Physical significance of the black hole quasinormal mode spectra instability

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(Received 11 April 2024; accepted 4 June 2024; published 8 July 2024)

It has been shown, via specific examples and a pseudospectrum analysis, that the black hole quasinormal spectra are unstable. The implication of such a result for gravitational-wave physics and of our understanding of black holes is, still, unclear. The purpose of this work is twofold: (i) we show that some of the setups leading to instabilities are unphysical and triggered by exotic matter or extreme spacetimes; (ii) nevertheless, we also show simple examples of compelling physical scenarios leading to spectral instabilities. Our results highlight the importance of understanding the overtone content of time-domain waveforms, and their detectability.

DOI: 10.1103/PhysRevD.110.024016

I. INTRODUCTION

In a binary black hole (BH) coalescence process, as the two BHs merge they leave behind a "distorted" remnant, emitting gravitational waves (GWs) and evolving towards stationarity via a *ringdown* phase. During this phase, the GW signal is simply described by a superposition of exponentially damped sinusoids termed quasinormal modes (QNMs). Thus, BHs relax as most systems do, by oscillating in a set of characteristic frequencies [1–4].

In a way similar to the analysis of the hydrogen atom, the response of (nonspinning) BH spacetimes is decomposed in spherical harmonics $Y_{\ell m}$, and each QNM carries, in addition to the polar and azimuthal indices (ℓ, m), an overtone index $n = 0, 1, 2\ldots$ The intrinsic dissipative character of BH spacetimes implies that the modes are not stationary (energy is being radiated to large distances and towards the horizon), and characteristic frequencies are complex numbers. For each (ℓ, m), the dominant mode is the one with the lowest imaginary component of the characteristic frequency, and assigned the $n = 0$ label, the fundamental mode. In vacuum general relativity (GR), the characteristic QNM frequencies of a single mode are enough to characterize the underlying geometry of the remnant. BH spectroscopy is the science of extracting information about BH spacetimes and the underlying gravitational description from the relaxation of BHs. Detection of these modes from a binary BH merger process provides information about the object itself, its surrounding environment, and even potential deviations from general relativity [1–3].

The first detection of a fundamental QNM is established for the first GW event, GW150914 [5], where the fundamental quadrupolar $(\ell, m, n) = (2, 2, 0)$ mode is reported to be consistent with general relativity predictions [6]. The ringdown of the BH remnant has since been reported for a number of other events (see in particular Tables VIII of Ref. [7] and XI of Ref. [8]). Detection of higher overtones will require louder signals and careful analysis [9–13]. Tentative evidence for additional modes $(\ell, m, n) = (3, 3, 0)$ was reported for event GW190521 [14–17] and in some events of GWTC-3 [18] but is the subject of debate, given systematic uncertainties such as precession and eccentricity. The impact of nonlinearities has recently been discussed [11,19–23]. In other words, BH spectroscopy is now blossoming into a vibrant field, and upcoming years will see precision tests of general relativity from the relaxation of BHs [24–27].

The BH spectroscopy program anchors on linearized calculations of the vacuum BH spectrum in GR [1–4,24,25], and the paradigm implicitly assumes a reasonable physical robustness against "small" changes in the system. In colloquial terms, one does not expect a "flea" far away from the BH to perturb significantly its spectrum. Surprisingly, such a spectral instability has been observed in a variety of circumstances.

Indeed, in addition to his pioneering work on the description of ringdown dynamics [28], Vishveshwara also explicitly pointed out that small changes in the BH potential may lead to a destabilization of the QNM spectra [29].[1] The sensitivity of the QNMs under small

[1]When recollecting his journey along the BH trail [30], Vishveshwara mentions that "we have studied the sensitivity of the QNMs to scattering potentials. The motivation is to understand how any perturbing influence, such as another gravitating source, that might alter the effective potential would thereby affect the QNMs. Interestingly, we find that the fundamental mode is, in general, insensitive to small changes in the potential, whereas the higher modes could alter drastically. The fundamental mode would therefore carry the imprint of the BH, while higher modes might indicate the nature of the perturbing source."

modifications of the potential has been somewhat rediscovered recurrently over the past decades [2,4,19,31–39], and it has recently been incorporated into a more formal framework [37,40]. Unfortunately, with notable exceptions discussed below, spectral instabilities have been discussed in the context of *ad hoc* perturbations to the effective potential, the physical significance, if any, of which is not fully understood. The purpose of the present work is to shed *some* light on these issues.

Broadly speaking, one can identify two major classes of spectral instabilities, closely related to the "infrared" and "ultraviolet" effects reported in Ref. [40] (but we are aware of the dangers of any classification such as this one).

