Soft-photon spectra and the Low-Burnett-Kroll theorem

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The study of next-to-leading-power (NLP) corrections in soft emissions continues to attract interest both in quantum chromodynamics (QCD) and in quantum electrodynamics (QED). Soft-photon spectra in particular provide a clean case-study for the experimental verification of the Low-Burnett-Kroll (LBK) theorem. In this paper we study the consistency of the LBK theorem in the context of an ambiguity arising from momentum-conservation constraints in the computation of nonradiative amplitudes. We clarify that this ambiguity leads to various possible formulations of the LBK theorem, which are all equivalent up to power-suppressed effects (i.e., beyond the formal accuracy of the LBK theorem). We also propose a new formulation of the LBK theorem with a modified shifted kinematics which facilitates the numerical computation of nonradiative amplitudes with publicly available tools. Furthermore, we present numerical results for soft-photon spectra in the associated production of a muon pair with a photon, both in e^+e^- annihilation and proton-proton collisions.

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I. INTRODUCTION

Perturbative calculations are the cornerstone of theoretical predictions for high energy physics experiments. The expansion in the coupling constant is arguably the most important example, with the expansion terms denoted as leading order (LO), next-to-leading order (NLO), and so forth. For processes involving several scales, a larger number of dimensionless parameters can be small in particular kinematic limits, hence other expansions are possible. A case that has received substantial attention is the study of power corrections to the strict soft and/or collinear limit, whose expansion terms are conventionally denoted as leading power (LP), next-to-leading power (NLP), etc.

The theoretical foundations of NLP emissions date back to the theorems of Low, Burnett and Kroll (LBK) [1,2] (see also [3,4]), which continue to be reformulated and generalized also in the recent years[1] [12–37]. In particle phenomenology, these subleading effects have mainly attracted attention due to their potential relevance for QCD resummation.[2] Indeed, it is well known that infrared divergences due to unresolved soft and collinear radiation yield logarithms in the cross section that become large when approaching some kinematic threshold, thus spoiling the predictive power of finite-order perturbation theory. The goal of the traditional (i.e., LP) resummation program is to reorganize the towers of these logarithms at a given logarithmic accuracy to all-orders in perturbation theory. In this context, subleading corrections due to emissions of gluons (and quarks) give rise to NLP logarithms which, although power-suppressed in the threshold limit, could give significant contribution to the cross section. In the last decade, a considerable effort has been invested in this direction [42–50].

The soft limit in the photon bremsstrahlung [51–55] provides another probe of NLP effects. In this case, the study of the photon spectrum gives direct access to the individual terms of NLP soft theorems, unlike the QCD resummation case, where one is blind to the energy of the undetected gluon since its momentum is integrated over the whole phase space. In fact, although the LBK theorem is very old and the conditions that ensure the soft limit for a given process are known in terms of a well-defined

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[1]Soft theorems are an active field of research also at LP, see, e.g., [5–11].

[2]NLP effects are also relevant for the numerical stability of differential NNLO calculations, both in QCD [38,39] and in QED [40,41].

hierarchy of scales, to the best of our knowledge, no study in the literature has studied numerically what is the resolution in energy and momentum of a soft photon that one has to reach in order for NLP effects to be measurable.

This question is not a mere theoretical exercise. Soft-photon spectra in hadronic decays have been puzzling physicists for years. The discrepancy between the LP prediction and the observed yields of photons produced together with hadrons is outstanding and the results of the measurements remain not understood at present [56–64]. Moreover, there are plans for an upgrade of the ALICE detector at the Large Hadron Collider that would enable the possibility of scrutinizing photons at ultrasoft energies [65,66]. In order to shed light on these discrepancies and correctly interpret data from future measurements, it is therefore of the utmost importance to have reliable theoretical predictions, including also NLP corrections as first proposed in this context in [28]. With this long-term goal in mind, in this paper we study the tree-level form of the LBK theorem for the production of a photon in association with a $\mu^+\mu^-$ pair in e^+e^- and pp collisions.

To analyze the soft-photon spectrum, one has to evaluate the expressions given by the LBK theorem. In fact, several issues must be addressed, both analytically and numerically. Most notably, as it has been already pointed out in [28], the traditional form of the theorem expressed through derivatives of the nonradiative amplitude is not optimal. Indeed, the nonradiative amplitude depends on a set of unphysical momenta that violates momentum conservation when the soft-photon momentum $k \neq 0$. This is problematic since, by definition, photon spectra are calculated for a nonvanishing momentum k. To overcome this issue, the strategy proposed first for two massless legs in [27][3] and then generalized in [28] for an arbitrary number of (massless or massive) legs, appears promising. The strategy relies on removing the derivatives of the nonradiative amplitude by computing such amplitude on momenta which are slightly shifted in value. Remarkably, the sum of these momenta shifts is equal to the soft-photon momentum, so that momentum conservation is restored. The price to pay for this trick is that the shifted momenta do not fulfill the on-shell conditions. This issue prevents the calculation of the nonradiative amplitude with most of the available public tools which can be used for the numerical evaluation of matrix elements. It is one of the goals of this paper to explicitly show how the momenta of the external particle can be kept on-shell by proposing a modified version of the shifts that are equivalent to the ones discussed in [28] up to NNLP corrections.

The observation of the dependence of the amplitude on nonphysical momenta at NLP is an old one, and it was first discussed by Burnett and Kroll [2]. More recent and detailed discussions on this aspect can be found in [24] (see also [31]). Despite the long history and the large body of papers which studied, reformulated and generalized the LBK theorem, the issue of having nonphysical momenta in the nonradiative amplitude led some authors [35–37] to question the validity of all known formulations of the theorem and to propose a modified version. In this paper, we argue that such criticism has no valid foundation by explicitly showing that the formulation in [35] is equivalent at NLP to the ones previously derived in the literature. More generally, we prove the invariance of the LBK theorem at NLP under a specific transformation of the nonradiative amplitude, which leads to many equivalent formulations that differ by NNLP corrections. As a consequence, the ambiguities contained in the LBK theorem due to violation of momentum conservation in the nonradiative amplitude are power-suppressed beyond the formal validity of the theorem.

Besides the issue of evaluating the nonradiative processes using physical on-shell momenta, other technical aspects must be addressed in a numerical implementation. In fact, the integration over phase space becomes unstable in the soft limit. To overcome these instabilities, the numerical results of this work have been generated with a program specifically targeted to treat these extreme phase-space configurations. In addition, to obtain NLP predictions for an arbitrary process that can be compared with experimental data, one wishes to calculate the nonradiative amplitude using general-purpose event generators. Thus, in this work, we demonstrate that with our modified shifted momenta it is possible to obtain predictions for the radiative amplitude in the soft-photon limit, using nonradiative amplitudes that are automatically generated.

The structure of this paper is as follows. In Sec. II we discuss the LBK theorem in all formulations that will be relevant for the numerical implementation: the one with derivatives, the one with unmodified shifts and the one with modified shifts. In doing so, we thoroughly analyze the ambiguities in the computation of the nonradiative process when the theorem is expressed through derivatives of the nonradiative amplitude. Section III contains numerical results for the $e^+e^- \to \mu^+\mu^-\gamma$ and $pp \to \mu^+\mu^-\gamma$ processes. Specifically, after comparing numerical results based on the aforementioned three versions of the LBK theorem, we study the predictive power of the soft approximation at LP and NLP in various kinematic ranges. We conclude in Sec. IV with a brief discussion.

II. LBK THEOREM AND SHIFTED KINEMATICS

We begin this section with a compact review of known results on the LBK theorem. More specifically, in Sec. II A we recall the traditional form of the theorem in terms of derivatives of the nonradiative amplitude, while in Section II C we recall the equivalent form of the theorem with shifted kinematics, recently introduced in [27,28].

[3]See also [67] and the recent [68,69].

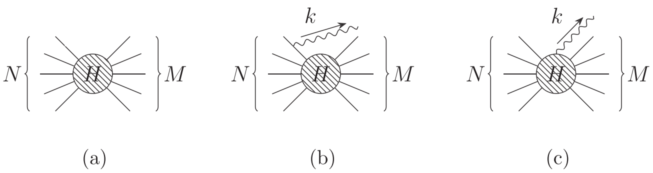

FIG. 1. Diagram (a) corresponds to the nonradiative amplitude $\mathcal{H}$, where the hard blob H represent an unknown hard interaction. Diagrams (b) and (c) represent respectively the external and internal contribution to the radiative amplitude $\mathcal{A}$.

The reason for reviewing these known forms of the theorem (apart from the sake of comprehensibility and the need to fix the notation) is twofold. On the one hand, we discuss how an intrinsic ambiguity in the computation of nonradiative processes does not invalidate the traditional formulation of the theorem, which has been recently questioned [35–37]. On the other hand, we want to stress the virtue of shifting the kinematics, which removes such ambiguity by restoring momentum conservation. We then present a new formulation of the theorem in Sec. II D, where the shifts are modified in order to keep the external lines on the mass shell.

A. Traditional LBK formulation

We consider a generic scattering amplitude $\mathcal{H}(p_1, \ldots, p_n)$ where N particles of hard momenta $p_1, \ldots, p_N$ scatter into M particles of hard momenta $p_{N+1}, \ldots, p_{N+M}$, with $n = N + M$. The particles interact via an unspecified hard dynamics which can be represented diagrammatically by the dashed blob $H(p_1, \ldots, p_n)$, as in Fig. 1. For spinning particles H is equal to the full scattering amplitude $\mathcal{H}$ stripped off of the external-state wave functions, while for scalars one trivially has $\mathcal{H} = H$.

In the radiative process $N \to M + \gamma$, the bremsstrahlung amplitude $\mathcal{A}(p_1, \ldots, p_n, k)$ includes a photon of momentum k in the final state. For reasons that will become clearer in the next sections, it is convenient to introduce a parameter $\eta = 1$ for initial particles $(1 \leq i \leq N)$ and $\eta = -1$ for final particles $(N < i \leq n)$, so that momentum conservation reads

$$\sum_{i=1}^{N} p_i - \sum_{i=N+1}^{N+M} p_i = \sum_{i=1}^{n} \eta_i p_i = k. \tag{2.1}$$

In this way, momenta are incoming for particles in the initial states and outgoing for particles in the final states. We also denote with Q_i the charge of the ith particle. In the following, we assume that the momenta p_i appearing both in the nonradiative (i.e., elastic) amplitude $\mathcal{H}(p_1, \ldots, p_n)$ and in the radiative (i.e., inelastic) $\mathcal{A}(p_1, \ldots, p_n, k)$ fulfill momentum conservation in the radiative configuration, as in Eq. (2.1). We will discuss this aspect in detail in Sec. II B.

The radiative process can be represented diagrammatically by two classes of diagrams, as depicted in Fig. 1. In the first one, the emitted photon is attached to one of the external lines. In the second one, the photon couples directly to some internal line of the unspecified hard subdiagram H. We denote the two corresponding radiative amplitudes [stripped off of the photon polarization vector $\epsilon_\mu(k)$] as $\mathcal{A}^\mu_{\text{ext}}$ and $\mathcal{A}^\mu_{\text{int}}$, respectively. We begin with the former.

Without loss of generality, we restrict the analysis to the case of an external emission from an initial-state fermion-antifermion pair of charge $Q_1 = -Q_2 \equiv Q$ and momenta p_1 and p_2, respectively, satisfying $p_1 + p_2 = k$. The sum of the diagrams corresponding to the two emissions reads

$$\begin{aligned}\mathcal{A}^\mu_{\text{ext}}(p_1, p_2, k) &= \bar{v}(p_2) H(p_1 - k, p_2) \frac{i(\not{p}_1 - \not{k} + m)}{(p_1 - k)^2 - m^2} (-iQ\gamma^\mu) u(p_1) \\ &\quad + \bar{v}(p_2)(-iQ\gamma^\mu) \frac{i(-\not{p}_2 + \not{k} + m)}{(p_2 - k)^2 - m^2} H(p_1, p_2 - k) u(p_1).\end{aligned} \tag{2.2}$$

In the limit where the photon momentum k is small compared to the hard momenta p_1 and p_2, we can expand[4] in k both the fermion propagators and the hard subdiagram H. After using the Dirac equation, enforcing the on-shell condition $k^2 = 0$ and neglecting terms proportional to k^μ that vanish by gauge invariance, we get

$$\begin{aligned}\mathcal{A}^\mu_{\text{ext}}(p_1, p_2, k) &= Q\bar{v}(p_2) H(p_1, p_2) \left(-\frac{p_1^\mu}{p_1 \cdot k} + \frac{p_2^\mu}{p_2 \cdot k} + \frac{ik_\nu S^{\mu\nu}}{p_1 \cdot k} \right) u(p_1) \\ &\quad + Q\bar{v}(p_2) \frac{ik_\nu S^{\mu\nu}}{p_2 \cdot k} H(p_1, p_2) u(p_1) + Q \frac{p_1^\mu}{p_1 \cdot k} k^\nu \bar{v}(p_2) \frac{\partial H(p_1, p_2)}{\partial p_1^\nu} u(p_1) \\ &\quad - Q \frac{p_2^\mu}{p_2 \cdot k} k^\nu \bar{v}(p_2) \frac{\partial H(p_1, p_2)}{\partial p_2^\nu} u(p_1) + \mathcal{O}(k),\end{aligned} \tag{2.3}$$

[4]The expansion in the four-momentum k is equivalent to the expansion in the photon energy ω_γ since all components of k scale homogeneously in the soft limit.

where we defined $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ and we exploited the functional dependence of H by setting

$$\frac{\partial H(p_1 - k, p_2)}{\partial k^\nu}\bigg|_{k=0} = -\frac{\partial H(p_1, p_2)}{\partial p_1^\nu}. \tag{2.4}$$

At this point we should note that in order to derive Eq. (2.3) we have expanded Eq. (2.2) in k while keeping all the other momenta fixed, in analogy with Low's derivation in [1]. Mathematically, one can regard the right-hand side of Eq. (2.2) as a function defined on the entire space spanned by the vectors $\{p_1, p_2, k\}$, where the vectors p_1 and p_2 are not restricted to the surface $p_1 + p_2 = k$. Although the result of such expansion is then defined for arbitrary momenta p_1 and p_2, eventually we are of course only interested in the physical value of the expanded function on the momentum-conservation surface. An alternative approach, followed, e.g., by Burnett and Kroll [2] and more recently by[5] [35–37], consists of expanding Eq. (2.2) on the momentum-conservation surface by inserting a dependence over k in the momenta $p_1(k)$ and $p_2(k)$. The parametrization of the momenta $p_1(k)$ and $p_2(k)$, which are not fixed, is then only constrained by $p_1(k) + p_2(k) = k$. The two approaches yield equivalent expressions on the momentum-conservation constraint up to power-corrections in the expansion.

