Tensor loop reduction via the Baikov representation and an auxiliary vector

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In this paper, we introduce a simple and efficient approach for the general reduction of one-loop integrals. Our method employs the introduction of an auxiliary vector and the identification of the tensor structure as an auxiliary propagator. This key insight allows us to express a wide range of one-loop integrals, encompassing both tensor structures and higher poles, in the Baikov representation. By establishing an integral-by-parts relation, we derive a recursive formula that systematically solves the one-loop reduction problem, even in the presence of various degenerate cases. Our proposed strategy is characterized by its simplicity and effectiveness, offering a significant advancement in the field of one-loop calculations.

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I. INTRODUCTION

Accurate calculations of higher-loop corrections play a crucial role in achieving precision physics at the Large Hadron Collider and future colliders. These calculations are essential for accurately predicting particle interactions and interpreting experimental data. However, the evaluation of loop integrals becomes increasingly challenging as the loop order and complexity of the integrals rise. In particular, the presence of intricate tensor structures and propagators raised to high powers introduces significant computational difficulties.

In recent years, remarkable progress has been made in both the computation and the understanding of the analytic structures of scattering amplitudes. Various powerful techniques have emerged to address the reduction of loop integrals at both the integrand and integral levels. Integration-by-parts (IBP) relations have proven to be highly effective in simplifying loop integrals by relating them to simpler master integrals [1–7]. Passarino-Veltman reduction [8] and Ossola-Papadopoulos-Pittau reduction [9–11] provide alternative approaches to simplify loop integrals, while unitarity-based methods [12–22] exploit the cutting equations to derive compact forms of loop integrals. Intersection number techniques [23–29] have also emerged as powerful tools for the reduction of loop integrals.

Several recent studies [30–33] have investigated these general one-loop integrals by contracting the loop momentum ℓ with an auxiliary vector R. Let us consider the one-loop r-rank tensor integral with n propagators

$$I_n^{\mu_1,\ldots,\mu_r} \equiv \int \frac{d^d\ell}{i(\pi)^{d/2}} \frac{\ell^{\mu_1}\ldots\ell^{\mu_r}}{\prod_{i=1}^n D_i}, \tag{1.1}$$

where the ith inverse propagator is $D_i = (\ell - q_i)^2 - m_i^2$, with $q_j = \sum_{i<j} p_i$ and $q_1 = 0$. By introducing an auxiliary vector R^μ,

$$I_n^{(r)} \equiv 2^r I_n^{\mu_1\cdots\mu_r} R_{\mu_1}\cdots R_{\mu_r} = \int \frac{d^d\ell}{i(\pi)^{d/2}} \frac{(2\ell\cdot R)^r}{\prod_{i=1}^n D_i}. \tag{1.2}$$

One can recover Eq. (1.1) by applying differential operators of R to the above expression,

$$I_n^{\mu_1,\ldots,\mu_r} = \frac{1}{r!2^r}\frac{d}{dR^{\mu_1}}\cdots\frac{d}{dR^{\mu_r}} I_n^{(r)}. \tag{1.3}$$

The introduction of R facilitates a more concise expression and enhances the efficiency of reduction in previous studies, such as [32,34–38] for one-loop integrals and [31,39] for higher loops. This technique can be combined with other methods, including differential operators [30–32], the syzygy equation in Baikov representation [37,40–51], and IBP in projective space [33]. In this paper, we introduce a simple approach that combines differential operators with respect to R and the IBP relation in Baikov representation. Baikov representation provides a systematic way to express loop integrals in terms of a set of master integrals. After introducing an auxiliary vector and recognizing the tensor structure as a new propagator with negative power, we can establish a simple recursive relation for the reduction of

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general one-loop integrals. This approach proves to be particularly advantageous in handling degenerate cases, where other methods face challenges.

One can certainly avoid introducing an auxiliary vector since the tensor structures of the n-point one-loop tensor integrals (1.1) are explicitly known from [52]. What remains are the single scalar integrals with shifted values of the propagator powers and the space-time dimension, which can be addressed using the recurrence relations outlined in [53]. The introduction of R serves to contract the tensor structure and ensure conciseness in the expression. However, it should be noted that this does not imply superiority over other approaches. The primary objective of this paper is to investigate an alternative method to avoid dimension shifting. We demonstrate the effectiveness and simplicity of our approach through several illustrative examples, including the reduction of tadpole, bubble, triangle, and pentagon integrals.

