvalue $\bar{d}$ of $r\mathcal{D} + pC^r$ has a minimum of $\bar{d} \simeq 3.4$ at $p \simeq -2.2$, that is, we cannot make $\bar{d}$ nonpositive with any choice of p. Hence we cannot make $\bar{d}$ nonpositive in this way.

For a second attempt, we observe that, with $\tilde{\beta}_{,r} = 0$ and $\tilde{\beta} = 0$ at regular center, β and its angular derivatives T_a and $\mathcal{T}$ vanish identically, as they must vanish at the center. If we experimentally take them out of the system, then $\bar{d} \simeq 3.1$. If we additionally bring $\omega(r) = r^p$ into play, $\bar{d}$ has a minimum of $\bar{d} \simeq 2.6$ at $p \simeq -1.2$. Again, this is not enough to make $\bar{d}$ nonpositive.

For a third attempt, we attempt to change the ratio d/c by a linear recombination of variables as in Eq. (77). We have already seen that the eigenvalues of $\mathcal{D}$ and C^t can change if A is not orthogonal. For the full system we cannot try out all possible matrices A by brute force, but we can for the scalar wave equation on Minkowski, as we shall see now.

From (108)–(111) we read off that on Minkowski $C^t \coloneqq C^u + C^r$ is the unit matrix, while

$$\mathcal{D} = D + D^\dagger = \frac{1}{r}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}. \tag{D3}$$

We have $\bar{d} = \sqrt{2} - 1 \simeq 0.4$. We have also just seen that with $p = 1$ we can make this $\bar{d} = 0$. However, we now focus on linear recombinations of the variables other than by an overall r-dependent factor.

Geometrically, it makes no sense to mix the components of the vector Q_a either with each other or with the scalars P and ψ, and we can fix an overall factor in A by leaving them completely unchanged, so A acts nontrivially only on the pair (P, ψ). Hence we can assume

$$A = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma & 0 & 0 & \delta \end{pmatrix}. \tag{D4}$$

Two of the eigenvalues $\tilde{\mathcal{D}} \coloneqq A^\dagger \mathcal{D} A$ are then always $-2/r$ for any A. As they are negative, this is not a problem. Furthermore, if we write $A = \bar{A}R$ where R is orthogonal, R does not change the eigenvalues of $\tilde{C}^t$ and $\tilde{\mathcal{D}}$. Hence we can choose the rotation R between P and ψ such that $\gamma = 0$, and without loss of generality we can then rescale ψ to set $\delta = 1$. The nontrivial eigenvalues of $\tilde{\mathcal{D}}$ are now $(\beta - 1) \pm \sqrt{(\beta - 1)^2 + \alpha^2}$. We have already used up the rotation between P and ψ that would allows us to set $\alpha = 0$, and hence one of the eigenvalues of $\tilde{\mathcal{D}}$ is always positive.

In summary, none of three relatively trivial ways of changing d/c in the estimate manage allows us to make it nonpositive, and as d/c is always proportional to $1/r$, we cannot let $r_0 \to 0$ in estimates based on our symmetric hyperbolic reduction.

APPENDIX E: NUMERICAL EXPERIMENTS

We have not been able to obtain an estimate (for the linearized Einstein equations) for the PDE problems we have been solving numerically in [13,18], where our time slices are outgoing null cones *emanating from a regular center*. The reason for this was that the factor $\exp d(r_0)(t_1 - t_0)$ in our estimates diverges as $r_0 \to 0$. However, we did not actually notice an instability in our code. In this appendix, we test the code a little harder by running it with small amplitude but random initial data, and see if there is a discrete norm that remains bounded in by its initial value. The control volume is as in Fig. 3, but with $r_0 = 0$, understood as a regular center.