To proceed further, one has to compute the internal emission contribution $\mathcal{A}_{\rm int}$. However, since in general one cannot know how the photon couples to the internal hard subdiagrams, one is seemingly prevented from an explicit calculation of $\mathcal{A}_{\rm int}$. However, gauge invariance comes to the rescue, since

$$k_\mu(\mathcal{A}^\mu_{\rm ext} + \mathcal{A}^\mu_{\rm int}) = 0. \tag{2.5}$$

From this, we deduce that

$$\mathcal{A}^\mu_{\rm int} = -\sum_{i=1}^{2} Q_i \bar{v}(p_2) \frac{\partial H(p_1, p_2)}{\partial p_{i\mu}} u(p_1) + K^\mu, \tag{2.6}$$

where K^μ is a gauge invariant term ($k \cdot K = 0$). A power counting analysis reveals that at the tree level K^μ is power-suppressed at NLP and can be set to zero.[6]

Therefore, combining Eqs. (2.3) and (2.6) we get the final form for the LBK theorem for the radiative amplitude $\mathcal{A}(p_1, p_2, k)$, which reads

$$\begin{aligned}\mathcal{A}(p_1, p_2, k) = &-\epsilon_\mu(k) \sum_{i=1}^{2} Q_i \frac{p_i^\mu}{p_i \cdot k} \mathcal{H}(p_1, p_2) \\ &- \epsilon_\mu(k) \sum_{i=1}^{2} Q_i \bar{v}(p_2) G_i^{\mu\nu} \frac{\partial H(p_1, p_2)}{\partial p_i^\nu} u(p_1) \\ &+ \epsilon_\mu(k) Q \bar{v}(p_2)\bigg[H(p_1, p_2)\frac{i k_\nu S^{\mu\nu}}{p_1 \cdot k} \\ &+ \frac{i k_\nu S^{\mu\nu}}{p_2 \cdot k} H(p_1, p_2)\bigg] u(p_1),\end{aligned} \tag{2.7}$$

where we have introduced the following tensor

$$G_i^{\mu\nu} = g^{\mu\nu} - \frac{(2p_i - k)^\mu k^\nu}{2 p_i \cdot k} = g^{\mu\nu} - \frac{p_i^\mu k^\nu}{p_i \cdot k} + \mathcal{O}(k). \tag{2.8}$$

The first term in Eq. (2.7) represents the well-known LP factorization in terms of the eikonal factor $p \cdot \epsilon/(p \cdot k)$. The remaining terms correspond to NLP corrections.

A generalization of the calculation above to an arbitrary number of initial or final state particles is straightforward, although the final result is not quite compact since one has to distinguish the four cases where the (anti)fermion is in the initial or final state. A short-hand notation that is quite common in the literature on scattering amplitudes [15–21] consists of factoring out the spin generator and the derivatives from the nonradiative amplitude, yielding

$$\mathcal{A}(p_1, \ldots, p_n, k) = (\mathcal{S}_{\rm LP} + \mathcal{S}_{\rm NLP}) \mathcal{H}(p_1, \ldots, p_n), \tag{2.9}$$

where

$$\mathcal{S}_{\rm LP} = -\sum_{i=1}^{n} \eta_i Q_i \frac{p_i \cdot \epsilon(k)}{p_i \cdot k}, \quad \mathcal{S}_{\rm NLP} = -\sum_{i=1}^{n} \eta_i Q_i \frac{i k_\nu J_i^{\mu\nu} \epsilon_\mu(k)}{p_i \cdot k}. \tag{2.10}$$

Here, $J_i^{\mu\nu} = S_i^{\mu\nu} + L_i^{\mu\nu}$ is the total angular momentum, while $L_i^{\mu\nu} = i(p_i^\mu \frac{\partial}{\partial p_{i\nu}} - p_i^\nu \frac{\partial}{\partial p_{i\mu}})$ is the orbital angular momentum which is related to the tensor $G^{\mu\nu}$ via $G_i^{\mu\nu} \frac{\partial}{\partial p_i^\nu} = i \frac{k_\nu}{p_i \cdot k} L_i^{\mu\nu}$. However, one should not be fooled by the simplicity of Eq. (2.9), since $J^{\mu\nu}$ is not a simple multiplicative factor but rather an operator that contains derivatives and gamma matrices. The derivatives act on the hard coefficient H only (not the full amplitude $\mathcal{H}$), while the spin generator must be inserted in the correct order within the spinors, as shown in Eq. (2.7) or the simple case of an initial state fermion-antifermion pair.

Things become much simpler for the squared unpolarized amplitude $\overline{|\mathcal{A}|^2}$, since all NLP corrections can be recast in terms of derivatives of the squared nonradiative amplitude, as first shown in [2]. This can be seen again by

[5]We thank O. Nachtmann for clarifying that in [35–37] this is how the expansion is performed.

[6]See [24] for a more detailed analysis.

considering the simple case of two charged incoming particles as in Eq. (2.7), where NLP corrections correspond to a derivative contribution (second term) and a spin contribution (third and fourth term). When squaring and averaging over the polarizations, the nonradiative amplitude reads simply

$$\overline{|\mathcal{H}(p_1,p_2)|}^2 = \mathrm{Tr}[(\not{p}_2 - m)H(p_1,p_2)(\not{p}_1 + m)\bar{H}(p_1,p_2))], \tag{2.11}$$

where we defined $\bar{H} = \gamma^0 H^\dagger \gamma^0$. For the radiative amplitude instead one has the following schematic structure at NLP

$$|\mathcal{A}(p_1,p_2,k)|^2 = |\mathcal{S}_{\mathrm{LP}}|^2 |\mathcal{H}(p_1,p_2)|^2 + 2\mathrm{Re}(\mathcal{S}_{\mathrm{LP}}\mathcal{H}(p_1,p_2)\mathcal{S}^\dagger_{\mathrm{NLP}}\mathcal{H}^\dagger(p_1,p_2)), \tag{2.12}$$

where $\mathcal{S}_{\mathrm{LP}}$ and $\mathcal{S}_{\mathrm{NLP}}$ have been defined in Eq. (2.9). The second term in Eq. (2.12) corresponds to the interference between the LP factor and either the derivative or the spin contribution as in Eq. (2.7). For an emission from the leg with momentum p_1, the spin term becomes

$$\mathrm{Tr}\left[(\not{p}_2 - m)H(p_1,p_2)\left(\frac{\not{k}\gamma_\mu}{2p_1\cdot k}(\not{p}_1 + m) + (\not{p}_1 + m)\frac{\gamma_\mu \not{k}}{2p_1\cdot k}\right)\bar{H}(p_1,p_2))\right]\sum_i \eta_i Q_i \frac{p_i^\mu}{p_i\cdot k}. \tag{2.13}$$

Up to terms proportional to k^μ one then has

$$\frac{\not{k}\gamma^\mu}{2p_1\cdot k}(\not{p}_1 + m) + (\not{p}_1 + m)\frac{\gamma^\mu\not{k}}{2p_1\cdot k} = -\gamma^\mu + \frac{p_1^\mu}{p_1\cdot k}\not{k} = -G_1^{\mu\nu}\frac{\partial}{\partial p_1^\nu}(\not{p}_1 + m). \tag{2.14}$$

Recalling that derivatives in Eq. (2.7) act on the hard function only, we conclude that both the spin and the orbital contribution combine into derivatives of the full squared nonradiative amplitude $\overline{|\mathcal{H}(p_1,p_2)|}^2$. Hence we obtain

$$\overline{|\mathcal{A}(p_1,p_2,k)|}^2 = Q^2\sum_{ij=1}^{2}\frac{p_i\cdot p_j}{(p_i\cdot k)(p_j\cdot k)}\overline{|\mathcal{H}(p_1,p_2)|}^2 + Q^2\sum_{ij=1}^{2}\frac{p_{i\mu}}{p_i\cdot k}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,p_2)|}^2. \tag{2.15}$$

Although so far we have considered fermions, the result above holds also in the case of spin 0 and spin 1 charged particles. In the former case, the spin generator vanishes, hence it is obvious that NLP terms include only the derivative contribution. For spin 1 one can exploit the gauge invariance of the amplitude to set $\sum_\lambda \epsilon_\mu^{(\lambda)}(k)\epsilon_\nu^{(\lambda)}(k) = -g_{\mu\nu}$, which does not depend on any momenta and therefore $\frac{\partial}{\partial p}|\mathcal{H}| = \frac{\partial}{\partial p}|H|$, leaving again only a derivative contribution.

Finally, we note that Eq. (2.15) can be trivially generalized to an arbitrary number n of external (charged or neutral) particles. One simply has to repeat the derivation above for each particle-antiparicle pair, paying special care to whether the particles are in the initial or final states. Thus, the general form for the LBK theorem in the traditional formulation reads

$$\overline{|\mathcal{A}(p_1,\dots,p_n,k)|}^2 = -\sum_{ij=1}^{n}\eta_i\eta_j Q_i Q_j\frac{p_i\cdot p_j}{(p_i\cdot k)(p_j\cdot k)}\overline{|\mathcal{H}(p_1,\dots,p_n)|}^2 - \sum_{ij=1}^{n}\eta_i Q_i Q_j\frac{p_{i\mu}}{p_i\cdot k}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\dots,p_n)|}^2, \tag{2.16}$$

where we used $\eta_j^2 = 1$. In Secs. II C and II D we will discuss two alternative forms of the theorem that do not involve derivatives. Before doing so, in the next section we analyze an important property of Eq. (2.16).

B. Nonradiative amplitude and unphysical momenta

In the traditional form of the theorem of Eq. (2.16), the nonradiative amplitude $\mathcal{H}(p_1,\dots,p_n)$ is affected by an ambiguity related to the fact that it must be evaluated outside the physical region. Indeed, in order for $\mathcal{H}(p_1,\dots,p_n)$ to represent a physical process with no photon radiation, the momenta p_i should fulfill $\sum_i \eta_i p_i = 0$. However, the momenta p_i that have been introduced in Eq. (2.1) fulfill momentum conservation in the radiative amplitude $\mathcal{A}(p_1,\dots,p_n,k)$, i.e., $\sum_i \eta_i p_i = k$. Therefore, in Eq. (2.16) we are in fact evaluating $\mathcal{H}$ using radiative momenta, which for $k \neq 0$ are unphysical for the nonradiative process and thus induce an unphysical ambiguity in the final result. It is the aim of this section to demonstrate that the use of unphysical momenta in the nonradiative amplitude does not invalidate the consistency of Eq. (2.16) at LP and NLP.

We start with the observation that every amplitude, and in particular the nonradiative amplitude of the LBK theorem $\mathcal{H}(p_1,\ldots,p_n)$, is intrinsically ambiguous if momentum conservation is not imposed. In fact, one can always find a function Δ such that the transformation

$$\mathcal{H}(p_1,\ldots,p_n)\to\mathcal{H}(p_1,\ldots,p_n)+\Delta(p_1,\ldots,p_n) \tag{2.17}$$

leads to the exact same physics, as long as Δ fulfills

$$\Delta(p_1,\ldots,p_n)\delta\left(\sum_i\eta_i p_i\right)=0. \tag{2.18}$$

We would like to exploit this property to show that Eq. (2.16) does not depend on Δ, up to NNLP corrections. To do so, we have to assign a scaling in k to Δ in Eq. (2.17). In this regard, we note that momentum conservation for the radiative amplitude must be fulfilled in order for Eq. (2.16) to give physical results. Therefore, what matters for the invariance of Eq. (2.16) under Eq. (2.17) is the value of Δ on the momentum-conservation surface $\sum_i\eta_i p_i=k$. By imposing this constraint, the momenta p_i can be effectively interpreted as functions $p_i(k)$, with an arbitrary functional dependence over k, constrained only by total momentum conservation. This induces an implicit dependence of $\Delta(p_1,\ldots,p_n)$ over k through the momenta $p_i(k)$, such that Δ can be expanded in k. Therefore, we can now check whether the right-hand side (rhs) of Eq. (2.16) is invariant at NLP under the transformation of Eq. (2.17) on the momentum-conserving surface $\sum_i\eta_i p_i=k$. Let us consider the LP and NLP cases separately. To simplify the discussion, we will first consider the form of the LBK theorem at the amplitude level as in Eq. (2.9) and Eq. (2.10) in the scalar case only. We will then discuss how the generalization for the squared amplitude (which is valid also in the spinning case) follows analogously.

