

Diffeomorphism invariant classical-quantum path integrals for Nordström gravity

Jonathan Oppenheim, Andrea Russo[✉], and Zachary Weller-Davies

*Department of Physics and Astronomy, University College London,
Gower Street, London WC1E 6BT, United Kingdom*

 (Received 22 January 2024; accepted 3 June 2024; published 3 July 2024)

When classical degrees of freedom and quantum degrees of freedom are consistently coupled, the former diffuse, while the latter undergo decoherence. Here, we construct a theory of quantum matter fields and Nordström gravity in which the space-time metric is treated classically. The dynamics is constructed via the classical-quantum path integral and is completely positive, trace preserving (CPTP) and respects the classical-quantum split. The weak-field limit of the model matches the Newtonian limit of the full covariant path integral, but it is easier to show that the theory is both diffeomorphism invariant and CPTP and has the appropriate classical limit.

DOI: [10.1103/PhysRevD.110.024007](https://doi.org/10.1103/PhysRevD.110.024007)

I. INTRODUCTION

The setting where gravity is effectively classical but matter retains its quantum properties belongs to the class of hybrid classical-quantum systems. These are systems whereby some parts of the system require a classical treatment while others are to be treated as quantum. Many challenges in modern physics, such as black hole evaporation, inflationary cosmology, and the measurement problem, occur in this regime, even if not all of them involve gravity directly. A good description of the classical-quantum regime is therefore paramount. While control of a quantum system through a classical variable is well understood, having a fully informative description of the classical-quantum regime hinges on properly treating the backreaction generated by the quantum system on the classical one. In gravity, a common approach to studying backreaction is via the semiclassical Einstein equations [1–3]. However, these are inconsistent when applied to all quantum states [4–6], in part because they ignore the correlations between the classical and the quantum degrees of freedom [7]. It has been shown in Refs. [8–15] that a classical-quantum treatment is possible if one allows for a stochastic coupling between the classical and quantum degrees of freedom. This stochasticity allows one to write down consistent evolution equations that are completely positive and norm preserving on the hybrid state, side-stepping the various no-go theorems and arguments about deterministic classical-quantum coupling [16–25]. The most general form of

consistent classical-quantum coupling was obtained in Refs. [7,26].

We call the resulting framework a “classical-quantum theory” (CQ theory). In this framework, the classical degrees of freedom act as a weak measurement apparatus, decohering the quantum system. The rate of this decoherence depends on the free parameters of the theory and is tied to the strength of the backreaction. The backreaction of the quantum degrees of freedom on the classical ones necessarily induces a stochastic diffusion process in the classical phase space trajectories, transforming classically deterministic evolution into probabilistic dynamics [27]. CQ dynamics can be expressed in multiple ways; it was originally formulated in the master equation formalism [8,10,26], but it has also been expressed as an unravelled set of stochastic partial differential equations (PDEs) [28,29] and via combined classical-quantum path integrals [30,31]. In this paper, we will focus on the path integral formulation of the framework [30,31].

Several models of gravitational CQ theories have been studied. These include Newtonian theories based on CQ master equations [32], measurement and feedback approaches [12–14], and relativistic theories based off of both master equations [7] and path integrals [31]. The weak-field regime of gravitational CQ theories was recently explored in Ref. [33]. A template was provided for constructing consistent gravitational classical-quantum dynamics in the Newtonian limit. It was found that the weak-field limit of the relativistic theories in Refs. [7,31] took the form of Ref. [13]. These theories can be experimentally tested via the decoherence-diffusion tradeoff [27], which is a relationship between the rate of decoherence, the strength of the backreaction, and the amount of stochasticity in the theory, which all consistent classical-quantum theories must obey.

In a CQ theory of gravity, the challenge presented by the presence of Hamiltonian constraints is not trivial [34], even

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

in the Newtonian limit. In the standard general relativistic treatment of the Newtonian limit, the dynamical gravitational degrees of freedom vanish, and Poisson’s equation for the Newtonian potential is found from the Hamiltonian constraint. When the Newtonian limit is considered in gravitational CQ dynamics, consistent coupling of the classical and quantum degrees of freedom necessarily introduces a diffusion process in the Newtonian potential. The effect is that of a stochastic Newtonian constraint, which requires a stochastic shift vector for the dynamics to be consistent [33]. Away from the Newtonian limit, the gravitational constraints in CQ theories are tightly related to notions of complete positivity and diffeomorphism invariance. Specifically, while the dynamics detailed in Ref. [7] are recognized as completely positive and norm preserving, their diffeomorphism invariance was not demonstrated. Conversely, the full gravity theory quoted in Ref. [31] is manifestly diffeomorphism invariant. However, its consistency has yet to be fully verified because the constraint contribution to the path integral does not look completely positive. Additionally, the relationship between the dynamics of Refs. [7,31] remains unclear, partly due to the ambiguity surrounding the constraint algebra [35].

Since Nordström gravity is a self-consistent and diffeomorphism invariant theory of gravity that does not require the Hamiltonian constraints, it can sidestep these complications. Furthermore, because the weak-field limit of Nordström gravity is the Newtonian theory, it provides us with a toy model that can be used to gain confidence in the weak-field limit of Refs. [7,31]. The present paper aims to construct a theory of Nordström gravity in the CQ framework through the path integral formalism. This gives a diffeomorphism invariant, though background-dependent, theory of CQ-gravity, where the conformal factor plays the role of the stochastic gravitational potential. Nevertheless, we recover the same Newtonian limit behavior as in Ref. [33], suggesting that the resolution of the constraints and the resulting dynamics was correct. The model also demonstrates no tension between diffeomorphism invariance and stochastic theories, nor any tension with the classical-quantum split. The theory provides a model in which various issues in quantum and classical-quantum gravity can be explored in a simpler form. It is also a manifestly Lorentz invariant theory of stochastic collapse [36–40], as are the models of Refs. [31,41].

The paper’s outline is as follows:

- (i) In Sec. II, we summarize the CQ framework through the path integral formalism. The CQ path integral was introduced in Refs. [30,31], in which it is shown it is possible to construct covariant path integrals for classical-quantum systems that give rise to completely positive evolution. A natural class of theories is then presented in which paths in CQ space are weighted according to a classical-quantum action,

where classical paths diffuse away from their averaged equations of motion, while simultaneously enforcing decoherence on the quantum system.

- (ii) In Sec. III, we review Nordström’s theory of gravity as a self-consistent theory of scalar gravity. The role of the dynamical field is played by a scalar conformal factor, which evolves in a flat background. We compare the theory’s advantages and shortcomings in relation to general relativity, and we discuss why it is useful as a toy model of classical-quantum gravity.
- (iii) In Sec. IV, we use the path integral approach to introduce and study a consistent diffeomorphism invariant theory of classical-quantum Nordström gravity. This provides a proof of principle that diffeomorphism invariant CQ dynamics can exist. In the Newtonian $c \rightarrow \infty$ limit, we find that the theory gives rise to the Newtonian interaction on average. Due to the decoherence-diffusion tradeoff, the dynamics necessarily involves diffusion away from the Newtonian solution by an amount lower bounded by the decoherence rate into mass eigenstates. Though this example is to be understood as a toy model, it provides an instance where we have full control over the symmetries of the theory and gives support to the treatment of a more complete theory.
- (iv) We conclude with a discussion in Sec. V.

II. PATH INTEGRAL FOR CQ DYNAMICS

This section introduces the CQ formalism, focusing on CQ path integrals. We outline how to define a classical-quantum state and how this state evolves through time according to completely positive CQ dynamics with interacting classical and quantum components. For a detailed description of the CQ dynamics, its derivation, and its properties, we point the reader to Refs. [7,26,29–31].

In the CQ framework, classical degrees of freedom live in a classical configuration space \mathcal{M} and are denoted by z . There are no restrictions on what the classical degrees of freedom can represent. The standard example is for them to be the position and momentum of a particle, in which case $\mathcal{M} = (\mathbb{R}^2)$. On the other hand, quantum degrees of freedom live in a Hilbert space \mathcal{H} . We denote the set of positive semidefinite operators living in this Hilbert space as $S(\mathcal{H})$. We define the CQ state of the system to be described at any time by the map $\varrho(z, t): \mathcal{M} \rightarrow S(\mathcal{H})$ such that the state of the system should always admit a decomposition into its classical and quantum parts $\varrho(z, t) = p(z, t)\sigma(z, t)$, where $p(z, t)$ is understood to be the probability density over the classical degrees of freedom z , while $\sigma(z, t)$ is a normalized density matrix representing the quantum state. Therefore, the CQ state $\varrho(z)$ is at every instant subject to a normalization constraint $\int_{\mathcal{M}} dz \text{Tr}_{\mathcal{H}}[\varrho] = 1$. To put it differently, we associate to each classical degree of freedom an unnormalized density operator $\varrho(z)$ such that $\text{Tr}_{\mathcal{H}}[\varrho] = p(z) \geq 0$

is a normalized probability distribution over the classical degrees of freedom and $\int_{\mathcal{M}} dz \varrho(z)$ is a normalized density operator on \mathcal{H} .