The rest of this paper is organized as follows: In Sec. II, we provide a review of two methods for tensor reduction: differential operators and syzygy equations. In Sec. III, we provide a detailed outline of our method and derive the general result for the reduction of one-loop integrals. Section IV presents a comprehensive set of examples, showcasing the application of our approach to various types of integrals. Finally, in Sec. V, we summarize our findings, discuss the implications of our method, and provide an outlook for future research in this area. The paper ends with an Appendix. In the Appendix, we present the results for the pentagon integral and compare the computation times of FIRE6 and our method.

II. REVIEW OF TWO METHODS

In this paper, we mainly consider the one-loop r-rank tensor integral with n propagators

$$I^{(r)}_{\boldsymbol{a}_n} \equiv \int \frac{d^d\ell}{i(\pi)^{d/2}} \frac{(2\ell\cdot R)^r}{\prod_{i=1}^n D_i^{a_j}}. \tag{2.1}$$

In this expression, $\boldsymbol{a}_n = \{a_1, a_2, \ldots, a_n\}$ represents the power list of the n propagators. The introduction of the auxiliary vector not only simplifies the reduction process but also helps us address the higher-pole case. One can see any general tensor structure can be recovered by applying differential operators of R on the standard expression. For example,

$$\int \frac{d^d\ell}{i(\pi)^{d/2}} \frac{\ell^2 \ell\cdot K}{\prod_{i=1}^n ((\ell-q_i)^2 - m_i^2)^{a_i}} \propto (K\cdot\partial_R)(\partial_R\cdot\partial_R) I^{(3)}_{\boldsymbol{a}_n}. \tag{2.2}$$

The more general case of tensor reductions for higher poles can be addressed by employing differential operators of m_i^2,

$$\int \frac{d^d\ell}{i(\pi)^{d/2}} \frac{(2\ell\cdot R)^r}{\prod_{i=1}^n D_i^{a_i}} \propto \left(\prod_i (\partial_{m_i^2})^{a_i-1}\right) \int \frac{d^d\ell}{i(\pi)^{d/2}} \frac{(2\ell\cdot R)^r}{\prod_{i=1}^n D_i}. \tag{2.3}$$

One can notice that any differential operator of mass can lift the power of the associated propagator by 1. Given the reduction results for the scalar integral class $I_{\{a_i=2\}}$, where $\{a_i = 2\}$ indicates all propagators' power $a_j = 1$ except $a_i = 2$, one can solve the general problem of reducing tensor integrals with higher poles. Therefore, for the sake of simplicity, we will focus solely on the integrals with simple poles and scalar integrals with single quadratic propagators, i.e., $I_n^{(r)} \equiv I^{(r)}_{\{1,1,\ldots,1\}}$ and $I_{\{a_i=2\}}$. Specifically, the standard scalar integrals $I_n \equiv I_n^{(0)}$.

A. Reduction by differential operators

In this subsection, we provide a review of utilizing differential operators for tensor reduction. In the original works [34,35], the authors note that there are two types of differential operators which can lower the rank r:

$$\mathcal{D}_i \equiv q_i\cdot\frac{\partial}{\partial R}, \quad i = 2,\ldots,n, \quad \mathcal{T} \equiv \eta^{\mu\nu}\frac{\partial}{\partial R^\mu}\frac{\partial}{\partial R^\nu}. \tag{2.4}$$

It is straightforward to determine the actions of these operators,

$$\begin{aligned} \mathcal{D}_i I_n^{(r)} &= r I^{(r-1)}_{n;\hat{1}} - r I^{(r-1)}_{n;\hat{i}} + r(m_1^2 + q_i^2 - m_i^2) I_n^{(r-1)}, \\ \mathcal{T} I_n^{(r)} &= 4r(r-1)m_1^2 I_n^{(r-2)} + 4r(r-1) I^{(r-2)}_{n;\hat{1}}, \end{aligned} \tag{2.5}$$

where $\hat{i}$ indicates that the ith propagator has been removed. We know that $I_n^{(r)}$ can be reduced to master integrals,

$$I_n^{(r)} = \sum_{\mathbf{b}_j} C^{(r)}_{n;\widehat{\mathbf{b}_j}} I_{n;\widehat{\mathbf{b}_j}}, \tag{2.6}$$

where $\mathbf{b}_j$ is the subset of $\{1,2,\ldots,n\}$. In a straightforward approach, given the results for the integrals $I^{(r')}_{m<n}$, the expressions for $C^{(r)}_{n\to n;\hat{\mathbf{b}}_j}$ can be determined by solving the n partial differential equations in (2.5), with the property