To check whether the LBK theorem in the form of Eqs. (2.9) and (2.10) is invariant under Eq. (2.17) at LP, one has to verify that

$$\left(\sum_i\eta_i Q_i\frac{p_i^\mu(k)}{p_i(k)\cdot k}\right)\Delta(p_1(k),\ldots,p_n(k))=\mathcal{O}(1), \tag{2.19}$$

or alternatively

$$\Delta(p_1(k),\ldots,p_n(k))=O(k). \tag{2.20}$$

The key point here is to notice that the limit $k\to 0$ implies the expression $\sum_i\eta_i p_i=0$. Then, from Eq. (2.18) one concludes that $\Delta\to 0$ for $k\to 0$. Since in this paper we are restricting the scope of our analysis to a tree-level calculation, where the absence of nonanalytic terms allows a Laurent expansion in k, we conclude that Δ is at worst of $\mathcal{O}(k)$ and hence Eq. (2.20) is fulfilled, thus validating the theorem at LP.

At NLP we have to modify the consistency condition of Eq. (2.19) as follows

$$\sum_i\eta_i Q_i\left[\frac{p_i^\mu(k)}{k\cdot p_i(k)}\Delta(p_1(k),\ldots,p_n(k)) + \eta_i G_i^{\mu\nu}\frac{\partial\Delta(p_1(k),\ldots,p_n(k))}{\partial p_i^\nu}\right]=\mathcal{O}(k), \tag{2.21}$$

where $\mathcal{O}(k)$ in the rhs represent NNLP corrections. In order to verify this condition, once again we introduce a k dependence in $\Delta(p_1,\ldots,p_n)$ via $p_i(k)$. Given that we have to deal with derivatives, it is convenient to make the dependence on k explicit by defining a new function $\tilde{\Delta}_\mu(p_1,\ldots,p_n,k)$ which is constrained on the momentum-conservation surface by

$$\begin{aligned}&k^\mu\tilde{\Delta}_\mu(p_1,\ldots,p_n,k)\delta\left(\sum_i\eta_i p_i-k\right)\\&\quad=\Delta(p_1(k),\ldots,p_n(k))\delta\left(\sum_i\eta_i p_i-k\right).\end{aligned} \tag{2.22}$$

By enforcing the delta constraints of Eq. (2.22), one can effectively substitute $k=k(p)=\sum_i\eta_i p_i$ in $k^\mu\tilde{\Delta}_\mu$. Therefore, the following relation between the derivatives of Δ and $\tilde{\Delta}_\mu$ can be found

$$\begin{aligned}\frac{\partial\Delta(p_1,\ldots,p_n)}{\partial p_j^\mu}&=\frac{d}{dp_j^\mu}(k^\nu(p)\tilde{\Delta}_\nu(p_1,\ldots,p_n,k(p)))\\&=\eta_j\tilde{\Delta}_\mu(p_1,\ldots,p_n,k(p))+k^\nu(p)\\&\quad\times\frac{\partial\tilde{\Delta}_\nu(p_1,\ldots,p_n,k(p))}{\partial p_j^\mu}\\&\quad+\eta_j k^\nu(p)\frac{\partial\tilde{\Delta}_\nu(p_1,\ldots,p_n,k(p))}{\partial k^\mu}\\&=\eta_j\tilde{\Delta}_\mu(p_1,\ldots,p_n,k(p))+\mathcal{O}(k),\end{aligned} \tag{2.23}$$

where in the last equality we dropped terms of order $\mathcal{O}(k)$, using the fact that $\tilde{\Delta}_\mu=\mathcal{O}(1)$. At this point Eq. (2.21) follows straightforwardly. Indeed, the left-hand side of Eq. (2.21) becomes

$$\sum_i\eta_i Q_i\left[\frac{p_i^\mu}{k\cdot p_i}k^\nu\tilde{\Delta}_\nu(p_1,\ldots,p_n,k) + \eta_i\left(g^{\mu\nu}-\frac{p_i^\mu k^\nu}{p_i\cdot k}\right)\eta_i\tilde{\Delta}_\nu(p_1,\ldots,p_n,k)\right]+\mathcal{O}(k). \tag{2.24}$$

For a given i, thanks to Eqs. (2.19) and (2.22), all terms in Eq. (2.24) are NLP. However, because $\eta_i^2 = 1$, there is a cancellation between the terms in Eq. (2.24), yielding

$$\left(\sum_i \eta_i Q_i\right)\tilde{\Delta}^\mu(p_1, \ldots, p_n, k) + \mathcal{O}(k). \tag{2.25}$$

Finally, given that $\sum_i \eta_i Q_i = 0$ by charge conservation, only a residual $\mathcal{O}(k)$ (i.e., NNLP) term remains, as required by Eq. (2.21). We then conclude that the NLP theorem at the amplitude level as in Eqs. (2.9) and (2.10) is invariant under Eq. (2.17) on the surface $\sum_i \eta_i p_i = k$ and thus it is consistent also when the corresponding nonradiative amplitude is evaluated with unphysical momenta.

A crucial step in the derivation above is the cancellation of NLP ambiguities between the LP term and the derivative term. To make the general arguments discussed above more concrete and see this cancellation explicitly, in Appendix A we consider the soft bremsstrahlung in a simple case of a $2 \to 2$ nonradiative process involving only scalar particles. This discussion is also meant to clarify the relation with the work of [35–37] where the validity of the traditional form of the LBK theorem has been questioned.

The generalization of the previous arguments to the squared-matrix elements of Eq. (2.16) straightforwardly carries over, by simply adjusting the correct power counting in k. More specifically, one has to check that Eq. (2.16) remains invariant under[7] $\overline{|\mathcal{H}|}^2 \to \overline{|\mathcal{H}|}^2 + \Delta$. At LP this is equivalent to showing that

$$\left(\sum_{ij} \eta_i \eta_j Q_i Q_j \frac{p_i(k) \cdot p_j(k)}{p_i(k) \cdot k p_j(k) \cdot k}\right)\Delta(p_1(k), \ldots, p_n(k)) = \mathcal{O}(k^{-1}), \tag{2.26}$$

while at NLP the consistency condition reads

$$\sum_{ij} \eta_i \eta_j Q_i Q_j \left(\frac{p_i(k) \cdot p_j(k)}{p_i(k) \cdot k p_j(k) \cdot k}\Delta(p_1(k), \ldots, p_n(k)) + \frac{p_{j\mu}(k)}{p_j(k) \cdot k}\eta_i G_i^{\mu\nu}\frac{\partial \Delta(p_1(k), \ldots, p_n(k))}{\partial p_i^\nu}\right) = \mathcal{O}(1). \tag{2.27}$$

Both conditions can be verified with the same arguments as outlined above, thus showing that the rhs of Eq. (2.16) does not depend on Δ at LP and NLP. Therefore, even though the nonradiative amplitude is evaluated with unphysical momenta, the formulation of the theorem as in Eq. (2.16) is consistent at NLP.

Finally, we note that the arbitrariness in the evaluation of the nonradiative function with nonphysical momenta was already observed by Burnett and Kroll in their original work [2]. In fact, Burnett and Kroll proposed a prescription to evaluate the nonradiative ampitude by shifting the unphysical momenta by an arbitrary quantity that restore momentum conservation in the elastic amplitude. The argument we have presented here, instead, is more general. By exploiting the invariance at NLP of the nonradiative amplitude under Eq. (2.17) we have proven that Eq. (2.16) is consistent without the need to restore momentum conservation. In fact, one could restrict the transformations of Eq. (2.17) to the special case of linear shifts on the external momenta. Then, the proposal of Burnett and Kroll would correspond to the specific case of shifts that fulfill momentum conservation in the elastic configuration. In order to shed light on the relation between the general argument of this section and the strategy of Burnett and Kroll, in Appendix B we discuss the invariance of Eq. (2.16) in the special case where Eq. (2.17) can be represented by linear transformations of the momenta.

[7]Note that although the definition of Δ here is different from the one in Eq. (2.17), it obeys Eq. (2.18).

C. From derivatives to shifts

In the previous section we have verified that the traditional form of the LBK theorem with derivatives of the nonradiative process is consistent at NLP, since nonphysical ambiguities arising in the computation of the nonradiative process are NNLP. Still, the dependence of Eq. (2.16) on an unphysical nonradiative amplitude seems unsatisfactory. In particular, if one intends to automatically generate the amplitude of the nonradiative process using publicly available tools, having a form of the theorem that is defined from scratch for physical amplitudes with momenta that fulfill momentum conservation is desirable. The nonradiative process is then computed for physical momenta and is thus unambiguous. Hence, it is natural to ask whether it is possible to find a simpler formulation of the theorem which is particularly suitable for numerical implementations.

The answer is yes, as proposed in [28], building on previous work in QCD [27]. It stems from the fact that since derivatives are the generators of translations, one can convert the term with derivatives in the LBK theorem into momentum shifts in the nonradiative amplitude. In fact, one can write Eq. (2.16) as

$$\overline{|\mathcal{A}(p_1, \ldots, p_n, k)|}^2 = \overline{|\mathcal{S}_{\mathrm{LP}}|}^2\left[1 + \sum_j \delta p_j^\nu \frac{\partial}{\partial p_j^\nu}\right]\overline{|\mathcal{H}(p_1, \ldots, p_n)|}^2, \tag{2.28}$$

where the shifts δp_i are to be determined, while from Eq. (2.10)

$$\overline{|\mathcal{S}_{\mathrm{LP}}|}^2 = -\sum_{ij=1}^{n} \eta_i \eta_j Q_i Q_j \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}. \tag{2.29}$$

By comparison with Eq. (2.16), we deduce

$$\delta p_j^\nu = Q_j \left(\sum\nolimits_{k,l} \eta_k \eta_l Q_k Q_l \frac{p_k \cdot p_l}{(p_k \cdot k)(p_l \cdot k)} \right)^{-1} \times \sum_i \left(\frac{\eta_i Q_i p_{i\mu}}{k \cdot p_i} \right) G_j^{\mu\nu}. \tag{2.30}$$

Thus, one obtains

$$\overline{|\mathcal{A}(p_1, \ldots, p_n, k)|^2} = \overline{|\mathcal{S}_{\rm LP}|^2}\,\overline{|\mathcal{H}(p_1 + \delta p_1, \ldots, p_n + \delta p_n)|^2}, \tag{2.31}$$

i.e., a form of the LBK theorem without derivatives and with a single LP soft factor.

We note immediately that $\delta p_j^\nu = \mathcal{O}(k)$, hence the shifts vanish at LP, as expected. Another crucial property that can be readily verified is that

$$\sum_j \eta_j \delta p_j^\mu = -k^\mu. \tag{2.32}$$

Therefore, recalling that $\sum_j \eta_j p_j^\mu = k^\mu$, we deduce that momentum conservation is restored in the nonradiative amplitude of Eq. (2.31), which can be then computed without the ambiguities discussed in the previous section.

Note also that by getting rid of the derivatives, we obtained a form of the theorem with just a single positive-defined term. Naturally, as long as the soft expansion is meaningful, we expect the derivative term in Eq. (2.16) to be small w.r.t. the LP term. Thus, for soft-photon momenta, also Eq. (2.16) remains positive, as expected for a cross section. Still, for a theorem whose scope is to extend the range of validity of the soft approximation to larger soft momenta, the formulation in Eq. (2.31) seems more elegant, since it ensures that the cross section remains positive. We will come back to this point in Sec. III.

Finally, one can easily verify that the momenta shifts are orthogonal to each momentum, i.e.,

$$\delta p_j \cdot p_j = 0. \tag{2.33}$$

This implies that

$$(p_j + \delta p_j)^2 = m_j^2 + \mathcal{O}(k^2), \tag{2.34}$$

thus fulfilling the on-shell condition up NLP. We notice however that the condition is violated already at NNLP. More precisely, one can verify that

$$(\delta p_j)^2 = Q_j^2 \left(\sum\nolimits_{k,l} \eta_k \eta_l Q_k Q_l \frac{p_k \cdot p_l}{(p_k \cdot k)(p_l \cdot k)} \right)^{-1} \neq 0, \tag{2.35}$$

hence masses do get shifted by a nonzero NNLP amount for nonvanishing k. This feature might be problematic when using automatically generated amplitudes, since most of the public tools typically require momenta to be exactly on-shell. In the next section we discuss how to overcome this problem.