Fundamental mode instabilities, which are associated with the introduction of a different scale in the problem. These include shells of matter for example, modeled as "potential bumps" far away from the BH [4,19,41], or abrupt cuts in the large range of the potential [31,32]. As we argue below, the new scale means that matter is present either asymptotically far or asymptotically close to the BH horizon, leading in either case to low-frequency echoes. These do indeed destabilize the spectrum, by introducing a longer lived second family of modes.

The time-domain response is only affected after the prompt ringdown emission, which is governed by the vacuum QNMs.

Overtone instabilities, caused by high-momentum fluctuations or seemingly near-horizon modifications to the geometry [4,19,33,37–40]. These affect the larger overtones, and potentially also affect promptly the time-domain signal [37], but a systematic understanding is lacking.

We will show how the existing body of work relates to previous known results, how the spectral instability can arise in physically interesting setups, and how some of the scenarios considered thus far require exotic matter or extreme spacetimes.

II. SETUP

To include in our analysis the relevant results reported in the literature, we consider a general, spherically symmetric spacetime,

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{b(r)} + r^2 d\Omega^2. \tag{1}$$

The mass function $m(r)$ is defined from the metric via

$$b(r) = 1 - \frac{2m(r)}{r}. \tag{2}$$

The energy density associated to the mass function is

$$\rho(r) = \frac{m'(r)}{4\pi r^2}. \tag{3}$$

The dynamical equations governing massless fields follow from the field equations and equation of state of matter. The details of the particular equation of state of matter are unnecessary. Indeed, all analyses in the literature focused exclusively on scalar fields and on only one sector of gravitational perturbations, for which the massless field does not couple to the background matter (assuming it is isotropic and dissipationless).

Both scalar and gravitational axial fluctuations can be expanded in harmonics of index ℓ, and are governed by a master wave function $\Psi = \Psi(t,r)$ which obeys the second order partial differential equation,

$$\frac{\partial^2\Psi}{\partial r_*^2} - \frac{\partial^2\Psi}{\partial t^2} - V\Psi = 0, \tag{4}$$

where the tortoise coordinate is defined by $\mathrm{d}r/\mathrm{d}r_* = \sqrt{ab}$.

For minimally coupled scalars $\Phi = \Psi/rY_{\ell m}$, one finds that the field is governed by [2]

$$V = a\left(\frac{\ell(\ell+1)}{r^2} + \frac{(ab)'}{2ar}\right). \tag{5}$$

In spherical symmetry, gravitational fluctuations can be decomposed into two sectors (axial or odd, and polar or even). The axial sector, for isotropic dissipationless fluids does not couple to the matter and obeys the following [42,43][2]:

$$V = a\left(\frac{\ell(\ell+1)}{r^2} - \frac{6m}{r^3} + \frac{m'}{r^2}\right). \tag{7}$$

III. SPECTRAL INSTABILITY OF THE FUNDAMENTAL MODE: A NEW SCALE

A. Double-bump potentials sourcing instabilities of the fundamental mode: Is it really a flea?

Spectral stability concerns the robustness of the QNM spectrum against slight deviations of the underlying system [4,29,31,37,40]. Some studies have focused on changes to the effective potential governing wave propagation, as [29,41,45]

[2]Note that Chandrasekhar and Ferrari [44] and Kokkotas find (M is total mass of star) [1]

$$V = a\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3} + \rho - P\right). \tag{6}$$

There are 4π factors difference between the two equations but these are due to the different convention in the field equations. Equation (7) uses $G_{\mu\nu} = 8\pi T_{\mu\nu}$, whereas Chandra uses $G_{\mu\nu} = 2T_{\mu\nu}$.

$$V_\epsilon = V + \epsilon V_{\rm bump}, \tag{8}$$

with V the "unperturbed" potential describing propagation in vacuum Schwarzschild BH spacetime (i.e., setting $m' = 0, m = M_{\rm BH}$ in (5)–(7). The focus was then on "bumpy" potentials $V_{\rm bump}$, such that V_ϵ admits a local maximum besides the global maximum of V.

Notice that the system will have distinct scales when the characteristic distance L of the bump away from the BH is much larger than any other scale in the problem, in this case the BH mass. In this circumstance, there is a cavity formed by the two potential peaks which supports a mode of frequency $\sim 1/L$, as long as it is trapped, $1/L^2 \ll \epsilon V_{\rm bump}$. In these circumstances, the BH fundamental QNM is destabilized [4,41,45]: for any arbitrarily small $\epsilon V_{\rm bump}$ there is a scale L at which an extremely low frequency mode appears, which dominates over the BH light ring mode [46,47] and which is never "close" to it.

