Tentative estimates of $\mathcal{B}(X(3872) \to \pi^0\pi^0\chi_{c1})$ and $\mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})$

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(Received 15 May 2024; accepted 26 June 2024; published 24 July 2024)

The rates of the $X(3872) \to \pi^0\pi^0\chi_{c1}$ and $X(3872) \to \pi^+\pi^-\chi_{c1}$ decays are estimated in the model of the triangle loop diagrams with charmed $D^*\bar{D}D$ and $\bar{D}^*D\bar{D}$ mesons in the loops. There are the triangle logarithmic singularities in the physical region of the $X(3872) \to \pi^0\pi^0\chi_{c1}$ decay which manifest themselves as narrow peaks in the $\pi^0\chi_{c1}$ mass spectrum near the $D^0\bar{D}^0$ threshold. The model predicts approximately the same branching fractions of the $X(3872) \to \pi^0\pi^0\chi_{c1}$ and $X(3872) \to \pi^+\pi^-\chi_{c1}$ decays at the level of about $(0.8–1.7) \times 10^{-4}$. A distinct prediction of the model is the value of the ratio $\mathcal{R} = \mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})/\mathcal{B}(X(3872) \to \pi^0\pi^0\chi_{c1}) \approx 1.1$. It weakly depends on the $X(3872)$ resonance parameters and indicates a significant violation of the isotopic symmetry according to which one would expect $\mathcal{R} = 2$.

DOI: 10.1103/PhysRevD.110.016023

I. INTRODUCTION

The modern studies of the first candidate for exotic charmoniumlike states $X(3872)$ or $\chi_{c1}(3872)$ [1] advance in the line increasing the data accuracy and expanding the nomenclature of it production and decay channels [1–13]. For example, the BESIII [2] and Belle [3] collaborations obtained information about the rate for the isospin-violating decay $X(3872) \to \pi^0\chi_{c1}$. Also the Belle [12] collaboration and recently the BESIII [13] collaboration obtained upper limits on the probability of the $X(3872) \to \pi^+\pi^-\chi_{c1}$ decay which formally preserves G parity.

According to the Belle collaboration [12] and the Particle Data Group [1] $\mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1}) < 7 \times 10^{-3}$ at the 90% confidence level (CL). According to the BESIII data [2,13]

$$\mathcal{R}_1 = \frac{\mathcal{B}[X(3872) \to \pi^+\pi^-\chi_{c1}]}{\mathcal{B}[X(3872) \to \pi^+\pi^- J/\psi]} < 0.18(90\%\,\mathrm{CL})\,[13] \quad \text{and}$$

$$\mathcal{R}_2 = \frac{\Gamma(X(3872) \to \pi^+\pi^-\chi_{c1})}{\Gamma(X(3872) \to \pi^0\chi_{c1})} < 0.2[2,13]. \tag{1}$$

The BESIII result [13] for $\mathcal{R}_1$ is consistent with the measurement from the Belle collaboration [12]. An upper limit on the ratio $\mathcal{R}_2$ turned out to be two orders of magnitude smaller than the value of $\frac{\Gamma(2^3P_1 \to \pi^+\pi^-\chi_{c1})}{\Gamma(2^3P_1 \to \pi^0\chi_{c1})} \approx 25$ expected under a pure charmonium 2^3P_1 assumption for the $X(3872)$ [14]. Therefore, Ref. [13] concluded that the BESIII data favor the nonconventional charmonium nature of the $X(3872)$ state. But this is not quite true. The point is that the large theoretical value for $\mathcal{R}_2$ found in Ref. [14] is entirely due to the tiny ($\simeq$0.06 keV) decay width of $2^3P_1 \to \pi^0\chi_{c1}$, calculated in this work under the assumption of the two-gluon production mechanism of the π^0, which is not a consequence of the hypothesis about the nature of the $X(3872)$. The mechanism of the isospin-violating decay of $2^3P_1 \to gg\chi_{c1} \to \pi^0\chi_{c1}$ considered in Ref. [14] is not a single one, and much less the leading one, for the 2^3P_1 charmonium state with a mass of 3872 MeV. The now known value for the decay width $\Gamma(X(3872) \to \pi^0\chi_{c1}) = (0.04 \pm 0.02)$ MeV [1,2] can be explained, for example, by the mechanism of the 2^3P_1 $c\bar{c}$ $X(3872)$ state transition into $\pi^0\chi_{c1}$ via the intermediate $D^*\bar{D}D^*$ and $\bar{D}^*D\bar{D}^*$ mesonic loops, see Ref. [15] and references herein. Thus, the results of the BESIII collaboration [13] have yet to be compared with the assumed possible variants for the nature of the $X(3872)$ state.

In anticipation of future experiments on the decays $X(3872) \to \pi^0\pi^0\chi_{c1}$ and $X(3872) \to \pi^+\pi^-\chi_{c1}$, it is interesting to estimate their probabilities and, accordingly, the deviation from the relation $\mathcal{B}(X(3872) \to \pi^0\pi^0\chi_{c1}) = \frac{1}{2}\mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})$ that takes place in the unbroken isotopic symmetry. These estimates are the subject of this work.

Earlier in the work [16], with the use a combination of the heavy hadron chiral perturbation theory and effective

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field theory for the X(3872), the following results were obtained:

$$\left(\frac{\mathcal{B}[X(3872)\to\pi^0\pi^0\chi_{c1}]}{\mathcal{B}[X(3872)\to\pi^0\chi_{c1}]}\right)_{\mathrm{LO}} = 6.1\times10^{-1},$$
$$\left(\frac{\mathcal{B}[X(3872)\to\pi^+\pi^-\chi_{c1}]}{\mathcal{B}[X(3872)\to\pi^0\chi_{c1}]}\right)_{\mathrm{LO}} \approx \mathcal{O}(10^{-3}). \quad (2)$$

These estimates were performed with accounting the contributions of the leading order (LO) diagrams for the amplitudes of the transitions $D^0\bar{D}^{*0}\to\pi^0\chi_{c1}$ and $D^0\bar{D}^{*0}\to\pi\pi\chi_{c1}$ [16]. Subsequently, the value of the ratio $\left(\frac{\mathcal{B}[X(3872)\to\pi^0\pi^0\chi_{c1}]}{\mathcal{B}[X(3872)\to\pi^0\chi_{c1}]}\right)_{\mathrm{LO}}$ was adjusted towards its decrease by two orders of magnitude as a result of recalculation of the $X(3872)\to\pi^0\pi^0\chi_{c1}$ amplitude [17]:

$$\left(\frac{Br[X(3872)\to\chi_{c1}\pi^0\pi^0]}{Br[X(3872)\to\chi_{c1}\pi^0]}\right)_{\mathrm{LO}} = 2.9\times10^{-3}. \quad (3)$$

