Scalar induced gravitational waves in metric teleparallel gravity with the Nieh-Yan term

Fengge Zhang,[1,2] Jia-Xi Feng,[2] and Xian Gao[2,*]

[1]Henan Academy of Sciences, Zhengzhou 450046, Henan, China
[2]School of Physics and Astronomy, Sun Yat-sen University, Zhuhai 519082, China

(Received 9 April 2024; accepted 17 June 2024; published 22 July 2024)

We investigate the scalar induced gravitational waves (SIGWs) in metric teleparallel gravity with the Nieh-Yan (NY) term, which results in parity violation during the radiation-dominated era. By solving the equations of motion of linear scalar perturbations from both the metric and the tetrad fields, we obtain the corresponding analytic expressions. Then, we calculate the SIGWs in metric teleparallel gravity with the NY term and evaluate the energy density of SIGWs with a monochromatic power spectrum numerically. We find that the spectrum of the energy density of SIGWs in metric teleparallel gravity with the NY term is significantly different from that in general relativity (GR), which makes metric teleparallel gravity distinguishable from GR.

DOI: 10.1103/PhysRevD.110.023537

I. INTRODUCTION

Gravitational waves (GWs) play an important role in exploring the early universe. The successful detection of GWs generated from the merger of compact objects by the Laser Interferometer Gravitational-Wave Observatory (LIGO) scientific collaboration and the Virgo collaboration [1–10] opens a new window to probe the nature of gravity in the strong gravitational field and nonlinear regime. It also marks the dawn of multimessenger astronomy. The scalar induced gravitational waves (SIGWs) originated during the early universe due to the nonlinear interaction between scalar and tensor perturbation, which contribute to the stochastic gravitational wave background, have attracted much attention recently [11–32]. The frequency of SIGWs varies widely, and SIGWs can be detected by space-based GW detectors like the Laser Interferometer Space Antenna (LISA) [33,34], Taiji [35], TianQin [36,37], the Deci-hertz Interferometer Gravitational-Wave Observatory (DECIGO) [38], as well as by the pulsar timing array (PTA) [39–42] and the Square Kilometer Array [43]. The recent stochastic GW signal captured by PTA [44–50] can also be explained with SIGWs [51–61].

The gravity theory with parity-violating (PV) terms attracted significant attention recently [62–78]. On the one hand, the violation of parity symmetry in a weak interaction [79,80] prompts the investigation of whether such parity violation occurs in gravitational interaction. On the other hand, the recent studies on galaxy trispectrum and the cross-correlation of the E and B mode polarization of the cosmic microwave background (CMB) [81–84] have hinted at the existence of parity violation in our universe. The PV scalar trispectrum was also studied in [85–89].

The simplest PV term in the Riemannian geometry is the Chern-Simons (CS) term, which is quadratic in the Riemann tensor. CS gravity was initially proposed in four-dimensional spacetime in [90], and has since been extensively studied in cosmology, GWs, and primordial non-Gaussianity [91–102]. Recently, the SIGWs in CS gravity have also been studied [103,104]. However, CS gravity suffers from Ostrogradsky instability [105] and propagates ghost modes [101,102] due to the presence of higher-derivative field equations. As a result, it can only be treated as a low-energy effective theory. To cure this issue, CS gravity was generalized to ghost-free PV gravity [105].

Considering gravity theory beyond Riemann geometry, various gravity theories based on teleparallel geometry have been proposed [106–127]. Interestingly, the symmetric teleparallel gravity with the method of spatially covariant gravity was also studied [128] recently. Within the framework of metric teleparallel gravity, similar to the CS gravity, the simplest PV term is $T_{A\mu\nu}\mathcal{T}^{A\mu\nu}$ [129], where $T_{A\mu\nu}$ represents the torsion tensor and $\mathcal{T}^{A\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}T^{A}{}_{\rho\sigma}/2$ is the dual of the torsion tensor. This term is a generalization of the Nieh-Yan (NY) term, initially proposed in Riemann-Cartan geometry [130,131], and the thermal Nieh-Yan anomaly in Weyl superfluids was also studied [132]. The simplest metric teleparallel gravity with the PV term was constructed by adding the NY term to the metric teleparallel equivalent Einstein-Hilbert action. Furthermore, the linear cosmological perturbations in this theory have also been investigated [129,133–135].

*Contact author: gaoxian@mail.sysu.edu.cn

Unlike CS gravity, the equations of motion (EOMs) in the above-mentioned gravity model do not involve higher-order derivatives, and thus the Ostrogradsky instability is effectively avoided. In this paper, we concentrate on the nonlinear perturbation in metric teleparallel gravity with the NY term, specifically investigating the behavior of the SIGWs. However, when the perturbations beyond linear order are taken into account, inconsistencies may arise in the simplest PV metric teleparallel gravity mentioned above. This situation is similar to what we encountered in symmetric teleparallel gravity with the PV term when calculating SIGWs [136]. Briefly, the theory contains extra scalar degrees of freedom due to the PV term, which, however, do not manifest themselves at the linear order around a homogeneous and isotropic background. This is reminiscent of the so-called strong coupling problem in the study of Hořava gravity [137–141] and $f(T)$ gravity [142–147].

In this paper, we will demonstrate that the simplest metric teleparallel gravity with the NY term also suffers from such a strong coupling problem. Specifically, the scalar perturbations arising from the tetrads do not possess their own linear EOMs, while appearing in the EOM of the SIGWs. To avoid this problem, we replace the metric teleparallel equivalent Einstein-Hilbert action with a general linear combination of quadratic monomials of the torsion tensor. We then derive the EOMs governing the perturbations originating from the tetrads and determine their solutions during the radiation-dominated era. Utilizing these results, we calculate the contribution of both the NY term and the scalar perturbations in the tetrads to the energy density of SIGWs in our model, respectively.

This paper is organized as follows. In Sec. II, we briefly introduce the metric teleparallel gravity with the NY term. In Sec. III, we give the EOMs for both the background evolution and the linear scalar perturbations, and we then solve these EOMs during the radiation-dominated era. In Sec. IV, we derive the EOM of SIGWs and calculate the power spectra of the SIGWs. To analyze the feature of SIGWs, we compute the energy density of SIGWs with the monochromatic power spectrum of primordial curvature perturbation. Our results are summarized in Sec. V. The analytic expressions of the integral kernel are included in the Appendix.

II. THE METRIC TELEPARALLEL GRAVITY WITH THE NIEH-YAN TERM

In this section, we review the metric teleparallel gravity. In this paper, we use the following conventions: the flat space metric is $\eta_{AB} = \mathrm{diag}(+1,-1,-1,-1)$, and lowercase Latin letters $(i, j, \ldots)$ denote purely spatial indices, while capital Latin letters $(A, B, \ldots)$ and Greek alphabet letters $(\mu, \nu, \ldots)$ are used for Lorentz indices and spacetime indices, respectively. The metric tensor is produced by the tetrads $e^A{}_\mu$ and their inverses $e_A{}^\mu$,

$$g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu \quad \text{and} \quad g^{\mu\nu} = \eta^{AB} e_A{}^\mu e_B{}^\nu, \tag{1}$$

where these tetrads satisfy $e_A{}^\mu e^B{}_\mu = \delta^B{}_A$ and $e_A{}^\mu e^B{}_\nu = \delta^\mu{}_\nu$.

In metric teleparallel geometry, the curvature vanishes,

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho} + \Gamma^\sigma{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\alpha}\Gamma^\alpha{}_{\mu\rho} = 0. \tag{2}$$

The gravitational effects are described in terms of the torsion tensor, which is defined by the antisymmetric part of the affine connection

$$T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}, \tag{3}$$

where the affine connection in the Weitzenböck gauge is [148]

$$\Gamma^\rho{}_{\mu\nu} = e_A{}^\rho \partial_\mu e^A{}_\nu. \tag{4}$$

Considering the following action:

$$S = S_g + S_{\mathrm{NY}} + S_m, \tag{5}$$

where S_g is the gravitational action

$$S_g = \frac{1}{2}\int \mathrm{d}^4 x e \mathbb{T}, \tag{6}$$

with $e = \det(e^A{}_\mu) = \sqrt{-g}$,

$$\mathbb{T} = \frac{1}{2} S_\alpha{}^{\mu\nu} T^\alpha{}_{\mu\nu}, \tag{7}$$

and the superpotential $S_\alpha{}^{\mu\nu}$ is [149]

$$S_\alpha{}^{\mu\nu} = t_1 T_\alpha{}^{\mu\nu} + t_2 T^{[\mu}{}_\alpha{}^{\nu]} + t_3 \delta_\alpha{}^{[\mu} T^{\nu]}, \tag{8}$$

where $T^\mu = T^{\nu\mu}{}_\nu$ and t_1, t_2, and t_3 are three constants.

The PV term in action (5) is

$$S_{\mathrm{NY}} = \int \mathrm{d}^4 x e \frac{g(\theta)}{4} \mathcal{L}_{\mathrm{NY}}, \tag{9}$$

where

$$\mathcal{L}_{\mathrm{NY}} = T_{A\mu\nu} \mathcal{T}^{A\mu\nu} - \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \tag{10}$$

is the NY term,

$$\mathcal{T}^{A\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} T^A{}_{\rho\sigma}/2 \tag{11}$$

is the dual of the torsion tensor, and $T^A{}_{\mu\nu} = e^A{}_\rho T^\rho{}_{\mu\nu}$, $\varepsilon^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$ is the Levi-Civita tensor, with $\epsilon^{\mu\nu\rho\sigma}$ the antisymmetric symbol. The NY term is a topological term and was first proposed in Riemann-Cartan geometry [130,131]. In the framework of teleparallelism, i.e., $R_{\mu\nu\rho\sigma} = 0$, the NY term coupled to a scalar field was extended to metric teleparallel geometry and added to the metric teleparallel equivalent Einstein-Hilbert Lagrangian [129].

The last term in Eq. (5) effectively describes the matter filled in the universe,

$$S_m = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}\nabla_\mu\theta\nabla^\mu\theta - V(\theta)\right]. \tag{12}$$

In conclusion, the action we consider in this paper is

$$S = \int \mathrm{d}^4xe\left[\frac{1}{2}\mathbb{T} + \frac{g(\theta)}{4}T_{A\mu\nu}\mathcal{T}^{A\mu\nu}\right] + \int d^4x\sqrt{-g}\left[\frac{1}{2}\nabla_\mu\theta\nabla^\mu\theta - V(\theta)\right]. \tag{13}$$

Note that if we choose the parameters to be

$$t_1 = 1/4, \qquad t_2 = 1/2, \qquad t_3 = -1, \tag{14}$$

then the first term in action (5) becomes the teleparallel equivalent Einstein-Hilbert Lagrangian up to a surface term

$$\mathbb{T} = -T^\mu T_\mu + \frac{1}{4}T^{\rho\sigma\mu}T_{\rho\sigma\mu} + \frac{1}{2}T^{\mu\sigma\rho}T_{\rho\sigma\mu} = -\mathring{R} - 2\mathring{\nabla}_\mu T^\mu, \tag{15}$$

where $\mathring{R}$ is the Ricci scalar corresponding to the Levi-Civita connection and $\mathring{\nabla}$ is the metric-compatible covariant derivative. Linear cosmological perturbations, including linear gravitational waves, were studied [110,134,150] with the parameter set (14). However, this model suffers from the strong coupling problem beyond linear orders, which we will show in the next section. Nevertheless, this problem can be avoided by choosing a suitable parameter set instead of (14).

III. THE BACKGROUND AND LINEAR SCALAR PERTURBATION

In this section, we calculate the evolution of the background and linear cosmological scalar perturbations. To this end, we first provide the perturbed tetrads and metric up to cubic order.

A. The tetrads and the corresponding metric

We consider the background spacetime to be a spatially flat Friedmann-Robertson-Walker universe. The background tetrads can be parametrized as

$$\bar{e}^A{}_\mu = \mathrm{diag}(a, a, a, a). \tag{16}$$

By substituting the tetrads into (1), we obtain the background metric

$$ds^2 = a^2(\mathrm{d}\tau^2 - \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j). \tag{17}$$

The parametrization for the linearly perturbed tetrad field is [144,151]

$$e^0{}_0 = a(1+\phi), \qquad e^0{}_i = a\partial_i\beta, \qquad e^a{}_0 = a\delta^{ai}\partial_i\gamma,$$
$$e^a{}_i = a\delta^{aj}\left[(1-\psi)\delta_{ij} + \partial_i\partial_jE + \epsilon_{ijk}\partial^k\lambda + \frac{1}{2}h_{ij}\right], \tag{18}$$

where we consider only the linear scalar and tensor perturbations. Substituting the parametrized tetrads into (1), we obtain the perturbed metric up to linear order,

$$g_{00} = a^2(1+2\phi), \qquad g_{0i} = -a^2\partial_iB,$$
$$g_{ij} = -a^2[(1-2\psi)\delta_{ij} + 2\partial_i\partial_jE + h_{ij}], \tag{19}$$

where $B = \gamma - \beta$. From the expressions of the linearly perturbed metric, we can observe that in the Newtonian gauge, where $B = 0$ and $E = 0$, the scalar perturbations from tetrads satisfy $\gamma = \beta$ and $E = 0$. We will use the Newtonian gauge for the remainder of this paper.