We have been discussing features asymptotically far from the BH, but similar phenomenology occurs if new scales show up close to the horizon, under the form of matter of simply new boundary conditions [46,47]. In all cases, the time-domain response is smooth: the prompt response corresponds to a vacuum BH signal, but at late times the double-bump cavity filters out the high-frequency component, giving rise to a sequence of "echoes" [46–48].

An open question is what physics gives rise to such a potential. Is it really a "small" spacetime perturbation? To establish an equivalence between a bumpy potential and the potential (5) arising from a generic spherically symmetric spacetime, one can take the eikonal limit $\ell \to \infty$. In this limit, potential (5) asymptotes to

$$V = a\frac{\ell(\ell+1)}{r^2}, \qquad \ell \to \infty. \tag{9}$$

The requirement that V has extrema corresponds to the requirement that $2a = ra'$ at some point in the exterior of the horizon. But this requirement is equivalent to the existence of a light ring at that location [49]. Thus, the spacetime would have *two* light rings, and one cannot consider $\epsilon V_{\rm bump}$ to be a small perturbation by any means.

The above reasoning assumes that $(ab)'/r$ decays faster than $1/r^2$, which it should for regularity reasons for all cases. The only exception concerns massive scalar fields of mass μ, which indeed can be looked at as a special case of all of the above. For such fields the true effective potential is $V + a\mu^2$; we discuss it below.

B. Massive fields: A true fundamental-mode instability

A very clear example of spectral instabilities concerns massive fields. What we will now discuss affects both massive tensor, vector and scalar fields, but for concreteness we focus on scalars, whose dynamics are governed by the Klein-Gordon equation,

$$\Box\Phi = \mu^2\Phi. \tag{10}$$

The mass parameter μ is related to the physical boson mass m_B via $m_B = \mu\hbar$, in our units.

We focus for simplicity on a Schwarzschild background. For massless fields, $\mu = 0$, the fundamental $\ell = 0, 1$ mode is given by $M\omega = 0.1104 - i0.1049,\ 0.2929 - i0.0977$, respectively.

For very small μ (the regime most interesting to us here), two things happen. First, a *new* family of modes, so-called quasibound states, appears [50–53] which in the small μ limit is well described by [see Eq. (C12) in Ref. [54]]

$$M\omega = M\mu - \frac{(M\mu)^2}{2(\ell+n+1)^2} - iM\omega_I. \tag{11}$$

For $\ell = 1$ for example, $M\omega_I = (M\mu)^{10}/12$. This is a true spectral instability: for any arbitrarily small μ the fundamental mode is changed by a large amount. For an *arbitrarily small* mass μ there are new modes of arbitrarily small frequency. In other words, we argue that among all "bumpy potentials" the only physically relevant concerns endowing a small mass to interaction carriers.

The second interesting property is that the massless family (the usual QNM spectrum when $\mu = 0$) is altered [51,52,55]. As an example, the fundamental $n = 0$ mode changes as

$$\frac{\omega_\mu^\ell - \omega_0^\ell}{\mu^2} = 0.225 + i0.823, \quad \ell = 0, \tag{12}$$

$$= 0.447 + i0.268, \qquad \ell = 1. \tag{13}$$

For small μ the larger the overtone the smaller the deviation from the $\mu = 0$ mode.

As discussed at length elsewhere, the most immediate effect of such a destabilization is drastic changes to the late-time behavior [4,41,46,47]. The impact on prompt ringdown is poorly studied, but it is not our focus here.

C. A holistic view on spectral instability of the fundamental mode: Soft changes and couplings to matter

The massive scalar case just discussed is a proxy for more general setups: whenever the effective potential governing propagation is changed, the spectrum changes. If the changes are "soft" (induced by a parametrically small quantity) but there are different scales in the problem, then as a rule new families of modes set in, and large scales control low frequency modes. For very small perturbations with scale separation, the vacuum family continues to exist, with slight induced corrections. In addition to the massive scalar field discussed above, these features have been observed for simple two-barrier toy models [4] or for asymptotically de Sitter spacetimes [56].