The time evolution of the hybrid state has been expressed in three distinct, but ultimately equivalent, ways: via master equations in Refs. [7,26], unravelings [28,29], and path integrals [30,31]. Regardless of one's choice, consistency requires the dynamics of the hybrid system to be linear in the density matrix, completely positive, and trace preserving. These conditions are necessary to preserve the density matrix's statistical interpretation and give rise to positive probabilities when acting on half of an entangled state. The dynamics are also generally assumed to be time local, which is a sensible assumption that must be made to entertain the possibility of the CQ theory describing a fundamentally classical gravitational field [7].

The original formulation of the CQ framework was carried out in the master equation picture. The advantage of this approach is that the complete positivity and the general consistency of the evolution are manifest. Unfortunately, the master equation is ideal neither when dealing with constrained systems [34] nor with numerical simulations. Moreover, it is hard to properly impose space-time symmetries directly at the master equation level. However, the path integral formulation of the framework works well for these tasks. First of all, constraints can be directly imposed on trajectories through the use of delta functionals, allowing one to consider only the paths of the system that live on the constraint surface. Second, there exist many efficient numerical methods capable of simulating path integrals. For example, those involving Monte Carlo methods find great applicability everywhere from finance [42–44] to lattice gauge theories and quantum mechanics [45,46]. Lastly, the path integral formulation allows one to impose space-time symmetries and gauge symmetries through an action. This formalism makes it easier to formulate CQ dynamics in a covariant manner [31] and to enforce the necessary principles when studying effective field theories [47].

The classical-quantum state can be expanded in terms of its components. Denoting a continuous quantum degree of freedom by ϕ , we have

$$\varrho(z, t) = \int d\phi^+ d\phi^- \varrho(z, \phi^+, \phi^-, t) |\phi^+\rangle \langle \phi^-|, \quad (1)$$

where $\varrho(z, \phi^+, \phi^-, t) = \langle \phi^+ | \varrho(z, t) | \phi^- \rangle$. Here, we double the degrees of freedom in order to define the density matrix—the ϕ^- field is the ket field, while the ϕ^+ field is the bra field. A general CQ configuration space path integral takes the form described in Refs. [30,31],

$$\begin{aligned} \varrho(z_f, \phi_f^+, \phi_f^-, t_f) &= \int \mathcal{D}z \mathcal{D}\phi^+ \mathcal{D}\phi^- \mathcal{N} e^{\mathcal{I}[z, \phi^+, \phi^-, t_i, t_f]} \\ &\times \varrho(z_i, \phi_i^+, \phi_i^-, t_i), \end{aligned} \quad (2)$$

where $z = (z_1, \dots, z_n)$ are the classical degrees of freedom, \mathcal{I} is the CQ action, and it is implicitly understood that boundary conditions are to be imposed at t_f . The \mathcal{N} normalization factor is included in case the action does not preserve the norm of the state. A derivation of path integrals from CPTP master equations that explicitly find the normalization conditions on the action can be found in Ref. [30].

According to the main result of Ref. [31], time-local CQ path integrals with actions of the form

$$\begin{aligned} \mathcal{I}(z, \phi^+, \phi^-, t_i, t_f) &= \mathcal{I}_{\text{CQ}}(z, \phi^+, t_i, t_f) + \mathcal{I}_{\text{CQ}}^*(z, \phi^-, t_i, t_f) \\ &\quad - \mathcal{I}_C(z, t_i, t_f) + \int_{t_i}^{t_f} dt d\vec{x} \sum_{\gamma} c^{\gamma}(z, x, t) L_{\gamma}(\phi^+) L_{\gamma}^*(\phi^-) \end{aligned} \quad (3)$$

define completely positive CQ dynamics. In Eq. (3), \mathcal{I}_{CQ} determines the CQ interaction on each branch, and $\mathcal{I}_C(z, t_i, t_f)$ is a purely classical Fokker-Plank-like action [48,49] taking real values. We assume that the path integral defined by Eq. (3) is convergent, which enforces that \mathcal{I}_C is positive (semi)definite, the real part of \mathcal{I}_{CQ} is negative (semi)definite, and $c^{\gamma} \geq 0$. Any path integral with an action of the form in Eq. (3) is completely positive [31], and the scalar gravity CQ theory we consider in Sec. IV is of this type.

One can notice that in Eq. (3) $\phi^+ \phi^-$ cross-terms are contained only in the last term. This term is responsible for the violation of the purity of the quantum system. However, we can work in a regime where we can take the $c^{\gamma} = 0$ so that pure quantum states are mapped to pure quantum states and there is no loss of quantum information. Conditioned on the classical trajectory, the quantum state evolution is deterministic, which provides a natural mechanism for wave-function collapse if the classical degrees of freedom are taken to be fundamentally classical. This regime coincides with the saturation of the decoherence-diffusion tradeoff [27], which physically implies the presence of an exact inverse relationship between the amount of diffusion introduced in the classical system and the amount of decoherence introduced in the quantum system when the two are coupled.

A natural class of theories introduced in Refs. [30,31] are those derivable from a classical-quantum protoaction $W_{\text{CQ}}[z, \phi] = \int dt d\vec{x} (\mathcal{L}_c[z] - \mathcal{V}_I[z, \phi])$. Here, and in the rest of the paper, we consider z to be a configuration space variable. \mathcal{L}_c is the Lagrangian density of the classical action $\mathcal{S}_c[z] = \int dt d\vec{x} \mathcal{L}_c[z]$, and \mathcal{V}_I is the interaction potential density $V_I[z, \phi] = \int dt d\vec{x} \mathcal{V}_I[z, \phi]$, which in the case of general relativity is just the Lagrangian for the matter degrees of freedom. The action is given by

$$\begin{aligned}
 \mathcal{I}(z, \phi^+, \phi^-, t_i, t_f) &= i\mathcal{S}_Q[z, \phi^+] - i\mathcal{S}_Q[z, \phi^-] \\
 &\quad + i\mathcal{S}_{\text{FV}}[z, \phi^+, \phi^-] - \mathcal{S}_{\text{diff}}[z, \phi^+, \phi^-] \\
 &= \int_{t_i}^{t_f} dt d\vec{x} \left[i\mathcal{L}_Q[z, \phi^+] - i\mathcal{L}_Q[z, \phi^-] \right. \\
 &\quad - \frac{1}{2} \frac{\delta \Delta W_{\text{CQ}}}{\delta z_\alpha} D_{0,\alpha\beta}[z(x)] \frac{\delta \Delta W_{\text{CQ}}}{\delta z_\beta} \\
 &\quad \left. - \frac{1}{2} \frac{\delta \bar{W}_{\text{CQ}}}{\delta z_\alpha} D_{2,\alpha\beta}^{-1}[z(x)] \frac{\delta \bar{W}_{\text{CQ}}}{\delta z_\beta} \right]. \quad (4)
 \end{aligned}$$

Equation (4) takes the form of Eq. (3) [31] and is motivated by the study of canonical form CPTP master equations in Ref. [30]. We now explain the notation in this action. Both $D_0^{\alpha\beta}$ and $D_{2,\alpha\beta}^{-1}$ are positive definite matrices. They are related by the so-called diffusion-decoherence tradeoff [27]

$$4D_0^{\alpha\beta} \succeq D_{2,\alpha\beta}^{-1}. \quad (5)$$

We impose the saturation of the tradeoff through the matrix restriction $4D_0 = D_2^{-1}$. This ensures that the action takes the form in Eq. (3) with all the $c^\gamma = 0$. Hence, the dynamics will be completely positive, and the path integral will also preserve the purity of the quantum system, conditioned on the classical degrees of freedom [29].

In Eq. (4), we have introduced the notation for the \pm averaged interaction

$$\bar{W}_{\text{CQ}}[z, \phi] = \frac{1}{2} (W_{\text{CQ}}[z, \phi^+] + W_{\text{CQ}}[z, \phi^-]), \quad (6)$$

and the difference in the interaction along the \pm branches

$$\Delta W_{\text{CQ}}[z, \phi] = W_{\text{CQ}}[z, \phi^+] - W_{\text{CQ}}[z, \phi^-]. \quad (7)$$

\mathcal{L}_Q denotes instead the purely quantum evolution together with any interaction terms between the quantum and classical degrees of freedom. For example, it could be any quantum field theory Lagrangian. The coefficient of the difference in the interaction along the two branches is denoted as D_0 . It regulates the decoherence of the system by suppressing paths that have different values along the ϕ^+ branch and ϕ^- branches. By taking a covariant protoaction, the dynamics described by Eq. (4) will be covariant [31]. As such, they can be used to construct examples of relativistic spontaneous collapse models [36–40]. We shall see an explicit example of diffeomorphism invariant CQ dynamics in Sec. IV. Examples of Lorentz invariant Lindbladian dynamics can be found in Ref. [41], in contrast to the Lorentz covariant dynamics found in Ref. [50] (see also Ref. [51]).