We now construct a hypothesis to test by modifying the estimate (180) as follows. $r = 0$ is not actually a boundary, but an interior world line, and no free data can be imposed there, so the term at $r = r_0$ disappears. In the setup of our code we are not interested in controlling the energy leaving the control volume through $v = v_1$, so we drop that term from the estimate (180). The hypothesis we want to test is that the function norms on $u = 0$ and $u = u_1$ are the ones that we derived but that an unknown function $K(u_1 - u_0)$ takes the place of $\exp d(r_0)(u_1 - u_0)$. Hence our hypothesis is

$$\int_{u=u_1} \mathbf{P}^\dagger \mathbf{P} \le K(u_1 - u_0) \int_{u=u_0} \mathbf{P}^\dagger \mathbf{P}, \tag{E1}$$

where $\mathbf{P}^\dagger \mathbf{P}$ was defined above in (163).

Restricting to vacuum in twist-free axisymmetry, and translated into the notation of our code, the two terms in $\mathbf{P}^\dagger \mathbf{P}$ are

$$\frac{1}{2} P^{ab} P_{ab} \coloneqq \frac{1}{2} q^{ac} q^{bd} (r\tilde{h}_{ab})_{,r} (r\tilde{h}_{cd})_{,r} \tag{E2}$$

$$= (r\tilde{h}_{\theta\theta})^2_{,r} \tag{E3}$$

$$= 4S^2 (rf)^2_{,r} \tag{E4}$$

and

$$P^a P_a \coloneqq q^{ab} (r\nabla^c \tilde{h}_{ac})_{,r} (r\nabla^d \tilde{h}_{bd})_{,r} \tag{E5}$$

$$= \left[r(\tilde{h}_{\theta\theta,\theta} + 2\cot\theta \tilde{h}_{\theta\theta})\right]^2_{,r} \tag{E6}$$

$$= 4S\left[S(rf)_{,ry} - 8y(rf)_{,r}\right]^2. \tag{E7}$$

(Recall $y \coloneqq -\cos\theta$ and $S \coloneqq 1 - y^2$.) Note, however, that δH forms part of the variables $\mathbf{Q}$, that these are not coming in through $r = 0$, and we do not control their leaving through $v = v_1$. Hence instead of our full estimate we can use Frittelli's truncated estimate, with only the term (E4) in the integrand.

In the linearized equations, the scalar matter field decouples from the metric perturbations, and we conjecture that it obeys a similar estimate with integrand

$$P^2 \coloneqq (r\psi)^2_{,r}. \tag{E8}$$

Our integration measure in axisymmetry is

$$\int dS = \int d\Omega\, dR = 2\pi \int_{-1}^{1} dy \int_0^{x_{\max}} R_{,x}\, dx, \tag{E9}$$

and we now drop the factor 2π. Our code requires the outer boundary to be null or future spacelike, so we cannot literally run in Bondi gauge $R = x$. However, running in lsB gauge and setting $x_0 = x_{\max}$ will turn the outer boundary $x = x_{\max}$ into an ingoing null In weak gravity, this will result in $R = R(u,x) \simeq c(u)x$, with $c(u)$ determined so that $x = x_0$ is ingoing null. We can then identify $x = x_0$ with $v = v_1$. See [18] for details.

To test our conjecture, we therefore evaluate the norms (at time u) defined by

$$\|(r\psi)_{,r}\|^2 \coloneqq \int_{-1}^{1} dy \int_0^{x_{\max}} \frac{[(R\psi)_{,x}]^2}{R_{,x}} dx, \tag{E10}$$

$$\|(rf)_{,r}\|_1^2 \coloneqq \int_{-1}^{1} (1 - y^2) dy \int \ldots f \ldots \tag{E11}$$

and their discretizations

$$\|(r\psi)_{,r}\|_1^2 \simeq 2 \sum_{j=1}^{N_y} A^{(0)}_{l=0,j} \sum_{i=1}^{N_x} \frac{(R_i \psi_{i,j} - R_{i-1}\psi_{i-1,j})^2}{R_i - R_{i-1}}, \tag{E12}$$

$$\|(rf)_{,r}\|_1^2 \simeq 2 \sum A^{(0)}_{l=0,j} (1 - y_j^2) \sum \ldots f \ldots, \tag{E13}$$

where R_i has only one index as $R = R(u,x)$ in lsB gauge as R does not depend on y. We have also used that $1/2 \int \psi dy$ is the $l = 0$ component of ψ, and have used the analysis matrix $A^{(0)}$ of our pseudospectral framework to determine it. Our radial grid starts at $i = 1$, but in the formulas we use $R = 0$ at $i = 0$. See again [18] for details.