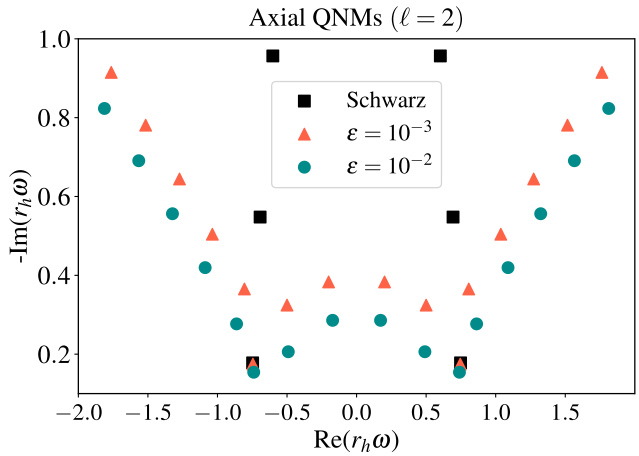

FIG. 1. Axial QNMs (angular mode $\ell = 2$) for potential with a Pöschl-Teller secondary bump [45] located at $L = 10r_h$. Black squares show the first three modes of a pure vacuum Schwarzschild spacetime. Triangles and circles are results when a Pöschl-Teller bump of height $\epsilon = 10^{-2}, 10^{-3}$ is introduced. New modes appear, some of low frequency. Higher overtones are exponentially sensitive in overtone number n and scale L; see the main text.

Figure 1 summarizes what happens when a soft change in the potential is introduced. We take the effective potential for axial gravitational modes and modify it by adding a Pöschl-Teller secondary bump [45]. The three lowest modes of the pure vacuum potential have frequencies [2]

$$\begin{aligned} r_h\omega/2 &= 0.3737 - i0.08896, \qquad n = 0, \\ &= 0.3467 - i0.2739, \qquad n = 1, \\ &= 0.3010 - i0.4783, \qquad n = 2, \end{aligned}$$

shown as black squares in the figure. It is clear that a small bump introduces new modes in the problem, some of them of lower frequency, a clear sign of instability of the spectrum. Nevertheless the fundamental mode of the vacuum family is only slightly disturbed. This can be understood from a perturbation theory point of view. The introduction of a small disturbing potential affects the modes as [4,57]

$$\delta\omega \sim \epsilon\langle\psi|V_{\rm bump}|\psi\rangle \sim \epsilon e^{2L\omega_I}, \qquad (14)$$

with L the scale at which the bump is located, and ω_I the imaginary component of the frequency of the mode in question. Thus, to be in the perturbative regime, one has to require that $\epsilon \ll e^{-2L\omega_I}$. But if $L = 10r_h$, $e^{-2L\omega_I} = 0.17$, 4×10^{-3}, 7×10^{-5} for $n = 0, 1, 2$. This explains why overtones are exponentially sensitive to any change in the potential. We have explicitly verified this exponential sensitivity for the double barrier potential in Ref. [4].

On the other hand, if there are "hard" changes, for instance, an abrupt change of boundary conditions, then as a rule the prompt ringdown, vacuum BH mode is still dominant in the time response, but it does not feature in the spectrum [46,47,58].

From the discussion of Sec. III A, one might be tempted to incorrectly conclude that instability of the fundamental mode occurs only in highly contrived, nonphysical situations. Asymptotically de Sitter spacetimes are a clear example where the instability is triggered, by the introduction of a new scale. The small cosmological Λ constant of our Universe adds a new family of low-frequency $\omega \propto \sqrt{\Lambda}$ modes to the BH spectrum [56,59].

Most importantly, the majority of physically relevant setups are not included in a simple, decoupled second-order partial differential equation of the form (4). Indeed, a wide class of problems couples gravitational degrees of freedom to matter modes [60]. Since matter moves at subliminal speeds, their characteristic frequencies are also lower. The universal coupling of gravity to matter then produces a spectral instability, no matter how low the matter density [60]. Because of the difference of scales associated to the instability of the fundamental mode, we expect that the prompt ringdown remains essentially unaffected for this class of perturbations.

IV. SPECTRAL INSTABILITY OF OVERTONES

We now focus on the second class of spectral instabilities, affecting BH overtones. As we show below, some of the setups used to argue for this type of instability are unphysical. Nevertheless, we show that the mechanism itself was known for some time, and can arise in physically relevant situations. With a few notable exceptions [37], the impact of such an instability in time-domain signals is largely unexplored, and understanding its detectability in more realistic scenarios requires further work.

A. Charged BHs: A true instability

The existence of a spectral instability should be no surprise, since it was indirectly known in a rather straightforward setup for at least two decades in BH physics [2,34–36,61]. Consider the asymptotic spectrum of nonspinning, neutral BHs, in the large overtone limit, and for any massless field

$$M\omega \sim \frac{\log 3 - i(2n+1)\pi}{8\pi} + \mathcal{O}(n^{-1/2}), \qquad Q = 0, \qquad (15)$$

a behavior which attracted considerable attention in relation to possible BH area quantization [2]. Frequencies are equally spaced in their imaginary component, whereas real part asymptotes to a constant $\log 3/(8\pi M)$.