D. Modified shifted kinematics

We seek another expression for δp_i that ensures that masses are not shifted, without spoiling the NLP terms of the LBK theorem. Hence we require the new definition for δp_i to fulfill the following conditions:

(i) it conserves momentum to all orders in k, i.e.,

$$\sum_i \eta_i \delta p_i + k = 0, \tag{2.36}$$

(ii) it fulfills the on-shell condition to all orders in k, i.e.,

$$(p_i + \delta p_i)^2 = m_i^2, \tag{2.37}$$

(iii) it reduces to Eq. (2.30) up to NNLP corrections, i.e.,

$$\delta p_j^\nu = -Q_j (\overline{|\mathcal{S}_{\rm LP}|^2})^{-1} \sum_i \left(\frac{\eta_i Q_i}{k \cdot p_i} \right) \left(p_i^\nu - \frac{p_i \cdot p_j}{p_j \cdot k} k^\nu \right) + \mathcal{O}(k^2). \tag{2.38}$$

We can find such definition by considering the following ansatz,

$$\delta p_i^\mu = \sum_j A_{ij} p_j^\mu + B_i k^\mu, \tag{2.39}$$

and, by imposing the constraints (i)–(iii), subsequently determine the unknown coefficients A_{ij} and B_i. It turns out that these conditions are not too restrictive and one is free to select a single solution. Details of this calculation can be found in Appendix C. The final result reads

$$\delta p_i^\mu = A Q_i \sum_j \frac{\eta_j Q_j}{k \cdot p_j} p_{j\nu} G_i^{\nu\mu} + \frac{1}{2} \frac{A^2 Q_i^2 \overline{|\mathcal{S}_{\rm LP}|^2}}{p_i \cdot k} k^\mu, \tag{2.40}$$

with

$$A = \frac{1}{\chi} \left(\sqrt{1 - \frac{2\chi}{\overline{|\mathcal{S}_{\rm LP}|^2}}} - 1 \right), \qquad \chi = \sum_i \frac{\eta_i Q_i^2}{p_i \cdot k}, \tag{2.41}$$

and $\overline{|\mathcal{S}_{\rm LP}|^2}$ defined in Eq. (2.29). It is straightforward to check that the conditions of Eqs. (2.36), (2.37) and (2.38) are satisfied by this solution. Indeed, Eq. (2.40) reduces to Eq. (2.30) up to NNLP corrections and therefore it still correctly reproduces the LBK theorem at NLP. Moreover, both momentum conservation and the on-shell condition hold to all-orders in the soft expansion.

The price to pay is that we need to introduce spurious NNLP terms in the hard momenta, which unavoidably affect the numerical evaluation of the nonradiative amplitude $\mathcal{H}$, hence the prediction for the photon spectra. However, one should bear in mind that the sensitivity of $\mathcal{H}$ to NNLP effects is not a feature that belongs only to the modified kinematics. We encountered it also in the other two versions of the LBK theorem. Specifically, in the traditional form with derivatives, $\mathcal{H}$ is not uniquely defined. Thus, even though the NLP ambiguities cancel, as we showed in Sec. II B, NNLP spurious terms do survive. In the formulation with unmodified shifted kinematics, although there are no unphysical ambiguities due to violation of momentum conservation, when going from Eqs. (2.28) to (2.31) we are implicitly adding spurious NNLP terms. Hence Eq. (2.31) is also valid only up to NLP.

More generally, we note that a residual arbitrariness in the final result due to missing higher-order terms is a feature common to all perturbative expansions. In fact, choosing a specific definition for the modified shifts in Eq. (2.31) corresponds to the choice of a "scheme," which is specified by the inclusion of power-suppressed (i.e., beyond NLP) terms. In this regard, we note that the modified shifts make this scheme-dependence transparent, since the choice is process-independent. Instead, in the traditional formulation of the LBK theorem of Eq. (2.16), $\mathcal{H}$ is not univocally determined and thus the (hidden) scheme-dependence corresponds to the choice of a specific functional form for the amplitude. This choice is obviously process-dependent.

The question that remains is what is the role of these NNLP effects in a numerical computation of photon spectra, i.e., what is the version of the LBK theorem that gives the best approximation of the exact radiative process. Among other things, we investigate these aspects in the next section.

III. NUMERICAL PREDICTIONS FOR $e^+e^- \to \mu^+\mu^-\gamma$ AND $pp \to \mu^+\mu^-\gamma$

In the following, we present a numerical study of the spectra of soft photons produced in association with a muon pair in e^+e^- and pp collisions. The cross sections for the $ij \to \mu^+\mu^-(+\gamma)$ processes ($ij = e^+e^-, q\bar{q}$) are calculated at the tree level, including both Z and γ exchange. We consider the e^+e^- collisions at the center-of-mass (c.m.) energy of 91 GeV, i.e., the LEP1 collision energy, at which the measurements of photon spectra were carried out by the DELPHI collaboration [57]. The pp collisions are considered at 14 TeV c.m. energy. To ensure that we are not sensitive to any infrared effects other than that related to the soft photon, we impose kinematical cuts on the transverse momentum of the muons, $p_{T,\mu} > 10$ GeV, pseudorapidity of all final state particles, $|\eta_i| < 2.5$, as well as the photon-muon separation, $\Delta R > 0.4$. The photon distributions are computed with an in-house code using the VEGAS+ algorithm [70] for performing the phase-space integration. In the case of pp collisions, we make use of LHAPDF6 [71] and choose to evaluate the cross sections with NNPDF4.0 [72] LO set of parton distribution functions. The tree-level amplitudes for the nonradiative and (exact) radiative amplitudes are either generated by MadGraph5@NLO [73] or calculated analytically.[8] All exact (i.e., obtained without imposing the soft-photon approximation) results have been cross-checked against numerical results generated using the SHERPA event generator.[9]

The exact predictions and the predictions to which we refer to as NLP are obtained by integrating the exact matrix elements or their particular NLP approximation over the full 3-particle phase space. At this point, we note that one could also consider the expansion of the phase-space factor in powers of the soft momentum and truncate it at LP or NLP depending on whether the matrix elements are evaluated at NLP or LP, respectively. However, as discussed in the last section, the soft approximation of the matrix elements based on the LBK theorem in all its forms receives NNLP contributions. Therefore, integrating over the full phase space leads to the same level of accuracy. On the other hand, our LP predictions are obtained by imposing momentum conservation on all external particles other than the photon, which effectively corresponds to truncating the expansion of the phase-space factor at LP and calculating the LP term of the nonradiative amplitude on such external momenta.

We begin with a comparison of numerical predictions for the $e^+e^- \to \mu^+\mu^-\gamma$ process obtained using the NLP approximations of the amplitude derived in the previous chapter, i.e., the traditional form with derivatives of Eq. (2.16), the form of Eq. (2.31) with the off-shell momenta shifts defined in Eq. (2.30), and the form of Eq. (2.31) with on-shell shifts defined in Eq. (2.40). To this aim, we use the analytic result for the nonradiative amplitude to calculate the derivatives in Eq. (2.16) analytically, as well as to compute Eq. (2.31) involving the off-shell momenta shifts. We also compare the NLP predictions to the full result where no soft approximation has been applied. The corresponding differential distributions in photon energy ω_γ are shown in Fig. 2, in the range of 1–500 MeV (left plot) and 0.1–10 GeV (right plot). As expected, we observe that all three approaches converge to the exact result in the limit of small ω_γ, and depart from it with growing photon energy. However, the approximation of the exact result provided by the formulation of the LBK

[8] The analytical expression for the nonradiative amplitude used in this section is specified by $\mathcal{H}(p_1, p_2, p_3, p_4) = \mathcal{H}(s(p_1, p_2), t(p_1, p_3)))$.

[9] Numerical checks with MadGraph5 were also performed. The results of MadGraph5 for the $e^+e^- \to \mu^+\mu^-\gamma$ process appear to depend on the chosen integration strategy and the way of grouping the Feynman diagrams for calculations. We thank the MadGraph team for clarifying that point.

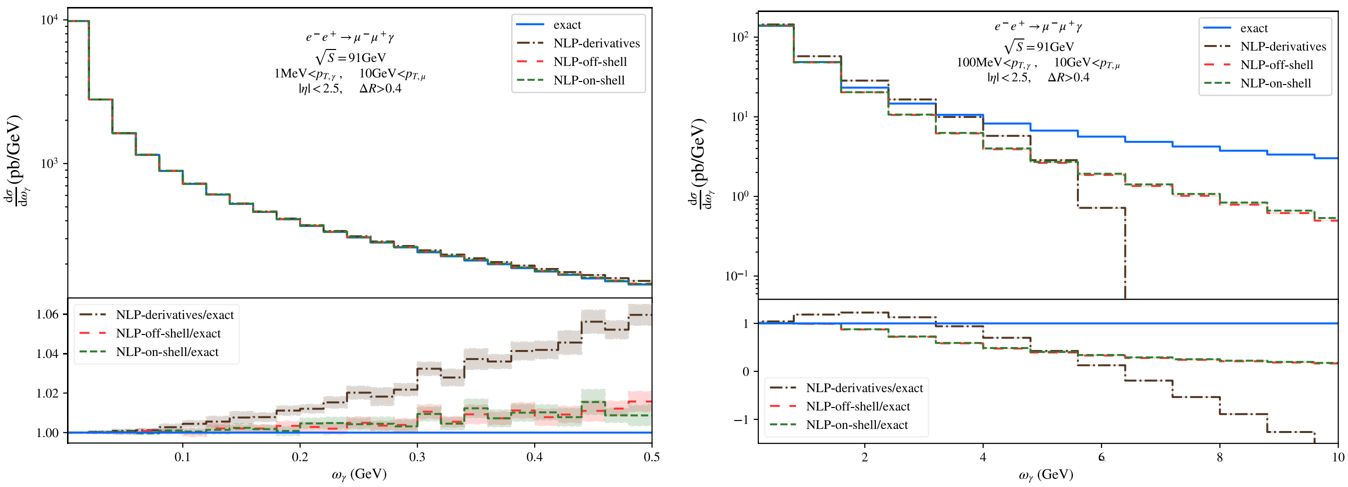

FIG. 2. Comparison of the soft-photon energy spectra calculated using three formulations of the LBK theorem discussed in this paper with the exact (i.e., no soft expansion) result for the process $e^-e^+ \to \mu^-\mu^+\gamma$ at $\sqrt{s} = 91$ GeV. The error bands in this figure (as well as in Figs. 3–6) show the statistical uncertainties of the Monte Carlo integration.

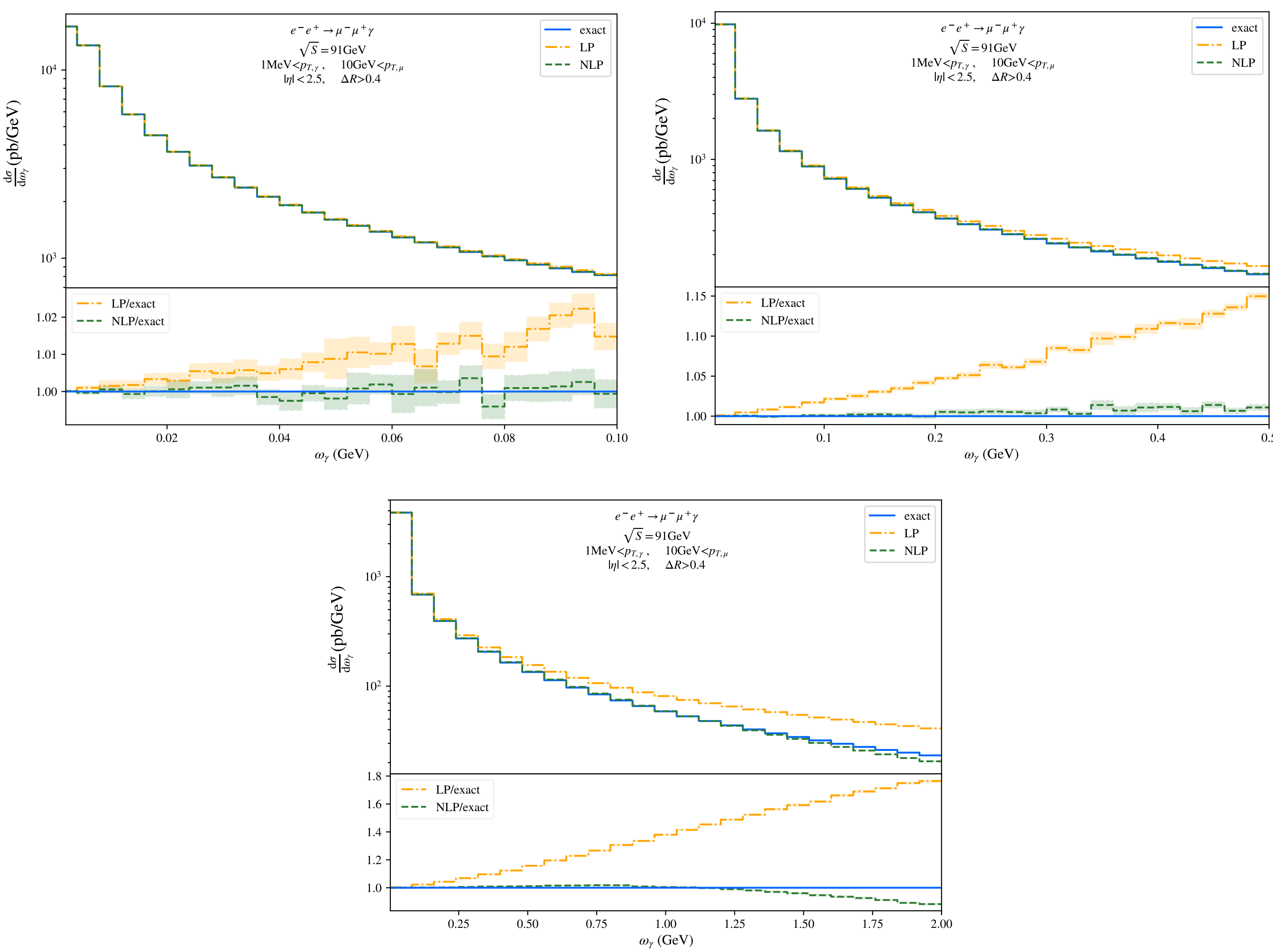

FIG. 3. The LP and NLP approximations and the exact result for the energy spectrum of the photon in the $e^-e^+ \to \mu^-\mu^+\gamma$ process at $\sqrt{s} = 91$ GeV. Error bands as in Fig. 2.