We have evolved with noise of amplitude 10^{-10} in f and ψ, at $N_x = 64 \ldots 1024$ and $N_y = 17 \ldots 65$. We set $x_{\max} = 1.1$ and $x_0 = 1$, and evolve to $u = 0.95$, when the range of R has gone down from its initial value of 0.5 by a factor of $1 - 0.95 = 0.05$. We do not filter out high frequencies, other than the components of f with $l = N_x + 1$ and N_x (see [18] for why we do this.) At each x_i, we also set all spherical harmonic components with $l > 2i$ to zero (see again [18] for why we do this.)

The discrete L^2 norms of ψ and f themselves do not decrease with u, but the discretized L^2 norms $\|(r\psi)_{,r}\|$ and $\|(rf)_{,r}\|_1$ given in (E12), (E13) do. From about $u = 0.01$, both decrease approximately as $u^{-0.35}$. Near the end, $\|(r\psi)_{,r}\|$ decreases much more rapidly, while $\|(rf)_{,r}\|$ increases a bit in an apparently random manor, then decreases again. In summary, our numerical experiments are easily compatible with the estimate (E1) for $K(u_1 - u_0) = 1$ and including only $P^{ab} P_{ab}$, that is

$$\int_{u=u_1} (rf)^2_{,r} dr\, dy \le \int_{u=u_0} (rf)^2_{,r} dr\, dy, \tag{E14}$$

and a similar estimate with ψ instead of f.

Looking at individual spherical harmonic components $f_l(u,x)$ and $\psi_l(u,x)$ of $f(u,x,y)$ and $\psi(u,x,y)$, we see that the random initial dominated by the grid frequency data quickly smooth out into well-resolved data on frequencies lower than the grid frequency. Hence our method seems to be quite dissipative. In a second phase we then see that these appear to be "stretched" in x, as the grid in fact zooms in on them, with $R_{\max}$ shrinking linearly in u.

For definiteness, we have implemented a specific continuum norm and its discretization, but we should stress that this was just an informed guess. In particular, we have arbitrarily chosen $dV = d\Omega dR$, rather than, for example, $dV = d\Omega R^p dR$. The "correct" choice would depend on what (if any) estimate can be proved. We have also not gone to particularly large values of N_x and N_y, but only ones that we have also used in physics simulations. (A soft upper limit on N_x is set by computing time, while a hard limit $N_y \leq 128$ is set by the accuracy of our spectral method in y.) Hence our results should only be considered as a slightly more challenging stability test of our code motivated by the results of this paper, not a numerical test of well-posedness.

[1] Y. Fourès-Bruhat, Théorème d'existence pour certains systèmes d'èquations aux dérivées partielles non linéaires, Acta Math. **88**, 141 (1952); see also Gen. Relativ. Gravit. **54**, 35 (2022) for an English translation.

[2] F. Pretorius, Evolution of binary black-hole spacetimes, Phys. Rev. Lett. **95**, 121101 (2005).

[3] H. Friedrich, On the hyperbolicity of Einstein's and other gauge field equations, Commun. Math. Phys. **100**, 525 (1985).

[4] C. Gundlach, G. Calabrese, I. Hinder, and J. M. Martín-García, Constraint damping in the Z4 formulation and harmonic gauge, Classical Quantum Gravity **22**, 3767 (2005).

[5] A. D. Rendall, Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations, Proc. Roy. Soc. A **427**, 1872 (1990).