In the present work (as in Refs. [15,18–22]), we consider the $X(3872)$ meson as a $\chi_{c1}(2P)$ charmonium state, which has the equal coupling constants with the $D^{*0}\bar{D}^0$ and $D^{*+}D^-$ channels owing to the isotopic symmetry. Its decay into $D^*\bar{D}$ + c.c. occurs [similarly to, for example, the $\psi(3770)\to D\bar{D}$ decay] by picking up of a light $q\bar{q}$ pair from vacuum quark-antiquark fluctuations, $c\bar{c}\to(c\bar{q})(q\bar{c})\to D^*\bar{D}$ + c.c. Undoubtedly, the main feature of the $X(3872)$ resonance is that it is located directly at the threshold of its main decay channel into $D^{*0}\bar{D}^0$ + c.c. $\to D^0\bar{D}^0\pi^0$ [1]. This circumstance ensures the smallness of its width (it is ∼1 MeV) and clear violation of the isotopic symmetry against a background of the kinematically closed decay channel of the $X(3872)$ into $D^{*+}D^-$ + c.c. (the thresholds of the $D^{*0}\bar{D}^0$ and $D^{*+}D^-$ channels are separated by 8.23 MeV). Section II considers the kinematics of the decays $X(3872)\to\pi^0\pi^0\chi_{c1}$ and $X(3872)\to\pi^+\pi^-\chi_{c1}$. Section III discusses hadronic loop diagrams, which we use to estimate the branching fractions of these processes. The estimates themselves are given in Sec. IV. Conclusions from the analysis performed are presented in Sec. V.

II. KINEMATICS OF THE $X(3872)\to\pi\pi\chi_{c1}$ DECAYS

Let us use the Particle Data Group data [1] and put a mass of the $X(3872)$ state equal to $m_X = 3871.65$ MeV, and also $m_{\chi_{c1}} = 3510.67$ MeV, $m_{\pi^+} = 139.57039$ MeV, and $m_{\pi^0} = 134.9768$ MeV. The invariant phase volumes (PV) [23] for the three-body decays $X(3872)\to\pi^0\pi^0\chi_{c1}$ and $X(3872)\to\pi^+\pi^-\chi_{c1}$ are equal to

$$\mathrm{PV}(m_X; m_{\pi^0}, m_{\pi^0}, m_{\chi_{c1}}) = 0.0049718\ \mathrm{GeV}^2,$$
$$\mathrm{PV}(m_X; m_{\pi^+}, m_{\pi^-}, m_{\chi_{c1}}) = 0.00407956\ \mathrm{GeV}^2, \quad (4)$$

respectively. For comparison, we point out that the invariant phase volumes for the decays $X(3872)\to D^0\bar{D}^0\pi^0$ and $X(3872)\to\pi^+\pi^-J/\psi$ are equal to 0.0000686751 and 0.225852 GeV^2, respectively. The energy release in $X(3872)\to\pi^0\pi^0\chi_{c1}$ is $\mathrm{T}_n = m_X - 2m_{\pi^0} - m_{\chi_{c1}} =$ 91.0264 MeV, and that in $X(3872)\to\pi^+\pi^-\chi_{c1}$ $\mathrm{T}_c = m_X - 2m_{\pi^\pm} - m_{\chi_{c1}} = 81.8392$ MeV. The invariant mass of the $\pi^0\chi_{c1}$ system, $m_{\pi^0\chi_{c1}}$, varies from $m_{\chi_{c1}} + m_{\pi^0}$ to $m_X - m_{\pi^0}$, i.e., in the near-threshold region with a width of 91.0264 MeV, and the invariant mass of the $\pi^\pm\chi_{c1}$ system, $m_{\pi^\pm\chi_{c1}}$, varies from $m_{\chi_{c1}} + m_{\pi^\pm}$ to $m_X - m_{\pi^\mp}$, i.e., in that with a width of 81.8392 MeV. It is quite natural to believe that in these regions the production amplitudes of the $\pi^0\chi_{c1}$ and $\pi^\pm\chi_{c1}$ pairs will be dominated by contributions from the corresponding lower partial waves.

Let us denote the four-momenta of the particles in the decay $X(3872)\to\pi\pi\chi_{c1}$ as $p_X = p_1$, $p_{\chi_{c1}} = p_2$, $p_{\pi_1} = p_3$, $p_{\pi_2} = p_4$, where $\pi_1 = \pi_1^0$ or π^+ and $\pi_2 = \pi_2^0$ or π^-, and the polarization four-vectors of the $X(3872)$ and χ_{c1} mesons as $\varepsilon_X = \varepsilon_1$ and $\varepsilon_{\chi_{c1}} = \varepsilon_2$. The matrix element $\mathcal{M}$ of the decay $X(3872)\to\pi\pi\chi_{c1}$ is described in general case by five independent invariant amplitudes $b_{i=1,\ldots,5}$ and it can be written as

$$\begin{aligned}\mathcal{M} &= \varepsilon_1^\mu\varepsilon_2^{\nu*}M_{\mu\nu}\\ &= \varepsilon_1^\mu\varepsilon_2^{\nu*}(g_{\mu\nu}b_1 + p_{2\mu}p_{1\nu}b_2 + \Delta_\mu\Delta_\nu b_3\\ &\quad + \Delta_\mu p_{1\nu}b_4 + p_{2\mu}\Delta_\nu b_5),\end{aligned} \quad (5)$$

where $\Delta = p_3 - p_4$; $b_i = b_i(m_X^2; s, t, u)$, $s = (p_2+p_3)^2 = (p_1-p_4)^2$, $t = (p_2+p_4)^2 = (p_1-p_3)^2$, $u = (p_3+p_4)^2 = (p_1-p_2)^2$, and $s+t+u = m_X^2 + m_{\chi_{c1}}^2 + 2m_\pi^2$. Here we indicated the dependence of the invariant amplitudes from m_X^2 because in what follows we will need to replace m_X^2 in $\mathcal{M}$ with the variable quantity S_1 meaning the invariant mass squared of the virtual $X(3872)$ state.

The $\pi\pi$ system in the $X(3872)\to\pi\pi\chi_{c1}$ decay has the positive C parity. As a consequence, only even orbital moments are allowed in this system and states with the isospin $I = 1$ are forbidden. It is clear that the matrix element $\mathcal{M}$ must be an even function of Δ, i.e., should not change with the permutation of p_3 and p_4, and the invariant amplitudes must possess the following crossing properties: $b_{1,2,3}(m_X^2;s,t,u) = b_{1,2,3}(m_X^2;t,s,u)$ and $b_{4,5}(m_X^2;s,t,u) = -b_{4,5}(m_X^2;t,s,u)$. In the following, we will denote the matrix elements for the decays $X(3872)\to\pi^0\pi^0\chi_{c1}$ and $X(3872)\to\pi^+\pi^-\chi_{c1}$ as $\mathcal{M}_n$ and $\mathcal{M}_c$, respectively.