$$\begin{aligned} \mathcal{D}_i &= 2s_{0i}\frac{\partial}{\partial s_{00}} + \sum_{j=1}^{n-1} s_{ij}\frac{\partial}{\partial s_{0j}}, \\ \mathcal{T} &= 2D\frac{\partial}{\partial s_{00}} + 4s_{00}\frac{\partial^2}{\partial s_{00}^2} + 4\sum_{i=1}^{n-1} s_{0i}\frac{\partial}{\partial s_{0i}}\frac{\partial}{\partial s_{00}} \\ &\quad + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}\frac{\partial}{\partial s_{0i}}\frac{\partial}{\partial s_{0j}}, \end{aligned} \tag{2.7}$$

where

$$s_{00} \equiv R \cdot R, \quad s_{0i} \equiv R \cdot q_{i+1}, \quad s_{ij} \equiv q_{i+1} \cdot q_{j+1}, \quad \forall \; i, \; j = 1, \ldots, n. \tag{2.8}$$

However, a more efficient approach exists. The key idea utilized in [34] is to expand the reduction coefficients based on their tensor structure

$$C^{(r)}_{n;\widehat{\mathbf{b}}_j} = \sum_{2a_0 + \sum_{k=1}^{n-1} a_k = r} \left\{ c^{(a_0,\ldots,a_{n-1})}_{n;\widehat{\mathbf{b}}_j}(r) \prod_{k=0}^{n-1} s_{0k}^{a_k} \right\}. \tag{2.9}$$

By substituting this expansion into the n partial differential equations, one can derive n recursion relations for the *expansion coefficients* $c^{(a_0,\ldots,a_{n-1})}_{n;\hat{\mathbf{b}}j}(r)$. Moreover, these recursion relations can be solved through an iterative approach. Finally, by collecting all expansion coefficients, one can compose the desired reduction coefficients. Regarding degenerate cases, these can be addressed through singularity analysis, as shown in [32].

B. Reduction with syzygy in Baikov representation

For the one-loop integrals in Baikov representation [54–59], we denote the inverse propagators and integral as

$$z_0 = (2\ell \cdot R), \quad z_i = (\ell - q_i)^2 - m_i^2, \quad i = \{1, 2, \ldots, n\},$$
$$I^{(r)}_{\mathbf{a}_n} \equiv I^{(r)}_{\{a_1, a_2, \ldots, a_n\}} \equiv \frac{\mathcal{K}^{-(d-n-1)/2}}{(4\pi)^{n/2}\Gamma((d-n)/2)} \times \int_{\mathcal{C}} d^{n+1}z \frac{\mathcal{G}(\{z\})^{(d-n-2)/2} z_0^r}{\prod_{i=1}^n z_i^{a_i}}, \tag{2.10}$$

where the prefactors ($\mathcal{K}$ involving the external momenta) do not depend on $\{z\}$ and can be ignored for our subsequent discussions. The $\mathcal{G}(\{z\})$ is another Gram determinant, which depends on both the loop momentum and the external momenta,

$$\mathcal{G}(\{z\}) = \det G(\ell, q_2, \ldots, q_n, R). \tag{2.11}$$

Here, the Gram matrix G is defined as

$$G(k_1, \ldots, k_n) \equiv (k_i \cdot k_j)_{n\times n} \equiv \begin{pmatrix} k_1 \cdot k_1 & k_1 \cdot k_2 & \cdots & k_1 \cdot k_n \\ k_2 \cdot k_1 & k_2 \cdot k_2 & \cdots & k_2 \cdot k_n \\ \vdots & & \ddots & \vdots \\ k_n \cdot k_1 & k_n \cdot k_2 & \cdots & k_n \cdot k_n \end{pmatrix}. \tag{2.12}$$

Introducing the syzygy module [37] and considering the IBP relation

$$\begin{aligned} 0 &= \int d^{n+1}z \sum_{a=0}^{n} \left[\partial_{z_a} \left(P_a \frac{z_0^r \mathcal{G}\{z\}^{(d-n-2)/2}}{\prod_{i=1}^n z_i^{b_i}} \right) \right] \\ &= \int d^{n+1}z \sum_{a=0}^{n} \left[\frac{d-n-2}{2} P_a \partial_{z_a}\mathcal{G} + (\partial_{z_a} P_a + r z_0^{-1} P_0 - b_i z_i^{-1} P_i)\mathcal{G} \right] \frac{z_0^r \mathcal{G}\{z\}^{((d-2)-n-2)/2}}{\prod_{i=1}^n z_i^{b_i}}, \end{aligned} \tag{2.13}$$

where P_a are polynominals of z_a. It should be noted that the power of $\mathcal{G}$ is dependent on the dimension d; therefore, to avoid dimension shifting in the reduction coefficients, it is advantageous to select P_a judiciously to satisfy the syzygy equation.