On the other hand, an analysis of the asymptotic behavior of BHs charged with an arbitrary charge Q yields the condition

$$e^{\omega/T} + 2 + 3e^{Q^4\omega/(Tr_+^4)} = 0, \qquad Q \neq 0, \tag{16}$$

$$r_+ \equiv M + \sqrt{M^2 - Q^2}, \qquad T \equiv \frac{\sqrt{M^2 - Q^2}}{2\pi r_+^2}. \tag{17}$$

In the limit that $Q \to 0$, one finds then

$$M\omega \sim \frac{\log 5 - i(2n+1)\pi}{8\pi} + \mathcal{O}(n^{-1/2}), \qquad Q \to 0. \tag{18}$$

As a consequence for any arbitrarily small charge Q the BH spectrum differs by $\mathcal{O}(Q^0)$ from the spectrum of a Schwarzschild BH.[3] This is a clear example of a spectral instability in the large overtone regime.

B. High-momentum fluctuations

Recently, another class of fluctuations leading to large-overtone spectral instability has been studied, and concerns a modification to the BH (of mass M_{BH}) potential in the form [40]

$$V_\epsilon = V + \frac{a}{r^2}\delta V, \tag{19}$$

with $|\delta V| \sim \epsilon$ a small perturbation.

Comparing Eqs. (7) and (19), it is straightforward to establish a relation between the mass aspect $m(r)$ and the potential modification $\delta V(r)$ via

$$\frac{6m(r)}{r} - m'(r) = \frac{6M_{\mathrm{BH}}}{r} - \delta V(r). \tag{20}$$

A careful study of the asymptotic limit $\lim_{r\to\infty} m(r)$ reveals that the condition

$$\lim_{r\to\infty} \delta V(r) = 0 \tag{21}$$

is necessary for a regular mass function.

The suggested models capturing high-momentum fluctuations employ $\delta V = \epsilon \cos(2\pi k r_h/r)$ [40], with $k \in \mathbb{N}$. This choice, however, does not satisfy the regularity condition (21). However, the perturbing potential,

$$\delta V(r) = \epsilon \sin\left(2\pi k \frac{r_h}{r}\right), \tag{22}$$

satisfies the regularity condition (21) and is as well motivated as the original. We verified that Eq. (22) yields the same qualitative behavior for the QNM instability [37].

[3]Hence also the mantra that the theory of the limit need not be the limit of the theory. A common, if trivial, example concerns the function $n\epsilon/(1+n\epsilon)$. At $\epsilon = 0$ this yields zero at any n, even though the large n limit is unity.

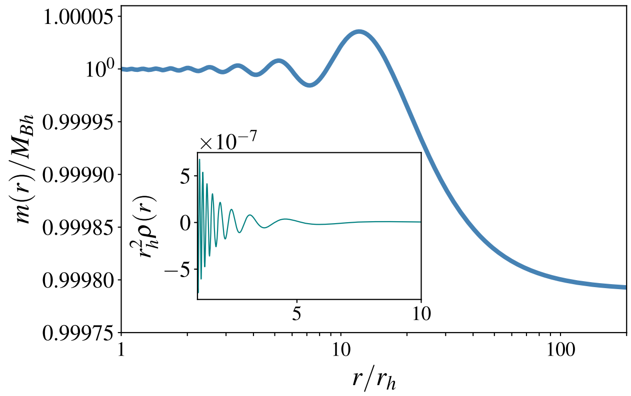

FIG. 2. Physical quantities associated to the potential $\delta V = \epsilon \sin(2\pi k r_h/r)$ which leads to spectral instabilities. Mass $m(r)$ and density $\rho(r)$ profiles are well defined in the BH exterior region $r \in [r_h, \infty)$. But $\rho(r)$ assumes negative values, which prevents models with a single wave number k from being realistic at a classic level. Here, $\epsilon = 10^{-5}$, $k = 10$.

With this choice, Eq. (20) is straightforwardly solved, with integration constant fixed to ensure $m(r)$'s regularity as $r \to \infty$. The corresponding energy density (3) reads

$$r_h^2\rho = -\frac{9r^3\epsilon}{8\pi^6 k^5 r_h^3} + \frac{\epsilon(\pi^2k^2r_h^2 - 3r^2)^2}{4\pi^5k^4r^2r_h^2}\sin\left(2\pi k\frac{r_h}{r}\right) + \frac{3\epsilon(2\pi^4k^4r_h^4 - 6\pi^2k^2r^2r_h^2 + 3r^4)}{8\pi^6k^5rr_h^3}\cos\left(2\pi k\frac{r_h}{r}\right). \tag{23}$$

The density decays like $\mathcal{O}(r^{-5})$ at large distances and is finite at $r = r_h$. Figure 2 shows the mass function and associated energy density (inset) related to a potential modification in the form (22) with $\epsilon = 10^{-5}$ and $k = 10$. Even though the mass function $m(r)$ is well defined in the outer domain $r \in [r_h, \infty)$, the energy density oscillates around the vacuum and assumes negative values. Therefore, these modifications are not realistic at a classic level either.