For the rates of the decays $X(3872)\to\pi\pi\chi_{c1}$, the exact isotopic symmetry predicts the following relation: $\mathcal{B}(X(3872)\to\pi^0\pi^0\chi_{c1}) = \frac{1}{2}\mathcal{B}(X(3872)\to\pi^+\pi^-\chi_{c1})$. As will be shown below, it can be significantly broken in the real situation.

III. HADRONIC LOOP DIAGRAMS FOR $X(3872) \to \pi\pi\chi_{c1}$

Currently, the mechanism of triangle loop diagrams with charmed mesons in the loops is considered as a main one of the two-body decay of $X(3872) \to \pi^0\chi_{c1}$, see in this regard Refs. [14–17,24–27] and references herein. We assume that in the three-body decay $X(3872) \to \pi\pi\chi_{c1}$ the final $\pi\chi_{c1}$ system is produced mainly in a lower partial wave. This is quite natural in the region near the $\pi\chi_{c1}$ threshold. Then, the decay of $X(3872) \to \pi\pi\chi_{c1}$ can be considered as a quasi-two-body process and applied to its description the mechanism of the triangle loop diagrams. Examples of such diagrams are shown in Figs. 1 and 2. These diagrams (not all) contain so-called triangle logarithmic singularities [28–33]. The literature is rich in examples showing that such singularities lead to various enhancements in two-body and three-body mass spectra in the decays of resonances, see, for example, Refs. [21,30,32,34–42] and references herein.

The logarithmic singularities in Figs. 1(a) and 1(b) lie along the solid curves shown in Figs. 3(a) and 3(b), respectively. The dependences of S_1 on s given by these curves follow from the equation $2x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - 1 = 0$ [29,31,32,40], where $x_1 = (S_1 - m_1^2 - m_2^2)/(2m_1m_2)$, $x_2 = (s - m_2^2 - m_3^2)/(2m_2m_3)$, and $x_3 = (m_\pi^2 - m_1^2 - m_3^2)/(2m_1^2m_3^2)$, in the solution of which it is necessary to substitute specific values of the masses (m_1, m_2, m_3) of particles in the loops [see notations in Fig. 1(a)] and the mass of the outgoing π meson. At singularity points, all three particles in the loops simultaneously are on the mass shell [28,29,32,33,40]. Of course, this requires that at least one of the particles corresponding to the internal lines of the diagram is unstable [32,33,40]. Horizontal and vertical dotted lines in Fig. 3(a) mark the thresholds for the $\sqrt{S_1}$ and $\sqrt{s}$ variables (i.e., the values of $\sqrt{S_1} = m_{D^{*0}} + m_{D^0} =$ 3.87169 GeV and $\sqrt{s} = 2m_{D^0} = 3.72968$ GeV) above which the matrix element $\mathcal{M}_n = \mathcal{M}_n(S_1; s, t, u)$ (see Sec. II) corresponding to Fig. 1(a) has the imaginary parts on the S_1 and s (or t) variables. Intervals containing the curve of singularities, $m_1 + m_2 < \sqrt{S_1} < \sqrt{m_1^2 + m_2^2 + m_2m_3 + m_2(m_1^2 - m_\pi^2)/m_3}$ and $m_2 + m_3 < \sqrt{s} < \sqrt{m_2^2 + m_3^2 + m_1m_2 + m_2(m_3^2 - m_\pi^2)/m_1}$, are bounded by the points, where this curve touches the above lines (see, for example, Ref. [40]). The horizontal dashed line in Fig. 3(a) marks the nominal mass of the $X(3872)$ state $m_X = 3.87165$ GeV [1]. Since the width of the $X(3872)$, Γ_X, is not less than 1 MeV [1,6,8,11], and the available values of $\sqrt{s}$ lie in the range from $m_{\chi_{1c}} + m_{\pi^0} = 3.64565$ GeV to $m_X - m_{\pi^0} = 3.73667$ GeV, then the locus of logarithmic singularities of triangle in Fig. 1(a) completely falls into the physical region of the $X(3872) \to \pi^0\pi^0\chi_{c1}$ decay.

Let us move on to Fig. 3(b) associated with Fig. 1(b). The threshold values of $\sqrt{S_1} = m_{D^{*+}} + m_{D^-} =$ 3.87992 GeV and $\sqrt{s} = 2m_{D^\pm} = 3.73932$ GeV marked by horizontal and vertical dotted lines lie 8.23 and 9.64 MeV above the thresholds of the $D^{*0}\bar{D}^0$ and $D^0\bar{D}^0$ channels, respectively. It is clear that the triangle singularities of the diagrams with charged charmed D^* and D mesons in the loops are located outside the physical region of the $X(3872) \to \pi^0\pi^0\chi_{c1}$ decay. However, the contribution of Fig. 1(b), as will be shown in the next section, turns out to be important and must be taken into account.

Let us now consider the diagrams in Fig. 2 for the decay $X(3872) \to \pi^+\pi^-\chi_{c1}$. In Fig. 2(a), there are no triangle singularities, since the decay channel of the D^{*0} into π^-D^+ is closed ($m_{D^{*0}} = 2.00685$ GeV, $m_{D^+} + m_{\pi^-} = 2.00923$ GeV, and $\Gamma_{D^{*0}} \simeq 55.6$ keV [21]). Figure 2(b) have triangle

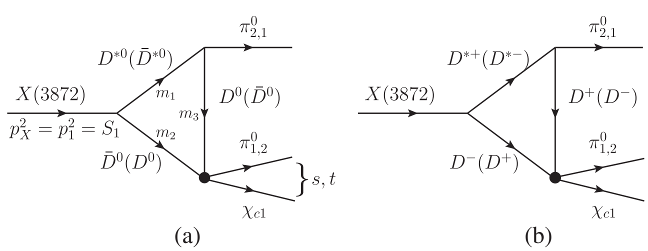

FIG. 1. Eight triangle loop diagrams for the transition $X(3872) \to \pi^0\pi^0\chi_{c1}$. (a), as well as (b), involves four diagrams taking into account two charge-conjugate states in the loops ($D^*\bar{D}D$ and $\bar{D}^*D\bar{D}$) and the permutation of identical π^0 mesons.