In a given coordinate system, we can always write the tetrad field as $e^A{}_\mu = \bar{e}^A{}_\nu e^\nu{}_\mu$, and thus use the exponential expansion [146,152]

$$e^\nu{}_\mu = \exp^{(\delta e^\nu{}_\mu)} = \delta^\nu{}_\mu + \delta e^\nu{}_\mu + \frac{1}{2}\delta e^\nu{}_\rho\delta e^\rho{}_\mu + \frac{1}{6}\delta e^\nu{}_\rho\delta e^\rho{}_\sigma\delta e^\sigma{}_\mu + \cdots, \tag{20}$$

where $\delta e^\nu{}_\mu$ is the perturbation of $e^\nu{}_\mu$ and $e^\nu{}_\mu = \bar{e}_A{}^\nu e^A{}_\nu$. Then the perturbed tetrad field up to the cubic order is[1]

$$e^0{}_0 = a\left[1 + \phi + \frac{1}{2}\phi^2 + \frac{1}{2}\partial_i\gamma\partial^i\gamma + \frac{1}{12}h_{ij}\partial^i\gamma\partial^j\gamma\right], \tag{21}$$

$$e^0{}_i = a\left[\partial_i\gamma + \frac{1}{2}(\phi-\psi)\partial_i\gamma - \frac{1}{2}\epsilon_{ijk}\partial^j\lambda\partial^k\gamma + \frac{1}{4}h_{ij}\partial^j\gamma + \frac{1}{12}(\phi-2\psi)h_{ij}\partial^j\gamma - \frac{1}{12}(\epsilon_{lij}h^l{}_k + \epsilon_{klj}h^l{}_i)\partial^j\lambda\partial^k\gamma\right], \tag{22}$$

$$e^a{}_0 = a\delta^{ai}\left[\partial_i\gamma + \frac{1}{2}(\phi-\psi)\partial_i\gamma - \frac{1}{2}\epsilon_{ijk}\partial^k\lambda\partial^j\gamma + \frac{1}{4}h_{ij}\partial^j\gamma + \frac{1}{12}(\phi-2\psi)h_{ij}\partial^j\gamma - \frac{1}{12}(\epsilon_{klj}h^k{}_i - \epsilon_{ijk}h^k{}_l)\partial^j\lambda\partial^l\gamma\right], \tag{23}$$

[1]For our purpose of calculating SIGWs, we only keep cubic terms that contain two scalar modes and one tensor mode for notational simplicity.

$$
\begin{aligned}
e^{a}{}_{i} = a\delta^{aj}\bigg[&(1-\psi)\delta_{ij} + \epsilon_{ijk}\partial^k\lambda + \frac{1}{2}h_{ij} + \frac{1}{2}\delta_{ij}\psi^2 \\
&+ \frac{1}{2}\partial_i\gamma\partial_j\gamma - \epsilon_{ijk}\psi\partial^k\lambda - \frac{1}{2}(\delta_{ij}\partial_l\lambda\partial^l\lambda - \partial_i\lambda\partial_j\lambda) \\
&- \frac{1}{2}h_{ij}\psi - \frac{1}{4}\epsilon^{k}{}_{il}h_{jk}\partial^l\lambda - \frac{1}{4}\epsilon_{jkl}h^{k}{}_{i}\partial^l\lambda + \frac{1}{8}h_{jk}h^{k}{}_{i} \\
&+ \frac{1}{4}h_{ij}\psi^2 - \frac{1}{4}(\epsilon_{ikl}h^{k}{}_{j} - \epsilon_{jkl}h^{k}{}_{i})\psi\partial^l\lambda \\
&+ \frac{1}{12}(h_{jk}\partial_i\gamma\partial^k\gamma + h_{ik}\partial_j\gamma\partial^k\gamma) \\
&- \frac{1}{12}(h_{ij}\partial_k\lambda\partial^k\lambda - h_{kl}\partial^k\lambda\partial^l\lambda\delta_{ij})\bigg]. \qquad (24)
\end{aligned}
$$

The inverse tetrad field $e_{A}{}^{\mu}$ can be obtained with the relation $e_{A}{}^{\mu}e^{A}{}_{\nu} = \delta^{\mu}_{\nu}$,

$$
e_{0}{}^{0} = \frac{1}{a}\left[1 - \phi + \frac{1}{2}\phi^2 + \frac{1}{2}\partial_i\gamma\partial^i\gamma - \frac{1}{12}h_{ij}\partial^i\gamma\partial^j\gamma\right], \qquad (25)
$$

$$
\begin{aligned}
e_{0}{}^{i} = \frac{1}{a}\bigg[&-\partial^i\gamma + \frac{1}{2}(\phi-\psi)\partial^i\gamma - \frac{1}{2}\epsilon^{i}{}_{jk}\partial^k\lambda\partial^j\gamma + \frac{1}{4}h^{ij}\partial_j\gamma \\
&- \frac{1}{12}(\phi-2\psi)h^{ij}\partial_j\gamma + \frac{1}{12}(\epsilon^{i}{}_{lj}h^{l}{}_{k} + \epsilon_{lkj}h^{li})\partial^j\lambda\partial^k\gamma\bigg], \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
e_{a}{}^{0} = \frac{1}{a}\delta_{a}{}^{i}\bigg[&-\partial_i\gamma + \frac{1}{2}(\phi-\psi)\partial_i\gamma + \frac{1}{2}\epsilon_{ijk}\partial^k\lambda\partial^j\gamma + \frac{1}{4}h_{ij}\partial^j\gamma \\
&- \frac{1}{12}(\phi-2\psi)h_{ij}\partial^j\gamma - \frac{1}{12}(\epsilon_{klj}h^{k}_{i} + \epsilon_{ikj}h^{k}_{l})\partial^j\lambda\partial^l\gamma\bigg], \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
e_{a}{}^{i} = \frac{1}{a}\delta^{il}\delta_{a}{}^{j}\bigg[&\left(1+\psi+\frac{1}{2}\psi^2\right)\delta_{lj} - \epsilon_{jlk}\partial^k\lambda - \frac{1}{2}h_{jl} \\
&+ \frac{1}{2}\partial_l\gamma\partial_j\gamma - \epsilon_{jlm}\psi\partial^m\lambda - \frac{1}{2}(\partial_k\lambda\partial^k\lambda\delta_{lj} - \partial_l\lambda\partial_j\lambda) \\
&- \frac{1}{2}\psi h_{jl} + \frac{1}{4}\epsilon_{klm}h^{k}{}_{j}\partial^m\lambda - \frac{1}{4}\epsilon^{k}{}_{jm}h_{lk}\partial^m\lambda + \frac{1}{8}h_{lk}h^{k}{}_{j} \\
&- \frac{1}{4}h_{jl}\psi^2 - \frac{1}{12}(h_{jk}\partial_l\gamma\partial^k\gamma + h_{lk}\partial_j\gamma\partial^k\gamma) \\
&+ \frac{1}{4}(\epsilon_{klm}h^{k}{}_{j} + \epsilon_{jkm}h^{k}{}_{l})\psi\partial^m\lambda \\
&+ \frac{1}{12}(h_{jl}\partial_k\lambda\partial^k\lambda - h_{km}\partial^k\lambda\partial^m\lambda\delta_{jl})\bigg]. \qquad (28)
\end{aligned}
$$

The corresponding perturbed metric up to the cubic order is

$$
g_{00} = a^2\left[1 + 2\phi + 2\phi^2 - \frac{1}{3}h_{ij}\partial^i\gamma\partial^j\gamma\right], \qquad (29)
$$

$$
\begin{aligned}
g_{0i} = a^2\bigg[&(\phi+\psi)\partial_i\gamma - \frac{1}{2}h_{ij}\partial^j\gamma + \psi h_{ji}\partial^j\gamma \\
&+ \frac{1}{3}\left(\epsilon_{kil}h^{k}{}_{j} - \frac{1}{2}\epsilon_{jkl}h^{k}{}_{i}\right)\partial^j\gamma\partial^l\lambda\bigg], \qquad (30)
\end{aligned}
$$

and

$$
\begin{aligned}
g_{ij} = -a^2\bigg[&(1-2\psi+2\psi^2)\delta_{ij} + h_{ij} - 2\psi h_{ij} \\
&- \frac{1}{2}(\epsilon_{kil}h^{k}{}_{j} + \epsilon_{kjl}h^{k}{}_{i})\partial^l\lambda + \frac{1}{2}h_{ki}h^{k}{}_{j} \\
&+ \left(\epsilon_{ilk}h^{k}{}_{j}\psi\partial^l\lambda + \frac{1}{6}h_{ik}\partial^k\gamma\partial_j\gamma + i\leftrightarrow j\right) \\
&+ 2h_{ij}\psi^2 - \bigg(\frac{2}{3}h_{ij}\partial_k\lambda\partial^k\lambda + \frac{1}{3}\delta_{ij}h_{kl}\partial^k\lambda\partial^l\lambda \\
&- \frac{1}{2}h_{jk}\partial^k\lambda\partial_i\lambda - \frac{1}{2}h_{ik}\partial^k\lambda\partial_j\lambda\bigg)\bigg]. \qquad (31)
\end{aligned}
$$

With the relation $g^{\nu\rho}g_{\rho\mu} = \delta^{\nu}{}_{\mu}$, the inverse metric is

$$
g^{00} = \frac{1}{a^2}\left[1 - 2\phi + 2\phi^2 + \frac{1}{3}h_{ij}\partial^i\gamma\partial^j\gamma\right], \qquad (32)
$$

$$
\begin{aligned}
g^{0i} = \frac{1}{a^2}\bigg[&(\phi+\psi)\partial^i\gamma - \frac{1}{2}h^{ij}\partial_j\gamma - \psi h^{ij}\partial_j\gamma \\
&+ \frac{1}{3}\left(\epsilon_{k}{}^{i}{}_{l}h^{k}{}_{j} - \frac{1}{2}\epsilon_{jkl}h^{ki}\right)\partial^j\gamma\partial^l\lambda\bigg], \qquad (33)
\end{aligned}
$$

and

$$
\begin{aligned}
g^{ij} = -\frac{1}{a^2}\bigg[&(1+2\psi+2\psi^2)\delta^{ij} - h^{ij} - 2\psi h^{ij} \\
&+ \frac{1}{2}(\epsilon_{l}{}^{j}{}_{k}h^{li} + \epsilon_{l}{}^{i}{}_{k}h^{lj})\partial^k\lambda + \frac{1}{2}h_{k}{}^{i}h^{jk} - 2\psi^2 h^{ij} \\
&+ \left(\epsilon^{i}{}_{lk}h^{kj}\psi\partial^l\lambda - \frac{1}{6}h^{ik}\partial_k\gamma\partial^j\gamma + i\leftrightarrow j\right) \\
&+ \bigg(\frac{2}{3}h^{ij}\partial_k\lambda\partial^k\lambda + \frac{1}{3}\delta^{ij}h_{kl}\partial^k\lambda\partial^l\lambda \\
&- \frac{1}{2}h^{jk}\partial_k\lambda\partial^i\lambda - \frac{1}{2}h^{ik}\partial_k\lambda\partial^j\lambda\bigg)\bigg]. \qquad (34)
\end{aligned}
$$

We also have

$$
\begin{aligned}
e = \sqrt{-g} = a^4\bigg(&1 + \phi - 3\psi + \frac{1}{2}\phi^2 - 3\phi\psi \\
&+ \frac{9}{2}\psi^2 + \frac{1}{4}h_{ij}h^{ij}\bigg). \qquad (35)
\end{aligned}
$$

B. The background equations of motion and linear scalar perturbations

Expanding action (5) to linear order, we obtain

$$S^{(1)} = \int d^3x d\tau a^2 \bigg[-\frac{3}{2}\mathcal{C}_1\mathcal{H}^2\phi - \left(\frac{1}{2}(\theta')^2 + a^2V\right)\phi - \frac{9}{2}\mathcal{C}_1\mathcal{H}^2\psi - \frac{3}{2}((\theta')^2 - 2a^2V)\psi - 3\mathcal{C}_1\mathcal{H}\psi' - a^2V_\theta\delta\theta + \delta\theta'\theta' \bigg], \tag{36}$$

where $\mathcal{C}_1 = 2t_1 + t_2 + 3t_3$ and the prime represents the derivative with respect to conformal time τ.

Varying the above action with respect to scalar perturbations, we obtain the background EOMs as follows:

$$-\frac{3}{2}\mathcal{C}_1\mathcal{H}^2 = \frac{1}{2}(\theta')^2 + a^2V, \tag{37}$$

$$\mathcal{C}_1(\mathcal{H}^2 + 2\mathcal{H}') = (\theta')^2 - 2a^2V, \tag{38}$$

$$\theta'' + 2\mathcal{H}\theta' + a^2V_\theta = 0. \tag{39}$$

It is obvious that when $t_1 = 1/4$, $t_2 = 1/2$, and $t_3 = -1$, yielding $\mathcal{C}_1 = -2$, we recover the results of general relativity (GR).

The quadratic action is

$$S^{(2)}_{SS} = \int d^3x d\tau a^2 \bigg[\frac{1}{2}(\delta\theta')^2 - \frac{1}{2}a^2V_{\theta\theta}\delta\theta^2 - a^2V_\theta\delta\theta(\phi - 3\psi) - (\phi + 3\psi)\delta\theta'\theta' - \frac{1}{2}\partial_i\delta\theta\partial^i\delta\theta + \frac{3}{2}\mathcal{C}_1(\psi')^2 + \frac{1}{4}(9\psi^2 + \phi^2)(3\mathcal{C}_1\mathcal{H}^2 + (\theta')^2 - 2a^2V) + 3\mathcal{C}_1\mathcal{H}\phi\psi' + 9\mathcal{C}_1\mathcal{H}\psi\psi' - \frac{1}{2}\mathcal{C}_2\partial_i\phi\partial^i\phi - \mathcal{C}_3\partial_i\psi\partial^i\psi + 2t_3\partial_i\psi\partial^i\phi - \mathcal{C}_2\psi\partial^2\gamma' - 2g_\theta\mathcal{H}\delta\theta\partial^2\lambda - 2g_\theta\theta'\psi\partial^2\lambda + \mathcal{C}_2\partial^i\phi\partial_i\gamma' - \frac{1}{2}\mathcal{C}_2\partial_i\gamma'\partial^i\gamma' + \frac{1}{2}\mathcal{C}_2\partial^2\gamma\partial^2\gamma + \mathcal{C}_4\partial_i\lambda'\partial^i\lambda' - \mathcal{C}_4\partial^2\lambda\partial^2\lambda \bigg], \tag{40}$$

where $\mathcal{C}_2 = 2t_1 + t_2 + t_3$, $\mathcal{C}_3 = 2t_1 + t_2 + 2t_3$, $\mathcal{C}_4 = 2t_1 - t_2$, $g_\theta = dg/d\theta$, and ∂^2 represents $\partial^i\partial_i$.