C. The continued-fraction parametrization

1. The spacetime and some of its properties

Finally, we discuss one other instance of reported high-overtone instability. This concerns a generic parametrization of an axisymmetric spacetime, which in the nonrotating limit reduces to parametrizing the metric functions (1) via [62,63]

$$a = \left(1 - \frac{r_h}{r}\right)A, \qquad b = \frac{(1 - \frac{r_h}{r})A}{B^2}. \tag{24}$$

The functions A and B are expressed in terms of a compact radial coordinate centered at the horizon $x = 1 - r_h/r$, as

$$A = 1 - \epsilon(1-x) + (a_0 - \epsilon)(1-x)^2 + \tilde{A}(1-x)^3 \tag{25}$$

$$B = 1 + b_0(1-x) + \tilde{B}(1-x)^2. \tag{26}$$

The functions $\tilde{A}(x)$ and $\tilde{B}(x)$ are expressed via continued fractions

$$\tilde{A} = \frac{a_1}{1 + \frac{a_2 x}{1 + \frac{a_3 x}{1 + \cdots}}}, \qquad \tilde{B} = \frac{b_1}{1 + \frac{b_2 x}{1 + \frac{b_3 x}{1 + \cdots}}}. \tag{27}$$

These definitions ensure that the BH horizon is located at $r = r_h$. The Hawking temperature reads

$$T = \frac{(1 + a_0 + a_1 - 2\epsilon)}{4\pi r_h (1 + b_0 + b_1)}. \tag{28}$$

The matter content is anisotropic, with a stress tensor of the form $\mathrm{diag}(-\rho, p_r, p_t, p_t)$. At the horizon,

$$8\pi r_h^2 \rho = -8\pi r_h^2 p_r = 1 - \frac{1 + a_0 + a_1 - 2\epsilon}{(1 + b_0 + b_1)^2}. \tag{29}$$

The tangential pressure p_t looks slightly more cumbersome and we refrain from writing it here.

2. Spectral properties

We focus on three specific models, two of those explored in Ref. [39], summarized in Table I. In particular, model BH0 corresponds to the Schwarzschild spacetime, whereas BH2 is spectrally unstable as we discuss shortly. We list the Hawking temperature (28) of each of these spacetimes.

We first reproduce the scalar QNMs reported in the original work [39]. Figure 3 summarizes our results for quadrupolar $\ell = 2$ modes. We employ the hyperboloidal framework, with Chebyshev spectral methods to calculate the QNMs [40,64], and ensure that the numerical resolution provides enough accuracy. We reproduce scalar QNMs exactly for all models BH0, BH1, BH2, confirming the correct implementation of the hyperboloidal framework.[4]

TABLE I. Parametrized BH spacetimes studied in this work. All other parameters (such as b_0, b_1, a_4) are set to zero, while $\epsilon = a_0$. The vacuum Schwarzschild spacetime corresponds to model BH0. Models BH1 and BH2 follow Ref. [39], with BH2 spectrally unstable. BH3 assumes conditions (30) to make it physically interesting, but is still spectrally unstable.

Model	$4\pi r_h$ T	a_0	a_1	a_2	a_3
BH0	1	0	0		
BH1	1.0001	0	10^{-4}	-10^3	1001
BH2	1.5	0	0.5	10^2	0
BH3	1	10^{-4}	10^{-4}	-2	10^3

We have extended the analysis to axial gravitational fluctuations, in Fig. 3. Overtone instabilities are present in models BH2 and BH3, clearly seen in Fig. 3. An inspection of (29) shows that $8\pi r_h^2 \rho$ goes from -10^{-4} for BH1 to -0.5 to model BH2, hardly a small number, and *negative*. Thus, models BH1 and BH2 should be considered unphysical.

3. A realistic model

As might be anticipated from the discussion on charged BHs, there are physically relevant situations for which the spectrum is unstable. To search for these we consider the following, somewhat stringent, assumptions:

$$a_0 = a_1 = \epsilon, \qquad a_2 = -2, \qquad b_0 = b_1 = 0. \tag{30}$$

These conditions ensure that there is no modification to the BH temperature, $4\pi r_h T = 1$, that the density, tangential and radial pressure vanish at the horizon, that the geometry agrees with observational constraints on Post-Newtonian parameters, and that $\lim_{r\to\infty} m(r) = M_{\mathrm{ADM}}$.