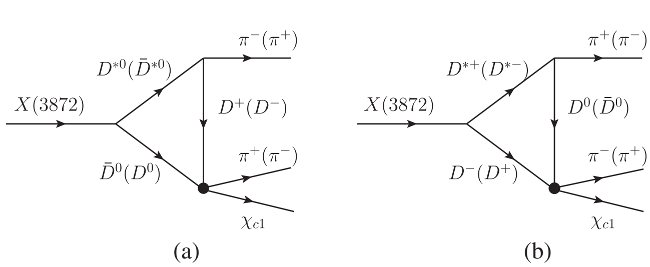

FIG. 2. Four triangle loop diagrams for the transition $X(3872) \to \pi^+\pi^-\chi_{c1}$. (a), as well as (b), involves two diagrams with charge-conjugate states in the loops ($D^*\bar{D}D$ and $\bar{D}^*D\bar{D}$).

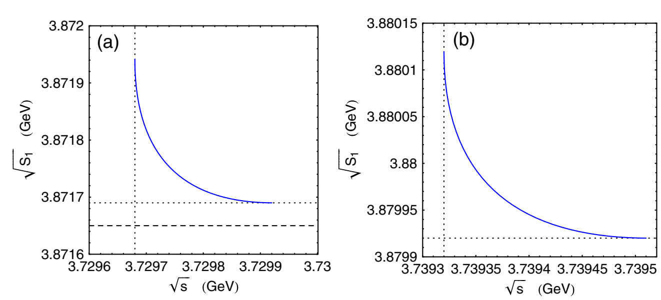

FIG. 3. Solid curves in (a) and (b) show loci of logarithmic singularities in the $(\sqrt{s}, \sqrt{S_1})$ plain for (a) and (b) in Fig. 1, respectively. The singularities are located in (a) in the intervals 3.87169 GeV $< \sqrt{S_1} <$ 3.87194 GeV and 3.72968 GeV $< \sqrt{s} <$ 3.72992 GeV, and in (b) in the intervals 3.87992 GeV $< \sqrt{S_1} <$ 3.88012 GeV and 3.73932 GeV $< \sqrt{s} <$ 3.73951 GeV.

singularities. But they lie in the region of 3.87992 GeV $< \sqrt{S_1} <$ 3.88014 GeV and 3.7345 GeV $< \sqrt{s} <$ 3.73471 GeV which on the $\sqrt{S_1}$ variable starts 8.27 MeV above the nominal mass of the $X(3872)$ and on the $\sqrt{s}$ variable 2.42 MeV to the right of the maximum permissible value of $\sqrt{s} = m_X - m_{\pi^-} = 3.73208$ GeV in this decay. The values $\sqrt{S_1} = 3.87992$ GeV and $\sqrt{s} = 3.7345$ GeV indicate the thresholds of the $D^{*+}D^-$ and D^0D^- channels, respectively. Thus, both of these channels are closed in the $X(3872) \to \pi^+\pi^-\chi_{c1}$ decay and the amplitude for Fig. 2(b) turns out to be purely real (if neglect by the tiny value of $\Gamma_{D^{*+}}$ in the D^{*+} meson propagator). How the contributions of Figs. 2(a) and 2(b) correlate to each other, we will find out in the next section.

IV. ESTIMATES OF $\mathcal{B}(X(3872) \to \pi\pi\chi_{c1})$

To estimate $\mathcal{B}(X(3872) \to \pi\pi\chi_{c1})$ we restrict ourselves to the contributions of the diagrams presented in Figs. 1 and 2. First of all, consider the amplitude of the subprocess $D\bar{D} \to \pi\chi_{c1}$ which is a component part of the matrix element $\mathcal{M}$. We will estimate it on the mass shell near the $D\bar{D}$ threshold and then use the found value as an effective "coupling constant" characterizing the $D\bar{D}\pi\chi_{c1}$ vertex in the triangular loops. The isotopic invariance of strong interactions and the P-parity conservation allow us to write down a number of useful relations for the reaction $D\bar{D} \to \pi\chi_{c1}$:

$$I_f = 1 = I_i, \quad G_f = -1 = G_i = (-1)^{I_i + l_i}, \quad l_i = 0, 2, \ldots,$$
$$J = l_i, \quad P_i = (-1)^{l_i} = P_f = -(-1)^{l_f}, \quad l_f = 1, 3, \ldots, \tag{6}$$

where the indices i and f indicate the belonging of quantum numbers to the initial $D\bar{D}$ and final $\pi\chi_{c1}$ states, respectively; I, G, l, J, and P are the isospin, G parity, orbital moment, total moment, and P parity, respectively. For $l_i = 0$ ($J = 0$) there is only one possible value of $l_f = 1$, and for each $l_i \geq 2$ two values $l_f = l_i \pm 1$ are allowed. The partial amplitude of the process $D\bar{D} \to \pi\chi_{c1}$ with $l_i = 0$ ($J = 0$) and $l_f = 1$ experiences a minimal suppression caused by the threshold factors near the threshold. This amplitude has the form

$$f^{J=0}_{D\bar{D}\pi\chi_{c1}} = g_{D\bar{D}\pi\chi_{c1}}(\vec{p}_{\chi_{c1}}(s), \vec{\xi}^*), \tag{7}$$

where $\vec{p}_{\chi_{c1}}(s)$ is the momentum of the χ_{c1} meson in the $D\bar{D}$ center-of-mass system, and $\vec{\xi}$ is the polarization vector of the χ_{c1} in its rest frame (see Ref. [43]); $|\vec{p}_{\chi_{c1}}(s)| = \sqrt{s^2 - 2s(m^2_{\chi_{c1}} + m^2_\pi) + (m^2_{\chi_{c1}} - m^2_\pi)^2}/(2\sqrt{s})$. It is quite natural to assume that the factor $g_{D\bar{D}\pi\chi_{c1}}$ near the threshold is a smooth function of $\sqrt{s}$. We will calculate it for $\sqrt{s} = 2m_D$ assuming that the reaction $D\bar{D} \to \pi\chi_{c1}$ (near the threshold) proceeds via D^* exchanges in its t and u channels. In this simple model we have

$$g_{D\bar{D}\pi\chi_{c1}} = g_{D^*D\pi}g_{\chi_{c1}D^*\bar{D}}\frac{4m_D}{m_{\chi_{c1}}}\frac{3 + m^2_D/m^2_{D^*}}{2m^2_D + 2m^2_{D^*} - m^2_{\chi_{c1}}}$$
$$= g_{D^*D\pi}g_{\chi_{c1}D^*\bar{D}} \times (3.05696\ \text{GeV}^{-2}), \tag{8}$$