Then the EOMs for scalar perturbations can be obtained by varying the above quadratic action with respect to the corresponding scalar perturbations

$$3\mathcal{C}_1\mathcal{H}(\psi' + \mathcal{H}\phi) - 2t_3\partial^2\psi + \mathcal{C}_2\partial^2(\phi - \gamma') = -(\theta')^2\phi + \delta\theta'\theta' + a^2V_\theta\delta\theta, \tag{41}$$

$$3\mathcal{C}_1\mathcal{H}^2(3\psi - 2\phi) - 2t_3\partial^2\phi + 2\mathcal{C}_3\partial^2\psi - \mathcal{C}_2\partial^2\gamma' - 3\mathcal{C}_1\mathcal{H}'(3\psi + \phi) - 3\mathcal{C}_1\mathcal{H}(2\psi' + \phi') - 3\mathcal{C}_1\psi'' - 2g_\theta\theta'\partial^2\lambda - 3\delta\theta'\theta' + 3a^2V_\theta\delta\theta + 9(\theta')^2\psi = 0, \tag{42}$$

$$\mathcal{C}_3\psi - t_3\phi = g_\theta\theta'\lambda, \tag{43}$$

$$\delta\theta'' + 2\mathcal{H}\delta\theta' - \partial^2\delta\theta + a^2V_{\theta\theta}\delta\theta - \theta'(\phi' + 3\psi') + 2a^2V_\theta\phi = -2g_\theta\mathcal{H}\partial^2\lambda, \tag{44}$$

$$g_\theta(\mathcal{H}\delta\theta + \theta'\psi) = \mathcal{C}_4(\lambda'' + 2\mathcal{H}\lambda' - \partial^2\lambda), \tag{45}$$

$$(2\mathcal{H}\phi + \phi' + 2\mathcal{H}\psi + \psi') = (\gamma'' + 2\mathcal{H}\gamma' - \partial^2\gamma), (\mathcal{C}_2 \neq 0). \tag{46}$$

From the above EOMs, we can see that if we choose the parameter set defined in (14), which corresponds to the teleparallel equivalent of GR with $\mathcal{C}_2 = \mathcal{C}_4 = 0$, the EOM for the scalar perturbation γ disappears. However, γ exists in the EOM of SIGWs, which leads to a strong coupling problem. Note that even if we choose the parameter set as (14), the linear perturbation λ from tetrads exists in the EOMs (43) and (44), resulting in the EOMs of the perturbations from the metric and scalar field being different from those in GR. Obviously, the extra scalar degrees of freedom exist in this model, whose nature and characterization need to be further studied, and we will leave this to our future work.

C. The evolution of background and linear scalar perturbations during the radiation-dominated era

During the radiation-dominated era, the equation of state is $\bar{P}/\bar{\rho} = 1/3$, with

$$\bar{P} = \frac{(\theta')^2}{2a^2} - V, \qquad \bar{\rho} = \frac{(\theta')^2}{2a^2} + V, \tag{47}$$

being the pressure and energy density of the background universe. We also have $\delta P/\delta\rho = 1/3$, where

$$\delta\rho = -a^{-2}\theta'(\phi\theta' - \delta\theta') + V_\theta\delta\theta, \quad \delta P = -V_\theta\delta\theta + a^{-2}\theta'(\delta\theta' - \phi\theta') \tag{48}$$

are the perturbations of the energy density and pressure.

Combining the above two equations (47) and the first two equations for the background evolution (37) and (38), we obtain

$$\mathcal{H} = \frac{1}{\tau}, \qquad \theta' = \pm\sqrt{-2\mathcal{C}_1}\frac{1}{\tau}. \tag{49}$$

From the above equation, we can see that the evolution of the Hubble parameter is independent of $\mathcal{C}_1$.

Obviously, for any choice of parameters t_1, t_2, and t_3, the EOMs for the linear scalar perturbations, (41)–(46), are very difficult to solve analytically. This poses significant challenges for us in analyzing the behavior of SIGWs in our model. In this paper, we primarily focus on the contributions from the PV term and the perturbations from the tetrads, denoted as λ and γ, to SIGWs. We expect the background evolution to be the same as that of GR, implying the selection of $\mathcal{C}_1 = 2t_1 + t_2 + 3t_3 = -2$. Furthermore, we aim to minimize the differences between the scalar perturbations from the metric in GR and teleparallel gravity with the NY term. In the case of GR, $\phi = \psi$ without consideration of anisotropic stress. We maintain this assumption in our model, setting $\phi = \psi$. Additionally, in (45), $\mathcal{H}\delta\theta + \theta'\psi = \mathcal{R}$ is the gauge-invariant curvature perturbation. As introduced in Sec. II, the curvature vanishes in teleparallel gravity, so we set $\mathcal{C}_4 = 0$ to ensure $\mathcal{R} = 0$.

With the above assumptions, $\mathcal{C}_1 = -2$, $\mathcal{C}_4 = 0$, and $\phi = \psi$, the EOMs for the linear perturbations (41)–(46) reduce to

$$\psi'' + 3\mathcal{H}\psi' + (\mathcal{H}^2 + 2\mathcal{H}')\psi = -\mathcal{H}(\psi' + \mathcal{H}\psi) + \mathbb{C}\partial^2\psi, \tag{50}$$

$$\mathcal{C}_2\psi = g_\theta \theta' \lambda, \tag{51}$$

$$\mathcal{H}\delta\theta + \theta'\psi = 0, \tag{52}$$

$$2(2\mathcal{H}\psi + \psi') = (\gamma'' + 2\mathcal{H}\gamma' - \partial^2\gamma), (\mathcal{C}_2 \neq 0), \tag{53}$$

with

$$\mathbb{C} = \frac{2t_3 - \mathcal{C}_2}{3\mathcal{C}_1}. \tag{54}$$

Here, Eq. (50) is derived by combining Eqs. (41)–(43) and (48).

Note that in the case of GR, $\mathbb{C} = 1/3$, we obtain the solution easily,

$$\mathcal{C}_2 = 0, \tag{55}$$

which results in the disappearance of EOM for γ (53), giving rise to the strong coupling problem. Thus, we require $\mathbb{C} \neq 1/3$ to avoid the strong coupling problem. Besides, $\mathbb{C}$ relates to the propagating speed of perturbation ψ, we assume $\mathbb{C}$ is a real number, and $\mathbb{C} \leq 1$.

For late convenience to calculate the SIGWs, we split the perturbations into the primordial perturbation and the transfer functions as follows:

$$\psi(\boldsymbol{k}, \tau) = \frac{2}{3}\zeta(\boldsymbol{k}) T_\psi(x), \tag{56}$$

$$\gamma(\boldsymbol{k}, \tau) = \frac{2}{3}\zeta(\boldsymbol{k}) \frac{1}{k} T_\gamma(x), \tag{57}$$

where ζ is the primordial curvature perturbation generated during the inflationary era and $x = k\tau$. It is worth noting that we assume $\mathcal{R} = 0$ in teleparallel gravity. However, teleparallel gravity can only be viewed as a low-energy theory. During inflation, gravity may be described by another UV-complete theory. Thus, we expect ζ to be nonzero, resulting in observable effects in the CMB.

Recalling the evolution of the conformal Hubble parameter, the EOMs for the transfer functions can be written as

$$T_\psi^{**}(x) + \frac{4}{x}T_\psi^*(x) + \mathbb{C}T_\psi(x) = 0, \tag{58}$$

$$T_\gamma^{**}(x) + \frac{2}{x}T_\gamma^*(x) + T_\gamma(x) = 2T_\psi^*(x) + \frac{4}{x}T_\psi(x), \tag{59}$$

where "$*$" represents the derivative with respect to the argument. Then we can solve the above EOMs easily,

$$T_\psi(x) = \frac{3}{x^2\mathbb{C}}\left(\frac{\sin(x\sqrt{\mathbb{C}})}{x\sqrt{\mathbb{C}}} - \cos(x\sqrt{\mathbb{C}})\right), \tag{60}$$

$$\begin{aligned} T_\gamma(x) = \frac{1}{2\mathbb{C}^{3/2}x^2}\{ & -2i\mathbb{C}^{3/2}c_1 x\sin(x) + \mathbb{C}^{3/2}c_2 x\sin(x) + 2\mathbb{C}^{3/2}c_1 x\cos(x) \\ & - i\mathbb{C}^{3/2}c_2 x\cos(x) + 3(\mathbb{C}+1)x\cos(x)[\mathrm{Ci}(x + \sqrt{\mathbb{C}}x) - \mathrm{Ci}(x - \sqrt{\mathbb{C}}x)] \\ & + 3(\mathbb{C}+1)x\sin(x)[\mathrm{Si}(\sqrt{\mathbb{C}}x + x) - \mathrm{Si}(x - \sqrt{\mathbb{C}}x)] - 6\sin(\sqrt{\mathbb{C}}x)\}, \end{aligned} \tag{61}$$

where

$$\mathrm{Si}(x) = \int_0^x \mathrm{d}y \frac{\sin y}{y}, \qquad \mathrm{Ci}(x) = -\int_x^\infty \mathrm{d}y \frac{\cos y}{y} \tag{62}$$

are sine integral and cosine integral, respectively.

If we choose the parameter set (14), then $\mathbb{C} = 1/3$, and the transfer function T_ψ is the same as that in GR. However, from EOM (52), the fluctuation of the scalar field $\delta\theta$ still differs from that in GR. There are two integral constants, c_1 and c_2, in the transfer function T_γ. We expect that T_γ is a real function and finite as $x \to 0$. Then, we can obtain

$$c_1 = \frac{1}{4\mathbb{C}^{3/2}}\left(6\sqrt{\mathbb{C}} + 3(1+\mathbb{C})\log\left(\frac{1-\sqrt{\mathbb{C}}}{1+\sqrt{\mathbb{C}}}\right)\right), \tag{63}$$

$$c_2 = \frac{i}{2\mathbb{C}^{3/2}}\left(6\sqrt{\mathbb{C}} + 3(1+\mathbb{C})\log\left(\frac{1-\sqrt{\mathbb{C}}}{1+\sqrt{\mathbb{C}}}\right)\right). \tag{64}$$

By substituting c_1 and c_2 into Eq. (61), the transfer function T_γ becomes

$$\begin{aligned} T_\gamma(x) = \frac{1}{2\mathbb{C}^{3/2}x^2}\Bigg\{ & x\cos(x)\left[6\sqrt{\mathbb{C}} + 3(1+\mathbb{C})\log\left(\frac{1-\sqrt{\mathbb{C}}}{1+\sqrt{\mathbb{C}}}\right)\right] - 6\sin(\sqrt{\mathbb{C}}x) \\ & + 3(\mathbb{C}+1)x\cos(x)[\mathrm{Ci}(x+\sqrt{\mathbb{C}}x) - \mathrm{Ci}(x-\sqrt{\mathbb{C}}x)] \\ & + 3(\mathbb{C}+1)x\sin(x)[\mathrm{Si}(x+\sqrt{\mathbb{C}}x) - \mathrm{Si}(x-\sqrt{\mathbb{C}}x)]\Bigg\}. \end{aligned} \tag{65}$$

It seems that T_γ is singular when $\mathbb{C}\to 1$ due to the logarithmic divergence. In fact, in the limit $\mathbb{C}\to 1$, we have

$$\begin{aligned} T_\gamma(x)|_{\mathbb{C}\to 1} = \frac{3}{x^2}[& -x\cos x(E_\gamma - 1 - \mathrm{Ci}(2x) + \log(2x)) \\ & + \sin x(x\mathrm{Si}(2x) - 1)], \end{aligned} \tag{66}$$

where E_γ is the Euler Gamma constant.

Recalling the assumptions we made above, $\mathcal{C}_1 = -2$ and $\mathcal{C}_4 = 0$, only t_1 is a free parameter; the others can be expressed as

$$\begin{aligned} & t_2 = 2t_1, \qquad t_3 = -\frac{4t_1+2}{3}, \qquad \mathcal{C}_2 = \frac{2}{3}(4t_1 - 1), \\ & \mathcal{C}_3 = \frac{4}{3}(t_1 - 1), \qquad \mathbb{C} = \frac{1}{9}(8t_1 + 1). \end{aligned} \tag{67}$$

IV. THE SCALAR INDUCED GRAVITATIONAL WAVES

In this section, we first derive the EOMs for SIGWs, and then we calculate the power spectrum and the energy density of SIGWs. Expanding action (5) to cubic order, we obtain

$$S_{\mathrm{GW}} = S^{(2)}_{TT} + S^{(3)}_{TT}, \tag{68}$$

where

$$S^{(2)}_{TT} = \int \mathrm{d}^3x\mathrm{d}\tau \frac{a^2}{8}[\mathcal{C}_5(h'_{ij}h'^{ij} - \partial_k h_{ij}\partial^k h^{ij}) + g'\epsilon_{ijk}h^{li}\partial^j h_l{}^k] \tag{69}$$

and

$$S^{(3)}_{TT} = \int \mathrm{d}^3x\mathrm{d}\tau a^2(\mathcal{L}_{ij} + \mathcal{L}^{\mathrm{PV}}_{ij})h^{ij}, \tag{70}$$

with

$$\begin{aligned} \mathcal{L}_{ij} = & \frac{1}{2}\partial_i\delta\theta\partial_j\delta\theta + \mathcal{C}_4\partial_i\partial_j\lambda\partial^2\lambda - \frac{1}{2}g_{\theta\theta}\theta'\delta\theta\partial_i\partial_j\lambda - \frac{1}{2}g_\theta\delta\theta'\partial_i\partial_j\lambda - \frac{1}{2}g_\theta\delta\theta\partial_i\partial_j\lambda' + g_\theta\theta'\psi\partial_i\partial_j\lambda + \frac{1}{2}\mathcal{C}_2\partial_i\phi\partial_j\phi - 2t_3\partial_i\phi\partial_j\psi \\ & + \mathcal{C}_3\partial_i\psi\partial_j\psi - \frac{1}{2}\mathcal{C}_2\partial_i\gamma'\partial_j\phi - t_3\mathcal{H}\partial_i\gamma\partial_j\phi + \frac{1}{4}\mathcal{C}_2\partial_i\gamma\partial_j\phi' + \mathcal{C}_2\partial_i\gamma'\partial_j\psi + \frac{7}{2}\mathcal{C}_2\mathcal{H}\partial_i\gamma\partial_j\psi - \frac{1}{4}(5\mathcal{C}_5 + t_3)\psi'\partial_i\partial_j\gamma \\ & - \frac{1}{2}\theta'\partial_i\gamma\partial_j\delta\theta - \frac{1}{2}\mathcal{C}_2\mathcal{H}\partial_i\gamma'\partial_j\gamma - \frac{1}{4}\mathcal{C}_2\partial_i\gamma\partial_j\gamma'' + \frac{1}{6}\mathcal{C}_1\mathcal{H}^2\partial_i\gamma\partial_j\gamma - \frac{1}{6}\mathcal{C}_1\mathcal{H}'\partial_i\gamma\partial_j\gamma - \frac{1}{4}\mathcal{C}_2\partial_i\partial_j\gamma\partial^2\gamma + \frac{1}{6}(\theta')^2\partial_i\gamma\partial_j\gamma \end{aligned} \tag{71}$$

and

$$\begin{aligned} \mathcal{L}^{\mathrm{PV}}_{ij} = \epsilon_{klj}\Bigg[& -\frac{1}{2}\mathcal{C}_5\partial^k\partial_i\lambda\partial^l\phi + \frac{1}{2}\mathcal{C}_5\partial^k\partial_i\lambda\partial^l\psi \\ & - \frac{1}{4}g_\theta\theta'\partial^k\lambda\partial^l\partial_i\lambda - \mathcal{C}_4\partial^k\lambda'\partial^l\partial_i\gamma \\ & - \frac{1}{2}g_\theta\partial^k\partial_i\gamma\partial^l\delta\theta - \frac{1}{4}g_\theta\theta'\partial^k\gamma\partial^l\partial_i\gamma\Bigg], \end{aligned} \tag{72}$$

where $\mathcal{C}_5 = 2t_1 + t_2 = 4t_1$.