As a toy example, we can take a single classical degree of freedom describing a scalar field $z = \{q(x)\}$ coupled to a quantum scalar degree of freedom with spatial dependence (i.e., a quantum field)

$$W_{\text{CQ}}[q, \phi] = \int dt d\vec{x} \left(-\frac{1}{2} \partial_\mu q \partial^\mu q - q(x) \phi(x) \right). \quad (8)$$

Then, choosing for simplicity $D_0(q)^{\alpha\beta} = D_0 \eta^{\alpha\beta}$,

$$i\mathcal{S}_{\text{FV}}[q, \phi] := -\frac{1}{2} \int dt d\vec{x} \left(\frac{\delta \Delta W_{\text{CQ}}}{\delta z_\alpha} D_0^{\alpha\beta}(z) \frac{\delta \Delta W_{\text{CQ}}}{\delta z_\beta} \right) \quad (9)$$

$$= -\frac{1}{2} D_0 \int dt d\vec{x} (\phi^-(x) - \phi^+(x))^2 \quad (10)$$

acts like a Feynman-Vernon term [52], which causes decoherence. The diagonal of the density matrix of Eq. (1) occurs when $\phi^+ = \phi^-$, and on these components of the density matrix, this term does nothing. Conversely, the greater the difference between the bra and ket fields, the more the paths are suppressed by the term in Eq. (10).

In a similar manner, D_2 tunes the averaged interaction term $\mathcal{S}_{\text{diff}}$, which is related to the diffusion of the classical system; paths that deviate from the Euler-Lagrange equations of motion, which are derived from the protoaction W_{CQ} , are suppressed. In the simple example of Eq. (8), one obtains

$$\begin{aligned}
 \mathcal{S}_{\text{diff}}[q, \phi] &= \frac{1}{2} \int dt d\vec{x} \left(\frac{\delta \bar{W}_{\text{CQ}}}{\delta z_\alpha} D_{2,\alpha\beta}^{-1}(z) \frac{\delta \bar{W}_{\text{CQ}}}{\delta z_\beta} \right) \\
 &= \frac{1}{2D_2} \int dt d\vec{x} \left(-\partial_\mu \partial^\mu q(x) + \frac{\phi^+(x) + \phi^-(x)}{2} \right)^2, \quad (11)
 \end{aligned}$$

where the force on the classical field is produced by the average of the bra and ket quantum fields. This term allows for fluctuations around the force but acts to suppress large deviations from them.

It is possible to recognize how the CQ path integral is connected to the path integral formulation of open quantum systems. If the action \mathcal{I} was only constructed out of the quantum degrees of freedom $\mathcal{I}[\phi^+, \phi^-, t_i, t_f] = i\mathcal{S}[\phi^+, t_i, t_f] - i\mathcal{S}[\phi^-, t_i, t_f] + i\mathcal{S}_{\text{FV}}[\phi^\pm, t_i, t_f]$, we would recover the standard decoherence functional of open quantum systems. If the Feynman-Vernon term [52] were not present ($\mathcal{S}_{\text{FV}} = 0$), we would then recover standard unitary quantum mechanics. Much like open systems have a path integral formulation of their master equation version, the CQ path integral can be directly thought of as coming from the master equation formulation [7]. Nevertheless, according to Ref. [31], we can take the path integral as the starting point of the CQ framework. This allows for a simpler definition of the path integral, given that deriving a clean form from the master equation requires one to map between configuration space and canonical coordinates [30].

In this section, we have introduced the formalism of the CQ framework. We have summarized how a positive path

integral version of the dynamics can be written and interpreted and how the classical-quantum path integral incorporates the backreaction between the quantum and classical systems, leading to decoherence and diffusion effects. In Ref. [29], it was also shown that when conditioned on the classical degrees of freedom a path integral that saturated the decoherence-diffusion tradeoff [27] preserves the purity of the quantum system. In the next section, we will apply it to a diffeomorphism invariant theory of scalar classical-quantum gravity and study its weak-field limit. We will find that, in the Newtonian limit, the theory predicts diffusion about the standard semi-classical Newtonian solution by an amount that is lower bounded by the decoherence rate into mass eigenstates according to Ref. [33].

III. NORDSTRÖM GRAVITY

So far, we have reviewed how to couple classical and quantum degrees of freedom via CQ path integrals. No explicit choice of classical and quantum systems has been made yet, and the equations presented so far hold in general. We will now discuss the classical system of interest in this paper, before integrating it into the CQ framework in Sec. IV.

Nordström gravity [53,54] was a first attempt at merging Newtonian gravity with relativity and ultimately led to the formulation of GR as it currently stands [55]. In its final formulation, Nordström gravity can be thought of as a self-consistent scalar theory of gravity. It was the first metric theory of gravity; hence, it obeys the equivalence principle. The classical theory is described through a conformally flat space-time background which couples to matter via the equations,

$$\mathcal{R} = \frac{24\pi G}{c^4} T, \quad (12)$$

$$C_{\mu\nu\rho\sigma} = 0, \quad (13)$$

where \mathcal{R} is the Ricci scalar, T denotes the trace of the stress-energy tensor for the matter degrees of freedom ϕ_m , and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. Equation (12) is the dynamical equation of motion linking the Ricci scalar to the trace of the stress-energy tensor. The vanishing of the Weyl tensor in Eq. (13) implies that the metric is conformally flat and always takes the form

$$g_{\mu\nu} = e^{\frac{2\Phi}{c^2}} \eta_{\mu\nu}, \quad (14)$$

where Φ is the conformal degree of freedom and $\eta_{\mu\nu}$ is the Minkowsky metric.

Nordström gravity merges relativistic ideas of causality with Newtonian gravity but lacks many of the properties required for a full description of gravitational phenomena. One can immediately notice that the conformal

metric couples only to the *trace* of the stress-energy tensor. However, some forms of energy and momentum, like the stress-energy tensor for electromagnetic radiation, are traceless. Therefore, Nordström gravity lacks the ability to describe some gravitational phenomena. In particular, it does not predict the bending of light, in direct contrast with gravitational lensing effects observed by astronomers [56,57]. With regard to other effects, Nordström gravity correctly predicts the result of the Pound-Rebka experiment for gravitational frequency shift but fails to predict the correct time delay factor and is missing subleading corrections to the acceleration of static test particles.

However, much like General Relativity, Nordström's theory is *diffeomorphism invariant*, by which we mean that (g, Φ, ϕ_m) is a solution to the equations of motion if and only if (g^*, Φ^*, ϕ_m^*) is also a solution to the same equation's of motion, where $*$ denotes the transformed variables after a diffeomorphism. In particular, conformal flatness is preserved under diffeomorphisms. The conformal flatness condition does not fix the conformal factor, which is the dynamical gravitational degree of freedom. For example, given the form of the metric in Eq. (14), the Ricci scalar reads

$$\mathcal{R} = -\frac{6\tilde{\square} e^{\frac{\Phi}{c^2}}}{c^2} e^{-\frac{3\Phi}{c^2}}, \quad (15)$$

where $\tilde{\square} = \partial_\mu \partial_\nu \eta^{\mu\nu}$ is the *flat* space D'Alabertian. In the vacuum state ($T = 0$), the field equation is the wave equation for the scalar field $\tilde{\square} e^{\frac{\Phi}{c^2}} = 0$. Therefore, the theory has a propagating conformal scalar degree of freedom, but this kind of gravitational wave differs from those predicted by General Relativity as they are scalar waves and do not have a spin-2 mode.

Nonetheless, despite being diffeomorphism invariant, the Nordström theory is *intuitively* background dependent. It has a preferred frame given by the Minkowski metric due to the condition that the metric be conformally flat. In particular, it admits a background-dependent formulation (which is still diffeomorphism invariant), where we first stipulate that the metric takes the form of Eq. (14), with the dynamics determined by Eq. (12). For a more detailed discussion on the relationship between diffeomorphism invariance and background independence, we refer the reader to Ref. [58]. We now study a CQ version of Nordström gravity. This provides a diffeomorphism invariant, self-consistent theory of CQ gravity that has a sensible Newtonian limit.

IV. CQ PATH INTEGRAL FOR NORDSTRÖM GRAVITY

In this section, we will first introduce CQ gravitational path integrals for general relativity, outlining the tension

between complete positivity of the dynamics and the gravitational constraints. Then, we construct the CQ path integral for Nordström gravity. Our choice of classical system will be the space-time metric, while the matter degrees of freedom will have a quantum nature.