With this setup, the small parameter ϵ provides only an overall rescale on the amplitudes of ρ, p_r, and p_t. We complement conditions above with $a_4 = 0$, reducing the parameter space to the quantities (ϵ, a_3). For this model, which we dub BH3 in Table I and Fig. 3, the density distribution reads

$$8\pi r_h^2 \rho = \epsilon \left(1 - \frac{r_h}{r}\right)^2 \left(\frac{r_h}{r}\right)^4 \frac{2(a_3^2 + a_3 - 2) r r_h + (a_3 - 1)^2 r^2 - 3(a_3 - 2) a_3 r_h^2}{(r(1 - a_3) + (a_3 - 2) r_h)^2}. \tag{31}$$

To ensure $\rho(r) \geq 0$ for $r > r_h$, we take $a_3 \geq 1$.

[4]Note, however, that we find a new overtone, $n = 2$, absent in Ref. [39]. The real part of this overtone is very close to the previous one: $\mathrm{Re}(r_h\omega_1) = 0.890084$, and $\mathrm{Re}(r_h\omega_2) = 0.895286$. Our method computes the QNM as the eigenvalues of the numerically discretized hyperboloidal operator, and therefore, it calculates all overtones at once. Alternative methods, such as Leaver's continued fraction [65], are based on a root search algorithm, for which an initial seed relatively close to a given overtone n is required. Since the overtones $n = 1$ and $n = 2$ are rather close to each other, it is not unexpected that the root search may converge to $n = 1$ and miss the new $n = 2$, unless a fine-tuned initial seed is specified. We have verified that the new QNM satisfies Leaver's criteria defining a QNM [65].

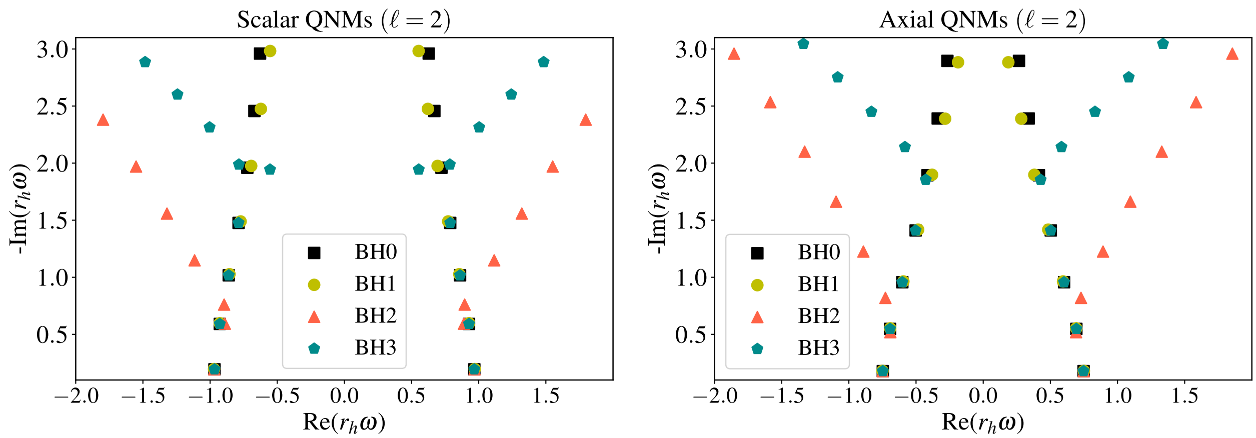

FIG. 3. Left panel: scalar QNMs for the BH models with parameters given in Table I. Overtone instability is triggered in models BH2 and BH3, in particular with the appearance of a new $n = 2$ overtone for BH2 not reported in Ref. [39]. Right panel: axial QNMs for the BH models with parameters given in Table I. As in the scalar sector, overtone instability is triggered for the models BH2 and BH3.

In particular, the density profile has a peak $8\pi r_h^2 \rho|_{\text{peak}} = 16\epsilon/243$ exactly at the photon sphere $r = 3r_h/2$ when $a_3 = 1$. As $a_3 \to \infty$, the peak moves slightly towards the horizon up until $r = (\sqrt{97} - 5)r_h/4 \approx 1.21 r_h$, where it takes the value $8\pi r_h^2 \rho = (512(17 - \sqrt{97})\epsilon)/(\sqrt{97} - 5)^6 = 2.82 \times 10^{-1}\epsilon$. Besides, we also observe that the corresponding potential for axial perturbations does not develop any further peak, and that its relative difference with respective to the Regge-Wheeler potential has a maximum of order $\sim 10^{-1}\epsilon$ around $r \sim 2r_h$.