A. The EOM for SIGWs

Varying the cubic action for SIGWs (68) with respective to h_{ij}, the EOM for SIGWs is

$$\begin{aligned} & -\frac{\mathcal{C}_5}{4}(h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij}) \\ & + \frac{1}{8}g'(\epsilon_{lki}\partial^l h^k{}_j + \epsilon_{lkj}\partial^l h^k{}_i) = \mathcal{T}^{lm}{}_{ij}s_{lm}, \end{aligned} \tag{73}$$

where

$$s_{ij} = -\frac{1}{2}(\mathcal{L}_{ij} + \mathcal{L}_{ji} + \mathcal{L}_{ij}^{\mathrm{PV}} + \mathcal{L}_{ji}^{\mathrm{PV}}). \tag{74}$$

$\mathcal{T}^{lm}{}_{ij}$ is the projection tensor.

We decompose h_{ij} into circularly polarized modes as

$$h_{ij}(\boldsymbol{x},\tau) = \sum_{A=R,L}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} p_{ij}^A h_{\boldsymbol{k}}^A(\tau), \tag{75}$$

where the circular polarization tensors are defined as

$$p_{ij}^R = \frac{1}{\sqrt{2}}(\mathbf{e}_{ij}^{+} + i\mathbf{e}_{ij}^{\times}), \qquad p_{ij}^L = \frac{1}{\sqrt{2}}(\mathbf{e}_{ij}^{+} - i\mathbf{e}_{ij}^{\times}). \tag{76}$$

The plus and cross polarization tensors can be expressed as

$$\begin{aligned} \mathbf{e}_{ij}^{+} &= \frac{1}{\sqrt{2}}(\mathbf{e}_i\mathbf{e}_j - \bar{\mathbf{e}}_i\bar{\mathbf{e}}_j), \\ \mathbf{e}_{ij}^{\times} &= \frac{1}{\sqrt{2}}(\mathbf{e}_i\bar{\mathbf{e}}_j + \bar{\mathbf{e}}_i\mathbf{e}_j), \end{aligned} \tag{77}$$

where $\mathbf{e}_i(\boldsymbol{k})$ and $\bar{\mathbf{e}}_i(\boldsymbol{k})$ are two basis vectors which are orthogonal to each other and perpendicular to the wave vector $\boldsymbol{k}$, i.e., satisfying $\boldsymbol{k}\cdot\mathbf{e} = \boldsymbol{k}\cdot\bar{\mathbf{e}} = \mathbf{e}\cdot\bar{\mathbf{e}} = 0$ and $|\mathbf{e}| = |\bar{\mathbf{e}}| = 1$.

The projection tensor extracts the transverse and trace-free part of the source, of which the definition is

$$\mathcal{T}^{lm}{}_{ij} s_{lm}(\boldsymbol{x},\tau) = \sum_{A=R,L}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} p_{ij}^A p^{Alm} \tilde{s}_{lm}(\boldsymbol{k},\tau), \tag{78}$$

where $\tilde{s}_{ij}$ is the Fourier transformation of the source s_{ij}.

With the above settings, we can now rewrite the EOM for SIGWs in Fourier space as

$$u_{\boldsymbol{k}}^{A''} + \left(\omega_A^2 - \frac{a''}{a}\right)u_{\boldsymbol{k}}^A = -\frac{4a}{\mathcal{C}_5} S_{\boldsymbol{k}}^A, \tag{79}$$

where $u^A = ah^A$ and

$$\omega_A^2 = k^2\left(1 - \frac{\lambda^A M_{\mathrm{PV}}}{k}\right), \qquad (\lambda^R = 1, \lambda^L = -1), \tag{80}$$

here $M_{\mathrm{PV}} = g'/\mathcal{C}_5$, the source

$$S_{\boldsymbol{k}}^A = p^{Aij}\tilde{s}_{ij}(\boldsymbol{k},\tau). \tag{81}$$

The source $S_{\boldsymbol{k}}^A$ can be divided into four parts,

$$S_{\boldsymbol{k}}^A = S_{\boldsymbol{k}}^{A(\mathrm{PC1})} + S_{\boldsymbol{k}}^{A(\mathrm{PC2})} + S_{\boldsymbol{k}}^{A(\mathrm{PV1})} + S_{\boldsymbol{k}}^{A(\mathrm{PV2})}, \tag{82}$$

where $S_{\boldsymbol{k}}^{A(\mathrm{PC1})}$ and $S_{\boldsymbol{k}}^{A(\mathrm{PV1})}$ do not contain the contribution from γ, representing the parity-conserved and parity-violating parts, respectively, while $S_{\boldsymbol{k}}^{A(\mathrm{PC2})}$ and $S_{\boldsymbol{k}}^{A(\mathrm{PV2})}$ represent the parts that contain the contribution from γ,

$$\begin{aligned} S_{\boldsymbol{k}}^{A(\mathrm{PC1})} &= \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^{3/2}} p^{Aij}k_i'k_j'\zeta(\boldsymbol{k}')\zeta(\boldsymbol{k}-\boldsymbol{k}')f_{\mathrm{PC1}}(u,v,x), \\ S_{\boldsymbol{k}}^{A(\mathrm{PC2})} &= \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^{3/2}} p^{Aij}k_i'k_j'\zeta(\boldsymbol{k}')\zeta(\boldsymbol{k}-\boldsymbol{k}')f_{\mathrm{PC2}}(u,v,x), \\ S_{\boldsymbol{k}}^{A(\mathrm{PV1})} &= \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^{3/2}} p^{Aij}k_i'k_j'\zeta(\boldsymbol{k}')\zeta(\boldsymbol{k}-\boldsymbol{k}')f_{\mathrm{PV1}}^A(k,u,v,x), \\ S_{\boldsymbol{k}}^{A(\mathrm{PV2})} &= \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^{3/2}} p^{Aij}k_i'k_j'\zeta(\boldsymbol{k}')\zeta(\boldsymbol{k}-\boldsymbol{k}')f_{\mathrm{PV2}}^A(k,u,v,x), \end{aligned} \tag{83}$$

where $u = k'/k$, $v = |\boldsymbol{k}-\boldsymbol{k}'|/k$, and

$$p^{Aij}k_i'k_j' = \frac{1}{2}k'^2\sin^2(\vartheta)\mathrm{e}^{2i\lambda^A\ell}, \tag{84}$$

with ϑ being the angle between $\boldsymbol{k}'$ and $\boldsymbol{k}$ and ℓ being the azimuthal angle of $\boldsymbol{k}'$. The function $f_{\mathrm{PC1}}(u,v,x)$, $f_{\mathrm{PC2}}(u,v,x)$, $f_{\mathrm{PV1}}^A(u,v,x)$, and $f_{\mathrm{PV2}}^A(u,v,x)$ are defined as

$$\begin{aligned} f_{\mathrm{PC1}}(u,v,x) = -\frac{2}{9}\bigg[&\frac{4t_1+8}{3}T_\psi(ux)T_\psi(vx) - \frac{2}{3}(4t_1-1) \\ &\times\frac{uk}{\mathcal{H}}T_\psi^*(ux)T_\psi(vx) + u\leftrightarrow v\bigg], \end{aligned} \tag{85}$$

$$\begin{aligned} f_{\mathrm{PC2}}(u,v,x) = -\frac{2}{9}\bigg[&\frac{4t_1-1}{3}T_\gamma^*(ux)T_\psi(vx) + \frac{16t_1-1}{3}\frac{v}{u} \\ &\times T_\gamma(ux)T_\psi^*(vx) + \frac{32t_1+1}{3}\frac{\mathcal{H}}{uk}T_\gamma(ux) \\ &\times T_\psi(vx) - \frac{1}{3}(4t_1-1)\frac{\mathcal{H}}{vk}T_\gamma^*(ux)T_\gamma(vx) \\ &-\frac{1}{6}(4t_1-1)\frac{v}{u}T_\gamma(ux)T_\gamma^{**}(vx) \\ &-\frac{1}{6}(4t_1-1)\frac{v}{u}T_\gamma(ux)T_\gamma(vx) + u\leftrightarrow v\bigg], \end{aligned} \tag{86}$$

$$f^A_{\mathrm{PV1}}(u,v,x)=\frac{2}{9}\frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\left(\frac{u}{4}T_\psi(ux)T_\psi(vx)+u\leftrightarrow v\right), \tag{87}$$

$$f^A_{\mathrm{PV2}}(u,v,x)=\frac{2}{9}\frac{\lambda^A M_{\mathrm{PV}}}{k}\mathcal{C}_5\left[\frac{uk}{2v\mathcal{H}}T_\psi(ux)T_\gamma(vx)+\frac{1}{2v}T_\gamma(ux)T_\gamma(vx)+u\leftrightarrow v\right]. \tag{88}$$

We have used the equations for the background and linear perturbations to simplify the above expressions.

Equation (79) can be solved by the method of Green's function,

$$h^A_{\boldsymbol{k}}(\tau)=-\frac{4}{\mathcal{C}_5 a(\tau)}\int^\tau \mathrm{d}\bar{\tau}G^A_k(\tau,\bar{\tau})a(\bar{\tau})S^A_{\boldsymbol{k}}(\bar{\tau}), \tag{89}$$

where Green's function $G^A_k(\tau,\bar{\tau})$ satisfies the equation

$$G^{A''}_k(\tau,\bar{\tau})+\left(\omega_A^2-\frac{a''}{a}\right)G^A_k(\tau,\bar{\tau})=\delta(\tau-\bar{\tau}). \tag{90}$$

The coupling function g characterizes the deviation of Green's function from standard GR. For an arbitrary form of g, ω_A defined in Eq. (80) is a complex function of both the wave number k and the conformal time τ. Consequently, it is challenging to solve Eq. (90) and obtain the expression for Green's function analytically. On the one hand, for our purpose of studying the contributions from the scalar perturbations to the SIGWs, we assume that the change in Green's function is as minimal as possible relative to that in GR. On the other hand, since ω_A is associated with the propagation speed of the GWs, we assume that ω_A is approximately time-independent and depends only on the wave number during the generation of SIGWs. We will consider an exponential form of the coupling function

$$g(\theta)=g_0\mathrm{e}^{\alpha\theta}, \tag{91}$$

which renders ω_A independent of time and allows us to obtain an analytical solution to Eq. (90).

Recalling the background equations (49), the solution for the scalar field is found to be

$$\theta=2\beta\ln(\tau/\tau_0)+\theta_0, \tag{92}$$

where θ_0 is the value of θ at τ_0 and $\beta=\pm1$, corresponding to $\theta'=\pm2/\tau$, respectively. Substituting Eqs. (91) and (92) into the coupling function g, we obtain

$$g'=\frac{2\alpha\beta g_0\mathrm{e}^{\alpha\theta_0}}{\tau_0^{2\alpha\beta}}\tau^{2\alpha\beta-1}. \tag{93}$$

From Eq. (93), it is evident that if we set $2\alpha\beta-1=0$, then g' becomes a constant. Consequently, M_{PV} defined above becomes

$$M_{\mathrm{PV}}=\frac{g_0\mathrm{e}^{\alpha\theta_0}}{\mathcal{C}_5\tau_0}, \tag{94}$$

which is independent of time, as is ω_A. With these assumptions, we can analytically solve Eq. (90) to obtain the expression for Green's function,

$$G^A_k(\tau,\bar{\tau})=\frac{\sin[\omega_A(\tau-\bar{\tau})]}{\omega_A}\Theta(\tau-\bar{\tau}), \tag{95}$$

where Θ is the Heaviside step function.

The constant M_{PV} defined in Eq. (94) has the dimension of energy, which can be interpreted as the characteristic energy scale of parity violation in our model. Therefore, it is of interest to estimate M_{PV} based on current observations. The recent observations from GW170817 [153] and GRB170817A [154] constrain the propagating speed of GWs to be

$$-3\times10^{-15}\le c_{\mathrm{gw}}-1\le 7\times10^{-16}. \tag{96}$$

Using the definition of ω_A in Eq. (80), we have

$$c_{\mathrm{gw}}=\frac{\omega_A}{k}=\left(1-\frac{\lambda^A M_{\mathrm{PV}}}{k}\right)^{1/2}\simeq 1-\frac{\lambda^A M_{\mathrm{PV}}}{2k}, \tag{97}$$

which means

$$\frac{|M_{\mathrm{PV}}|}{k}<1.4\times10^{-15}. \tag{98}$$

Therefore, the typical energy scale of parity violation is much smaller than the wave numbers that we are interested in.

Besides, the constraint on the parity-violating energy scale M_{PV} from the GW events of binary black hole mergers in the LIGO-Virgo catalogs GWTC-1 and GWTC-2 is $M_{\mathrm{PV}}<6.4\times10^{-42}$ Gev at 90% confidence level [107], which corresponds to $M_{\mathrm{PV}}\sim\mathcal{O}(10^{-3})$ Mpc^{-1}. Since SIGWs are generated on small scales where $k\gg1$ Mpc^{-1}, this also implies $M_{\mathrm{PV}}/k\ll1$.

From Eq. (87), it might appear that the term $f^A_{\mathrm{PV1}}\propto(k/M_{\mathrm{PV}})\mathcal{C}_2^2/\mathcal{C}_5$ could be very large and potentially violate perturbation theory. However, we will show that this term is negligible.

The dimensions of scalar perturbations ψ and λ are $[\psi]=k^0$ and $[\lambda]=k^{-1}$, while the coefficient $[\mathcal{C}_2]=k^0$. Furthermore, according to (18), we have $\psi\sim k\lambda$. Taking into account relation (51), we find that

$$\frac{\mathcal{C}_2}{g_\theta \theta'} k = \frac{k\lambda}{\psi} \sim 1, \tag{99}$$

which implies

$$\mathcal{C}_2 \sim \frac{g_\theta \theta'}{k} \sim \frac{M_{\rm PV}}{k} \mathcal{C}_5 \ll \mathcal{C}_5. \tag{100}$$

Recalling the relation $\mathcal{C}_2 = 2/3(\mathcal{C}_5 - 1)$, we have $\mathcal{C}_5 \sim 1$.

Consequently, the coefficient in Eq. (87) can be estimated as

$$\frac{k}{M_{\rm PV}} \frac{\mathcal{C}_2^2}{\mathcal{C}_5} \sim \frac{M_{\rm PV}}{k} \mathcal{C}_5 \ll 1. \tag{101}$$

Considering the above analysis, we can conclude that the contribution from the PV term to SIGWs is negligible.