A. CQ general relativity

When writing the path integral for General relativity, we follow Ref. [31] and write a manifestly covariant path integral over 4-geometries g , of the form given by Eq. (4),

$$\begin{aligned} \mathcal{Q}(\Sigma_f, \phi_f^+, \phi_f^-, t_f) &= \int \mathcal{D}g \mathcal{D}\phi^+ \mathcal{D}\phi^- \mathcal{N} e^{\mathcal{I}_{\text{CQ}}[g, \phi^+, \phi^-, t_i, t_f]} \\ &\times \mathcal{Q}(\Sigma_i, \phi_i^+, \phi_i^-, t_i), \end{aligned} \quad (16)$$

with

$$\begin{aligned} \mathcal{I}_{\text{CQ}}[g, \phi^\pm] &= \int_{t_i}^{t_f} dt d\vec{x} \left[i(\mathcal{L}_{\mathcal{Q}}[g, \phi^+] - [g, \phi^-]) \right. \\ &\quad - \frac{\text{Det}[-g]}{8} (T^{\mu\nu}[\phi^+] - T^{\mu\nu}[\phi^-]) \\ &\quad \times D_{0,\mu\nu\rho\sigma}[g] (T^{\rho\sigma}[\phi^+] - T^{\rho\sigma}[\phi^-]) \\ &\quad - \frac{\text{Det}[-g] c^8}{128\pi^2 G_N^2} \left(G^{\mu\nu} - \frac{8\pi G_N}{c^4} \bar{T}^{\mu\nu}[\phi^+, \phi^-] \right) \\ &\quad \left. \times D_{0,\mu\nu\rho\sigma}[g] \left(G^{\rho\sigma} - \frac{8\pi G_N}{c^4} \bar{T}^{\rho\sigma}[\phi^+, \phi^-] \right) \right], \end{aligned} \quad (17)$$

where $\mathcal{L}_{\mathcal{Q}}[g, \phi^\pm]$ is the Lagrangian for the quantum matter in curved space-time and we have suppressed the metric dependence of all terms in Einstein equations for clarity. Σ_i and Σ_f are the initial and final spatial surfaces. As a quick note, introducing initial and final boundary conditions may seem to reduce the set of diffeomorphisms to those that are foliation preserving at the spacelike boundaries. While this is true in the case when initial and final conditions are imposed on the path integral, in general, we expect that there should be a representation of the diffeomorphism group that acts on CQ observables, forming a CQ representation of the diffeomorphism group. We expect this representation to be nonunitary to incorporate the nonunitary CQ dynamics. For example, the generator of time translations is described by the generator of a CQ master equation. As such, we also expect the generators to be related to the CQ action appearing in the path integral. Studying CQ representations of space-time symmetries beyond that of initial-/final-state path integrals is beyond the scope of the current work. The term $\bar{T}^{\mu\nu}[\phi^+, \phi^-]$ indicates the average of the bra and ket stress-energy tensors, as in Eq. (7),

$$\bar{T}^{\mu\nu}[\phi^+, \phi^-] = \frac{1}{2} (T^{\mu\nu}[\phi^+] + T^{\mu\nu}[\phi^-]). \quad (18)$$

Here, we have assumed that the decoherence-diffusion tradeoff is saturated ($4D_0 = D_2^{-1}$). This is a manifestly diffeomorphism invariant hybrid path integral for General Relativity, and it is fully characterized by the tensor density $D_{0,\mu\nu\rho\sigma}[g]$. As explained in Ref. [31], if we were to choose D_0 such that it is positive semidefinite, we would have a consistent treatment of semiclassical General Relativity. Unfortunately, choosing a positive semidefinite D_0 and capturing the transverse part of the Einstein equations are not possible in Lorentzian signature. For example, one can consider

$$D_{0,\mu\nu\rho\sigma} = \frac{D_0(x)}{2\sqrt{-g}} (g_{\mu\rho}g_{\nu\sigma} + g_{\nu\rho}g_{\mu\sigma} - 2\beta g_{\mu\nu}g_{\rho\sigma}), \quad (19)$$

with D_0 a positive constant. In Lorentzian signature, this is not positive semidefinite. However, one can choose

$$D_{0,\mu\nu\rho\sigma} = \frac{D_0(x)}{\sqrt{-g}} g_{\mu\nu}g_{\rho\sigma}. \quad (20)$$

This will lead to a positive semidefinite path integral describing suppressed trajectories as they diffuse away from the *trace* of Einstein's equations.

Alternatively, if one is happy with a consistent diffeomorphism invariant toy model of gravity in the CQ framework, one can instead start with a scalar theory of gravity. In particular, Nordström gravity is an ideal candidate as a self-consistent theory of gravity that allows us to study the gravitational backreaction of space-time and quantum matter without worrying about the constraints of General relativity or the positivity of the full CQ path integral for GR. We also take this chance to address the presence of higher-order derivatives in the path integral action. The reader may be concerned about such actions being associated with Hamiltonians unbounded from below, as shown by Ostrogradsky [59]. However, problems arise only when assuming that the action generates deterministic evolution. Onsager-Machlup path integrals for stochastic processes often contain higher-order derivatives when presented in configuration space (see Chapter 12 of the work by Feynman-Hibbs [60]). The action is already composed of the equation of motion. Its variation should be interpreted as the most probable path between the set initial and final points of configuration space. For example, a stochastic harmonic oscillator will have a variation of its Onsager-Machlup action, which results in an apparent runaway of the solutions due to increasing the oscillation amplitudes. However, this increase is necessary to reach points that would normally be outside the range of the equations of motion (EOM) for fixed initial position and velocity. More about the relation between Ostrogradsky and CQ path integrals is analysed in Ref. [61].

B. CQ Nordström

To construct the CQ theory of Nordström gravity, we let ϕ_m denote the quantum matter degrees of freedom such that $\mathcal{L}_Q[g, \phi_m]$ is the matter Lagrangian (inclusive of the appropriate metric determinant factor). Nordström gravity can be derived classically from the action principle in the Jordan frame defined in Ref. [62], which we summarize in Appendix. In that derivation, a Lagrange multiplier is used to impose the conformal flatness of the space-time (14). Here, we are faced with two choices. We could insert the constraint in the protoaction directly or impose it through a delta functional. Given that in Nordström gravity matter fields do not couple to the Weyl tensor, we do not expect the backreaction of the quantum degrees of freedom to break the conformal flatness of the metric. After some consideration, one can realize that it is more sensible to choose the latter, imposing the constraint in a way that is more akin to a gauge fixing of the classical degrees of freedom. Therefore, we construct the protoaction for Nordström gravity with matter as

$$W_{\text{CQ}}[g_{\mu\nu}, \phi_m] = -\frac{c^4}{48\pi G_N} \int d^4x \sqrt{-g} \mathcal{R} + \int d^4x \mathcal{L}_Q[g, \phi_m]. \quad (21)$$

While this action might look similar to the Einstein-Hilbert action, one should notice the different coefficients of the gravitational sector and the different relative signs between the gravitational and matter part; both are required to obtain the correct Nordström dynamical equation. We are now in the position to write down the CQ path integral. We choose the trace realization of the decoherence coefficient in Eq. (20) and impose the conformal flatness constraint as a delta functional through a Lagrange multiplier $\lambda_{\mu}^{\nu\rho\sigma}$,

$$\begin{aligned} & \varrho(\Sigma_f, \phi_{m,f}^+, \phi_{m,f}^-, t_f) \\ &= \int \mathcal{D}g \mathcal{D}\phi_m^+ \mathcal{D}\phi_m^- \mathcal{D}\lambda_{\mu}^{\nu\rho\sigma} \mathcal{N} e^{\mathcal{I}_{\text{CQ}}(g_{\mu\nu}, \phi_m^+, \phi_m^-, t_f, t_i)} \\ & \quad \times \varrho(\Sigma_i, \phi_{m,i}^+, \phi_{m,i}^-, t_i), \end{aligned} \quad (22)$$

where the CQ action is

$$\begin{aligned} & \mathcal{I}_{\text{CQ}}(g_{\mu\nu}, \phi_m^+, \phi_m^-, t_f, t_i) \\ &= \int_{t_i}^{t_f} dt d\vec{x} \left[i(\mathcal{L}_Q[g, \phi_m^+] - \mathcal{L}_Q[g, \phi_m^-]) \right. \\ & \quad - \frac{D_0(x)}{2\sqrt{-g}} \frac{\delta \Delta W_{\text{CQ}}}{\delta g^{\mu\nu}} g_{\mu\nu} g_{\rho\sigma} \frac{\delta \Delta W_{\text{CQ}}}{\delta g^{\rho\sigma}} \\ & \quad \left. - \frac{2D_0(x)}{\sqrt{-g}} \frac{\delta \bar{W}_{\text{CQ}}}{\delta g^{\mu\nu}} g_{\mu\nu} g_{\rho\sigma} \frac{\delta \bar{W}_{\text{CQ}}}{\delta g^{\rho\sigma}} - i\lambda_{\mu}^{\nu\rho\sigma} C_{\nu\rho\sigma}^{\mu} \right], \end{aligned} \quad (23)$$