$$\sum_{a=0}^{n} (P_a \partial_{z_a} \mathcal{G}) + P_{n+1}\mathcal{G} = 0. \tag{2.14}$$

Another consideration is that the term $b_i z_i^{-1} P_i$ may increase the power of the ith propagator, which is undesirable. To preclude this, we require P_i to be divisible by z_i. With these constraints in place, one can identify solutions to the syzygy equation by searching for syzygy modules $\langle \partial_{z_0}\mathcal{G}, \ldots, \partial_{z_n}\mathcal{G}, \mathcal{G} \rangle$ subject to the requirement $\langle P \rangle \equiv \langle P_0, P_1, \ldots, P_n \rangle = \langle d_0, d_1, \ldots, d_n \rangle$, where

$$\begin{aligned} d_1 &= \{z_1, 0, \ldots, 0, 0, 0\} \\ &\vdots \\ d_n &= \{0, 0, \ldots, z_n, 0, 0\} \\ d_0 &= \{0, 0, \ldots, 0, 1, 0\} \\ d_{n+1} &= \{0, 0, \ldots, 0, 0, 1\} \end{aligned} \tag{2.15}$$

III. COMBINED METHOD

Differential operators and syzygy equations both require extensive algebraic computation. While solving the syzygy equation is typically straightforward at the one-loop level, our objective is to explore a reduction method that does not rely on this approach. Instead, we aim to develop a method that can be applied more generally, starting with the one-loop case as a preliminary step. We introduce a simple approach that combines differential operators with respect to R and the IBP relation in Baikov representation. First, we examine the elements of the Gram matrix given in (2.12). The Gram matrix contains elements that represent the Lorentz invariant products of loop momenta, external momenta, and auxiliary vectors. Specifically, the matrix elements are constructed from scalar products of the form $\ell \cdot \ell$, $\ell \cdot q_i$, $\ell \cdot R$, $q_i \cdot q_j$, $q_i \cdot R$, and $R \cdot R$, where

$$\ell \cdot \ell = m_1^2 + z_1, \qquad \ell \cdot R = z_0/2,$$
$$\ell \cdot q_i = \frac{1}{2}(m_1^2 - m_i^2 + z_1 - z_i - q_i^2). \tag{3.1}$$

It is evident that the variables $\{z\}$ only appear in the first column and row of matrix G, indicating that $\mathcal{G}$ must be a quadratic expression of $\{z\}$. To simplify the expressions, we introduce the notation

$$\mathcal{G} = \mathcal{G}_{00} z_0^2 + \sum_{i=1}^{n} \mathcal{G}_{0i} z_0 z_i + \mathcal{G}_0 z_0 + \sum_{i \le j}^{n} \mathcal{G}_{ij} z_i z_j + \sum_{i=1}^{n} \mathcal{G}_i z_i + \mathcal{G}_c. \tag{3.2}$$

Since only $\mathcal{G}_{00}$, $\mathcal{G}_0$, and $\mathcal{G}_c$ are associated with singularities, we will focus exclusively on presenting their specific results,

$$\mathcal{G}_{00} = -\frac{1}{4}\det\left[G(\{z\})_{\cancel{1(n+1)};\cancel{1(n+1)}}\right],$$
$$\mathcal{G}_0 = -\det\left[G(\{z\}=0)_{\cancel{(n+1)};\cancel{1}}\right], \quad \mathcal{G}_c = \det G(\{z\}=0), \tag{3.3}$$

where $\cancel{i}\cancel{j};\cancel{l}\cancel{k}$ indicates removing the ith and jth rows as well as the lth and kth columns. In addition, the $\{z\}=0$ is to set $z_a = 0,\ \forall\ a = 0, 1, \ldots, n$. We consider the following integral relation derived from the IBP:

$$0 = \int_{\mathcal{C}} d^{n+1} z \partial_{z_0}\left[\mathcal{G}(\{z\})^{(d-n-2)/2}\mathcal{G}(\{z\})\frac{z_0^{r-1}}{\prod_{i=1}^{n} z_i}\right]$$
$$= \int_{\mathcal{C}} d^{n+1} z\left[\left(\frac{d-n}{2} z_0 \partial_{z_0}\mathcal{G} + (r-1)\mathcal{G}\right) \times \mathcal{G}(\{z\})^{(d-n-2)/2}\frac{z_0^{r-2}}{\prod_{i=1}^{n} z_i}\right]. \tag{3.4}$$