B. The power spectrum of SIGWs

The solutions of the circularly polarized modes can be written as

$$h^A_{\boldsymbol{k}}(\tau) = \frac{4}{\mathcal{C}_5} \int \frac{\mathrm{d}^3 \boldsymbol{k}'}{(2\pi)^{3/2}} p^{Aij} k'_i k'_j \zeta(\boldsymbol{k}') \zeta(\boldsymbol{k} - \boldsymbol{k}') \frac{1}{k^2} I^A(k, u, v, x), \tag{102}$$

where

$$\begin{aligned} I^A(k, u, v, x) &= -\int_0^x \mathrm{d}\bar{x} \frac{a(\bar{\tau})}{a(\tau)} k G^A_k(\tau, \bar{\tau}) \sum_{i=1,2} (f_{\mathrm{PC}i}(u, v, \bar{x}) + f^A_{\mathrm{PV}i}(k, u, v, \bar{x})) \\ &= \sum_{i=1,2} (I^A_{\mathrm{PC}i}(k, u, v, x) + I^A_{\mathrm{PV}i}(k, u, v, x)), \end{aligned} \tag{103}$$

with

$$I^A_{\mathrm{PC}i}(k, u, v, x) = -\int_0^x \mathrm{d}\bar{x} \frac{a(\bar{\tau})}{a(\tau)} k G^A_k(\tau, \bar{\tau}) f_{\mathrm{PC}i}(u, v, \bar{x}) \tag{104}$$

and

$$I^A_{\mathrm{PV}i}(k, u, v, x) = -\int_0^x \mathrm{d}\bar{x} \frac{a(\bar{\tau})}{a(\tau)} k G^A_k(\tau, \bar{\tau}) f^A_{\mathrm{PV}i}(u, v, \bar{x}). \tag{105}$$

The analytic expressions for $I^A_{\mathrm{PC}1}$ and $I^A_{\mathrm{PV}1}$ can be found in the Appendix. The remaining parts, $I^A_{\mathrm{PC}2}$ and $I^A_{\mathrm{PV}2}$, cannot be calculated analytically, so we will compute them numerically.

The power spectra of the SIGWs $\mathcal{P}^A_h$ are defined by

$$\langle h^A_{\boldsymbol{k}} h^C_{\boldsymbol{k}'} \rangle = \frac{2\pi^2}{k^3} \delta^3(\boldsymbol{k} + \boldsymbol{k}') \delta^{AC} \mathcal{P}^A_h(k). \tag{106}$$

With the definition of $\mathcal{P}^A_h$ (106) and the solution of SIGWs, we can obtain the power spectra of the SIGWs[2]

$$\begin{aligned} \mathcal{P}^A_h(k, x) = \frac{4}{\mathcal{C}_5^2} \int_0^\infty \mathrm{d}u \int_{|1-u|}^{1+u} \mathrm{d}v (\mathcal{J}(u, v) I^A(u, v, x)^2 \\ \times \mathcal{P}_\zeta(uk) \mathcal{P}_\zeta(vk)), \end{aligned} \tag{107}$$

where

$$\mathcal{J}(u, v) = \left[\frac{4u^2 - (1 + u^2 - v^2)^2}{4uv}\right]^2 \tag{108}$$

and $\mathcal{P}_\zeta$ is the power spectrum of primordial curvature perturbation.

The fractional energy density of the SIGWs is

$$\begin{aligned} \Omega_{\rm GW}(k, x) &= \frac{1}{12} \left(\frac{k}{\mathcal{H}}\right)^2 \sum_{A=R,L} \overline{\mathcal{P}^A_h(k, x)} = \frac{x^2}{12} \sum_{A=R,L} \overline{\mathcal{P}^A_h(k, x)} \\ &= \frac{1}{3\mathcal{C}_5^2} \int_0^\infty \mathrm{d}u \int_{|1-u|}^{1+u} \mathrm{d}v \mathcal{J}(u, v) \sum_{A=R,L} \overline{\tilde{I}^A(k, u, v, x)^2} \mathcal{P}_\zeta(uk) \mathcal{P}_\zeta(vk), \end{aligned} \tag{109}$$

[2] We have assumed that ζ is Gaussian to derive Eq. (107). For the non-Gaussian contributions, please refer to Refs. [89,155–158] and references therein.

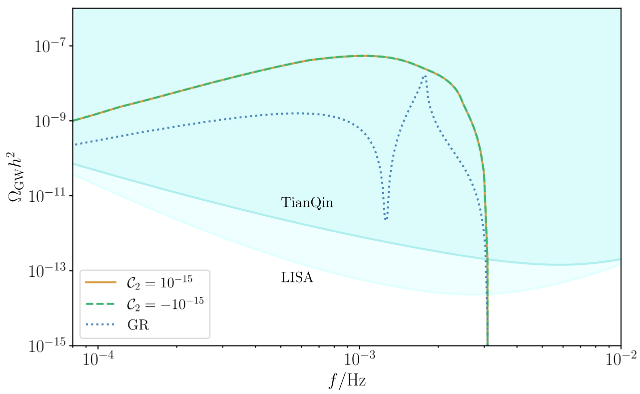

FIG. 1. The energy density of SIGWs from GR (dotted line) and our model (solid and dashed lines). The peak scale is $k_p = 10^{12}\ \mathrm{Mpc}^{-1}$, which corresponds to the maximum sensitivity of TianQin and LISA. The amplitude of the power spectrum is fixed to be $\mathcal{A}_\zeta = 10^{-2}$.

where the overline represents the time average and $\overline{\tilde{I}^A(k,u,v,x)^2} = \overline{I^A(k,u,v,x)^2}x^2$.

The GWs behave as free radiation, thus the fractional energy density of the SIGWs at the present time $\Omega_{\rm GW,0}$ can be expressed as [16]

$$\Omega_{\rm GW,0}(k) = \Omega_{\rm GW}(k, \eta \to \infty)\Omega_{r,0}, \tag{110}$$

where $\Omega_{r,0} \simeq 9 \times 10^{-5}$ is the current fractional energy density of the radiation [159].

To analyze the behavior of SIGWs in our model, in the following part of this section, we compute the energy density of SIGWs with a concrete power spectrum of primordial curvature perturbations. Considering the SIGWs induced by the monochromatic power spectrum,

$$\mathcal{P}_\zeta(k) = \mathcal{A}_\zeta \delta(\ln(k/k_p)), \tag{111}$$

then we obtain the energy density of SIGWs at the present time,

$$\Omega_{\rm GW,0}(k) = \frac{1}{3\mathcal{C}_5^2}\Omega_{r,0}\mathcal{A}_\zeta^2 \tilde{k}^{-2}\mathcal{J}(\tilde{k}^{-1},\tilde{k}^{-1}) \times \sum_{A=R,L} \overline{\tilde{I}^A(k,\tilde{k}^{-1},\tilde{k}^{-1},x\to\infty)^2}\,\Theta(2-\tilde{k}), \tag{112}$$

where $\tilde{k} = k/k_p$. We perform numerical calculations to determine the energy density of SIGWs, and the result is shown in Fig. 1.

Recalling the discussion in Sec. III, the energy density of SIGWs may exhibit discrepancies between PV metric teleparallel gravity and GR. This discrepancy arises from the extra scalar perturbation in tetrads γ, as well as the distinct evolution of scalar perturbation from metric ψ and the fluctuation of the scalar field $\delta\theta$, which deviate from their counterparts in GR. From Fig. 1, as expected, it is evident that the spectrum of the energy density of SIGWs differs significantly between GR and our model. For the SIGWs from GR, there is a divergence at $\tilde{k} = 2/\sqrt{3}$ due to the resonant amplification [11,15]. In contrast, the energy density of SIGWs in our model is regular across all frequencies. This feature makes metric teleparallel gravity distinguishable from GR.

It is difficult to completely analyze the behavior of the spectrum of the energy density of SIGWs due to the absence of analytic expressions for I^A. We only consider the analytic part of I^A, especially the contribution from $I^A_{\rm PC1}$, which is analytic. From the expression (A11) in the Appendix, the possible divergence in $I^A_{\rm PC1}$ comes from a logarithmic term, $\log|w - \sqrt{\mathbb{C}}(u+v)|$, which is similar to GR. In the case of SIGWs induced by the monochromatic power spectrum, the term in $I^A_{\rm PC1}$ that contains this logarithmic divergence is given by

$$I^A_{\rm PC1} \supset \frac{3}{4}\tilde{k}^3\left(\frac{9\tilde{k}(8(5t_1+1)(8t_1+1)-3(14t_1+1)\tilde{k}^2)}{(8t_1+1)^3} - \frac{8(16t_1+5)}{(8t_1+1)^{3/2}}\right)\log(|-9\tilde{k}^2+32t_1+4|). \tag{113}$$

For convenience of discussion, we have approximated $w = \omega_k/k \simeq 1$, taking into account the observational constraints on the propagating speed of GWs (96) and (97). The above expression vanishes even when $-9\tilde{k}^2 + 32t_1 + 4 = 0$; thus $I^A_{\rm PC1}$ does not contribute a divergent term.

V. CONCLUSION

SIGWs are a useful tool to test gravitational theory and probe the early universe. In this paper, we calculate the SIGWs from metric teleparallel gravity with the NY term in teleparallel geometry. By replacing the teleparallel equivalent Einstein-Hilbert Lagrangian with the general torsion scalar $\mathbb{T}$, Eq. (7), the strong coupling problem was avoided effectively only if $\mathcal{C}_2 \neq 0$.

In the context of teleparallelism, we assumed $\mathcal{C}_4 = 0$ to maintain vanishing curvature perturbation. Furthermore, we aimed to minimize deviations in the background evolution and the scalar perturbation from the metric compared to those in GR, so that we can focus on the contribution from scalar perturbations in the tetrad field and the PV source. With this consideration, we solved EOMs for the background and the linear scalar perturbations during the radiation-dominated era and obtained the analytic solutions. We had chosen the coupling function of the PV term to be the exponential form, which ensures that the speed of SIGWs is independent of time. We further derived the analytical expression for Green's function of the tensor perturbations. With these analytic solutions, we calculated the power spectrum and the energy density of SIGWs. Then, we evaluated numerically the energy density of SIGWs with a monochromatic power spectrum of the primordial curvature perturbation. Considering the observational constraints on the propagating speeds of the GWs, we conclude that the effect of the PV term on the SIGWs is negligible. Nevertheless, the spectrum of the energy density of SIGWs in our model still differs from that in GR. Crucially, there is no divergence in the energy density of SIGWs in our model, which makes teleparallel gravity distinguishable from GR.

ACKNOWLEDGMENTS

F. Z. thanks Yumin Hu for helpful discussions. This work was supported by the National Natural Science Foundation of China (NSFC) under Grants No. 12305075 and No. 11975020.

APPENDIX: THE INTEGRAL KERNEL

In this appendix, we derive the integral kernel I^A_{PC1} and I^A_{PV1}. With Green's function (90) and the definition of I^A, we can express the integral kernel as follows;

$$I^A_1(u,v,x)=\frac{\sin(wx)}{wx}I^A_{1s}(u,v,x)+\frac{\cos(wx)}{wx}I^A_{1c}(u,v,x), \tag{A1}$$

where the subscripts "s" and "c" stand for contributions involving the sine and cosine functions, respectively, and $w=\omega_A/k$. We also write

$$\begin{aligned} I^A_{1s}(u,v,x)&=\mathcal{I}^A_{1s}(u,v,x)-\mathcal{I}^A_{1s}(u,v,0),\\ I^A_{1c}(u,v,x)&=\mathcal{I}^A_{1c}(u,v,x)-\mathcal{I}^A_{1c}(u,v,0), \end{aligned} \tag{A2}$$

where $\mathcal{I}^A_{1s}$ and $\mathcal{I}^A_{1c}$ are defined by

$$\begin{aligned} \mathcal{I}^A_{1s}(u,v,y)&=-\int \mathrm{d}y\cos wy f(u,v,y)y,\\ \mathcal{I}^A_{1c}(u,v,y)&=\int \mathrm{d}y\sin wy f(u,v,y)y. \end{aligned} \tag{A3}$$

After lengthy calculations, we obtain the kernel of PC part

$$\begin{aligned} \mathcal{I}^A_{\mathrm{PC1}s}(u,v,y)=&-\frac{1}{9\mathbb{C}^3u^3v^3y^4}[6\mathbb{C}(14t_1+1)uvy^2\cos(wy)\cos(\sqrt{\mathbb{C}}uy)\cos(\sqrt{\mathbb{C}}vy)+\sqrt{\mathbb{C}}vy(-2t_1(9\mathbb{C}y^2(u^2-v^2)\\ &-7w^2y^2+42)+9\mathbb{C}y^2(v^2-u^2)+w^2y^2-6)\cos(wy)\sin(\sqrt{\mathbb{C}}uy)\cos(\sqrt{\mathbb{C}}vy)+\sqrt{\mathbb{C}}uy(2t_1(9\mathbb{C}y^2(u^2-v^2)\\ &+7(w^2y^2-6))+9\mathbb{C}y^2(u^2-v^2)+w^2y^2-6)\cos(wy)\cos(\sqrt{\mathbb{C}}uy)\sin(\sqrt{\mathbb{C}}vy)\\ &+(2t_1(9\mathbb{C}y^2(u^2+v^2)-7w^2y^2+42)+y^2(9\mathbb{C}(u^2+v^2)-w^2)+6)\cos(wy)\sin(\sqrt{\mathbb{C}}uy)\sin(\sqrt{\mathbb{C}}vy)\\ &-2\mathbb{C}(14t_1+1)uvwy^3\sin(wy)\cos(\sqrt{\mathbb{C}}uy)\cos(\sqrt{\mathbb{C}}vy)+2\sqrt{\mathbb{C}}(14t_1+1)vwy^2\sin(wy)\sin(\sqrt{\mathbb{C}}uy)\\ &\times\cos(\sqrt{\mathbb{C}}vy)+2\sqrt{\mathbb{C}}(14t_1+1)uwy^2\sin(wy)\cos(\sqrt{\mathbb{C}}uy)\sin(\sqrt{\mathbb{C}}vy)+wy(-2-11\mathbb{C}(u^2+v^2)y^2\\ &+w^2y^2-2t_1(14+23\mathbb{C}(u^2+v^2)y^2-7w^2y^2))\sin(wy)\sin(\sqrt{\mathbb{C}}uy)\sin(\sqrt{\mathbb{C}}vy)]\\ &-\frac{1}{36\mathbb{C}^3u^3v^3}[B_1(\mathrm{Ci}(A_3y)+\mathrm{Ci}(A_4y)-\mathrm{Ci}(|A_2|y)-\mathrm{Ci}(A_1y))+B_2(\mathrm{Ci}(A_3y)-\mathrm{Ci}(A_4y)+\mathrm{Ci}(|A_2|y)\\ &-\mathrm{Ci}(A_1y))+B_3(\mathrm{Ci}(|A_2|y)-\mathrm{Ci}(A_1y)-\mathrm{Ci}(A_3y)+\mathrm{Ci}(A_4y))], \end{aligned} \tag{A4}$$

where

$$A_1=w+\sqrt{\mathbb{C}}(u+v),\qquad A_2=w-\sqrt{\mathbb{C}}(u+v),\qquad A_3=w+\sqrt{\mathbb{C}}(u-v),\qquad A_4=w-\sqrt{\mathbb{C}}(u-v), \tag{A5}$$

and

$$B_1=9\mathbb{C}^2(2t_1+1)(u^2-v^2)^2+12\mathbb{C}(5t_1+1)w^2(u^2+v^2)-(14t_1+1)w^4, \tag{A6}$$

$$B_2 = 4\mathbb{C}^{3/2}(5+16t_1)u^3 w, \qquad B_3 = 4\mathbb{C}^{3/2}(5+16t_1)v^3 w. \tag{A7}$$