and we have saturated the decoherence diffusion tradeoff:

$$4D_0[g] = D_2^{-1}[g]. \quad (24)$$

Once we integrate over the Lagrange multiplier, the delta function will ensure that we only sum over 4-geometries that are conformally flat $C_{\nu\rho\sigma}^{\mu} = 0$. Any conformally flat metric can be written as $g_{\mu\nu} = e^{2\frac{\Phi}{c^2}} \eta_{\mu\nu}$ for some Φ by definition. Therefore, we find the Nordström hybrid path integral

$$\begin{aligned} & \varrho(\Sigma_f, \phi_{m,f}^+, \phi_{m,f}^-, t_f) \\ &= \int \mathcal{D}\Phi \mathcal{D}\phi_m^+ \mathcal{D}\phi_m^- \mathcal{N} \exp \left[\int dt d\vec{x} i(\mathcal{L}_Q[\phi_m^+] \right. \\ & \quad - \mathcal{L}_Q[\phi_m^-]) - \frac{\sqrt{-g} D_0(x)}{8} (T[\phi_m^+] - T[\phi_m^-])^2 \\ & \quad \left. - \frac{\sqrt{-g} c^8 D_0(x)}{1152\pi^2 G_N^2} \left(\mathcal{R} - \frac{24\pi G_N}{c^4} \bar{T}[\phi_m^+, \phi_m^-] \right)^2 \right] \\ & \quad \times \varrho(\Sigma_i, \phi_{m,i}^+, \phi_{m,i}^-, t_i), \end{aligned} \quad (25)$$

where we have suppressed the Φ dependence in \mathcal{R} , T , and \mathcal{L}_Q to lighten the notation. When integrating over conformally flat metrics, we include any Jacobian factor in the measure $\mathcal{D}g^{C=0} \sim \frac{2}{c^2} e^{\frac{2\Phi}{c^2}} \mathcal{D}\Phi$, as it will not be relevant to the Newtonian limit of interest in this paper. In particular, to leading order, we have that $Dg^{C=0} \sim \frac{2}{c^2} \mathcal{D}\Phi$. Since the action in Eq. (25) contains quantum terms proportional to the square of the stress-energy tensor, a sufficient condition for the path integral to be normalized is that the purely quantum part of the action $\mathcal{L}_Q[g, \phi^{\pm}]$ contains higher-derivative kinetic terms $\sim \dot{\phi}^2$ [63], which is suggestive that Eq. (25) describes an effective theory [47].

The action has the effect of diffusing away from the bra/ket averaged Nordström equations, while simultaneously decohering the quantum system according to the stress-energy tensor of the matter and the coupling $D_0[\Phi]$. Treated classically, the action is manifestly diffeomorphism invariant, which also includes the case where the diffeomorphism $\sigma: M \rightarrow M$ is dependent on the classical and quantum trajectories $\sigma[\Phi, \phi_m^{\pm}]$. However, just as for the classical theory, the CQ theory is not background independent and has a preferred frame given by the requirement that the metric is conformally flat.

We now wish to compute the Newtonian limit of the theory in order to gain insight into the Newtonian limit of more general classical-quantum theories and compare and contrast it with Ref. [33]. We recall that the final goal is to search for low-energy experimental signatures of CQ by treating the gravitational field classically. To that end, we shall take the quantum degrees of freedom to be described by a pressureless dust distribution $\hat{T}^{\mu\nu} = \hat{m}(x) U^{\mu} U^{\nu}$ where $\hat{m}(x)^{\pm}$ is a (smeared) quantum mass density.

Using the conformally flat metric and our choice of matter, we can rewrite some of the quantities in the path integral as

$$\sqrt{-g} = ce^{\frac{4\Phi}{c^2}}, \quad \mathcal{R}[\Phi] = -\frac{6\tilde{\square}e^{\frac{\Phi}{c^2}}}{c^2}e^{-\frac{3\Phi}{c^2}},$$

$$T^\pm[\Phi, m^\pm] = -e^{\frac{2\Phi}{c^2}}m^\pm(x), \quad (26)$$

and the path integral takes the form

$$q(\Sigma_f, m_f^+, m_f^-, t_f)$$

$$= \int \mathcal{D}\Phi \mathcal{D}m^+ \mathcal{D}m^- \mathcal{N} \exp \left[\int_{t_i}^{t_f} dt d\vec{x} i(\mathcal{L}_Q[\Phi, m^+] - \mathcal{L}_Q[\Phi, m^-]) - \frac{cD_0(x)e^{\frac{6\Phi}{c^2}}}{8}(m^+(x) - m^-(x))^2 - \frac{c^7 D_0(x)}{192\pi^2 G_N^2} \left(-e^{\frac{\Phi}{c^2}} \tilde{\square} e^{\frac{\Phi}{c^2}} + \frac{4\pi G_N e^{\frac{6\Phi}{c^2}}}{c^2} \bar{m}(x) \right)^2 \right]$$

$$\times q(\Sigma_i, m_i^+, m_i^-, t_i), \quad (27)$$

where $\bar{m}(x) = \frac{1}{2}(m^+(x) + m^-(x))$.

We take the Newtonian $c \rightarrow \infty$ limit of the metric perturbations. Carrying out the transformations, we get

$$\sqrt{-g} = ce^{\frac{4\Phi}{c^2}} \approx c \left(1 + \frac{4\Phi}{c^2} \right) + \mathcal{O}\left(\frac{1}{c^3}\right), \quad (28)$$

$$e^{\frac{\Phi}{c^2}} \tilde{\square} e^{\frac{\Phi}{c^2}} \approx \frac{\tilde{\square}\Phi}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right), \quad (29)$$

$$\frac{4\pi G_N e^{\frac{6\Phi}{c^2}}}{c^2} \bar{m}(x) \approx \frac{4\pi G_N}{c^2} \bar{m}(x) + \mathcal{O}\left(\frac{1}{c^4}\right). \quad (30)$$

To leading order in c , we then arrive at the Newtonian limit of the CQ scalar gravity theory

$$q(\Sigma_f, m_f^+, m_f^-, t_f) = \int \mathcal{D}\Phi \mathcal{D}m^+ \mathcal{D}m^- \mathcal{N} e^{\mathcal{I}_{\text{CQ}}[\Phi, m^+, m^-, t_i, t_f]}$$

$$\times q(\Sigma_i, m_i^+, m_i^-, t_i), \quad (31)$$

with CQ action given by

$$\mathcal{I}_{\text{CQ}}[\Phi, m^+, m^-, t_i, t_f] = \int_{t_i}^{t_f} dt d\vec{x} \left[i(\mathcal{L}_Q[m^+] - \mathcal{V}_I[\Phi, m^+] - \mathcal{L}_Q[m^-] + \mathcal{V}_I[\Phi, m^-]) - \tilde{D}_0(x)(m(x)^+ - m(x)^-)^2 - \frac{c^2 \tilde{D}_2^{-1}(x)}{6} \left(\frac{\tilde{\square}\Phi}{4\pi G} - \bar{m}(x) \right)^2 \right], \quad (32)$$

where $\mathcal{V}_I[\Phi, m^\pm]$ is the interaction potential coming from the expansion of the metric determinant in the quantum

Lagrangians. For example, in the spirit of Ref. [33], one could have $\mathcal{V}_I = \Phi m^\pm$. We have defined $\tilde{D}_0(x) = \frac{cD_0(x)}{8}$, $\tilde{D}_2^{-1}(x) = \frac{cD_2(x)}{2}$, which are related to the decoherence and diffusion coefficients of the Newtonian potential. With these redefinitions, these coefficients relate to physically observable quantities: \tilde{D}_0 quantifies the suppression of quantum trajectories away from $m(x)^+ = m(x)^-$, which is the decohered trajectory. We point out that the \mathcal{V}_I is a quantum term with classical control; it does not contain information about the backreaction of the classical on the quantum degrees of freedom, which is instead present in the decoherence term. On the hand, $\tilde{D}_2(x) = \frac{1}{2}\sigma_\Phi^2$, where σ_Φ quantifies the variance away from the semiclassical Newtonian solution. They saturate the decoherence-diffusion relation:

$$4\tilde{D}_0(x) = \tilde{D}_2^{-1}(x). \quad (33)$$

In Eq. (32), we have explicitly kept the d'Alembert operator to highlight the fact that, differently from Ref. [33], ϕ is in principle still a dynamical variable as it is not constrained by the Arnowitt–Deser–Misner (ADM) constraints. Keeping the d'Alembert operator is also required for normalization [63]. However, in the slow-moving limit, we recover the randomly sourced Poisson equation and exactly match the Newtonian limit of Ref. [33].