To prevent the power of the function $\mathcal{G}$ from changing when differentiating, we append an additional factor of $\mathcal{G}$ [60]. Since z_0 is independent of the propagators, taking the derivative ∂_{z_0} does not increase the power of the propagators. We substitute (3.2) into it and omit the summation symbol. Recall that any z_i represents a certain propagator and z_0 represents the tensor structure; it is then direct to recognize the equation as a reduction relation at the integral level as below,

$$(r-1+d-n)\mathcal{G}_{00} I_n^{(r)} + \left(r-1+\frac{d-n}{2}\right) \times (\mathcal{G}_0 I_n^{(r-1)} + \mathcal{G}_{0i} I_{n;\hat{i}}^{(r-1)}) + (r-1)(\mathcal{G}_c I_n^{(r-2)} + \mathcal{G}_i I_{n;\hat{i}}^{(r-2)} + \mathcal{G}_{ij} I_{n;\widehat{ij}}^{(r-2)}) = 0. \tag{3.5}$$

For the case $i \neq j$, the integral $I_{n;\widehat{ij}}^{(r)}$ simply represents the subsector with two different propagators removed. However, when $i = j$, special care must be taken to keep z_i in the numerator of $I_{n;\widehat{ij}}^{(r-2)}$. The terms, except for the first one, have lower rank than the initial $I_n^{(r)}$. Hence, with the seed scalar integrals, we can use this relation to construct r-rank tensor integrals. To enable the recursion relation to progress smoothly, it is critical to properly handle this $i = j$ case. One approach is to transfer the propagator z_i to the differential operator directly using the fact that $l^2 \propto \partial_R \cdot \partial_R$, $l \cdot q_i \propto q_i \cdot \partial_R$. Since the reduction result of the subsector is assumed to be known, one can easily apply it to any differential operators. Alternatively, we can write z_i as

$$(z_i - z_{i+1}) + z_{i+1} = 2\ell \cdot (q_{i+1} - q_i) + f_{i,i+1} + z_{i+1}, \tag{3.6}$$

with $n+1 \equiv 1$, $f_{ij} \equiv m_j^2 - m_i^2 + q_i^2 - q_j^2$. There is only one linear term in ℓ, so only one differential operator ∂_R is needed. The z_{i+1} term that appears can be canceled by the denominator. Applying this logic leads to the simplified result

$$\begin{aligned}
I_{n;\widehat{ii}}^{(r-2)} &= \int \frac{d^d\ell (2\ell \cdot R)^{(r-2)}}{i(\pi)^{d/2}} \frac{z_i}{\prod_{j=1, j\neq i}^{n} z_j} \\
&= \int \frac{d^d\ell (2\ell \cdot R)^{(r-2)}}{i(\pi)^{d/2}} \frac{2\ell \cdot (q_{i+1} - q_i) + f_{i,i+1} + z_{i+1}}{\prod_{j=1, j\neq i}^{n} z_j} \\
&= \frac{(q_{i+1} - q_i) \cdot \partial_R}{r-1} \int \frac{d^d\ell (2\ell \cdot R)^{(r-1)}}{i(\pi)^{d/2}} \frac{1}{\prod_{j=1, j\neq i}^{n} z_j} \\
&\quad + \int \frac{d^d\ell (2\ell \cdot R)^{(r-2)}}{i(\pi)^{d/2}} \frac{f_{i,i+1}}{\prod_{j=1, j\neq i}^{n} z_j} \\
&\quad + \int \frac{d^d\ell (2\ell \cdot R)^{(r-2)}}{i(\pi)^{d/2}} \frac{z_{i+1}}{\prod_{j=1, j\neq i}^{n} z_j} \\
&= \frac{(q_{i+1} - q_i) \cdot \partial_R}{r-1} I_{n;\hat{i}}^{(r-1)} + f_{i,i+1} I_{n;\hat{i}}^{(r-2)} + I_{n;\widehat{i,i+1}}^{(r-2)}.
\end{aligned} \tag{3.7}$$