We also get

$$\begin{aligned}
\mathcal{I}^A_{\mathrm{PC1}c}(u,v,y) = {}& \frac{1}{9\mathbb{C}^3 u^3 v^3 y^4}[6\mathbb{C}(14t_1+1)uvy^2\sin(wy)\cos(\sqrt{\mathbb{C}}uy)\cos(\sqrt{\mathbb{C}}vy) \\
& + \sqrt{\mathbb{C}}vy(-2t_1(9\mathbb{C}y^2(u^2-v^2)-7w^2y^2+42) \\
& + 9\mathbb{C}y^2(v^2-u^2)+w^2y^2-6)\sin(wy)\sin(\sqrt{\mathbb{C}}uy)\cos(\sqrt{\mathbb{C}}vy) \\
& + \sqrt{\mathbb{C}}uy(2t_1(9\mathbb{C}y^2(u^2-v^2)+7(w^2y^2-6)) \\
& + 9\mathbb{C}y^2(u^2-v^2)+w^2y^2-6)\sin(wy)\cos(\sqrt{\mathbb{C}}uy)\sin(\sqrt{\mathbb{C}}vy) \\
& + (2t_1(9\mathbb{C}y^2(u^2+v^2)-7w^2y^2+42) \\
& + y^2(9\mathbb{C}(u^2+v^2)-w^2)+6)\cos(wy)\sin(\sqrt{\mathbb{C}}uy)\sin(\sqrt{\mathbb{C}}vy) \\
& + 2\mathbb{C}(14t_1+1)uvwy^3\cos(wy)\cos(\sqrt{\mathbb{C}}uy)\cos(\sqrt{\mathbb{C}}vy) \\
& + 2\sqrt{\mathbb{C}}(14t_1+1)vwy^2\cos(wy)\sin(\sqrt{\mathbb{C}}uy)\cos(\sqrt{\mathbb{C}}vy) \\
& + 2\sqrt{\mathbb{C}}(14t_1+1)uwy^2\cos(wy)\cos(\sqrt{\mathbb{C}}uy)\sin(\sqrt{\mathbb{C}}vy) \\
& + wy(2+11\mathbb{C}(u^2+v^2)y^2-w^2y^2 \\
& + 2t_1(14+23\mathbb{C}(u^2+v^2)y^2-7w^2y^2))\cos(wy)\sin(\sqrt{\mathbb{C}}uy)\sin(\sqrt{\mathbb{C}}vy)] \\
& - \frac{1}{36\mathbb{C}^3u^3v^3}[B_1(\mathrm{Si}(A_1y)+\mathrm{Si}(A_2y)-\mathrm{Si}(A_3y)-\mathrm{Si}(A_4y)) \\
& - B_2(\mathrm{Si}(A_3y)+\mathrm{Si}(A_2y)-\mathrm{Si}(A_1y)-\mathrm{Si}(A_4y)) \\
& + B_3(\mathrm{Si}(A_1y)+\mathrm{Si}(A_3y)-\mathrm{Ci}(A_2y)-\mathrm{Ci}(A_4y))].
\end{aligned} \tag{A8}$$

We have the following limit:

$$\mathcal{I}^A_{\mathrm{PC1}s}(u,v,x\to\infty) = 0 \tag{A9}$$

and

$$\begin{aligned}
\mathcal{I}^A_{\mathrm{PC1}s}(u,v,x\to 0) = {}& -\frac{9\mathbb{C}(2t_1+1)(u^2+v^2)+(14t_1+1)w^2}{9\mathbb{C}^2u^2v^2} \\
& + \frac{1}{36\mathbb{C}^3u^3v^3}\left(B_1\log\left|\frac{A_1A_2}{A_3A_4}\right| - B_2\log\left|\frac{A_2A_3}{A_1A_4}\right| - B_3\log\left|\frac{A_2A_4}{A_1A_3}\right|\right),
\end{aligned} \tag{A10}$$

thus

$$\begin{aligned}
I^A_{\mathrm{PC1}s}(u,v,x\to\infty) = {}& \frac{9\mathbb{C}(2t_1+1)(u^2+v^2)+(14t_1+1)w^2}{9\mathbb{C}^2u^2v^2} \\
& - \frac{1}{36\mathbb{C}^3u^3v^3}\left(B_1\log\left|\frac{A_1A_2}{A_3A_4}\right| - B_2\log\left|\frac{A_2A_3}{A_1A_4}\right| - B_3\log\left|\frac{A_2A_4}{A_1A_3}\right|\right).
\end{aligned} \tag{A11}$$

We also have

$$\mathcal{I}^A_{\mathrm{PC1}c}(u,v,x\to\infty) = \frac{B_1-B_2-B_3}{36\mathbb{C}^3u^3v^3}\pi\Theta(-A_2) \tag{A12}$$

and

$$\mathcal{I}^A_{\mathrm{PC1}c}(u,v,0)=0; \tag{A13}$$

thus

$$I^A_{\mathrm{PC1}c}(u,v,x\to\infty)=\frac{B_1-B_2-B_3}{36\mathbb{C}^3u^3v^3}\pi\Theta(-A_2). \tag{A14}$$

As for the PV part, we have the following expressions:

$$\begin{aligned}\mathcal{I}^A_{\mathrm{PV1}s}(u,v,y)=&\frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{u+v}{12\mathbb{C}^3u^3v^3y^4}\Big[6\mathbb{C}uvy^2\cos(wy)\cos\left(\sqrt{\mathbb{C}}uy\right)\cos\left(\sqrt{\mathbb{C}}vy\right)\\&-2\mathbb{C}uvwy^3\sin(wy)\cos\left(\sqrt{\mathbb{C}}uy\right)\cos\left(\sqrt{\mathbb{C}}vy\right)\\&+2\sqrt{\mathbb{C}}vwy^2\sin(wy)\sin\left(\sqrt{\mathbb{C}}uy\right)\cos\left(\sqrt{\mathbb{C}}vy\right)\\&+2\sqrt{\mathbb{C}}uwy^2\sin(wy)\sin\left(\sqrt{\mathbb{C}}vy\right)\cos\left(\sqrt{\mathbb{C}}uy\right)\\&+\sqrt{\mathbb{C}}vy(3\mathbb{C}y^2(v^2-u^2)+w^2y^2-6)\cos(wy)\sin\left(\sqrt{\mathbb{C}}uy\right)\cos\left(\sqrt{\mathbb{C}}vy\right)\\&+\sqrt{\mathbb{C}}uy(3\mathbb{C}y^2(u^2-v^2)+w^2y^2-6)\cos(wy)\cos\left(\sqrt{\mathbb{C}}uy\right)\sin\left(\sqrt{\mathbb{C}}vy\right)\\&+(3\mathbb{C}y^2(u^2+v^2)-w^2y^2+6)\cos(wy)\sin\left(\sqrt{\mathbb{C}}uy\right)\sin\left(\sqrt{\mathbb{C}}vy\right)\\&+wy(w^2y^2-5\mathbb{C}y^2(u^2+v^2)-2)\sin(wy)\sin\left(\sqrt{\mathbb{C}}uy\right)\sin\left(\sqrt{\mathbb{C}}vy\right)]\\&-\frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{u+v}{48\mathbb{C}^3u^3v^3}[D_1(\mathrm{Ci}(A_1y)+\mathrm{Ci}(|A_2|y)-\mathrm{Ci}(A_3y)-\mathrm{Ci}(A_4y))\\&-D_2(\mathrm{Ci}(A_3y)+\mathrm{Ci}(|A_2|y)-\mathrm{Ci}(A_1y)-\mathrm{Ci}(A_4y))\\&-D_3(\mathrm{Ci}(|A_2|y)+\mathrm{Ci}(A_4y)-\mathrm{Ci}(A_1y)-\mathrm{Ci}(A_3y))\Big],\end{aligned} \tag{A15}$$

where

$$D_1=3\mathbb{C}^2(u^2-v^2)^2+6\mathbb{C}w^2(u^2+v^2)-w^4,\qquad D_2=8\mathbb{C}^{3/2}u^3w,\qquad D_3=8\mathbb{C}^{3/2}v^3w. \tag{A16}$$

We have the following limits:

$$\mathcal{I}^A_{\mathrm{PV1}s}(u,v,x\to\infty)=0 \tag{A17}$$

and

$$\begin{aligned}\mathcal{I}^A_{\mathrm{PV1}s}(u,v,y\to0)=&\frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{(u+v)(3\mathbb{C}(u^2+v^2)+w^2)}{12\mathbb{C}^2u^2v^2}\\&-\frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{u+v}{48\mathbb{C}^3u^3v^3}\left(D_1\log\left|\frac{A_1A_2}{A_3A_4}\right|-D_2\log\left|\frac{A_2A_3}{A_1A_4}\right|-D_3\log\left|\frac{A_2A_4}{A_1A_3}\right|\right),\end{aligned} \tag{A18}$$

thus

$$\begin{aligned}I^A_{\mathrm{PV1}s}(u,v,x\to\infty)=&-\frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{(u+v)(3\mathbb{C}(u^2+v^2)+w^2)}{12\mathbb{C}^2u^2v^2}\\&+\frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{u+v}{48\mathbb{C}^3u^3v^3}\left(D_1\log\left|\frac{A_1A_2}{A_3A_4}\right|-D_2\log\left|\frac{A_2A_3}{A_1A_4}\right|-D_3\log\left|\frac{A_2A_4}{A_1A_3}\right|\right).\end{aligned} \tag{A19}$$

We also have

$$\begin{aligned}
\mathcal{I}^A_{\mathrm{PV1}c}(u,v,y) = {} & \frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{u+v}{12\mathbb{C}^3u^3v^3y^4}\Big[6\mathbb{C}uvy^2\cos(wy)\cos\big(\sqrt{\mathbb{C}}uy\big)\cos\big(\sqrt{\mathbb{C}}vy\big) \\
& - 2\mathbb{C}uvwy^3\sin(wy)\cos\big(\sqrt{\mathbb{C}}uy\big)\cos\big(\sqrt{\mathbb{C}}vy\big) \\
& + 2\sqrt{\mathbb{C}}vwy^2\sin(wy)\sin\big(\sqrt{\mathbb{C}}uy\big)\cos\big(\sqrt{\mathbb{C}}vy\big) \\
& + 2\sqrt{\mathbb{C}}uwy^2\sin(wy)\sin\big(\sqrt{\mathbb{C}}vy\big)\cos\big(\sqrt{\mathbb{C}}uy\big) \\
& + \sqrt{\mathbb{C}}vy(3\mathbb{C}y^2(v^2-u^2)+w^2y^2-6)\cos(wy)\sin\big(\sqrt{\mathbb{C}}uy\big)\cos\big(\sqrt{\mathbb{C}}vy\big) \\
& + \sqrt{\mathbb{C}}uy(3\mathbb{C}y^2(u^2-v^2)+w^2y^2-6)\cos(wy)\cos\big(\sqrt{\mathbb{C}}uy\big)\sin\big(\sqrt{\mathbb{C}}vy\big) \\
& + (3\mathbb{C}y^2(u^2+v^2)-w^2y^2+6)\cos(wy)\sin\big(\sqrt{\mathbb{C}}uy\big)\sin\big(\sqrt{\mathbb{C}}vy\big) \\
& + wy(w^2y^2-5\mathbb{C}y^2(u^2+v^2)-2)\sin(wy)\sin\big(\sqrt{\mathbb{C}}uy\big)\sin\big(\sqrt{\mathbb{C}}vy\big)\big] \\
& + \frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{u+v}{48\mathbb{C}^3u^3v^3}[D_1(\mathrm{Si}(A_1y)+\mathrm{Si}(A_2y)-\mathrm{Si}(A_3y)-\mathrm{Si}(A_4y)) \\
& + D_2(\mathrm{Si}(A_1y)-\mathrm{Si}(A_2y)-\mathrm{Si}(A_3y)+\mathrm{Si}(A_4y)) \\
& + D_3(\mathrm{Si}(A_1y)-\mathrm{Si}(A_2y)+\mathrm{Si}(A_3y)-\mathrm{Si}(A_4y))\Big].
\end{aligned} \tag{A20}$$

We have the following limits:

$$\mathcal{I}^A_{\mathrm{PV1}c}(u,v,x\to\infty) = -\frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{u+v}{48\mathbb{C}^3u^3v^3}(D_1-D_2-D_3)\pi\Theta(-A_2) \tag{A21}$$

and

$$\mathcal{I}^A_{\mathrm{PV1}c}(u,v,x\to 0) = 0, \tag{A22}$$

thus

$$I^A_{\mathrm{PV1}c}(u,v,x\to\infty) = -\frac{\lambda^A k}{M_{\mathrm{PV}}}\frac{\mathcal{C}_2^2}{\mathcal{C}_5}\frac{u+v}{48\mathbb{C}^3u^3v^3}(D_1-D_2-D_3)\pi\Theta(-A_2). \tag{A23}$$

[1] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW151226: Observation of gravitational waves from a 22-solar-mass binary black hole coalescence, Phys. Rev. Lett. **116**, 241103 (2016).

[2] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of gravitational waves from a binary black hole merger, Phys. Rev. Lett. **116**, 061102 (2016).

[3] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170608: Observation of a 19-solar-mass binary black hole coalescence, Astrophys. J. Lett. **851**, L35 (2017).