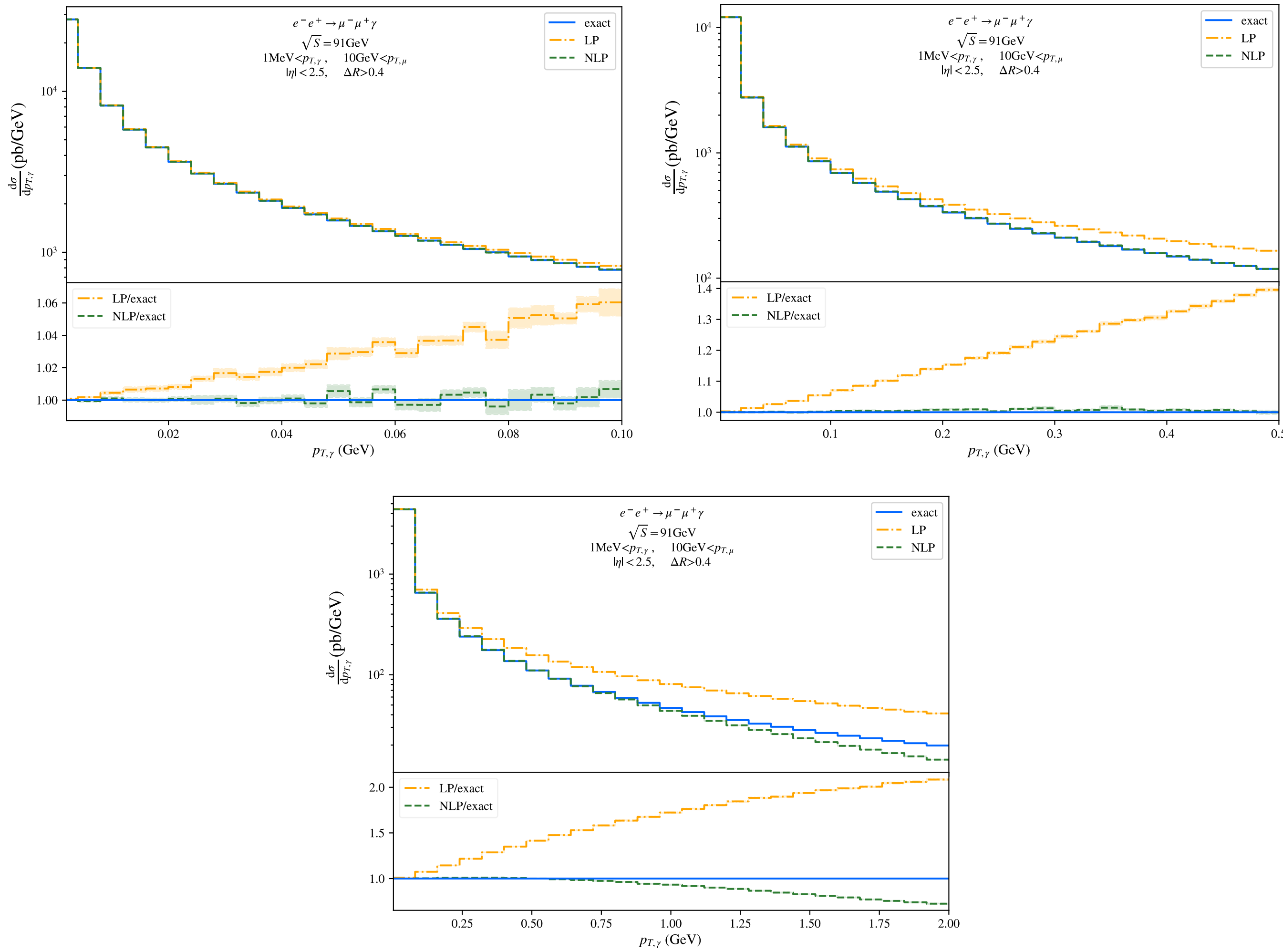

FIG. 4. The LP and NLP approximations and the exact result for the transverse momentum spectrum of the photon in the $e^-e^+ \to \mu^-\mu^+\gamma$ process at $\sqrt{s} = 91$ GeV. Error bands as in Fig. 2.

theorem involving derivatives, Eq. (2.16), is distinctively worse than those based on formulations involving shifting of momenta. While the latter agree with the exact result within 1%–2% for $\omega_\gamma \lesssim 500$ MeV, where one naively expects the soft approximation to work, the former differs from the exact predictions within the same range by up to 6%. Besides, its behavior is qualitatively different: at $\omega_\gamma \gtrsim 6$ GeV, the derivative approach gives a nonphysical negative result, in contrast to the other predictions which stay positive. This can be understood from Eq. (2.16) which is a sum of two not-positive-defined terms and as such can get negative when the expansion breaks down. As the difference between various NLP approximations is due to the NNLP terms, these results clearly show the relevance of the subleading terms beyond the formal accuracy of the LBK theorem.

We also see that the NLP approximations of the photon spectrum calculated using the nonradiative amplitude with momenta shifted on-shell or off-shell perform equally well for the photon energies considered here, indicating that the NNLP effects introduced in the on-shell shifts are not significant. Since the formulation with momenta shifted on-shell enables sourcing the amplitude subroutines from a wide range of public tools, we employ this formulation in further studies.

The NLP distributions in photon energy ω_γ and transverse momentum $p_{T,\gamma}$, obtained using the radiative amplitude with on-shell momenta Eq. (2.40), are then compared to the LP and exact predictions in Figs. 3 and 4, respectively. In particular, we show distributions for very soft photons with 1 MeV $< \omega_\gamma, p_{T,\gamma} <$ 100 MeV. As discussed above, the NLP formula relying on shifting momenta on-shell returns predictions which provide a very good approximation of the exact result. Up to the scale of 100 MeV, the difference between the two predictions is at a few per mille level and grows to a 1%–2% level for ω_γ or $p_{T,\gamma}$ of up to ca. 1 GeV. In contrast, the LP approximation differs from the exact result by up to ca. 2% (6%) and up to ca. 40% (70%) in these two ranges of ω_γ ($p_{T,\gamma}$), correspondingly.

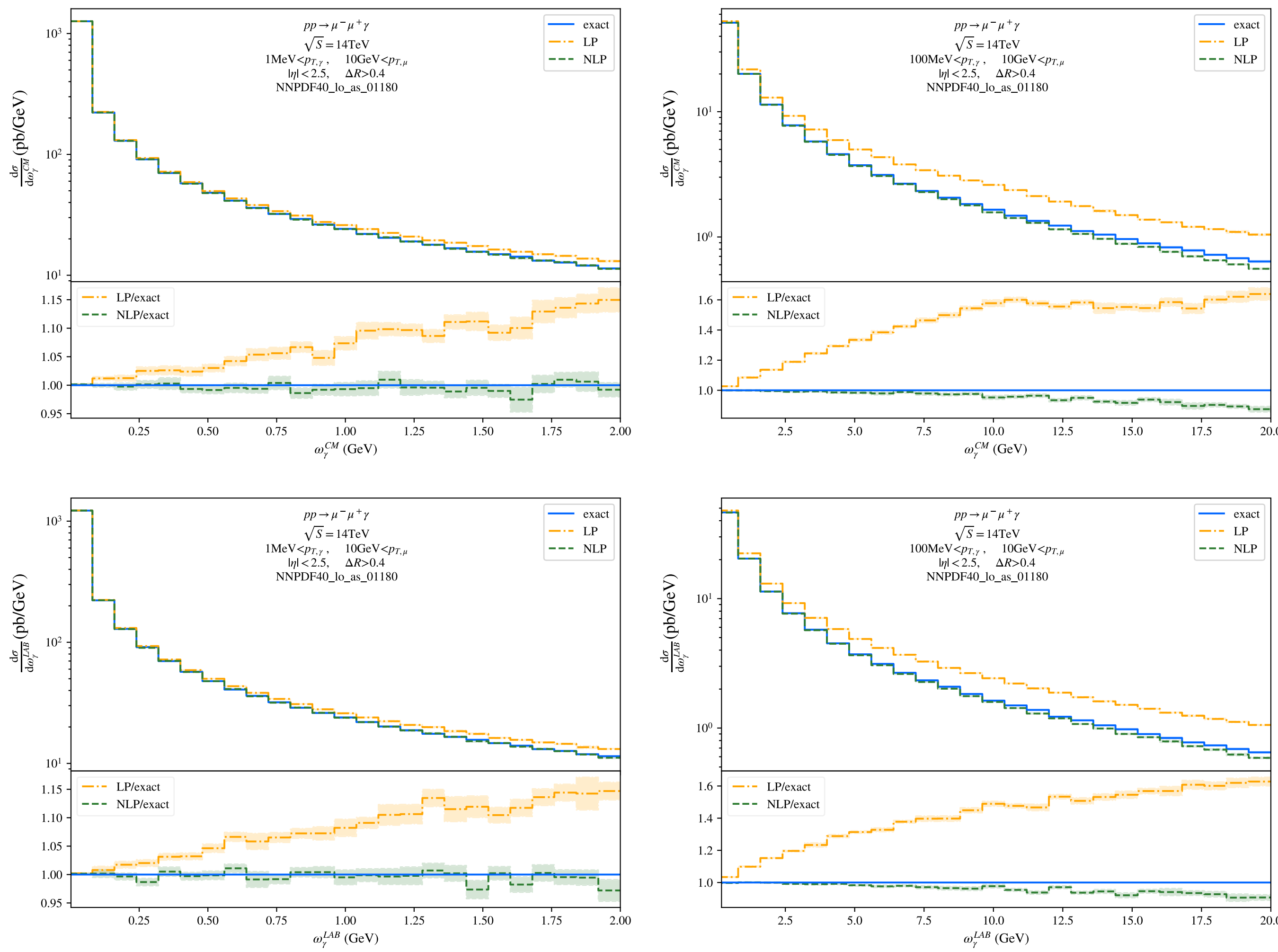

FIG. 5. The LP and NLP approximations and the exact result for the energy spectrum of the photon in the partonic c.m. frame (top row) and the laboratory frame (bottom row) in the $pp \to \mu^-\mu^+\gamma$ process at $\sqrt{s} = 14$ TeV. Error bands as in Fig. 2.

Next, we study the soft-photon spectra in the process $pp \to \mu^+\mu^-\gamma$. The differential distributions in photon energy and transverse momentum are shown in Figs. 5 and 6, respectively. The photon energy spectrum in Fig. 5 is presented in both the partonic c.m. frame and the laboratory frame. No perceptible difference is observed between the results in the two frames for our choice of kinematical cuts. Perhaps not surprisingly, the behavior of the LP and NLP approximations is very similar to the one found for the e^+e^- collisions. Quantitatively, however, in the ranges of ω_γ and $p_{T,\gamma}$ studied here, the LP and NLP predictions appear to be relatively closer to the exact results than in the e^+e^- case. To be more precise, within an accuracy of roughly 10%, the LP spectrum deviates from the exact result for $\omega_\gamma \gtrsim 400$ MeV, $p_{T,\gamma} \gtrsim 150$ MeV (e^+e^-) and $\omega_\gamma \gtrsim 1$ GeV, $p_{T,\gamma} \gtrsim 500$ MeV (pp). The NLP predictions reach this level of deviation only at $p_{T,\gamma} \gtrsim 1$ GeV in the e^+e^- case, and at $p_{T,\gamma} \gtrsim 10$ GeV in the pp case, i.e., outside of the soft regime.