Plugging the expression into (3.5), we obtain

$$\mathsf{A}_{n,r} I_n^{(r)} + \mathsf{B}_{n,r} I_n^{(r-1)} + \mathsf{C}_{n,r} I_n^{(r-2)} + \mathsf{B}_{n,r;\hat{i}} I_{n;\hat{i}}^{(r-1)} + \mathsf{C}_{n,r;\hat{i}} I_{n;\hat{i}}^{(r-2)} + \mathsf{C}_{n,r;\widehat{ij}} I_{n;\widehat{ij}}^{(r-2)} = 0, \tag{3.8}$$

where the coefficients are

$$\mathsf{A}_{n,r} = (r-1+d-n)\mathcal{G}_{00}, \tag{3.9}$$

$$\mathsf{B}_{n,r} = \left(r-1+\frac{d-n}{2}\right)\mathcal{G}_0,$$
$$\mathsf{B}_{n,r;\hat{i}} = \left(r-1+\frac{d-n}{2}\right)\mathcal{G}_{0i} + \mathcal{G}_{ii}(q_{i+1}-q_i)\cdot\partial_R, \tag{3.10}$$

$$\mathsf{C}_{n,r} = (r-1)\mathcal{G}_c, \qquad \mathsf{C}_{n,r;\hat{i}} = (r-1)(\mathcal{G}_i + f_{i,i+1}\mathcal{G}_{ij}),$$
$$\mathsf{C}_{n,r;\widehat{ij}} = (r-1)(\mathcal{G}_{ij} + \delta_{j,i+1}\mathcal{G}_{ii}). \tag{3.11}$$

In fact, all degenerate cases are captured by $\mathcal{G}_{00}$, $\mathcal{G}_0$, and $\mathcal{G}_c$, as we discuss below. For the nondegenerate case $\mathcal{G}_{00} \neq 0$, dividing (3.8) by the prefactor $\mathsf{A}_{n,r}$ and introducing simplified notations, we can derive the recursion relation that governs $I_n^{(r)}$,

$$I_n^{(r)} = \frac{-1}{\mathsf{A}_{n,r}}\left(\mathsf{B}_{n,r}^{-}I_n^{(r-1)} + \mathsf{C}_{n,r}I_n^{(r-2)} + \mathsf{B}_{n,r;\hat{i}}I_{n;\hat{i}}^{(r-1)} + \mathsf{C}_{n,r;\hat{i}}I_{n;\hat{i}}^{(r-2)} + \sum_{i<j}\mathsf{C}_{n,r;\widehat{ij}}I_{n;\widehat{ij}}^{(r-2)}\right). \tag{3.12}$$

As we can see, when $\mathcal{G}_{00} = 0$, i.e., $\mathsf{A}_{n,r}$, Eq. (3.12) is no longer applicable. We can divide both sides of (3.8) by $\mathsf{B}_{n,r}$ and shift $r-1$ to r, resulting in the following expression:

$$I_n^{(r)} = \frac{-1}{\mathsf{B}_{n,r+1}}\left(\mathsf{C}_{n,r+1}I_n^{(r-1)} + \mathsf{B}_{n,r+1;\hat{i}}I_{n;\hat{i}}^{(r)} + \mathsf{C}_{n,r+1;\hat{i}}I_{n;\hat{i}}^{(r-1)} + \sum_{i<j}\mathsf{C}_{n,r+1;\widehat{ij}}I_{n;\widehat{ij}}^{(r-1)}\right). \tag{3.13}$$

Setting $r=0$ leads all C_n to vanish, as evident from (3.11). This implies that I_n is no longer a master integral.

When the conditions $\mathcal{G}_{00} = 0$ and $\mathcal{G}_0 = 0$ are satisfied, meaning that $\mathsf{A}_{n,r} = 0$ and $\mathsf{B}_{n,r} = 0$, the previous recursion relation is no longer valid. To address this breakdown, we can divide both sides of (3.8) by the prefactor $\mathsf{C}_{n,r}$ and perform a shift by replacing $r-2$ with r. This gives the result

$$I_n^{(r)} = \frac{-1}{\mathsf{C}_{n,r+2}}\left(\mathsf{B}_{n,r+2;\hat{i}}I_{n;\hat{i}}^{(r+1)} + \mathsf{C}_{n,r+2;\hat{i}}I_{n;\hat{i}}^{(r)} + \sum_{i<j}\mathsf{C}_{n,r+2;\widehat{ij}}I_{n;\widehat{ij}}^{(r)}\right). \tag{3.14}$$