where $g_{D^*D\pi}$ and $g_{\chi_{c1}D^*\bar{D}}$ are the coupling constants in the interaction vertices $V_{D^*D\pi} = g_{D^*D\pi}(\varepsilon^*_{D^*}, p_\pi + p_D)$ and $V_{\chi_{c1}D^*\bar{D}} = g_{\chi_{c1}D^*\bar{D}}(\varepsilon_{D^*}, \varepsilon^*_{\chi_{c1}})$. When obtaining Eq. (8), we neglected the mass squared of the π meson, and also put $m_{D^{*+}} = m_{D^{*0}}$ and $m_{D^+} = m_{D^0}$. At the $D\bar{D}$ threshold, the virtuality of the exchanged D^* mesons (i.e., $m^2_{D^*} - q^2$, where q is the four-momentum of the D^*) is approximately 1.343 GeV^2. In order to take into account to some extent the internal structure and the off-mass-shell effect for the D^* meson, it is necessary to introduce the form factor into the each vertices of the D^* exchange: $\mathcal{F}(q^2, m^2_{D^*}) = \frac{\Lambda^2 - m^2_{D^*}}{\Lambda^2 - q^2}$ [15,44–47]. Here we orient on the typical value of the parameter $\alpha \approx 2$ [15] associated with the Λ by the relation $\Lambda = m_{D^*} + \alpha\Lambda_{\text{QCD}}$ [47], where $\Lambda_{\text{QCD}} = 220$ MeV. This form factor results in decreasing the effective coupling constant $g_{D\bar{D}\pi\chi_{c1}}$ by approximately 2.84 times in comparison with the estimate in Eq. (8); $g^2_{D\bar{D}\pi\chi_{c1}}$ decreases by a factor of 8.06 accordingly. Next we will use for $g_{D\bar{D}\pi\chi_{c1}}$ the value of $g_{D^*D\pi}g_{\chi_{c1}D^*\bar{D}} \times (1.07647\ \text{GeV}^{-2})$ obtained taking into account the form factor. From the isotopic symmetry for the coupling constants $g_{D^*D\pi}$ and the data on the decays $D^{*+} \to (D\pi)^+$ [1], it follows that $g_{D^{*0}D^0\pi^0} = g_{D^{*0}D^+\pi^-}/\sqrt{2} = g_{D^{*+}D^0\pi^+}/\sqrt{2} = -g_{D^{*+}D^+\pi^0} \approx 5.93$ [21]. The constant $g_{\chi_{c1}D^*\bar{D}}$ cannot be measured directly, but its value is predicted theoretically within the framework of the effective theory of heavy quarks [24,27,45–48]: $g_{\chi_{c1}D^*\bar{D}} = 2\sqrt{2}g_1\sqrt{m_D m_{D^*} m_{\chi_{c1}}} = (-21.45 \pm 1.68)$ GeV [21], where g_1 is an universal constant. As a result, we get $g_{D^0\bar{D}^0\pi^0\chi_{c1}} = -g_{D^+D^-\pi^0\chi_{c1}} = g_{D^+\bar{D}^0\pi^+\chi_{c1}}/\sqrt{2} = g_{D^0D^-\pi^-\chi_{c1}}/\sqrt{2} \approx 137\ \text{GeV}^{-1}$ and will use this value as a guide.

The above structure of the $D\bar{D}\pi\chi_{c1}$ vertex allows us to write the matrix element $\mathcal{M}_n(S_1; s, t, u)$ for the contribution of the eight diagrams in Fig. 1 as follows:

$$\mathcal{M}_n(S_1; s, t, u) = 2\frac{\bar{g}}{16\pi}\varepsilon^\mu_X[I_\mu(p_1, p_4)(\vec{p}_{\chi_{c1}}(s), \vec{\xi}^*)$$
$$+ I_\mu(p_1, p_3)(\vec{p}_{\chi_{c1}}(t), \vec{\xi}^*)$$
$$+ \tilde{I}_\mu(p_1, p_4)(\vec{p}_{\chi_{c1}}(s), \vec{\xi}^*)$$
$$+ \tilde{I}_\mu(p_1, p_3)(\vec{p}_{\chi_{c1}}(t), \vec{\xi}^*)], \tag{9}$$

where the common factor 2 arises owing to the equality of the contributions from the loops with the charge conjugated intermediate states, $\bar{g} = g_X g_{D^{*0}D^0\pi^0} g_{D^0\bar{D}^0\pi^0\chi_{c1}}$, g_X is the coupling constant of the $X(3872)$ to $D^{*0}\bar{D}^0$ in the vertex $V_{XD^{*0}\bar{D}^0} = g_X(\varepsilon_X, \varepsilon^*_{D^{*0}})$ (the values of g_X will be specified below); the amplitude $I_\mu(p_1, p_4)$ represents the following vector integral

$$I_\mu(p_1,p_4)=\frac{i}{\pi^3}\int\frac{(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m^2_{D^{*0}}})(2p_{4\nu}-k_\nu)d^4k}{(k^2-m^2_{D^{*0}}+i\epsilon)((p_1-k)^2-m^2_{\bar D^0}+i\epsilon)((k-p_4)^2-m^2_{D^0}+i\epsilon)}. \tag{10}$$

The amplitude $I_\mu(p_1,p_3)$, $\tilde I_\mu(p_1,p_4)$, and $\tilde I_\mu(p_1,p_3)$ have a similar form. In so doing, $I_\mu(p_1,p_4)$ and $I_\mu(p_1,p_3)$ correspond to Fig. 1(a) which differ in the permutation of identical π^0 mesons, and $\tilde I_\mu(p_1,p_4)$ and $\tilde I_\mu(p_1,p_3)$ correspond to similar Fig. 1(b). In Ref. [21] it was shown that the divergent part of a vector integral of type (10) is proportional to $p_{1\mu}$ [i.e., the four-momentum of the $X(3872)$ resonance] and it does not contribute to the matrix element $\mathcal{M}_n(S_1;s,t,u)$ because $(\varepsilon_X,p_1)=0$. It was also shown in Ref. [21] that its convergent part, $I^{\rm conv}_\mu(p_1,p_4)$, proportional to $p_{4\mu}$ is dominated by the amplitude of the scalar triangle diagram, which we denote here as $I(S_1,s)$, i.e., $I^{\rm conv}_\mu(p_1,p_4)=-2p_{4\mu}I(S_1,s)$, where

$$I(S_1,s)=\frac{i}{\pi^3}\int\frac{d^4k}{(k^2-m^2_{D^{*0}}+i\epsilon)((p_1-k)^2-m^2_{\bar D^0}+i\epsilon)((k-p_4)^2-m^2_{D^0}+i\epsilon)}. \tag{11}$$

As a result, Eq. (9) takes the form

$$\mathcal{M}_n(S_1;s,t,u)=-4\frac{\bar g}{16\pi}\{(\varepsilon_X,p_4)[I(S_1,s)+\tilde I(S_1,s)](\vec p_{\chi_{c1}}(s),\vec\xi^*)+(\varepsilon_X,p_3)[I(S_1,t)+\tilde I(S_1,t)](\vec p_{\chi_{c1}}(t),\vec\xi^*)\}. \tag{12}$$

About the contributions of the scalar amplitudes $I(S_1,s)$ and $\tilde I(S_1,s)$ one can speak as of the s contributions from Figs. 1(a) and 1(b), respectively, and about the contributions of the scalar amplitudes $I(S_1,t)$ and $\tilde I(S_1,t)$ one can speak as of the t contributions from Figs. 1(a) and 1(b) with permutation of identical π^0 mesons, respectively.