As a final remark, we note that the photon spectra of this section have been generated with tree-level calculations. Therefore, one could wonder to what extent these theoretical predictions can be compared with data and what is the role of radiative corrections in QED and QCD, respectively. In this regard, we note that the observables we considered are the transverse momentum $p_{T,\gamma}$ and the energy ω_γ of the emitted photon, and not of the muon pair. In the latter case, the (assumed undetected) soft photon emission would lead to the appearance of logarithmic QED corrections, generated by integrating the same LP and NLP factors studied in this paper over the full phase space of the photon. In this work, however, we only study the spectra of bremsstrahlung photons, whose momenta by definition are not fully integrated. The evaluation of the impact of multiple (undetected) photon emissions on the p_T of the muon pair would further require resummation of the corresponding QED logarithms [74–82]. In the case of proton-proton collisions, a full description of the p_T spectra of the muon pair at small

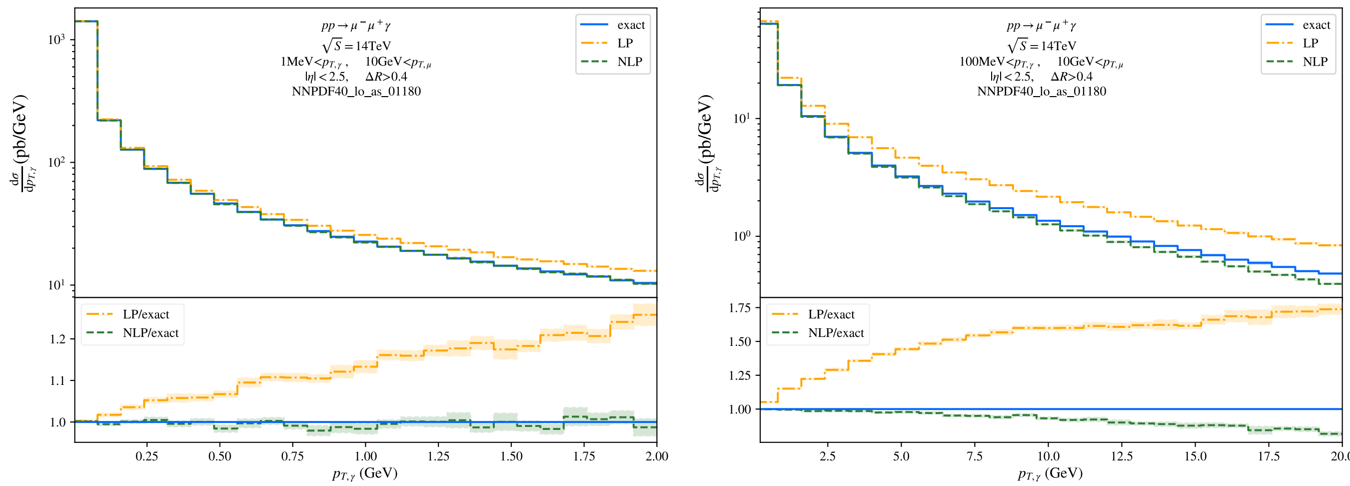

FIG. 6. The LP and NLP approximations and the exact result for the transverse momentum spectrum of the photon in the $pp \to \mu^-\mu^+\gamma$ process at $\sqrt{s} = 14$ TeV. Error bands as in Fig. 2.

p_T would additionally need to take into account QCD effects, most notably resummation of soft gluon emission corrections (see, e.g., [83,84] and references therein). Similarly, one could study the impact of multiple soft-photon emissions (and soft-gluon emissions in the case of pp collisions) on the spectra of the photon produced in association with the $\mu^+\mu^-$ pair. Such studies are, however, beyond the scope of the current work.

Finally, it is worth stressing that, on top of the higher-order corrections to the nonradiative amplitude, also the soft factors at LP and NLP multiplying the nonradiative cross section can be modified by real and virtual effects. In this regard we note that the absence of collinear singularities in massive QED makes the LP factor of Eq. (2.10) tree-level exact, while in QCD (and in QED with parametrically small masses) the LP factor receives corrections at higher loops [11]. The NLP factor in Eq. (2.10) on the other hand is modified already at one-loop, both in the massive case due to a nonvanishing soft-region [31] and in the massless case due to collinear effects [28]. In these cases the NLP factorization is modified with logarithmic corrections, that can be large and thus would invalidate the predictive power of finite-order results. The impact of these effects is left for future studies.

IV. CONCLUSIONS

In this paper, we have presented an extensive study of the LBK theorem and its implications on numerical predictions for the soft-photon spectra.

First, motivated by recent discussions in the literature [35–37], we have addressed the consistency of the LBK theorem. As known for a long time, in the original formulation of the theorem [1] the nonradiative amplitude is calculated on a set of unphysical momenta. Seen in the most general way, this violation of momentum conservation leads to an ambiguity in the functional form of the nonradiative amplitude. We have provided a proof that the aforementioned ambiguity can only affect the expansion of the radiative amplitude starting from NNLP in soft-photon energy, i.e., beyond the formal accuracy of the LBK theorem, thus proving the validity of the LBK theorem at NLP. In doing so, we have generalized the remark by Burnett and Kroll by observing an invariance of the theorem at NLP under a specific transformation of the nonradiative amplitude. The consequence of this invariance is the presence of many equivalent forms of the LBK theorem, which include the original formulation by Low as one possibility.

Among the different versions of the theorem, for practical reasons, it is particularly attractive to consider those that restore momentum conservation in the calculation of the nonradiative amplitude. Such restoration can be achieved by reformulating the LBK theorem in terms of the nonradiative amplitude calculated on momenta which values are modified with respect to the momenta of the radiative amplitude. The modification that has been put forward in the literature [27,28,67] relies on adding small shifts of the order of the soft-photon momentum k. Apart from reviewing the derivation of the reformulated LBK theorem in terms of shifted momenta, we have proposed expressions for the momenta shifts which not only restore momentum conservation, but also ensure that each of the shifted momenta is on-shell. In this way, we facilitate the generation and numerical calculation of the nonradiative amplitude with a wide range of publicly available tools.

We have also studied the quality of the soft approximation provided by the LBK theorem in its various formulations on the example of the $e^+e^- \to \mu^+\mu^-\gamma$ process, analyzed at LEP1 energies. We have found a remarkable improvement in the quality of the approximation for the formulations involving shifts with respect to the formulation of the theorem with derivatives, although we have considered the latter only for a specific form of the nonradiative amplitude. Depending on the required quality of the approximation, NNLP effects can be thus numerically relevant and the form of the LBK theorem used to calculate the NLP approximation of the soft-photon spectra needs to be chosen carefully. In this regard, our studies indicate that the formulations involving shifts should be preferred. Notably, for the implementation involving on-shell shifts, we have used the amplitudes generated by the MADGRAPH5 code, in this way demonstrating the feasibility of the calculations for a wide range of processes, given a corresponding phase-space integrating code is available.

Despite the long history of the LBK theorem, to the best of our knowledge, no study in the literature has explicitly identified when power-suppressed effects become visible in the soft-photon spectra. In order to address this question, we compared the exact, LP and NLP results for the $e^+e^- \to \mu^+\mu^-\gamma$ process at the LEP1 energy of 91 GeV and the $pp \to \mu^+\mu^-\gamma$ process at the LHC energy of 14 TeV. We have studied various ranges of photon energy and transverse momentum. For the cases studied here, the quality of the NLP approximation in a reasonably soft regime is of the order of, or better than, one percent, even though the specific results depend on the process, observable and the analysis set-up. The quality of the LP approximation is significantly worse, meaning that for measurements where a precision of a few percent can be reached in the soft regime, the LP might not provide a good enough approximation. Correspondingly, our results suggest that in order to access the power-suppressed terms, a percent-level precision is needed, especially in the case of the $pp \to \mu^+\mu^-\gamma$ process at the LHC. Obviously, this is only a crude estimate. A more precise statement would require a careful analysis of all theoretical and experimental uncertainties.

The results of this work open up the possibility of many follow-up analyses. The two processes we studied here involve simple leptonic final states and central rapidity photons. Given that the excess in the soft-photon spectrum was observed for e^+e^- collisions in hadronic final states, it would be interesting to investigate the impact of NLP corrections on the predictions for the associated photon production with jets. In this regard, it would also be interesting to extend the analysis by including QCD corrections at 1-loop [28], which will be relevant for both initial and final state hadrons. In the long run, the studies need to be extended to pp collisions resulting in various hadronic final states with photons at forward rapidities, as planned to be investigated with the ALICE 3 detector [66].

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APPENDIX A: CASE STUDY: 2 → 2 SCATTERING

In this section we apply the general arguments of Sec. II B concerning the traditional form of the LBK theorem with derivatives to the simple case of photon bremsstrahlung in pion scattering. In doing so, we compare with the analysis of Lebiedowicz, Nachtmann, Szczurek (LNS) [35], showing that their result is equivalent to the traditional form of the LBK theorem and with previous studies in the literature.

1. LBK theorem for $\pi^-\pi^0 \to \pi^-\pi^0\gamma$

Following [1,35], and using the notation established in [35], we consider the following process:

$$\pi^-(p_a)\pi^0(p_b) \to \pi^-(p_1')\pi^0(p_2')\gamma(k),$$

with

$$p_a + p_b = p_1' + p_2' + k.$$

For this process, the LBK theorem in the form of Eq. (2.7), adapted to the case of scalar particles, reads

$$\begin{aligned}
\mathcal{A}^\mu(p_a,p_b,p_1',p_2',k) &= e\left[\frac{p_a^\mu}{k\cdot p_a}\mathcal{H}(p_a,p_b,p_1',p_2') + G_a^{\mu\nu}\frac{\partial\mathcal{H}(p_a,p_b,p_1',p_2')}{\partial p_a^\nu}\right] \\
&\quad - e\left[\frac{p_1'^\mu}{k\cdot p_1'}\mathcal{H}(p_a,p_b,p_1',p_2') - G_1^{\mu\nu}\frac{\partial\mathcal{H}(p_a,p_b,p_1',p_2')}{\partial p_1'^\nu}\right] + \mathcal{O}(k) \\
&= e\left[\frac{p_a^\mu}{k\cdot p_a} - \frac{p_1'^\mu}{k\cdot p_1'} + G_a^{\mu\nu}\frac{\partial}{\partial p_a^\nu} + G_1^{\mu\nu}\frac{\partial}{\partial p_1'^\nu}\right]\mathcal{H}(p_a,p_b,p_1',p_2') + \mathcal{O}(k).
\end{aligned} \tag{A1}$$

To show the ambiguity in the calculation of the nonradiative amplitude with radiative kinematics, we consider the following two choices for $\mathcal{H}$:

$$\mathcal{H}_1 = \mathcal{H}(s_L', t_1), \qquad \mathcal{H}_2 = \mathcal{H}(s_L', t_2), \tag{A2}$$

where we defined

$$s_L' = p_a\cdot p_b + p_1'\cdot p_2', \qquad t_1 = (p_a - p_1')^2, \qquad t_2 = (p_b - p_2')^2. \tag{A3}$$

Here, $\mathcal{H}(s_L, t)$ is the amplitude for the nonradiative process $\pi^-(p_a)\pi^0(p_b) \to \pi^-(p_1)\pi^0(p_2)$. Obviously, $p_a + p_b = p_1 + p_2$, so in the elastic limit $t_1 = (p_a - p_1)^2$ and $t_2 = (p_b - p_2)^2$, hence we have $t_1 = t_2$, and thus $\mathcal{H}_1 = \mathcal{H}_2$.

Now we consider

$$\begin{aligned}
\frac{\partial\mathcal{H}_i}{\partial p_a^\nu} &= \frac{\partial\mathcal{H}_i}{\partial s_L'}\frac{\partial s_L'}{\partial p_a^\nu} + \frac{\partial\mathcal{H}_i}{\partial t_i}\frac{\partial t_i}{\partial p_a^\nu}, \\
\frac{\partial\mathcal{H}_i}{\partial p_1'^\nu} &= \frac{\partial\mathcal{H}_i}{\partial s_L'}\frac{\partial s_L'}{\partial p_1'^\nu} + \frac{\partial\mathcal{H}_i}{\partial t_i}\frac{\partial t_i}{\partial p_1'^\nu},
\end{aligned} \tag{A4}$$

where no sum over the index $i = 1, 2$ has been assumed. The derivatives of s_L' in Eq. (A4) read

$$\frac{\partial s_L'}{\partial p_a^\nu} = p_{b\nu}, \qquad \frac{\partial s_L'}{\partial p_1'^\nu} = p_{2\nu}'.$$

The terms with derivatives of t_i are different for $i = 1, 2$, and are given by

$$\frac{\partial t_1}{\partial p_a^\nu} = 2(p_a - p_1')_\nu, \qquad \frac{\partial t_1}{\partial p_1'^\nu} = -2(p_a - p_1')_\nu, \qquad \frac{\partial t_2}{\partial p_a^\nu} = \frac{\partial t_2}{\partial p_1'^\nu} = 0. \tag{A5}$$