Then we turn to discuss the reduction for the higher-pole case $I_{\{a_i=2\}}$. To lift the power of propagators, one can consider using ∂_{z_i}; then we translate z_i's appearing in the numerator to the differential operators of R acting on the standard tensor integrals $I_n^{(r)}$ that we have obtained. A little difference here is that there is no need to introduce z_0 anymore. The scalar one-loop integral in Baikov representation without tensor structure is

$$I_{\{1,\ldots,1\}} \equiv C_n(d)\mathcal{K}^{-(d-n)/2}\int_{\mathcal{C}} d^n z \mathcal{G}^{\text{scalar}}(\{z\})^{(d-n-1)/2}\frac{1}{\prod_{i=1}^n z_i}, \tag{3.15}$$

where the Gram determinant is $\mathcal{G}^{\text{scalar}}(\{z\}) = \det G(\ell, p_1, \ldots, p_E)$. Analogous to (3.4),

$$\begin{aligned} 0 &= \int_{\mathcal{C}} d^n z \partial_{z_i}\left[\mathcal{G}^{\text{scalar}}(\{z\})^{(d-n-1)/2}\mathcal{G}^{\text{scalar}}(\{z\})\frac{1}{\prod_{i=1}^n z_i}\right] \\ &= \int_{\mathcal{C}} d^n z\left[\left(\frac{d-n+1}{2}\partial_{z_i}\mathcal{G}^{\text{scalar}} - \frac{\mathcal{G}^{\text{scalar}}}{z_i}\right)\mathcal{G}^{\text{scalar}}(\{z\})^{(d-n-1)/2}\frac{1}{\prod_{i=1}^n z_i}\right]. \end{aligned} \tag{3.16}$$

Based on the definition of $G(\ell, q_2, \ldots, q_n)$, we know that $\mathcal{G}^{\text{scalar}}(\{z\})$ is a quadratic expression of $\{z\}$,

$$\mathcal{G}^{\text{scalar}} = \sum_{j\leq k}^{n}\mathcal{G}_{jk}^{\text{scalar}}z_j z_k + \sum_j \mathcal{G}_j^{\text{scalar}}z_j + \mathcal{G}_c^{\text{scalar}}. \tag{3.17}$$

Plugging this into (3.16),

$$\mathsf{H}_n^{i+}I_{\{a_i=2\}} + \mathsf{H}_n I_n + \mathsf{H}_{n;\hat{i}}I_{\{a_i=0\}} + \sum_{j\neq i}(\mathsf{H}_{n;\hat{j}}I_{\{a_j=0\}} + \mathsf{H}_{n;\widehat{jj}}^{i+}I_{\{a_i=2,a_j=-1\}} + \mathsf{H}_{n;\hat{j}}^{i+}I_{\{a_i=2,a_j=0\}}) = 0. \tag{3.18}$$

Paralleling the approach in (3.7),

$$I_{\{a_i=2,a_j=-1\}} = f_{ji} I_{\{a_i=2,a_j=0\}} + I_{\{a_j=0\}} + (q_i - q_j)\cdot \partial_R I^{(1)}_{\{a_i=2,a_j=0\}}. \tag{3.19}$$

Plugging this into (3.18),

$$\begin{aligned} &\mathsf{H}_n^{i+} I_{\{a_i=2\}} + \mathsf{H}_n I_n + \mathsf{H}_{n;\hat{i}} I_{\{a_i=0\}} + \sum_{j\neq i} [(\mathsf{H}_{n;\hat{j}} + \mathsf{H}^{i+}_{n;\widehat{jj}}) I_{\{a_j=0\}} \\ &+ (f_{ji}\mathsf{H}^{i+}_{n;\widehat{jj}} + \mathsf{H}^{i+}_{n;\hat{j}}) I_{\{a_i=2,a_j=0\}} + \mathsf{H}^{i+}_{n;\widehat{jj}} (q_i - q_j)\cdot \partial_R I^{(1)}_{\{a_i=2,a_j=0\}}] = 0, \end{aligned} \tag{3.20}$$

where the coefficients are

$$\begin{aligned} &\mathsf{H}_n^{i+} = -\mathcal{G}_c^{\mathrm{scalar}}, \qquad \mathsf{H}_n = \frac{d-n-1}{2}\mathcal{G}_i^{\mathrm{scalar}}, \qquad \mathsf{H}_{n;\hat{i}} = (d-n)\mathcal{G}_{ii}^{\mathrm{scalar}}, \\ &\mathsf{H}^{i+}_{n;\hat{j}} = -\mathcal{G}_j^{\mathrm{scalar}}, \qquad \mathsf{H}_{n;\hat{j}} = \frac{d-n-1}{2}\mathcal{G}_{ij}^{\mathrm{scalar}}, \qquad \mathsf{H}^{i+}_{n;\widehat{jj}} = -\mathcal{G}_{jj}^{\mathrm{scalar}}. \end{aligned} \tag{3.21}$$

The recursion relation displayed in (3.20) demonstrates that higher-pole integrals $I_{\{a_i=2\}}$ can be expressed in terms of the master integral $I_n, I_{\{a_i=0\}}$, and lower-sector integrals $I_{\{a_i=2,a_j=0\}}$, whose values are known from previous recursion relations.