To numerically calculate scalar triangle amplitudes, we use explicit formulas obtained in Refs. [24,40] within the framework of nonrelativistic formalism. We convinced that the results of such a calculation are in excellent agreement with what is given for these amplitudes the exact expressions through dilogarithms [49]. We take into account the finite width of the D^{*0} meson by replacing $m^2_{D^{*0}}$ in its propagator with $m^2_{D^{*0}}-im_{D^{*0}}\Gamma_{D^{*0}}$ and put $\Gamma_{D^{*0}}=55.6$ keV [21]. This leads to a significant smoothing and reduction in the contributions of triangle logarithmic singularities to $\mathcal{M}_n(S_1;s,t,u)$ as compared with the hypothetical case corresponding to $\Gamma_{D^{*0}}=0$. The finite width of the D^{*+} meson, $\Gamma_{D^{*+}}=83.6$ keV, is taken into account in a similar way.

The differential probability of the $X(3872)\to\pi^0\pi^0\chi_{c1}$ decay which determines the distribution of events in the Dalitz plot has the form [1]:

$$\frac{d^2\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1;s,t,u)}{dtds}=\frac{1}{3(2\pi)^3 32S_1^{3/2}}\sum_{\lambda\lambda'}|\mathcal{M}_n(S_1;s,t,u)|^2, \tag{13}$$

where summation over λ and λ' means summation over polarizations of the $X(3872)$ and χ_{c1} mesons, respectively.

We write the mass spectrum of the $\pi^0\chi_{c1}$ system over the $\sqrt{s}$ variable as

$$\frac{d\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1,s)}{d\sqrt{s}}=2\sqrt{s}\int_{t_-(S_1,s)}^{t_+(S_1,s)}\frac{d^2\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1;s,t,u)}{dtds}dt, \tag{14}$$

where $t_\pm(S_1,s)$ denote the boundaries of the physical region for the t variable for fixed values of s and S_1 [1]. Figure 4(a) shows examples of the $\pi^0\chi_{c1}$ unnormalized mass spectra near the $D^0\bar D^0$ threshold for several values of $\sqrt{S_1}$. These examples illustrate the resonantlike manifestations of the triangle singularities present in the amplitude $I(S_1,s)$. Figure 4(b) shows [in the same units as in Fig. 4(a)] all significant components of the $\pi^0\chi_{c1}$ mass spectrum at $\sqrt{S_1}=3.87172$ GeV throughout the accessible region of $\sqrt{s}$. Curves 1, 2, and 3 correspond to the contributions of the amplitudes $I(S_1,s)$ [Fig. 1(a)], $\tilde I(S_1,s)$ [Fig. 1(b)], and their sum $I(S_1,s)+\tilde I(S_1,s)$, respectively. Curve 4 corresponds to the contribution of the amplitude $I(S_1,t)+\tilde I(S_1,t)$ [from the sum of Figs. 1(a) and 1(b) with the transposed identical π^0 mesons]. The contributions of the amplitudes $I(S_1,t)$ and $\tilde I(S_1,t)$ are not shown separately so as not to clutter the figure. Curve 6 corresponds to the contribution of interference between the amplitudes $I(S_1,s)+\tilde I(S_1,s)$ and $I(S_1,t)+\tilde I(S_1,t)$ which differ by

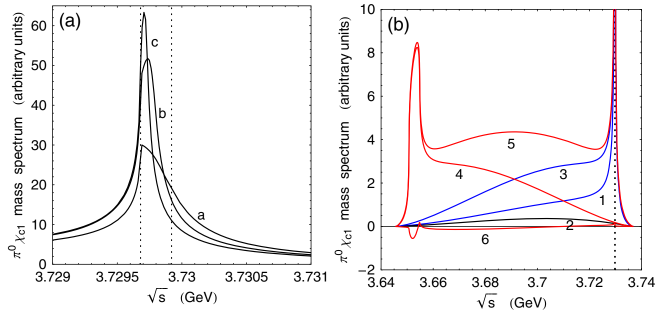

FIG. 4. (a) The solid curves a, b, and c show the examples of the $\pi^0\chi_{c1}$ mass spectrum $d\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1,s)/d\sqrt{s}$ in the region of the $D^0\bar{D}^0$ threshold calculated using Eq. (14) at $\sqrt{S_1}=3.87165$, 3.87172, and 3.87177 GeV, respectively. The the dotted vertical lines mark the $\sqrt{s}$ values between which the amplitude of the $X(3872)\to(D^{*0}\bar{D}^0+\bar{D}^{*0}D^0)\to\pi^0\pi^0\chi_{c1}$ decay contains the logarithmic singularities which manifest themselves in the $\pi^0\chi_{c1}$ mass spectrum as narrow peaks. (b) The components of the $\pi^0\chi_{c1}$ mass spectrum at $\sqrt{S_1}=3.87172$ GeV throughout the accessible region of $\sqrt{s}$; description of the curves see in the text.

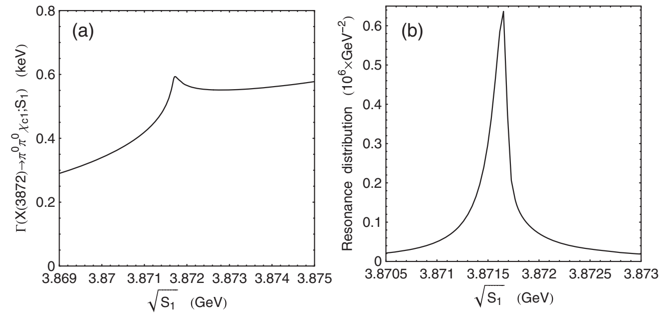

FIG. 5. (a) The solid curve shows the width $\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1)$ calculated using Eq. (15). The constructed example corresponds to $g_X^2/(16\pi)=0.25$ GeV^2. (b) An example of the resonance distribution $2S_1/(\pi|D_X(S_1)|^2)$ for the $X(3872)$ at $g_X^2/(16\pi)=0.25$ GeV^2 and $\Gamma_{\rm non}=1$ MeV [21].

permutation of identical π^0 mesons. It can be seen that the interference is small for all values of $\sqrt{s}$ and can be neglected. The total contribution to the $\pi^0\chi_{c1}$ mass spectrum from the amplitudes $(I(S_1,s)+\tilde{I}(S_1,s))$ and $(I(S_1,t)+\tilde{I}(S_1,t))$ in neglecting their interference is shown in Fig. 4(b) by curve 5. If the peak in the $\pi^0\chi_{c1}$ mass spectrum over the $\sqrt{s}$ variable in the vicinity of $\sqrt{s}\approx 2m_{D^0}\approx 3.72968$ GeV is due to triangle singularities in the amplitude $I(S_1,s)$, then the peak in the region 3.65 GeV $<\sqrt{s}<$ 3.6575 GeV is a manifestation in the distribution over $\sqrt{s}$ of the triangle singularities in the amplitude $I(S_1,t)$.