Putting the above equations together we obtain two seemingly distinct forms for the LBK theorem, depending on whether we take $\mathcal{H} = \mathcal{H}_1$ or $\mathcal{H} = \mathcal{H}_2$ in Eq. (A1). They read

$$\begin{aligned}
\mathcal{A}^\mu(p_a,p_b,p_1',p_2',k) &= e\left[\left(\frac{p_a^\mu}{k\cdot p_a} - \frac{p_1'^\mu}{k\cdot p_1'}\right) + \left(p_b^\mu - \frac{p_b\cdot k}{p_a\cdot k}p_a^\mu + p_2'^\mu - \frac{p_2'\cdot k}{p_1'\cdot k}p_1'^\mu\right)\frac{\partial}{\partial s_L'}\right. \\
&\quad \left. - 2(p_a - p_1')\cdot k\left(\frac{p_a^\mu}{k\cdot p_a} - \frac{p_1'^\mu}{k\cdot p_1'}\right)\frac{\partial}{\partial t_1}\right]\mathcal{H}_1 + \mathcal{O}(k), \\
\mathcal{A}^\mu(p_a,p_b,p_1',p_2',k) &= e\left[\left(\frac{p_a^\mu}{k\cdot p_a} - \frac{p_1'^\mu}{k\cdot p_1'}\right) + \left(p_b^\mu - \frac{p_b\cdot k}{p_a\cdot k}p_a^\mu + p_2'^\mu - \frac{p_2'\cdot k}{p_1'\cdot k}p_1'^\mu\right)\frac{\partial}{\partial s_L'}\right]\mathcal{H}_2 + \mathcal{O}(k).
\end{aligned} \tag{A6}$$

If one ignores the fact that $\mathcal{H}_1$ and $\mathcal{H}_2$ are not the same, the two expressions indeed yield different results, thus seemingly invalidating the LBK theorem. However, by relating $\mathcal{H}_1$ and $\mathcal{H}_2$, the difference disappears at NLP. In fact, one has

$$\mathcal{H}_2 = \mathcal{H}(s_L', t_2) = \mathcal{H}(s_L', t_1) + (t_2 - t_1)\frac{\partial\mathcal{H}(s_L', t_1)}{\partial t_1} + \mathcal{O}(k^2). \tag{A7}$$

Using then the expression

$$\begin{aligned} t_2 - t_1 &= (p_b - p_2')^2 - (p_a - p_1')^2 \\ &= (p_a - p_1' - k)^2 - (p_a - p_1')^2 \\ &= -2k \cdot (p_a - p_1') + \mathcal{O}(k^2), \end{aligned} \tag{A8}$$

we get

$$\mathcal{H}_2 = \mathcal{H}_1 - 2k \cdot (p_a - p_1') \frac{\partial \mathcal{H}_1}{\partial t_1} + \mathcal{O}(k^2), \tag{A9}$$

which accounts for the difference between the two previous expressions. This is the cancellation we saw in Sec. II B, where we proved it in the general case.

2. Comparison between LNS and the original work of Low

The soft bremsstrahlung in pion scattering has been computed also by the authors of Ref. [35]. Their final result, which is given by Eq. (3.27) of their paper, reads

$$\begin{aligned} \mathcal{A}^\mu &= e\mathcal{H}(s_L, t) \left[\frac{p_a^\mu}{p_a \cdot k} - \frac{p_1'^\mu}{p_1' \cdot k} \right] \\ &+ 2e \frac{\partial \mathcal{H}(s_L, t)}{\partial s_L} \left[p_b^\mu - \frac{p_b \cdot k}{p_a \cdot k} p_a^\mu \right] \\ &- 2e \frac{\partial \mathcal{H}(s_L, t)}{\partial t} [(p_a - p_1) \cdot k - p_a \cdot l_1] \\ &\times \left[\frac{p_a^\mu}{p_a \cdot k} - \frac{p_1^\mu}{p_1 \cdot k} \right] + \mathcal{O}(k), \end{aligned} \tag{A10}$$

where the nonradiative amplitude $\mathcal{H}$ is written as a function of the two variables s_L and t

$$s_L = p_a \cdot p_b + p_1 \cdot p_2, \qquad t = (p_a - p_1)^2 = (p_b - p_2)^2. \tag{A11}$$

The momenta fulfill the relations

$$p_a + p_b = p_1 + p_2 = p_1' + p_2' + k \tag{A12}$$

and l_i are defined as a shift between p and p' as follows:

$$l_i = p_i - p_i'. \tag{A13}$$

As done in [1], here we set $k^2 = 0$ and drop all terms proportional to k^μ since we assume all final states to be on-shell.

The authors of [35] compare then Eq. (A10) to the one derived by Low in Ref. [1], which they report in Eq. (3.29) of their paper. It reads

$$\begin{aligned} \tilde{\mathcal{A}}^\mu = e\mathcal{H}(s_L, t) \left[\frac{p_a^\mu}{p_a \cdot k} - \frac{p_1^\mu}{p_1 \cdot k} \right] + e \frac{\partial \mathcal{H}(s_L, t)}{\partial s_L} \\ \times \left[p_b^\mu - \frac{p_b \cdot k}{p_a \cdot k} p_a^\mu + p_2^\mu - \frac{p_2 \cdot k}{p_1 \cdot k} p_1^\mu \right] + \mathcal{O}(k). \end{aligned} \tag{A14}$$

They conclude that, while the LP terms agree, there is a discrepancy in the NLP terms. We first point out that (A14) does not precisely coincide with the result given by Low in equation (2.16) of [1]. In fact, by looking at Eqs. (2.1) and (2.16) in [1], one concludes that the correct expression should be[10]

$$\begin{aligned} \mathcal{A}^\mu = e\mathcal{H}(s_L', t_2) \left[\frac{p_a^\mu}{p_a \cdot k} - \frac{p_1'^\mu}{p_1' \cdot k} \right] + e \frac{\partial \mathcal{H}(s_L', t_2)}{\partial s_L'} \\ \times \left[p_b^\mu - \frac{p_b \cdot k}{p_a \cdot k} p_a^\mu + p_2'^\mu - \frac{p_2' \cdot k}{p_1' \cdot k} p_1'^\mu \right] + \mathcal{O}(k), \end{aligned} \tag{A15}$$

with s_L' and t_2 defined as in Eq. (A3). This form of the LBK theorem is indeed what we obtained in Eq. (A6).

A careful comparison of expressions Eqs. (A10) and (A15) shows that they are in perfect agreement with each other, up to $\mathcal{O}(k)$. To simplify the comparison, it is worth noticing that p and p' are equal up to $\mathcal{O}(k)$ corrections. Therefore, to order $\mathcal{O}(k)$, the difference between p and p' is only relevant for the first term, and Eq. (A15) can be rewritten as

$$\begin{aligned} \mathcal{A}^\mu = e\mathcal{H}(s_L', t_2) \left[\frac{p_a^\mu}{p_a \cdot k} - \frac{p_1'^\mu}{p_1' \cdot k} \right] + e \frac{\partial \mathcal{H}(s_L, t)}{\partial s_L} \\ \times \left[p_b^\mu - \frac{p_b \cdot k}{p_a \cdot k} p_a^\mu + p_2^\mu - \frac{p_2 \cdot k}{p_1 \cdot k} p_1^\mu \right] + \mathcal{O}(k). \end{aligned} \tag{A16}$$

To compare Eqs. (A10) with (A16) it is necessary to use the formula

$$\mathcal{H}(s_L', t_2) = \mathcal{H}(s_L, t) + \delta s_L' \frac{\partial \mathcal{H}}{\partial s_L} + \delta t_2 \frac{\partial \mathcal{H}}{\partial t}, \tag{A17}$$

where we defined

[10] In the final stages of writing this paper, we have been informed that the authors are aware of this mistake.

$$\begin{aligned}
\delta s'_L &= s'_L - s_L = p'_1 \cdot p'_2 - p_1 \cdot p_2 = (p_1 - l_1)\cdot(p_2 - l_2) - p_1\cdot p_2 \\
&= -(l_1\cdot p_2 + l_2\cdot p_1) + \mathcal{O}(k^2) = -(p_1+p_2)\cdot k + \mathcal{O}(k^2) \\
&= -(p_a + p_b)\cdot k + \mathcal{O}(k^2), \\
\delta t_2 &= t_2 - t = (p_b - p'_2)^2 - (p_b - p_2)^2 = (p_b - p_2 + l_2)^2 - (p_b - p_2)^2 \\
&= 2l_2\cdot(p_b - p_2) + \mathcal{O}(k^2) = -2(k - l_1)\cdot(p_a - p_1) + \mathcal{O}(k^2) \\
&= -2[(p_a - p_1)\cdot k - p_a\cdot l_1] + \mathcal{O}(k^2).
\end{aligned} \tag{A18}$$

Here we used the relations $p_1 \cdot l_2 = p_1 \cdot k$ and $p_2 \cdot l_1 = p_2 \cdot k$, which can be derived from the two relations $l_1 + l_2 = k$ and $p_i \cdot l_i = 0$ (i.e., Eqs. (3.17) and (3.22) in [35]).

Inserting now Eq. (A17) into Eq. (A16), we see that the term proportional to $\mathcal{H}$ is identical to the one in Eq. (A10). Additionally, the term proportional to $\frac{\partial\mathcal{H}}{\partial t}$ is given by

$$e\delta t_2 \frac{\partial\mathcal{H}}{\partial t}\left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1'^\mu}{p'_1\cdot k}\right] = -2e\frac{\partial\mathcal{H}}{\partial t}[(p_a - p_1)\cdot k - p_a\cdot l_1] \times \left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1'^\mu}{p'_1\cdot k}\right]. \tag{A19}$$

The rhs of Eq. (A19) coincides with the third term in Eq. (A10) after dropping the prime in the last parenthesis (this is possible because this term is already a NLP term, and thus the difference due to replacing p with p' is a NNLP effect). Finally, the term proportional to $\frac{\partial\mathcal{H}}{\partial s_L}$ has now the following expression

$$e\delta s'_L \frac{\partial\mathcal{H}}{\partial s_L}\left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1^\mu}{p_1\cdot k}\right] + e\frac{\partial\mathcal{H}}{\partial s_L}\left[p_b^\mu - \frac{p_b\cdot k}{p_a\cdot k}p_a^\mu + p_2^\mu - \frac{p_2\cdot k}{p_1\cdot k}p_1^\mu\right], \tag{A20}$$

where again the prime in the first term can dropped because $\delta s'_L$ is already of order $\mathcal{O}(k)$. After some algebra, it is easy to show that this term reads

$$2e\frac{\partial\mathcal{H}}{\partial s_L}\left[p_b^\mu - k\cdot p_b \frac{p_a^\mu}{p_a\cdot k}\right], \tag{A21}$$

in perfect agreement with the second term in Eq. (A10). Therefore, the LNS result [Eq. (A10)] and Low's original result [Eq. (A15)] are completely equivalent at NLP.

We note also that although here we focused on pion scattering, we expect analogous arguments to hold for the proton scattering discussed in [36,37].

3. Comparison between LNS and other literature

A comparison between the result of LNS, i.e., Eq. (A10), and the previous literature has been carried out in the appendix of [35]. It is claimed there that all previous known forms of the LBK theorem are problematic. In particular, LNS claim that the result of Ref. [24] is not consistent, since it depends on an arbitrary quantity, as discussed below. However, a careful analysis shows that this claim has no valid foundation, since the dependence on such quantity vanishes.

The argument in [35] is the following. In [24] Low's theorem is written in a form that depends on the following four quantities:

$$\begin{aligned}
I_1 &= \left(-l_1\cdot\frac{\partial}{\partial p_1} - l_2\cdot\frac{\partial}{\partial p_2} - k\cdot\frac{\partial}{\partial p_a}\right)\mathcal{H}(p_a,p_b,p_1,p_2), \\
I_2 &= \left(-l_1\cdot\frac{\partial}{\partial p_1} - l_2\cdot\frac{\partial}{\partial p_2} + k\cdot\frac{\partial}{\partial p_1}\right)\mathcal{H}(p_a,p_b,p_1,p_2), \\
I_3^\mu &= \frac{\partial\mathcal{H}(p_a,p_b,p_1,p_2)}{\partial p_{a\mu}}, \qquad I_4^\mu = \frac{\partial\mathcal{H}(p_a,p_b,p_1,p_2)}{\partial p_{1\mu}}.
\end{aligned} \tag{A22}$$

If one then restricts the analysis to an amplitude $\mathcal{H}$ that depends solely on the quantity $p_a^2 + p_1^2 - p_b^2 - p_2^2$, as done in [35], the elastic amplitude is given by a constant, since[11]

$$\begin{aligned}
\mathcal{H}(p_a,p_b,p_1,p_2) &= f(p_a^2 + p_1^2 - p_b^2 - p_2^2) \\
&= f(m_a^2 + m_1^2 - m_b^2 - m_2^2) \equiv f_0.
\end{aligned} \tag{A23}$$

Thus, the derivatives of the elastic amplitude vanish and the final result only depends on the value of f_0. However, if one considers the corresponding expressions in Eq. (A22), they become

$$\begin{aligned}
I_1 &= -2(k\cdot p_a)f'_0, \qquad I_2 = 2(k\cdot p_1)f'_0, \\
I_3^\mu &= 2p_a^\mu f'_0, \qquad I_4^\mu = 2p_1^\mu f'_0.
\end{aligned} \tag{A24}$$

[11]Note that the definition of η and ξ given by Eq. (14) in [24] implies that the elastic momenta are not on-shell. However, this detail is irrelevant for our discussion in this section.

TABLE I. Relation between the notations used in [24,35].

Gervais	Lebiedowicz, Nachtmann, Szczurek
$\frac{1}{\not{p}-m}$	$\frac{1}{p^2-m^2}$
γ_μ	$p_\mu + p'_\mu$
$u, v, \bar{u}, \bar{v}$	1
$p_1,\ p'_1$	p_a
$k_1,\ k'_1$	p_b
p_2	p'_1
k_2	p'_2
p'_2	p_1
k'_2	p_2
q	k
$\eta_1,\ \xi_1$	0
ξ_2	$-l_1$
η_2	$-l_2$

Therefore, in Eq. (A24) there is a dependence on f'_0 (the derivative of f evaluated on nonradiative momenta). f'_0 is an arbitrary quantity, which seems to invalidate the consistency of the result in [24].