Summarizing, BH3 seems to have all ingredients of a "reasonable" spacetime. Nevertheless, Fig. 3 shows clearly that it also turns the BH spectrum unstable, even when it should be only a minor modification away from a vacuum, Schwarzschild BH. Notice that the spacetime was constructed to be reasonable and forced to obey a number of unnecessary restrictions (coefficients after a_4 are set to zero, etc.). Thus, we believe that spectral instabilities of high overtones are a generic effect, and the arguments underlying the asymptotic analysis support this claim [2,34–36].

To study the role played by the parameters a_3 and ϵ in the instability, we define the critical mode n_c as the overtone for which the relativity difference to its Schwarzschild counterpart is larger than the small parameter ϵ. Specifically, the critical mode n_c is defined via the relation $\delta\omega_n > \epsilon$ for $n > n_c$, with

$$\delta\omega_n = \frac{|\omega_n^{\text{BH3}} - \omega_n^{\text{BH0}}|}{|\omega_n^{\text{BH0}}|}. \tag{32}$$

This definition captures the notion of QNM instability, and the model BH3 allows us to assess the two main important features underlying the phenomena of QNM instability. While ϵ controls the overall scale of how intense the perturbation is, a_3 impacts the opening of the QNM branches. Thus, the bigger either values, the lower the value n_c for the critical mode. For instance, when $\epsilon = 10^{-4}$, we observe $n_c = 6$ for $a_3 \sim 10$, $n_c = 5$ for $a_3 \sim 25$, $n_c = 4$ for $50 \lesssim a_3 \lesssim 250$, and $n_c = 3$ for $a_3 > 500$. Increasing ϵ allows one to obtain lower values of n_c, but their dependence on a_3 remains unaltered.

V. DISCUSSION

The purpose of this work is to shed light on spectral instabilities in BH spacetimes. One first important point is that some scenarios used to study this issue are not realistic. Matter is either extreme, or the spacetime is extreme and not a fluctuation away from the intended background.

More importantly, we argue that instabilities come in two broad classes [40] and both can be triggered with reasonable physics. Fundamental-mode instabilities, of direct interest to current detectors, involve a new scale in the problem. For many compelling scenarios these destabilize the spectra. We argued that these affect the signal only at very late times when the signal is already weak. However, a perturbation theory approach indicates that the changes in the frequency of the modes behave as $\delta\omega \sim e^{2L\omega_I}$, showing therefore an exponential dependence on the overtone number n and on the new length scale L. This argument indicates an urgent need to understand mode content in time-domain waveforms. The above concerns "soft changes," but near-horizon structure or hard conditions somewhere in the spacetime is known to result in echoes of significant amplitudes.

The second class of fluctuations has no obvious new physical scale associated with it, and changes the asymptotic overtone structure of the spectrum. As we remarked, a result well known for years (surprisingly not mentioned in this context in the literature) concerns slightly charged BHs. But there are other examples. The analysis and conclusions of Refs. [2,34–36] suggests that the asymptotic structure of the spectrum relates to the behavior of the potential close to the *singularity*, a remarkable statement bearing in mind that these are cloaked by horizons. In addition, mathematical arguments [66] suggest that spiked perturbations are more efficient in triggering BH QNM instabilities than smooth ones. The impact of the large overtone structure on our understanding of BH spectroscopy is unknown. In other words, some theories might leave the first few modes unaffected, possibly producing a ringdown signal identical to that of GR at late times. We do not have a precise understanding of the impact of the large overtone structure on time domain waveforms, but first attempts indicate the overtone instability is measurable in ideal highly accurate setups [37]. Thus, ours and all the recent results in the literature make this an important issue.

ACKNOWLEDGMENTS

We are thankful to Gregorio Carullo, Thomas Spieksma, and Jaime Redondo Yuste for a critical reading of the manuscript and for useful feedback. We are thankful to the Yukawa Institute for Theoretical Physics, for warm hospitality. We acknowledge support by VILLUM Foundation (Grant No. VIL37766) and the DNRF Chair program (Grant No. DNRF162) by the Danish National Research Foundation. V. C. acknowledges financial support provided under the European Union's H2020 ERC Advanced Grant "Black holes: gravitational engines of discovery" Grant Agreement No. Gravitas–101052587. Views and opinions expressed are however those of the author only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them. This project has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie Grant Agreements No. 101007855 and No. 101131233.

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