The width of the $X(3872)\to\pi^0\pi^0\chi_{c1}$ decay in the general case is determined by the expression

$$\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1)=\frac{1}{2}\int\limits_{(m_{\chi_{c1}}+m_{\pi^0})^2}^{(\sqrt{S_1}-m_{\pi^0})^2}ds\int\limits_{t_-(S_1,s)}^{t_+(S_1,s)}\frac{d^2\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1;s,t,u)}{dtds}dt, \tag{15}$$

where the factor 1/2 takes into account the identity of π^0 mesons. In Fig. 5(a), we presented the result of the calculation of $\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1)$ using the coupling constant $g_X^2/(16\pi)=0.25$ GeV^2 as a guide (see Refs. [15,18,21]). The maximum of the width $\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1)$ near the $D^{*0}D^0$ threshold is caused by the presence in the amplitude of the triangle singularities.

To estimate $\mathcal{B}(X(3872)\to\pi^0\pi^0\chi_{c1})$ it is necessary to weigh the energy dependent width $\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1)$ with the resonance distribution $2S_1/(\pi|D_X(S_1)|^2)$:

$$\mathcal{B}(X(3872)\to\pi^0\pi^0\chi_{c1})=\int\limits_{m_{\chi_{c1}}+2m_{\pi^0}}^{\infty}\frac{2\sqrt{S_1}}{\pi}\frac{\sqrt{S_1}\Gamma(X(3872)\to\pi^0\pi^0\chi_{c1};S_1)}{|D_X(S_1)|^2}d\sqrt{S_1}, \tag{16}$$

where $D_X(S_1)$ is the inverse propagator of the $X(3872)$ which we take from Refs. [18,21]. Note that the resonance distribution $2S_1/(\pi|D_X(S_1)|^2)$ has good analytical and unitary properties [18,21]. Figure 5(b) shows an example of this distribution calculated at $m_X=3871.65$ MeV, $g_X^2/(16\pi)=0.25$ GeV^2, and $\Gamma_{\rm non}=1$ MeV, where $\Gamma_{\rm non}$ approximately describes the width of the $X(3872)$ decay into all non-$(D^*\bar{D}+\bar{D}^*D)$ channels. Of course, the main contribution to the integral (16) comes from the narrow region of the resonance peak. The result of integration over the region 3.869 GeV $<\sqrt{S_1}<$ 3.875 GeV for the above parameter values gives $\mathcal{B}(X(3872)\to\pi^0\pi^0\chi_{c1})\approx 1.24\times10^{-4}$. Table I shows the estimates of $\mathcal{B}(X(3872)\to\pi^0\pi^0\chi_{c1})$ for different values of $g_X^2/(16\pi)$ and $\Gamma_{\rm non}$, which

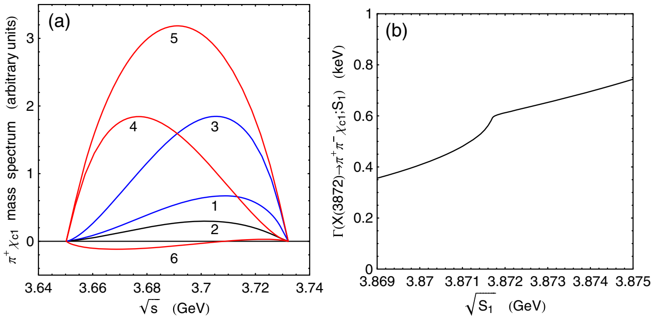

FIG. 6. (a) The components of the $\pi^+\chi_{c1}$ mass spectrum at $\sqrt{S_1} = 3.87172$ GeV throughout the accessible region of $\sqrt{s}$ in the same units as in Fig. 4; the curves here have the same meaning as the curves with the corresponding numbers in Fig. 4(b), which have been described in detail above in the text. (b) The width $\Gamma(X(3872) \to \pi^+\pi^-\chi_{c1}; S_1)$ as a function of $\sqrt{S_1}$. The constructed example corresponds to $g_X^2/(16\pi) = 0.25$ GeV^2.

we vary in a reasonable range taking into account the current (far from final) information about the $X(3872)$ obtained from the analyses of its main decay channels in Refs. [6,8,11,15,18,21].

Let us now consider the diagrams in Fig. 2 describing the decay of $X(3872) \to \pi^+\pi^-\chi_{c1}$. Although there are only four of such diagrams, and not eight as in Fig. 1, the factor of 2 in Eq. (9) is preserved also for the amplitude $\mathcal{M}_c(S_1; s, t, u)$ owing to the isotopic factors in the $D^*D\pi$ and $D\bar{D}\pi\chi_{c1}$ vertices, which are indicated above in the paragraph after Eq. (8). Thus, with taking into account the replacement of the particle masses in the loops and the masses of the final pions, as well as the necessary changes in designations and exclusion of the factor 1/2 from Eq. (15) when determining the width of the $X(3872) \to \pi^+\pi^-\chi_{c1}$ decay, we can use Eqs. (9)–(16) to calculate the $\pi^\pm\chi_{c1}$ mass spectra, $\Gamma(X(3872) \to \pi^+\pi^-\chi_{c1}; S_1)$, and $\mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})$. Figure 6(a) shows (in the same units as in Fig. 4) the main components of the $\pi^+\chi_{c1}$ mass spectrum in the $X(3872) \to \pi^+\pi^-\chi_{c1}$ decay at $\sqrt{S_1} = 3.87172$ GeV throughout the accessible region of $\sqrt{s}$. The curves here have the same meaning as the curves with the corresponding numbers in Fig. 4(b), which have been described in detail above in the text. In this case, there are no triangle singularities in the physical region of the decay and the $\pi^\pm\chi_{c1}$ mass spectra are smooth functions of $\sqrt{s}$. The energy dependent decay width $\Gamma(X(3872) \to \pi^+\pi^-\chi_{c1}; S_1)$ [see the example shown in Fig. 6(b)] has a characteristic break at the threshold of the $D^{*0}\bar{D}^0$ channel.