Although the scalar theory is a toy model, it is worth highlighting some of its appealing features which we expect to apply to more general CQ theories. First, as mentioned, we have both diffusion in the Newtonian potential and decoherence in the quantum system, by an amount quantified by Eq. (33). More generally, we expect that the amount of diffusion in the Newtonian potential will be lower bounded by (33), which is an experimental signature of classical-quantum theories [27]. Indeed, we see the same decoherence-diffusion relation between the diffusion of the Newtonian potential away from its averaged solution, and the decoherence rate into the mass eigenbasis, as for the Newtonian path integral of Ref. [33].

Second, expanding out Eqs. (25) and (32), we see that all the \pm cross-terms cancel so that the path integral preserves the purity of the quantum state even though the state decoheres into the mass eigenbasis. Such classical-quantum theories, therefore, provide a natural mechanism to describe wave function collapse via the interaction of a classical field with a quantum one. Moreover, the fact that the classical field is dynamical is enough to restore apparent diffeomorphism invariance in the theory, which can be seen via the diffeomorphism invariant action in (25). The fact that gravity interacts with matter through its stress-energy tensor provides an amplification mechanism by which small masses can maintain coherence, while macroscopic objects will be decohered. Indeed, if one disregards the classical degrees of freedom, then the

resulting dynamics are very similar to the dynamics of collapse theories [36–40], but we see that the full theory is diffeomorphism invariant due to the fact that we consider a dynamical classical field.

Let us finally comment on the continuity properties of the Nordström theory and its Newtonian limit. Typically, in path integral approaches, the path integral can be understood as an integral over paths that are (almost surely) continuous. The reason for this is that they typically involve kinetic terms $\frac{i}{\hbar} [\frac{1}{\delta t} (x_{t+\delta t} - x_t)]^2$ which give a highly oscillatory contribution to the path integral $\sim e^{\frac{i}{\hbar} \dots}$ if the paths are discontinuous. For the Nordström theory, we expect similar behavior for the gravitational field in the full path integral [Eq. (25)] due to the $-\frac{\mathcal{R}^2}{4D_2}$ term, which includes kinetic terms through $\tilde{\square}\Phi$. However, in the $c \rightarrow \infty$ limit, we are led to neglect such terms, leading to a discontinuous path integral in Eq. (32). This is a remnant of the approximation, and we expect that any physical measurable quantity of interest should be smeared over a timescale to reflect this.

V. DISCUSSION

In this work, we have constructed the covariant path integral of a full diffeomorphism invariant theory of CQ Nordström gravity and derived its Newtonian limit. The result matches the Newtonian limit obtained in Ref. [33], where we started from general relativity and proceeded with gauge fixing the Newtonian metric. In both cases, the nonrelativistic path integral describes a classical Newtonian gravitational field diffusing around Poisson’s equation of motion. At the same time, quantum matter degrees of freedom decohere into mass eigenstates due to the back-reaction of the classical geometry. For the dynamics to be completely positive, the amount of diffusion is lower bounded by the coherence time for superpositions of mass distributions. Differently from Ref. [33], our choice of classical system was Nordström’s scalar theory of gravity. While Nordström’s theory does not accurately describe all gravitational phenomena, it is nonetheless a self-consistent theory of relativistic gravitation. The main appeal of this choice is that it allows us to bypass any discussion or concern regarding gravitational constraints, serving as proof that there is no fundamental impediment to constructing a positive diffeomorphism invariant theory of CQ gravity. We expect the scalar theory to be an interesting toy model for exploring conceptual issues around classical-quantum gravity theories, for example, whether or not they can be made renormalizable [61].

One can also study the Nordström theory as a toy model for exploring experimental signatures of CQ gravity since the theory has the same Newtonian limit [33]. Tests of CQ gravity include applications to cosmology and galactic rotation curves [64] and proposals for the detection of gravitationally induced entanglement between masses in interferometric setups [65–73], c.f. [74], which may

become feasible in the next decade or two. There are also experiments that can exclude classical theories of space-time through the decoherence vs diffusion tradeoff [27], as in Eq. (33). So far, the upper bounds on diffusion have been found from precision mass experiments measuring the variation in acceleration experienced by mass in a Newtonian potential. Tighter bounds would follow from increased precision in the readings of acceleration experienced by the masses. On the other hand, improved decoherence times would lead to stronger lower bounds on D_2 and further squeeze theories in which gravity remains classical. As explained in detail in Ref. [27], since the tradeoff is in terms of the inverse Lindbladian coupling D_0^{-1} , one can also constrain classical theories of gravity by bounds on anomalous heating of the quantum system [75–88]. Other experimental proposals look for coherence or correlations in the gravitational interaction [89,90]. The variety of experimental proposals and new theoretical tools suggest that probing the quantum vs classical nature of space-time can be accomplished at low energy and is likely to shed light on attempts to reconcile quantum theory with general relativity.

ACKNOWLEDGMENTS

We would like to thank Isaac Layton, Emanuele Panella, Maite Arcos, and Andrew Svesko for valuable discussions. We especially thank Bill Unruh who suggested we look at Nordström gravity as an interesting toy model. J. O. is supported by an EPSRC Established Career Fellowship and a Royal Society Wolfson Merit Award. A. R. acknowledges financial support from EPSRC. This research was supported by the National Science Foundation under Grant No. NSF PHY11-25915 and by the Simons Foundation *It from Qubit* Network.

APPENDIX: NORDSTROM GRAVITY FROM ACTION VARIATION

In this Appendix, we summarize the derivation of the Nordström equation of motion from an action principle. This derivation is discussed in detail in Ref. [62]. To begin, one considers an action similar to that of Eq. (21),

$$\mathcal{S}[g_{\mu\nu}, \phi_m, \lambda_\mu^{\nu\rho\sigma}] = -\frac{c^4}{48\pi G_N} \int d^4x \sqrt{-g} (\mathcal{R} + \lambda_\mu^{\nu\rho\sigma} C_{\nu\rho\sigma}^\mu) + \int d^4x \mathcal{L}_Q[g, \phi_m], \quad (\text{A1})$$

where $\lambda_\mu^{\nu\rho\sigma} C_{\nu\rho\sigma}^\mu$ is the Weyl tensor which is constrained to vanish through the Lagrange multiplier $\lambda_\mu^{\nu\rho\sigma}$. Here, the Lagrange multiplier is directly inserted in the action. When this action is varied with respect to the matter degrees of freedom, the Lagrange multiplier, and the metric, one obtains for its extrema

$$\frac{\delta\mathcal{S}}{\delta\phi_m} = \nabla_\nu T_m^{\mu\nu} = 0, \quad \frac{\delta\mathcal{S}}{\delta\lambda_{\nu\rho\sigma}^\mu} = C_{\nu\rho\sigma}^\mu = 0, \quad (\text{A2})$$

which impose energy conservation and conformal flatness of the metric on the constraint surface $g_{\mu\nu} - e^{\frac{2\phi}{c^2}}\eta_{\mu\nu} \approx 0$. Lastly,

$$\frac{\delta\mathcal{S}}{\delta g^{\mu\nu}} = \begin{cases} \mathcal{R} - \frac{24\pi G_N}{c^4} T_m = 0 & \text{trace part} \\ \frac{\partial^\alpha \partial_\beta \lambda_{\mu\nu}^\beta}{\phi^2} = -\frac{24\pi G_N}{c^4} (T_{m,\mu\nu} - \mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}(T_m + \mathcal{R})) & \text{traceless part.} \end{cases} \quad (\text{A3})$$

This leaves the equation of Jordan's frame version of Nordström's final theory to be

$$\mathcal{R} - \frac{24\pi G_N}{c^4} T_m = 0, \quad \nabla_\nu T_m^{\mu\nu} = 0, \quad C_{\nu\rho\sigma}^\mu = 0. \quad (\text{A4})$$

The traceless part can be written in terms of flat covariant derivatives using the conformal flatness of the metric

$$\mathcal{R}_{\mu\nu} = -e^{-\frac{\phi}{c^2}} \partial_\mu \partial_\nu e^{\frac{\phi}{c^2}} - \eta_{\mu\nu} e^{-\frac{\phi}{c^2}} \tilde{\square} e^{\frac{\phi}{c^2}} + 4e^{-\frac{2\phi}{c^2}} \partial_\mu e^{\frac{\phi}{c^2}} \partial_\nu e^{\frac{\phi}{c^2}} - \eta_{\mu\nu} e^{-\frac{2\phi}{c^2}} \partial_\rho \partial^\rho e^{\frac{\phi}{c^2}} \quad (\text{A5})$$

$$\mathcal{R} = -\frac{6\tilde{\square} e^{\frac{\phi}{c^2}}}{c^2} e^{-\frac{3\phi}{c^2}}. \quad (\text{A6})$$

This is often found in the literature written using the notation $e^{\frac{\phi}{c^2}} = \phi$, which results in

$$\mathcal{R}_{\mu\nu} = -\frac{\partial_\mu \partial_\nu \phi}{\phi} - \eta_{\mu\nu} \frac{\tilde{\square} \phi}{\phi} + 4 \frac{\partial_\mu \phi \partial_\nu \phi}{\phi^2} - \eta_{\mu\nu} \frac{\partial_\rho \partial^\rho \phi}{\phi^2}, \quad (\text{A7})$$

$$\mathcal{R} = -\frac{6\tilde{\square} \phi}{c^2 \phi^3}. \quad (\text{A8})$$

Once the solution for the conformal factor ϕ is known, the traceless equation is an equation of motion for the Lagrange multiplier $\lambda_{\mu\nu\rho\sigma}$. However, the system is undefined as λ has 10 components (it has the same symmetries as the Weyl tensor), but the system of equation composed of (A2) and (A5), being traceless, has only 9. Classically, this is not a problem, as the Lagrange multiplier does not enter the equation of motion for the Nordström field ϕ . For the CQ equations considered in Sec. IV, the Lagrange multiplier enforces a delta function δC in the path integral. This allows us to only consider diffusion away from the trace of the equations of motion, which are exactly the Nordström field equations.