[4] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170817: Observation of gravitational waves from a binary neutron star inspiral, Phys. Rev. Lett. **119**, 161101 (2017).

[5] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170814: A three-detector observation of gravitational waves from a binary black hole coalescence, Phys. Rev. Lett. **119**, 141101 (2017).

[6] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170104: Observation of a 50-solar-mass binary black hole coalescence at redshift 0.2, Phys. Rev. Lett. **118**, 221101 (2017); **121**, 129901(E) (2018).

[7] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GWTC-1: A gravitational-wave transient catalog of compact binary mergers observed by LIGO and Virgo during the first and second observing runs, Phys. Rev. X **9**, 031040 (2019).
[8] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW190814: Gravitational waves from the coalescence of a 23 solar mass black hole with a 2.6 solar mass compact object, Astrophys. J. Lett. **896**, L44 (2020).
[9] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW190425: Observation of a compact binary coalescence with total mass $\sim 3.4 M_{\odot}$, Astrophys. J. Lett. **892**, L3 (2020).
[10] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW190412: Observation of a binary-black-hole coalescence with asymmetric masses, Phys. Rev. D **102**, 043015 (2020).
[11] K. N. Ananda, C. Clarkson, and D. Wands, The cosmological gravitational wave background from primordial density perturbations, Phys. Rev. D **75**, 123518 (2007).
[12] R. Saito and J. Yokoyama, Gravitational wave background as a probe of the primordial black hole abundance, Phys. Rev. Lett. **102**, 161101 (2009); **107**, 069901(E) (2011).
[13] T. Nakama, J. Silk, and M. Kamionkowski, Stochastic gravitational waves associated with the formation of primordial black holes, Phys. Rev. D **95**, 043511 (2017).
[14] S. Wang, Y.-F. Wang, Q.-G. Huang, and T. G. F. Li, Constraints on the primordial black hole abundance from the first Advanced LIGO observation run using the stochastic gravitational-wave background, Phys. Rev. Lett. **120**, 191102 (2018).
[15] K. Kohri and T. Terada, Semianalytic calculation of gravitational wave spectrum nonlinearly induced from primordial curvature perturbations, Phys. Rev. D **97**, 123532 (2018).
[16] J. R. Espinosa, D. Racco, and A. Riotto, A cosmological signature of the SM Higgs instability: Gravitational waves, J. Cosmol. Astropart. Phys. 09 (2018) 012.
[17] S. Kuroyanagi, T. Chiba, and T. Takahashi, Probing the Universe through the stochastic gravitational wave background, J. Cosmol. Astropart. Phys. 11 (2018) 038.
[18] J. Fumagalli, S. Renaux-Petel, and L. T. Witkowski, Oscillations in the stochastic gravitational wave background from sharp features and particle production during inflation, J. Cosmol. Astropart. Phys. 08 (2021) 030.
[19] M. Braglia, X. Chen, and D. K. Hazra, Probing primordial features with the stochastic gravitational wave background, J. Cosmol. Astropart. Phys. 03 (2021) 005.
[20] J. Lin, Q. Gao, Y. Gong, Y. Lu, C. Zhang, and F. Zhang, Primordial black holes and secondary gravitational waves from k and G inflation, Phys. Rev. D **101**, 103515 (2020).
[21] Y. Lu, A. Ali, Y. Gong, J. Lin, and F. Zhang, Gauge transformation of scalar induced gravitational waves, Phys. Rev. D **102**, 083503 (2020).
[22] G. Domènech, Scalar induced gravitational waves review, Universe **7**, 398 (2021).
[23] F. Zhang, J. Lin, and Y. Lu, Double-peaked inflation model: Scalar induced gravitational waves and primordial-black-hole suppression from primordial non-Gaussianity, Phys. Rev. D **104**, 063515 (2021); **104**, 129902(E) (2021).
[24] F. Zhang, Primordial black holes and scalar induced gravitational waves from the E model with a Gauss-Bonnet term, Phys. Rev. D **105**, 063539 (2022).
[25] T. Papanikolaou, C. Tzerefos, S. Basilakos, and E. N. Saridakis, Scalar induced gravitational waves from primordial black hole Poisson fluctuations in f(R) gravity, J. Cosmol. Astropart. Phys. 10 (2022) 013.
[26] Z. Yi and Q. Fei, Constraints on primordial curvature spectrum from primordial black holes and scalar-induced gravitational waves, Eur. Phys. J. C **83**, 82 (2023).
[27] Z.-Q. You, Z. Yi, and Y. Wu, Constraints on primordial curvature power spectrum with pulsar timing arrays, J. Cosmol. Astropart. Phys. 11 (2023) 065.
[28] Y. Lu, Y. Gong, Z. Yi, and F. Zhang, Constraints on primordial curvature perturbations from primordial black hole dark matter and secondary gravitational waves, J. Cosmol. Astropart. Phys. 12 (2019) 031.
[29] B.-M. Gu, F.-W. Shu, and K. Yang, Inflation with shallow dip and primordial black holes, arXiv:2307.00510.
[30] S. Choudhury, A. Karde, S. Panda, and M. Sami, Scalar induced gravity waves from ultra slow-roll Galileon inflation, arXiv:2308.09273.
[31] Z. Wang, S. Gao, Y. Gong, and Y. Wang, Primordial black holes and scalar-induced gravitational waves from the polynomial attractor model, Phys. Rev. D **109**, 103532 (2024).
[32] S. Choudhury, A. Karde, S. Panda, and M. Sami, Realisation of the ultra-slow roll phase in Galileon inflation and PBH overproduction, arXiv:2401.10925.
[33] K. Danzmann, LISA: An ESA cornerstone mission for a gravitational wave observatory, Classical Quantum Gravity **14**, 1399 (1997).
[34] P. Amaro-Seoane *et al.* (LISA Collaboration), Laser Interferometer Space Antenna, arXiv:1702.00786.
[35] W.-R. Hu and Y.-L. Wu, The Taiji program in space for gravitational wave physics and the nature of gravity, Natl. Sci. Rev. **4**, 685 (2017).
[36] J. Luo *et al.* (TianQin Collaboration), TianQin: A space-borne gravitational wave detector, Classical Quantum Gravity **33**, 035010 (2016).
[37] Y. Gong, J. Luo, and B. Wang, Concepts and status of Chinese space gravitational wave detection projects, Nat. Astron. **5**, 881 (2021).
[38] S. Kawamura *et al.*, The Japanese space gravitational wave antenna: DECIGO, Classical Quantum Gravity **28**, 094011 (2011).
[39] M. Kramer and D. J. Champion (EPTA Collaboration), The European Pulsar Timing Array and the Large European Array for pulsars, Classical Quantum Gravity **30**, 224009 (2013).
[40] G. Hobbs *et al.*, The international pulsar timing array project: Using pulsars as a gravitational wave detector, Classical Quantum Gravity **27**, 084013 (2010).
[41] M. A. McLaughlin, The North American Nanohertz Observatory for gravitational waves, Classical Quantum Gravity **30**, 224008 (2013).
[42] G. Hobbs, The parkes pulsar timing array, Classical Quantum Gravity **30**, 224007 (2013).
[43] C. J. Moore, R. H. Cole, and C. P. L. Berry, Gravitational-wave sensitivity curves, Classical Quantum Gravity **32**, 015014 (2015).

[44] G. Agazie *et al.* (NANOGrav Collaboration), The NANOGrav 15 yr data set: Evidence for a gravitational-wave background, Astrophys. J. Lett. **951**, L8 (2023).

[45] G. Agazie *et al.* (NANOGrav Collaboration), The NANOGrav 15 yr data set: Observations and timing of 68 millisecond pulsars, Astrophys. J. Lett. **951**, L9 (2023).

[46] A. Zic *et al.*, The parkes pulsar timing array third data release, Publ. Astron. Soc. Aust. **40**, e049 (2023).

[47] D. J. Reardon *et al.*, Search for an isotropic gravitational-wave background with the parkes pulsar timing array, Astrophys. J. Lett. **951**, L6 (2023).

[48] J. Antoniadis *et al.* (EPTA Collaboration), The second data release from the European Pulsar Timing Array—I. The dataset and timing analysis, Astron. Astrophys. **678**, A48 (2023).

[49] J. Antoniadis *et al.* (EPTA and InPTA Collaborations), The second data release from the European Pulsar Timing Array—III. Search for gravitational wave signals, Astron. Astrophys. **678**, A50 (2023).

[50] H. Xu *et al.*, Searching for the nano-hertz stochastic gravitational wave background with the Chinese pulsar timing array data release I, Res. Astron. Astrophys. **23**, 075024 (2023).

[51] A. Afzal *et al.* (NANOGrav Collaboration), The NANOGrav 15 yr data set: Search for signals from new physics, Astrophys. J. Lett. **951**, L11 (2023).

[52] J. Antoniadis *et al.* (EPTA Collaboration), The second data release from the European Pulsar Timing Array: IV. Implications for massive black holes, dark matter and the early Universe, Astron. Astrophys. **685**, A94 (2024).

[53] Z. Yi, Q. Gao, Y. Gong, Y. Wang, and F. Zhang, Scalar induced gravitational waves in light of pulsar timing array data, Sci. China Phys. Mech. Astron. **66**, 120404 (2023).

[54] Z. Yi, Z.-Q. You, and Y. Wu, Model-independent reconstruction of the primordial curvature power spectrum from PTA data, J. Cosmol. Astropart. Phys. 01 (2024) 066.

[55] Z. Yi, Z.-Q. You, Y. Wu, Z.-C. Chen, and L. Liu, Exploring the NANOGrav signal and planet-mass primordial black holes through Higgs inflation, arXiv:2308.14688.

[56] Y.-F. Cai, X.-C. He, X.-H. Ma, S.-F. Yan, and G.-W. Yuan, Limits on scalar-induced gravitational waves from the stochastic background by pulsar timing array observations, Sci. Bull. **68**, 2929 (2023).

[57] L. Liu, Z.-C. Chen, and Q.-G. Huang, Implications for the non-Gaussianity of curvature perturbation from pulsar timing arrays, Phys. Rev. D **109**, L061301 (2024).

[58] J.-H. Jin, Z.-C. Chen, Z. Yi, Z.-Q. You, L. Liu, and Y. Wu, Confronting sound speed resonance with pulsar timing arrays, J. Cosmol. Astropart. Phys. 09 (2023) 016.

[59] Z.-C. Chen, J. Li, L. Liu, and Z. Yi, Probing the speed of scalar-induced gravitational waves with pulsar timing arrays, Phys. Rev. D **109**, L101302 (2024).

[60] L. Liu, Y. Wu, and Z.-C. Chen, Simultaneously probing the sound speed and equation of state of the early Universe with pulsar timing arrays, J. Cosmol. Astropart. Phys. 04 (2024) 011.

[61] Z.-C. Chen and L. Liu, Can we distinguish the adiabatic fluctuations and isocurvature fluctuations with pulsar timing arrays?, arXiv:2402.16781.

[62] P. Horava, Quantum gravity at a Lifshitz point, Phys. Rev. D **79**, 084008 (2009).

[63] X. Gao and X.-Y. Hong, Propagation of gravitational waves in a cosmological background, Phys. Rev. D **101**, 064057 (2020).

[64] Y.-M. Hu and X. Gao, Covariant $3+1$ correspondence of the spatially covariant gravity and the degeneracy conditions, Phys. Rev. D **105**, 044023 (2022).

[65] Y.-M. Hu and X. Gao, Spatially covariant gravity with 2 degrees of freedom: Perturbative analysis, Phys. Rev. D **104**, 104007 (2021).

[66] T. Takahashi and J. Soda, Chiral primordial gravitational waves from a Lifshitz Point, Phys. Rev. Lett. **102**, 231301 (2009).

[67] Y. S. Myung, Chiral gravitational waves from $z=2$ Hořava-Lifshitz gravity, Phys. Lett. B **684**, 1 (2010).

[68] A. Wang, Q. Wu, W. Zhao, and T. Zhu, Polarizing primordial gravitational waves by parity violation, Phys. Rev. D **87**, 103512 (2013).

[69] T. Zhu, W. Zhao, Y. Huang, A. Wang, and Q. Wu, Effects of parity violation on non-gaussianity of primordial gravitational waves in Hořava-Lifshitz gravity, Phys. Rev. D **88**, 063508 (2013).

[70] D. Cannone, J.-O. Gong, and G. Tasinato, Breaking discrete symmetries in the effective field theory of inflation, J. Cosmol. Astropart. Phys. 08 (2015) 003.

[71] W. Zhao, T. Liu, L. Wen, T. Zhu, A. Wang, Q. Hu, and C. Zhou, Model-independent test of the parity symmetry of gravity with gravitational waves, Eur. Phys. J. C **80**, 630 (2020).

[72] W. Zhao, T. Zhu, J. Qiao, and A. Wang, Waveform of gravitational waves in the general parity-violating gravities, Phys. Rev. D **101**, 024002 (2020).

[73] J. Qiao, T. Zhu, W. Zhao, and A. Wang, Polarized primordial gravitational waves in the ghost-free parity-violating gravity, Phys. Rev. D **101**, 043528 (2020).

[74] J. Qiao, T. Zhu, W. Zhao, and A. Wang, Waveform of gravitational waves in the ghost-free parity-violating gravities, Phys. Rev. D **100**, 124058 (2019).

[75] J. Qiao, T. Zhu, G. Li, and W. Zhao, Post-Newtonian parameters of ghost-free parity-violating gravities, J. Cosmol. Astropart. Phys. 04 (2022) 054.

[76] C. Gong, T. Zhu, R. Niu, Q. Wu, J.-L. Cui, X. Zhang, W. Zhao, and A. Wang, Gravitational wave constraints on Lorentz and parity violations in gravity: High-order spatial derivative cases, Phys. Rev. D **105**, 044034 (2022).

[77] M. Zhu and Y. Cai, Parity-violation in bouncing cosmology, J. High Energy Phys. 04 (2023) 095.

[78] S. Akama and M. Zhu, Parity violation in primordial tensor non-Gaussianities from matter bounce cosmology, arXiv:2404.05464.

[79] T. D. Lee and C.-N. Yang, Question of parity conservation in weak interactions, Phys. Rev. **104**, 254 (1956).

[80] C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson, Experimental test of parity conservation in β decay, Phys. Rev. **105**, 1413 (1957).

[81] O. H. E. Philcox, Probing parity violation with the four-point correlation function of BOSS galaxies, Phys. Rev. D **106**, 063501 (2022).

[82] J. Hou, Z. Slepian, and R. N. Cahn, Measurement of parity-odd modes in the large-scale 4-point correlation function of Sloan Digital Sky Survey Baryon Oscillation Spectroscopic Survey twelfth data release CMASS and LOWZ galaxies, Mon. Not. R. Astron. Soc. **522**, 5701 (2023).