The estimates for $\mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})$ are given in Table I. We see that the model under discussion predicts the close values for $\mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})$ and $\mathcal{B}(X(3872) \to \pi^0\pi^0\chi_{c1})$ the absolute values of which turn out to be at the level of about 10^{-4}. Their ratio averaged over the variants in Table I, $\mathcal{R} = \mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})/\mathcal{B}(X(3872) \to \pi^0\pi^0\chi_{c1}) \approx 1.1$, indicates a noticeable violation of isotopic symmetry, according to which one would expect $\mathcal{R} = 2$.

TABLE I. $\mathcal{B}(X(3872) \to \pi\pi\chi_{c1})$ in units of 10^{-4}.

$g_X^2/(16\pi)$ (in GeV^2)	0.25	0.5	0.671	0.25	0.5	0.671
Γ_{non}	1 MeV			2 MeV		
$\mathcal{B}(X(3872) \to \pi^0\pi^0\chi_{c1})$	1.24	1.63	1.61	0.77	0.88	0.90
$\mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})$	1.51	1.77	1.73	0.86	0.97	0.99
$\mathcal{R} = \frac{\mathcal{B}(X(3872)\to\pi^+\pi^-\chi_{c1})}{\mathcal{B}(X(3872)\to\pi^0\pi^0\chi_{c1})}$	1.22	1.09	1.07	1.12	1.10	1.10

TABLE II. $\mathcal{B}(X(3872) \to \pi\pi\chi_{c1})$ (in units of 10^{-4}) only for Figs. 1(a) and 2(a).

$g_X^2/(16\pi)$ (in GeV^2)	0.25	0.5	0.671	0.25	0.5	0.671
Γ_{non}	1 MeV			2 MeV		
$\mathcal{B}(X(3872) \to \pi^0\pi^0\chi_{c1})$	0.662	0.741	0.731	0.338	0.393	0.404
$\mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})$	0.526	0.614	0.602	0.296	0.337	0.342
$\mathcal{R} = \frac{\mathcal{B}(X(3872)\to\pi^+\pi^-\chi_{c1})}{\mathcal{B}(X(3872)\to\pi^0\pi^0\chi_{c1})}$	0.795	0.829	0.824	0.876	0.858	0.847

The present calculation assumes that the $X(3872)$ is a pure charmonium, and this is reflected in the equal couplings of $X(3872) \to D^{*0}\bar{D}^0$ and $D^{*+}D^-$. In a molecular interpretation of the $X(3872)$, $X(3872)$ couples differently with $D^{*0}\bar{D}^0$ and $D^{*+}D^-$. For example, in Ref. [16], $X(3872) \to D^{*0}\bar{D}^0$ is considered while $X(3872) \to D^{*+}D^-$ neglected. In this regard, we present in Table II the values of the branching fractions corresponding only to Figs. 1(a) and 2(a). In a sense, this corresponds to the limiting variant of the molecular model when the $X(3872)$ is associated only with the $D^{*0}\bar{D}^0$ + c.c. channel.

V. CONCLUSION

We have obtained the tentative estimates for $\mathcal{B}(X(3872) \to \pi^0\pi^0\chi_{c1})$ and $\mathcal{B}(X(3872) \to \pi^+\pi^-\chi_{c1})$ in the model of the triangle loop diagrams with charmed $D^*\bar{D}D$ and $\bar{D}^*D\bar{D}$ mesons in the loops. The decay rates are predicted at the level of 10^{-4} at the reasonable values of the coupling constants. We would like to draw a special attention to the fact that in this model an important contribution to $\mathcal{B}(X(3872) \to \pi\pi\chi_{c1})$ is given by the ("heavy") charged $D^{*+}D^-$ + c.c. intermediate states, certainly, together with the ("light") neutral $D^{*0}\bar{D}^0$ + c.c. intermediate states. This is obvious from Figs. 4(b) and 6(a).

Within the framework of the considered model, the decay rates $X(3872) \to \pi^0\pi^0\chi_{c1}$ and $X(3872) \to \pi^+\pi^-\chi_{c1}$ are proportional to the same product of coupling constants. The existing uncertainties in these constants, as well as the remaining (so far) uncertainties in such characteristics of the $X(3872)$ resonance as its mass m_X and width

$\Gamma_{\rm non}$ [1,6,8,11,15] allow us only to hope (before the experiment) that the model correctly predicts the order of magnitude of the probabilities for the $X(3872) \to \pi^0\pi^0\chi_{c1}$ and $X(3872) \to \pi^+\pi^-\chi_{c1}$ decays. The ratio $\mathcal{R} = \mathcal{B}(X(3872) \to \pi^+\pi^+\chi_{c1})/\mathcal{B}(X(3872) \to \pi^0\pi^0\chi_{c1}) \approx 1.1$ does not depend on the product of coupling constants included in the vertices of triangle loops and, in general, weakly depends on the parameters of the $X(3872)$ resonance. Its value is a direct consequence of the kinematics of the loops determined by the masses of the internal particles. The isotopic symmetry prediction for $\mathcal{R}$ is noticeably broken. The value obtained for $\mathcal{R}$ is a specific prediction of the considered model, which gives an opportunity to verify it experimentally.

ACKNOWLEDGMENTS

We thank Professor Dr. Zhiqing Liu for his interest in our calculations of the $X(3872) \to \pi^+\pi^-\chi_{c1}$ decay rate. The work was carried out within the framework of the state contract of the Sobolev Institute of Mathematics, Project No. FWNF-2022-0021.

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[43] The reaction amplitude $D\bar{D} \to \pi\chi_{c1}$ can be represented as $\varepsilon^{\mu*}[(q_D + q_{\bar{D}})_\mu a + (q_D - q_{\bar{D}})_\mu b]$, where q_D and $q_{\bar{D}}$ are the four-momenta of the D and $\bar{D}$ mesons, respectively, and a and b are the invariant amplitudes. We write the polarization four-vector of the χ_{c1} as $\varepsilon^\mu = (\frac{(\vec{p}_{\chi_{c1}},\vec{\xi})}{m_{\chi_{c1}}}, \vec{\xi} + \frac{\vec{p}_{\chi_{c1}}(\vec{p}_{\chi_{c1}},\vec{\xi})}{m_{\chi_{c1}}(m_{\chi_{c1}}+p^0_{\chi_{c1}})})$. Let us neglect here the mass difference of D^0 and D^+ mesons, calculate the helicity amplitudes in the reaction center-of-mass system, and expand them by the partial waves. Taking into account Eq. (6) we obtain that near the $D\bar{D}$ threshold $a \sim 1/\mathrm{GeV}^2$ and $b \sim (|\vec{q}||\vec{p}_{\chi_{c1}}|\cos\theta)/\mathrm{GeV}^4$, where θ is the angle between the vectors $\vec{q}$ and $\vec{p}_{\chi_{c1}}$. This implies Eq. (7).

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