To see why this argument does not imply an inconsistency of the LBK theorem in the form given in Ref. [24], a more detailed analysis of the complete expressions is needed. A direct comparison between [24,35] is unfortunately not possible, since the processes under consideration are different ($e^-\pi^0 \to e^-\pi^0$ for ref. [24], and $\pi^-\pi^0 \to \pi^-\pi^0$ for LNS). The comparison thus requires a dictionary to relate the theorems with fermions and scalar fields, respectively, which is summarized in Table I. After carefully taking this translation into account, Low's theorem in the notation of Ref. [24] [see Eq. (20) there] reads

$$\mathcal{A}^\mu = e\left(\frac{p_1'^\mu}{p'_1\cdot k} - \frac{p_a^\mu}{p_a\cdot k}\right)\mathcal{H}(p_a, p_b, p_1, p_2) - e\left(\frac{p_a^\mu}{p_a\cdot k}I_1 - \frac{p_1^\mu}{p_1\cdot k}I_2 + I_3^\mu + I_4^\mu\right). \tag{A25}$$

In particular, it is worth noting that the expression for I_i in Eq. (A25) enter via the following combinations

$$p_a^\mu I_1 + (p_a\cdot k)I_3^\mu, \qquad p_1^\mu I_2 - (p_1\cdot k)I_4^\mu. \tag{A26}$$

Therefore, after inserting the expressions of Eq. (A24) in Eq. (A25), the dependence on f'_0 vanishes. Hence, for the case of Eq. (A23) studied in this section, Eq. (A25) is consistent with other formulations of the LBK theorem, such as LNS [Eq. (A10)] and Low's [Eq. (A15)].

APPENDIX B: LBK INVARIANCE UNDER MOMENTA TRANSFORMATION

We consider here the invariance under Eq. (2.17) in the case where Δ arises from linear transformations of the momenta in $\mathcal{H}$. Specifically, we prove that, at NLP, Eq. (2.16) is invariant under the following transformation,

$$\mathcal{H}(p_1,\ldots,p_n) \to \mathcal{H}(\tilde{p}_1,\ldots,\tilde{p}_n), \tag{B1}$$

with

$$\tilde{p}_i(k) = p_i + c_i k + \mathcal{O}(k^2), \tag{B2}$$

where the coefficients c_i are arbitrary. To verify the invariance, let us apply Eq. (B1) to Eq. (2.16). We get

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \sum_{ij=1}^n(-\eta_i\eta_j Q_i Q_j)\frac{p_i\cdot p_j}{p_i\cdot k p_j\cdot k}\overline{|\mathcal{H}(\tilde{p}_1,\ldots,\tilde{p}_n)|}^2 + \sum_{ij=1}^n(-\eta_i\eta_j Q_i Q_j)\frac{p_{i\mu}}{p_i\cdot k}\eta_j\left(g^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j\cdot k}\right)\frac{d}{dp_j^\nu}\overline{|\mathcal{H}(\tilde{p}_1,\ldots,\tilde{p}_n)|}^2, \tag{B3}$$

where partial derivatives have been replaced with total derivatives due to the nontrivial functional dependence inside the nonradiative amplitude. The function $\overline{|\mathcal{H}|}^2$ can be simply expanded in k by using the functional dependence of Eq. (B2), to get

$$\overline{|\mathcal{H}(\tilde{p}_1,\ldots,\tilde{p}_n)|}^2 = \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + k^\mu\sum_i c_i\frac{\partial}{\partial p_i^\mu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \mathcal{O}(k^2). \tag{B4}$$

To proceed further, we note that so far momentum conservation has not been imposed. We can do so by making the dependence over the momenta p_i explicit in the soft momentum, i.e., by enforcing $k^\mu \to k^\mu(p_1,\ldots,p_n) = \sum_i \eta_i p_i^\mu$. Subsequently, by differentiating Eq. (B4) we get

$$\frac{d}{dp_j^\nu}\overline{|\mathcal{H}(\tilde{p}_1,\ldots,\tilde{p}_n)|}^2 = \frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \delta_\nu^\mu\eta_j\sum_i c_i\frac{\partial}{\partial p_i^\mu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \mathcal{O}(k). \tag{B5}$$

Note that since Eq. (B4) is expanded up to $\mathcal{O}(k^2)$, we have to truncate Eq. (B5) to $\mathcal{O}(k)$ because differentiating k with respect to the momenta p_i reduces the order of the expansion. In particular, $\frac{d\mathcal{O}(k^2)}{dp_j^\nu} = \mathcal{O}(k)$. This is not a problem, since the lhs of Eq. (B5) is multiplied by an expression which is suppressed w.r.t. the other term in Eq. (B3) by one power of k. Therefore, plugging Eqs. (B4) and (B5) into Eq. (B3) we get

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \sum_{ij=1}^{n}(-\eta_i\eta_j Q_i Q_j)\frac{p_i\cdot p_j}{p_i\cdot k p_j\cdot k}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \sum_{ij=1}^{n}(-\eta_i\eta_j Q_i Q_j)\frac{p_{i\mu}}{p_i\cdot k}\eta_j\left(g^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j\cdot k}\right)\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + R(c_i), \tag{B6}$$

where

$$R(c_i) = \sum_{ij=1}^{n}(-\eta_i\eta_j Q_i Q_j)\frac{p_i\cdot p_j}{p_i\cdot k p_j\cdot k}k^\mu\sum_m c_m\frac{\partial}{\partial p_m^\mu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \sum_{ij=1}^{n}(-\eta_i\eta_j Q_i Q_j)\frac{p_\mu^i}{p_i\cdot k}\eta_j\left(g^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j\cdot k}\right)\eta_j\sum_m c_m\frac{\partial}{\partial p_m^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \mathcal{O}(1). \tag{B7}$$

To prove the invariance of Eq. (2.16) under Eq. (B1) at NLP, we have to show that the remainder term $R(c_i)$, which depends on the arbitrary coefficients c_i, is $\mathcal{O}(1)$ (i.e., NNLP). This follows straightforwardly by first noting that, thanks to $\eta_j^2 = 1$, the term in the first line of Eq. (B7) cancels with the analogous term in the second line. The term proportional to $g^{\mu\nu}$ then vanishes since $\sum_j \eta_j Q_j = 0$ by charge conservation, thus leaving $R(c_i) = \mathcal{O}(1)$, as desired.

Having established that the LBK theorem in the form of Eq. (2.16) is invariant under Eq. (B1) at NLP, the consistency of the theorem [i.e., the possibility to evaluate $\mathcal{H}$ on a set of unphysical momenta as in Eq. (2.16)] follows as a corollary. In fact, the invariance under Eq. (B1) guarantees that at NLP there is an infinite number of equivalent forms of the theorem, one for each choice of the coefficients c_i. In particular, the unphysical momenta of Eq. (2.16) corresponds to the trivial transformation with $c_i = 0$. On the other hand, we can choose c_i so that momentum conservation for nonradiative amplitude is restored (as in the strategy of Burnett and Kroll), i.e.,

$$\sum_i \eta_i c_i = -1, \qquad \sum_i \eta_i \tilde{p}_i = 0. \tag{B8}$$

Therefore, the form of the theorem in Eq. (2.16) where $\mathcal{H}$ is evaluated on a set of unphysical momenta is equivalent, up to NNLP corrections, to the form where momentum conservation is restored (and thus no ambiguity is present).

Note that although the argument presented in this Appendix is quite general, it fails when the invariance of the amplitude cannot be represented by linear shifts. A simple example is given by a constant amplitude that does not depend on the external momenta. In that case, for the consistency of Eq. (2.16) one has to rely on the more general argument of Sec. II B.

APPENDIX C: CALCULATION OF THE MODIFIED SHIFTS

We show here the calculation that leads to the expression in Eq. (2.40). We consider the conditions (i)–(iii) of Eqs. (2.36), (2.37) and (2.38). The most general form for δp_i reads

$$\delta p_i^\mu = \sum_j A_{ij}^{\mu\nu} p_{j\nu} + B_i^{\mu\nu} k_\nu. \tag{C1}$$

However, it is enough for our purposes to consider

$$\delta p_i^\mu = \sum_j A_{ij} p_j^\mu + B_i k^\mu. \tag{C2}$$

We can further restrict our ansatz by assuming the set $\{p_i^\mu, k^\mu\}$ to be linearly independent. Although clearly not true in general, this is not a problem since we are only interested in finding a single solution. With this assumption, we can now insert Eq. (C2) into Eqs. (2.36), (2.37) and (2.38) to determine the coefficients A_{ij} and B_i. We get

(i)

$$\sum_{i,j}\eta_i A_{ij} p_j^\mu + \left(\sum_i \eta_i B_i + 1\right)k^\mu = 0, \tag{C3}$$

which gives

$$\sum_i \eta_i A_{ij} = 0, \qquad \sum_i \eta_i B_i = -1. \tag{C4}$$

(ii)

$$\left(\sum\nolimits_j (\delta_{ij} + A_{ij}) p_j^\mu + B_i k^\mu\right)^2 = m^2, \tag{C5}$$

which gives

$$\sum_{j,k} (2\delta_{ij} A_{ik} + A_{ij} A_{ik})(p_j \cdot p_k) + 2\sum_j (\delta_{ij} B_i + A_{ij} B_i)(p_j \cdot k) = 0. \tag{C6}$$

(iii)

$$\begin{aligned} A_{ij} &= -Q_i(\overline{|\mathcal{S}_{\rm LP}|^2})^{-1} \frac{\eta_j Q_j}{k \cdot p_j} + \mathcal{O}(k^2), \\ B_i &= Q_i(\overline{|\mathcal{S}_{\rm LP}|^2})^{-1} \sum_j \left(\frac{\eta_j Q_j}{k \cdot p_j}\right) \frac{p_j \cdot p_i}{p_i \cdot k} + \mathcal{O}(k) \\ &= -\sum_j A_{ij} \frac{p_i \cdot p_j}{p_i \cdot k} + \mathcal{O}(k). \end{aligned} \tag{C7}$$

As we can see, the conditions given by Eqs. (C4), (C6) and (C7) are not too restrictive. Thus, we still have the freedom to select a single solution by introducing a scalar coefficient A such that we have

$$A_{ij} = A Q_i \frac{\eta_j Q_j}{k \cdot p_j}. \tag{C8}$$

Then, the condition $\sum_i \eta_i A_{ij} = 0$ is immediately satisfied by charge conservation. The remaining conditions now yield

$$\sum_i \eta_i B_i = -1, \tag{C9}$$

$$2AQ_i \sum_j \eta_j Q_j \frac{p_i \cdot p_j}{k \cdot p_j} - A^2 Q_i^2 \overline{|\mathcal{S}_{\rm LP}|^2} + 2B_i(p_i \cdot k) = 0, \tag{C10}$$

$$A = \frac{-1}{\overline{|\mathcal{S}_{\rm LP}|^2}} + \mathcal{O}(k^3) \qquad B_i = \frac{Q_i}{\overline{|\mathcal{S}_{\rm LP}|^2}} \sum_j \frac{\eta_j Q_j}{k \cdot p_j} \frac{p_i \cdot p_j}{p_i \cdot k} + \mathcal{O}(k). \tag{C11}$$

We can finally determine the coefficients A and B_i. Specifically, for the coefficients B_i we can use Eq. (C10), which yields

$$B_i = -AQ_i \sum_j \eta_j Q_j \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)} + \frac{1}{2} \frac{A^2 Q_i^2 \overline{|\mathcal{S}_{\rm LP}|^2}}{p_i \cdot k}. \tag{C12}$$

Assuming the behavior of A given in Eq. (C11), the second term in Eq. (C12) is $\mathcal{O}(k)$, so B_i have the correct limit given in Eq. (C11). To determine A, we can use Eq. (C9), which yields

$$\begin{aligned} 1 = -\sum_i \eta_i B_i &= A \sum_{i,j} \eta_i Q_i \eta_j Q_j \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)} - \frac{A^2 \overline{|\mathcal{S}_{\rm LP}|^2}}{2} \sum_i \frac{\eta_i Q_i^2}{p_i \cdot k} \\ &= -A\overline{|\mathcal{S}_{\rm LP}|^2} - \frac{A^2 \overline{|\mathcal{S}_{\rm LP}|^2}}{2} \sum_i \frac{\eta_i Q_i^2}{p_i \cdot k}, \end{aligned} \tag{C13}$$

which is a quadratic equation. Defining $\chi = \sum_i \frac{\eta_i Q_i^2}{p_i \cdot k}$, we find that

$$A = \frac{1}{\chi}\left(-1 \pm \sqrt{1 - \frac{2\chi}{\overline{|\mathcal{S}_{\rm LP}|^2}}}\right). \tag{C14}$$

Only the + solution has the correct behavior at low k. Combining thus Eqs. (C2), (C8), (C12) and (C14), we find the expression for the modified shifts as given by Eq. (2.40).

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