[83] Y. Minami and E. Komatsu, New extraction of the cosmic birefringence from the Planck 2018 polarization data, Phys. Rev. Lett. **125**, 221301 (2020).

[84] J. R. Eskilt and E. Komatsu, Improved constraints on cosmic birefringence from the WMAP and Planck cosmic microwave background polarization data, Phys. Rev. D **106**, 063503 (2022).

[85] T. Liu, X. Tong, Y. Wang, and Z.-Z. Xianyu, Probing P and *CP* violations on the cosmological collider, J. High Energy Phys. 04 (2020) 189.

[86] X. Niu, M. H. Rahat, K. Srinivasan, and W. Xue, Parity-odd and even trispectrum from axion inflation, J. Cosmol. Astropart. Phys. 05 (2023) 018.

[87] G. Cabass, S. Jazayeri, E. Pajer, and D. Stefanyszyn, Parity violation in the scalar trispectrum: No-go theorems and yes-go examples, J. High Energy Phys. 02 (2023) 021.

[88] C. Creque-Sarbinowski, S. Alexander, M. Kamionkowski, and O. Philcox, Parity-violating trispectrum from Chern-Simons gravity, J. Cosmol. Astropart. Phys. 11 (2023) 029.

[89] S. Garcia-Saenz, Y. Lu, and Z. Shuai, Scalar-induced gravitational waves from ghost inflation and parity violation, Phys. Rev. D **108**, 123507 (2023).

[90] R. Jackiw and S. Y. Pi, Chern-Simons modification of general relativity, Phys. Rev. D **68**, 104012 (2003).

[91] A. Lue, L.-M. Wang, and M. Kamionkowski, Cosmological signature of new parity violating interactions, Phys. Rev. Lett. **83**, 1506 (1999).

[92] M. Satoh, S. Kanno, and J. Soda, Circular polarization of primordial gravitational waves in string-inspired inflationary cosmology, Phys. Rev. D **77**, 023526 (2008).

[93] S. Saito, K. Ichiki, and A. Taruya, Probing polarization states of primordial gravitational waves with CMB anisotropies, J. Cosmol. Astropart. Phys. 09 (2007) 002.

[94] S. Alexander and N. Yunes, Chern-Simons modified general relativity, Phys. Rep. **480**, 1 (2009).

[95] N. Yunes, R. O'Shaughnessy, B. J. Owen, and S. Alexander, Testing gravitational parity violation with coincident gravitational waves and short gamma-ray bursts, Phys. Rev. D **82**, 064017 (2010).

[96] V. Gluscevic and M. Kamionkowski, Testing parity-violating mechanisms with cosmic microwave background experiments, Phys. Rev. D **81**, 123529 (2010).

[97] Y. S. Myung and T. Moon, Primordial massive gravitational waves from Einstein-Chern-Simons-Weyl gravity, J. Cosmol. Astropart. Phys. 08 (2014) 061.

[98] S. Kawai and J. Kim, Gauss–Bonnet Chern–Simons gravitational wave leptogenesis, Phys. Lett. B **789**, 145 (2019).

[99] R. Nair, S. Perkins, H. O. Silva, and N. Yunes, Fundamental physics implications for higher-curvature theories from binary black hole signals in the LIGO-Virgo catalog GWTC-1, Phys. Rev. Lett. **123**, 191101 (2019).

[100] A. Nishizawa and T. Kobayashi, Parity-violating gravity and GW170817, Phys. Rev. D **98**, 124018 (2018).

[101] N. Bartolo and G. Orlando, Parity breaking signatures from a Chern-Simons coupling during inflation: The case of non-Gaussian gravitational waves, J. Cosmol. Astropart. Phys. 07 (2017) 034.

[102] N. Bartolo, G. Orlando, and M. Shiraishi, Measuring chiral gravitational waves in Chern-Simons gravity with CMB bispectra, J. Cosmol. Astropart. Phys. 01 (2019) 050.

[103] F. Zhang, J.-X. Feng, and X. Gao, Circularly polarized scalar induced gravitational waves from the Chern-Simons modified gravity, J. Cosmol. Astropart. Phys. 10 (2022) 054.

[104] J.-X. Feng, F. Zhang, and X. Gao, Scalar induced gravitational waves from Chern-Simons gravity during inflation era, J. Cosmol. Astropart. Phys. 07 (2023) 047.

[105] M. Crisostomi, K. Noui, C. Charmousis, and D. Langlois, Beyond Lovelock gravity: Higher derivative metric theories, Phys. Rev. D **97**, 044034 (2018).

[106] A. Chatzistavrakidis, G. Karagiannis, and P. Schupp, Torsion-induced gravitational θ term and gravitoelectromagnetism, Eur. Phys. J. C **80**, 1034 (2020).

[107] Q. Wu, T. Zhu, R. Niu, W. Zhao, and A. Wang, Constraints on the Nieh-Yan modified teleparallel gravity with gravitational waves, Phys. Rev. D **105**, 024035 (2022).

[108] M. Långvik, J.-M. Ojanperä, S. Raatikainen, and S. Rasanen, Higgs inflation with the Holst and the Nieh–Yan term, Phys. Rev. D **103**, 083514 (2021).

[109] H. Rao, Parametrized post-Newtonian limit of the Nieh-Yan modified teleparallel gravity, Phys. Rev. D **104**, 124084 (2021).

[110] M. Li and D. Zhao, A simple parity violating model in the symmetric teleparallel gravity and its cosmological perturbations, Phys. Lett. B **827**, 136968 (2022).

[111] E. Battista and V. De Falco, First post-Newtonian generation of gravitational waves in Einstein-Cartan theory, Phys. Rev. D **104**, 084067 (2021).

[112] M. Li, Z. Li, and H. Rao, Ghost instability in the teleparallel gravity model with parity violations, Phys. Lett. B **834**, 137395 (2022).

[113] M. Li, Y. Tong, and D. Zhao, Possible consistent model of parity violations in the symmetric teleparallel gravity, Phys. Rev. D **105**, 104002 (2022).

[114] M. Hohmann and C. Pfeifer, Teleparallel axions and cosmology, Eur. Phys. J. C **81**, 376 (2021).

[115] F. Bombacigno, S. Boudet, G. J. Olmo, and G. Montani, Big bounce and future time singularity resolution in Bianchi I cosmologies: The projective invariant Nieh-Yan case, Phys. Rev. D **103**, 124031 (2021).

[116] D. Iosifidis and L. Ravera, Parity violating metric-affine gravity theories, Classical Quantum Gravity **38**, 115003 (2021).

[117] M. Hohmann and C. Pfeifer, Gravitational wave birefringence in spatially curved teleparallel cosmology, Phys. Lett. B **834**, 137437 (2022).

[118] A. Conroy and T. Koivisto, Parity-violating gravity and GW170817 in non-Riemannian cosmology, J. Cosmol. Astropart. Phys. 12 (2019) 016.

[119] D. Iosifidis, The full quadratic metric-affine gravity (including parity odd terms): Exact solutions for the affine-connection, Classical Quantum Gravity **39**, 095002 (2022).

[120] C. Pagani and R. Percacci, Quantum gravity with torsion and non-metricity, Classical Quantum Gravity **32**, 195019 (2015).

[121] Z. Chen, Y. Yu, and X. Gao, Polarized gravitational waves in the parity violating scalar-nonmetricity theory, J. Cosmol. Astropart. Phys. 06 (2023) 001.

[122] I. D. Gialamas and K. Tamvakis, Inflation in metric-affine quadratic gravity, J. Cosmol. Astropart. Phys. 03 (2023) 042.

[123] T. Papanikolaou, C. Tzerefos, S. Basilakos, and E. N. Saridakis, No constraints for f(T) gravity from gravitational waves induced from primordial black hole fluctuations, Eur. Phys. J. C **83**, 31 (2023).

[124] A. Salvio, Inflating and reheating the Universe with an independent affine connection, Phys. Rev. D **106**, 103510 (2022).

[125] I. D. Gialamas and H. Veermäe, Electroweak vacuum decay in metric-affine gravity, Phys. Lett. B **844**, 138109 (2023).

[126] C. Tzerefos, T. Papanikolaou, E. N. Saridakis, and S. Basilakos, Scalar induced gravitational waves in modified teleparallel gravity theories, Phys. Rev. D **107**, 124019 (2023).

[127] V. De Falco, E. Battista, D. Usseglio, and S. Capozziello, Radiative losses and radiation-reaction effects at the first post-Newtonian order in Einstein–Cartan theory, Eur. Phys. J. C **84**, 137 (2024).

[128] Y. Yu, Z. Chen, and X. Gao, Spatially covariant gravity with nonmetricity, Eur. Phys. J. C **84**, 549 (2024).

[129] M. Li, H. Rao, and D. Zhao, A simple parity violating gravity model without ghost instability, J. Cosmol. Astropart. Phys. 11 (2020) 023.

[130] H. T. Nieh and M. L. Yan, An Identity in Riemann-cartan Geometry, J. Math. Phys. (N.Y.) **23**, 373 (1982).

[131] H. T. Nieh, in *Conference in Honor of C. N. Yang's 85th Birthday: Statistical Physics, High Energy, Condensed Matter and Mathematical Physics* (2008), pp. 29–37, arXiv:1309.0915.

[132] J. Nissinen and G. E. Volovik, Thermal Nieh-Yan anomaly in Weyl superfluids, Phys. Rev. Res. **2**, 033269 (2020).

[133] M. Li, H. Rao, and Y. Tong, Revisiting a parity violating gravity model without ghost instability: Local Lorentz covariance, Phys. Rev. D **104**, 084077 (2021).

[134] R.-G. Cai, C. Fu, and W.-W. Yu, Parity violation in stochastic gravitational wave background from inflation in Nieh-Yan modified teleparallel gravity, Phys. Rev. D **105**, 103520 (2022).

[135] M. Li and H. Rao, Irregular universe in the Nieh-Yan modified teleparallel gravity, Phys. Lett. B **841**, 137929 (2023).

[136] F. Zhang, J.-X. Feng, and X. Gao, Scalar induced gravitational waves in symmetric teleparallel gravity with a parity-violating term, Phys. Rev. D **108**, 063513 (2023).

[137] D. Blas, O. Pujolas, and S. Sibiryakov, On the extra mode and inconsistency of Horava gravity, J. High Energy Phys. 10 (2009) 029.

[138] C. Charmousis, G. Niz, A. Padilla, and P. M. Saffin, Strong coupling in Horava gravity, J. High Energy Phys. 08 (2009) 070.

[139] D. Blas, O. Pujolas, and S. Sibiryakov, Consistent extension of Horava gravity, Phys. Rev. Lett. **104**, 181302 (2010).

[140] A. Papazoglou and T. P. Sotiriou, Strong coupling in extended Horava-Lifshitz gravity, Phys. Lett. B **685**, 197 (2010).

[141] D. Blas, O. Pujolas, and S. Sibiryakov, Comment on "Strong coupling in extended Horava-Lifshitz gravity", Phys. Lett. B **688**, 350 (2010).

[142] R. Ferraro and M. J. Guzmán, Hamiltonian formalism for f(T) gravity, Phys. Rev. D **97**, 104028 (2018).

[143] M. Li, R.-X. Miao, and Y.-G. Miao, Degrees of freedom of $f(T)$ gravity, J. High Energy Phys. 07 (2011) 108.

[144] K. Izumi and Y. C. Ong, Cosmological perturbation in f(T) gravity revisited, J. Cosmol. Astropart. Phys. 06 (2013) 029.

[145] A. Golovnev and M.-J. Guzmán, Foundational issues in f (T) gravity theory, Int. J. Geom. Methods Mod. Phys. **18**, 2140007 (2021).

[146] Y.-M. Hu, Y. Zhao, X. Ren, B. Wang, E. N. Saridakis, and Y.-F. Cai, The effective field theory approach to the strong coupling issue in f(T) gravity, J. Cosmol. Astropart. Phys. 07 (2023) 060.

[147] Y.-M. Hu, Y. Yu, Y.-F. Cai, and X. Gao, The effective field theory approach to the strong coupling issue in f(T) gravity with a non-minimally coupled scalar field, J. Cosmol. Astropart. Phys. 03 (2024) 025.

[148] S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. Levi Said, J. Mifsud, and E. Di Valentino, Teleparallel gravity: From theory to cosmology, Rep. Prog. Phys. **86**, 026901 (2023).

[149] J. Beltrán Jiménez, L. Heisenberg, and T. Koivisto, Coincident general relativity, Phys. Rev. D **98**, 044048 (2018).

[150] C. Fu, J. Liu, X.-Y. Yang, W.-W. Yu, and Y. Zhang, Explaining pulsar timing array observations with primordial gravitational waves in parity-violating gravity, Phys. Rev. D **109**, 063526 (2024).

[151] A. Golovnev and T. Koivisto, Cosmological perturbations in modified teleparallel gravity models, J. Cosmol. Astropart. Phys. 11 (2018) 012.

[152] C. Li, Y. Cai, Y.-F. Cai, and E. N. Saridakis, The effective field theory approach of teleparallel gravity, $f(T)$ gravity and beyond, J. Cosmol. Astropart. Phys. 10 (2018) 001.

[153] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170817: Observation of gravitational waves from a binary neutron star inspiral, Phys. Rev. Lett. **119**, 161101 (2017).

[154] B. P. Abbott *et al.* (LIGO Scientific, Virgo, Fermi-GBM, and INTEGRAL Collaborations), Gravitational waves and gamma-rays from a binary neutron star merger: GW170817 and GRB 170817A, Astrophys. J. Lett. **848**, L13 (2017).

[155] R.-g. Cai, S. Pi, and M. Sasaki, Gravitational waves induced by non-Gaussian scalar perturbations, Phys. Rev. Lett. **122**, 201101 (2019).

[156] C. Unal, Imprints of primordial non-Gaussianity on gravitational wave spectrum, Phys. Rev. D **99**, 041301 (2019).
[157] P. Adshead, K. D. Lozanov, and Z. J. Weiner, Non-Gaussianity and the induced gravitational wave background, J. Cosmol. Astropart. Phys. 10 (2021) 080.
[158] S. Garcia-Saenz, L. Pinol, S. Renaux-Petel, and D. Werth, No-go theorem for scalar-trispectrum-induced gravitational waves, J. Cosmol. Astropart. Phys. 03 (2023) 057.
[159] G. Sato-Polito, E. D. Kovetz, and M. Kamionkowski, Constraints on the primordial curvature power spectrum from primordial black holes, Phys. Rev. D **100**, 063521 (2019).