-
- [1] Iwao Sato, An attempt to unite the quantum theory of wave field with the theory of general relativity, *Sci. Rep. Tohoku Univ. Ser. 1: Phys. Chem. Astron.* **33**, 30 (1950).
- [2] Christian Møller, The energy-momentum complex in general relativity and related problems, *Théor. Relat. Gravit. Colloq. Int. CNRS* **91**, 2 (1962).
- [3] Leon Rosenfeld, On quantization of fields, *Nucl. Phys.* **40**, 353 (1963).
- [4] Nicolas Gisin, Stochastic quantum dynamics and relativity, *Helv. Phys. Acta* **62**, 363 (1989).
- [5] Nicolas Gisin, Weinberg's non-linear quantum mechanics and supraluminal communications, *Phys. Lett. A* **143**, 1 (1990).
- [6] Joseph Polchinski, Weinberg's nonlinear quantum mechanics and the Einstein-Podolsky-Rosen paradox, *Phys. Rev. Lett.* **66**, 397 (1991).
- [7] Jonathan Oppenheim, A postquantum theory of classical gravity?, *Phys. Rev. X* **13**, 041040 (2023).
- [8] Philippe Blanchard and Arkadiusz Jadczyk, On the interaction between classical and quantum systems, *Phys. Lett. A* **175**, 157 (1993).
- [9] P. Blanchard and Arkadiusz Jadczyk, Event-enhanced quantum theory and piecewise deterministic dynamics, *Ann. Phys. (Berlin)* **507**, 583 (1995).
- [10] Lajos Diosi, Quantum dynamics with two Planck constants and the semiclassical limit, [arXiv:quant-ph/9503023](https://arxiv.org/abs/quant-ph/9503023).
- [11] Lajos Diósi, Hybrid quantum-classical master equations, *Phys. Scr.* **2014**, 014004 (2014).
- [12] D. Kafri, J.M. Taylor, and G.J. Milburn, A classical channel model for gravitational decoherence, *New J. Phys.* **16**, 065020 (2014).
- [13] Antoine Tilloy and Lajos Diósi, Sourcing semiclassical gravity from spontaneously localized quantum matter, *Phys. Rev. D* **93**, 024026 (2016).
- [14] Antoine Tilloy and Lajos Diósi, Principle of least decoherence for Newtonian semiclassical gravity, *Phys. Rev. D* **96**, 104045 (2017).
- [15] David Poulin and John Preskill, Information loss in quantum field theories, *Frontiers of Quantum Information Physics* (KITP, 2017).
- [16] Niels Bohr and Leon Rosenfeld, On the question of the measurability of electromagnetic field quantities, edited by

- John A. Wheeler and Wojciech H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, NJ, 1983), pp. 479–522, originally published as “Zur Frage der Messbarkeit der Elektromagnetischen Feldgrößen” (Mat.-fys. Medd Dan. Vid Selsk. 12), translated by Aage Petersen.
- [17] Cécile M. DeWitt and Dean Rickles, *The Role of Gravitation in Physics: Report from the 1957 Chapel Hill Conference* (Epubli, Berlin, 2011), Vol. 5.
- [18] R. P. Feynman, *Feynman Lectures on Gravitation*, edited by F. B. Morinigo, W. G. Wagner, and B. Hatfield (Addison-Wesley, Reading, USA, 1996), pp. 10–11.
- [19] Bryce S. DeWitt, Definition of commutators via the uncertainty principle, *J. Math. Phys. (N.Y.)* **3**, 619 (1962).
- [20] Bryce S. DeWitt, Quantum theory of gravity. i. the canonical theory, *Phys. Rev.* **160**, 1113 (1967).
- [21] Kenneth Eppley and Eric Hannah, The necessity of quantizing the gravitational field, *Found. Phys.* **7**, 51 (1977).
- [22] Yakir Aharonov and Daniel Rohrlich, *Quantum Paradoxes: Quantum Theory for the Perplexed* (Wiley-VCH, New York, 2003), pp. 212–213.
- [23] Daniel R. Terno, Inconsistency of quantum–classical dynamics, and what it implies, *Found. Phys.* **36**, 102 (2006).
- [24] Chiara Marletto and Vlatko Vedral, Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity, *Phys. Rev. Lett.* **119**, 240402 (2017).
- [25] Thomas D. Galley, Flaminia Giacomini, and John H. Selby, Any consistent coupling between classical gravity and quantum matter is fundamentally irreversible, *Quantum* **7**, 1142 (2023).
- [26] Jonathan Oppenheim, Carlo Sparaciari, Barbara Šoda, and Zachary Weller-Davies, The two classes of hybrid classical-quantum dynamics, [arXiv:2203.01332](https://arxiv.org/abs/2203.01332).
- [27] Jonathan Oppenheim, Carlo Sparaciari, Barbara Šoda, and Zachary Weller-Davies, Gravitationally induced decoherence vs space-time diffusion: Testing the quantum nature of gravity, *Nat. Commun.* **14**, 7910 (2023).
- [28] Jonathan Oppenheim, Carlo Sparaciari, Barbara Šoda, and Zachary Weller-Davies, Objective trajectories in hybrid classical-quantum dynamics, *Quantum* **7**, 891 (2023).
- [29] Isaac Layton, Jonathan Oppenheim, and Zachary Weller-Davies, A healthier semi-classical dynamics, [arXiv:2208.11722](https://arxiv.org/abs/2208.11722).
- [30] Jonathan Oppenheim and Zachary Weller-Davies, Path integrals for classical-quantum dynamics, [arXiv:2301.04677](https://arxiv.org/abs/2301.04677).
- [31] Jonathan Oppenheim and Zachary Weller-Davies, Covariant path integrals for quantum fields back-reacting on classical space-time, [arXiv:2302.07283](https://arxiv.org/abs/2302.07283).
- [32] Lajos Diósi, The gravity-related decoherence master equation from hybrid dynamics, *J. Phys. Conf. Ser.* **306**, 012006 (2011).
- [33] Isaac Layton, Jonathan Oppenheim, Andrea Russo, and Zachary Weller-Davies, The weak field limit of quantum matter back-reacting on classical spacetime, *J. High Energy Phys.* **08** (2023) 163.
- [34] Jonathan Oppenheim and Zachary Weller-Davies, The constraints of post-quantum classical gravity, *J. High Energy Phys.* **02** (2022) 080.
- [35] Jonathan Oppenheim, The constraints of a continuous realisation of hybrid classical-quantum gravity (to be published).
- [36] G. C. Ghirardi, A. Rimini, and T. Weber, A model for a unified quantum description of macroscopic and microscopic systems, in *Quantum Probability and Applications*, edited by L. Accardi *et al.* (Springer, Berlin, 1985).
- [37] Philip M. Pearle, Combining stochastic dynamical state vector reduction with spontaneous localization, *Phys. Rev. A* **39**, 2277 (1989).
- [38] Gian Carlo Ghirardi, Philip Pearle, and Alberto Rimini, Markov processes in Hilbert space and continuous spontaneous localization of systems of identical particles, *Phys. Rev. A* **42**, 78 (1990).
- [39] Angelo Bassi and GianCarlo Ghirardi, Dynamical reduction models, *Phys. Rep.* **379**, 257 (2003).
- [40] G. C. Ghirardi, A. Rimini, and T. Weber, Unified dynamics for microscopic and macroscopic systems, *Phys. Rev. D* **34**, 470 (1986).
- [41] Jonathan Oppenheim and Zachary Weller-Davies, Lorentz invariant information destruction is not Lorentz covariant (unpublished).
- [42] Pietro Capuozzo, Emanuele Panella, Tancredi Schettini Gherardini, and Dimitri D. Vvedensky, Path integral Monte Carlo method for option pricing, *Physica (Amsterdam)* **581A**, 126231 (2021).
- [43] Guido Montagna, Oreste Nicosini, and Nicola Moreni, A path integral way to option pricing, *Physica (Amsterdam)* **310A**, 450 (2002).
- [44] Phelim P. Boyle, Options: A Monte Carlo approach, *J. Financ. Econ.* **4**, 323 (1977).
- [45] M. Creutz and B. Freedman, A statistical approach to quantum mechanics, *Ann. Phys. (N.Y.)* **132**, 427 (1981).
- [46] Michael Creutz, Laurence Jacobs, and Claudio Rebbi, Monte Carlo computations in lattice gauge theories, *Phys. Rep.* **95**, 201 (1983).
- [47] H. Georgi, Effective field theory, *Annu. Rev. Nucl. Part. Sci.* **43**, 209 (1993).
- [48] Hagen Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009).
- [49] L. M. Sieberer, M. Buchhold, and S. Diehl, Keldysh field theory for driven open quantum systems, *Rep. Prog. Phys.* **79**, 096001 (2016).
- [50] R. Alicki, M. Fannes, and A. Verbeure, Unstable particles and the poincare semigroup in quantum field theory, *J. Phys. A* **19**, 919 (1986).
- [51] Lajos Diósi, Is there a relativistic Gorini-Kossakowski-Lindblad-Sudarshan master equation?, *Phys. Rev. D* **106**, L051901 (2022).
- [52] R. P. Feynman and F. L. Vernon, The theory of a general quantum system interacting with a linear dissipative system, *Ann. Phys. (N.Y.)* **24**, 118 (1963).
- [53] G. Nordström, Zur theorie der gravitation vom standpunkt des relativitätssprinzips, *Ann. Phys. (N.Y.)* **42**, 533 (1913).
- [54] Finn Ravndal, Scalar gravitation and extra dimensions, *Comment. Phys. Math. Soc. Sci. Fenn.* **166**, 151 (2004).
- [55] John D. Norton, Einstein, Nordström and the early demise of scalar, Lorentz-covariant theories of gravitation, *Arch. Hist. Exact Sci.* **45**, 17 (1992).