[6] J. Luk, On the local existence for the characteristic initial value problem in general Relativity, Int. Math. Res. Not. **2012**, 4625 (2012).

[7] P. T. Chruściel, The existence theorem for the general relativistic Cauchy problem on the light-cone, Forum Math. Sigma **2**, e10 (2014).

[8] P. T. Chruściel and J. Jezierski, On free general relativistic initial data on the light cone, J. Geom. Phys. **62**, 578 (2012).

[9] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems, Proc. Roy. Soc. A **269**, 21 (1962).

[10] R. K. Sachs, Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time, Proc. R. Soc. A **270**, 103 (1962).

[11] T. Giannakopoulos, D. Hilditch, and M. Zilhão, Hyperbolicity of general relativity in Bondi-like gauges, Phys. Rev. D **102**, 064035 (2020).

[12] F. Siebel, J. A. Font, E. Müller, and P. Papadopoulos, Axisymmetric core collapse simulations using characteristic numerical relativity, Phys. Rev. D **67**, 124018 (2003).

[13] C. Gundlach, T. W. Baumgarte, and D. Hilditch, preceding paper, Simulations of gravitational collapse in null coordinates: II. Critical collapse of an axisymmetric scalar field, Phys. Rev. D **110**, 024019 (2024).

[14] S. Ma, J. Moxon, M. A. Scheel, K. C. Nelli, N. Deppe, M. S. Bonilla, L. E. Kidder, P. Kumar, G. Lovelace, W. Throwe, and N. L. Vu, Fully relativistic three-dimensional Cauchy-characteristic matching, arXiv:2308.10361.

[15] S. Frittelli, Well-posed first-order reduction of the characteristic problem of the linearized Einstein equations, Phys. Rev. D **71**, 024021 (2005).

[16] T. Giannakopoulos, N. T. Bishop, D. Hilditch, D. Pollney, and M. Zilhão, Gauge structure of the Einstein field equations in Bondi-like coordinates, Phys. Rev. D **105**, 084055 (2022).

[17] T. Giannakopoulos, N. T. Bishop, D. Hilditch, D. Pollney, and M. Zilhão, Numerical convergence of model Cauchy-characteristic extraction and matching, Phys. Rev. D **108**, 104033 (2023).

[18] C. Gundlach, D. Hilditch, and T. W. Baumgarte, this issue, Simulations of gravitational collapse in null coordinates: I. Formulation and weak field tests in Bondi-like gauges, Phys. Rev. D **110**, 024018 (2024)..

[19] S. Frittelli, Estimates for the characteristic problem of the first-order reduction of the wave equation, J. Phys. A **37**, 8639 (2004).

[20] S. Frittelli, Estimates for first-order homogeneous linear characteristic problems, J. Phys. A **38**, 4209 (2005).

[21] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations*, 2nd ed. (Springer, New York, 2004).

[22] B. Gustafsson, H.-O. Kreiss, and J. Oliger, *Time-Dependent Problems and Difference Methods* (Wiley, New York, 1995).

[23] R. Gómez, P. Papadopoulos, and J. Winicour, Null cone evolution of axisymmetric vacuum spacetimes, J. Math. Phys. (N.Y.) **35**, 4184 (1994).

[24] J. Winicour, Characteristic evolution and matching, Living Rev. Relativity **15**, 2 (2012).

[25] M. Dossa, Espaces de Sobolev non isotropes, à poids et problèmes de Cauchy quasi-linéaires sur un conoïde caractéristique, Ann. l'Inst. Henri Poincaré Phys. Théor. **66**, 37 (1997), http://eudml.org/doc/76748.

[26] H.-O. Kreiss and J. Lorenz, *Initial-Boundary Value Problems and the Navier-Stokes Equations* (SIAM, Philadelphia, 2004).

[27] T. Giannakopoulos, Characteristic formulations of general relativity and applications, Ph.D. thesis, Instituto Superior Tecnico, University of Lisbon, 2022, https://arxiv.org/abs/2308.16001.