- [56] F. W. Dyson, A. S. Eddington, and C. Davidson, A determination of the deflection of light by the sun's gravitational field, from observations made at the total eclipse of may 29, 1919, *Phil. Trans. R. Soc. A* **220**, 291 (1920).
- [57] Matthias Bartelmann, Gravitational lensing, *Classical Quantum Gravity* **27**, 233001 (2010).
- [58] Oliver Pooley, Background independence, diffeomorphism invariance, and the meaning of coordinates, *Einstein Stud.* **13**, 105 (2017).
- [59] Michael Ostrogradsky, *Mémoire sur les équations différentielles relatives au problème des isopérimètres* (Mem. Acad. St. Petersburg, 1850), 6e série, pp. 385–517.
- [60] Richard P. Feynman and Albert R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), Sect. 12.6 on Brownian motion.
- [61] Andrzej Grudka, Jonathan Oppenheim, Andrea Russo, and Muhammad Sajjad, Renormalisation of postquantum-classical gravity, [arXiv:2402.17844](https://arxiv.org/abs/2402.17844).
- [62] Nathalie Deruelle, Nordström's scalar theory of gravity and the equivalence principle, *Gen. Relativ. Gravit.* **43**, 3337 (2011).
- [63] Zachary Weller-Davies, Quantum gravity with dynamical wave-function collapse via a classical scalar field, [arXiv:2402.17024](https://arxiv.org/abs/2402.17024).
- [64] Jonathan Oppenheim and Andrea Russo, Anomalous contribution to galactic rotation curves due to stochastic spacetime, [arXiv:2402.19459](https://arxiv.org/abs/2402.19459).
- [65] Dvir Kafri and J. M. Taylor, A noise inequality for classical forces, [arXiv:1311.4558](https://arxiv.org/abs/1311.4558).
- [66] D. Kafri, G. J. Milburn, and J. M. Taylor, Bounds on quantum communication via Newtonian gravity, *New J. Phys.* **17**, 015006 (2015).
- [67] Sougato Bose, Anupam Mazumdar, Gavin W. Morley, Hendrik Ulbricht, Marko Toroš, Mauro Paternostro, Andrew A. Geraci, Peter F. Barker, M. S. Kim, and Gerard Milburn, Spin entanglement witness for quantum gravity, *Phys. Rev. Lett.* **119**, 240401 (2017).
- [68] C. Marletto and V. Vedral, Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity, *Phys. Rev. Lett.* **119**, 240402 (2017).
- [69] R. J. Marshman, Anupam Mazumdar, and Sougato Bose, Locality and entanglement in table-top testing of the quantum nature of linearized gravity, *Phys. Rev. A* **101**, 052110 (2020).
- [70] Julen S. Pedernales, Kirill Streltsov, and Martin B. Plenio, Enhancing gravitational interaction between quantum systems by a massive mediator, *Phys. Rev. Lett.* **128**, 110401 (2022).
- [71] Marios Christodoulou, Andrea Di Biagio, Markus Aspelmeyer, Časlav Brukner, Carlo Rovelli, and Richard Howl, Locally mediated entanglement in linearized quantum gravity, *Phys. Rev. Lett.* **130**, 100202 (2023).
- [72] Daine L. Danielson, Gautam Satishchandran, and Robert M. Wald, Gravitationally mediated entanglement: Newtonian field versus gravitons, *Phys. Rev. D* **105**, 086001 (2022).
- [73] Daniel Carney, Holger Müller, and Jacob M. Taylor, Using an atom interferometer to infer gravitational entanglement generation, *PRX Quantum* **2**, 030330 (2021).
- [74] Kirill Streltsov, Julen Simon Pedernales, and Martin Bodo Plenio, On the significance of interferometric revivals for the fundamental description of gravity, *Universe* **8**, 58 (2022).
- [75] T. Banks, M. E. Peskin, and L. Susskind, Difficulties for the evolution of pure states into mixed states, *Nucl. Phys.* **B244**, 125 (1984).
- [76] Gian Carlo Ghirardi, Alberto Rimini, and Tullio Weber, Unified dynamics for microscopic and macroscopic systems, *Phys. Rev. D* **34**, 470 (1986).
- [77] L. E. Ballentine, Failure of some theories of state reduction, *Phys. Rev. A* **43**, 9 (1991).
- [78] Philip Pearle, James Ring, Juan I. Collar, and Frank T. Avignone, The CSL collapse model and spontaneous radiation: An update, *Found. Phys.* **29**, 465 (1999).
- [79] Angelo Bassi, Emiliano Ippoliti, and Bassano Vacchini, On the energy increase in space-collapse models, *J. Phys. A* **38**, 8017 (2005).
- [80] Stephen L. Adler, Lower and upper bounds on CSL parameters from latent image formation and IGM heating, *J. Phys. A* **40**, 2935 (2007).
- [81] Kinjal Lochan, Suratna Das, and Angelo Bassi, Constraining continuous spontaneous localization strength parameter λ from standard cosmology and spectral distortions of cosmic microwave background radiation, *Phys. Rev. D* **86**, 065016 (2012).
- [82] Stefan Nimmrichter, Klaus Hornberger, and Klemens Hammerer, Optomechanical sensing of spontaneous wave-function collapse, *Phys. Rev. Lett.* **113**, 020405 (2014).
- [83] Mohammad Bahrami, Angelo Bassi, and Hendrik Ulbricht, Testing the quantum superposition principle in the frequency domain, *Phys. Rev. A* **89**, 032127 (2014).
- [84] Franck Laloë, William J. Mullin, and Philip Pearle, Heating of trapped ultracold atoms by collapse dynamics, *Phys. Rev. A* **90**, 052119 (2014).
- [85] Mohammad Bahrami, M. Paternostro, Angelo Bassi, and H. Ulbricht, Proposal for a noninterferometric test of collapse models in optomechanical systems, *Phys. Rev. Lett.* **112**, 210404 (2014).
- [86] Daniel Goldwater, Mauro Paternostro, and P. F. Barker, Testing wave-function-collapse models using parametric heating of a trapped nanosphere, *Phys. Rev. A* **94**, 010104 (2016).
- [87] Antoine Tilloy and Thomas M. Stace, Neutron star heating constraints on wave-function collapse models, *Phys. Rev. Lett.* **123**, 080402 (2019).
- [88] Sandro Donadi, Kristian Piscicchia, Catalina Curceanu, Lajos Diósi, Matthias Laubenstein, and Angelo Bassi, Underground test of gravity-related wave function collapse, *Nat. Phys.* **17**, 1 (2020).
- [89] Ludovico Lami, Julen S. Pedernales, and Martin B. Plenio, Testing the quantum nature of gravity without entanglement, *Phys. Rev. X* **14**, 021022 (2024).
- [90] Serhii Kryhin and Vivishek Sudhir, Distinguishable consequence of classical gravity on quantum matter, [arXiv:2309.09105](https://arxiv.org/abs/2309.09105).