IV. EXAMPLES

The upcoming tadpoles and bubbles subsections will present step-by-step calculations exemplifying our approach. By walking through specific examples, we aim to demonstrate the utilization of the method in practice for computing integral families of interest.

A. Tadpoles

In this subsection, we examine the reduction of tensor tadpoles while considering the simplest case. To begin, let us explicitly define the propagators involved,

$$z_1 = \ell^2 - m_1^2, \qquad z_0 = 2\ell\cdot R. \tag{4.1}$$

We can express the polynomial $\mathcal{G}$ in the Baikov representation as follows:

$$\mathcal{G} = \det\begin{pmatrix} m_1^2 + z_1 & z_0/2 \\ z_0/2 & s_{00} \end{pmatrix} = -\frac{z_0^2}{4} + s_{00} z_1 + s_{00} m_1^2. \tag{4.2}$$

Using this expression, we can compute the derivative of $\mathcal{G}$ with respect to z_0 as

$$\frac{\partial \mathcal{G}}{\partial z_0} = -\frac{z_0}{2}. \tag{4.3}$$

Consequently, we can rewrite the integrand of (3.4) as

$$\left(\mathsf{A}_{1,r}\frac{z_0^r}{z_1} + \mathsf{C}_{1,r}\frac{z_0^{r-2}}{z_1} + \mathsf{C}_{1,r;\hat{1}} z_0^{r-2}\right)\mathcal{G}^{(d-3)/2} = 0. \tag{4.4}$$

In the above equation, we have the coefficients

$$\mathsf{A}_{1,r} = -\frac{d+r-2}{4}, \qquad \mathsf{C}_{1,r} = (r-1)m_1^2 s_{00}. \tag{4.5}$$

The last term in the equation is zero, as there are no propagators in it. As a result, we obtain the following relation:

$$I_1^{(r)} = \frac{4(r-1)m_1^2 s_{00}}{d+r-2} I_1^{(r-2)}. \tag{4.6}$$

Consequently, the final result for $I_1^{(r)}$ can be summarized as

$$I_1^{(r)} = \begin{cases} 0 & r = \mathrm{odd}, \\ \frac{2^r (r-1)!! m_1^r R^r}{(d+r-2)!!/(d-2)!!} I_1 & r = \mathrm{even}. \end{cases} \tag{4.7}$$

The results for nonstandard tadpole integrals can be obtained through momentum shifting

$$\begin{aligned} &\int d^d\ell \frac{(2\ell\cdot R)^r}{(\ell-K)^2 - m^2} \\ &\overset{\ell\to\ell+K}{=\!=\!=} \int d^d\ell \frac{((2\ell+2K)\cdot R)^r}{\ell^2 - m^2} \\ &= \sum_{a=0}^{r}\binom{r}{a}(2R\cdot K)^a \int d^d\ell \frac{(2\ell\cdot R)^{r-a}}{\ell^2 - m^2}. \end{aligned} \tag{4.8}$$

To elucidate the methodology, we consider $r = 1$ as an example,

$$\int d^d\ell \frac{(2\ell\cdot R)}{(\ell-K)^2 - m^2} = I_1^{(1)} + (2R\cdot K)I_1 = (2R\cdot K)I_1. \tag{4.9}$$

B. Bubbles

Bubbles are the simplest case involving singularities. The explicit propagators are given by

$$z_1 = \ell^2 - m_1^2, \qquad z_2 = (\ell - q_2)^2 - m_2^2, \qquad z_0 = 2\ell \cdot R. \tag{4.10}$$

The polynomial $\mathcal{G}$ in the Baikov representation is defined as

$$\mathcal{G} = \det\begin{pmatrix} m_1^2 + z_1 & -(m_1^2 - m_2^2 + s_{11} + z_1 - z_2)/2 & z_0/2 \\ -(m_1^2 - m_2^2 + s_{11} + z_1 - z_2)/2 & s_{11} & s_{01} \\ z_0/2 & s_{01} & s_{00} \end{pmatrix}. \tag{4.11}$$

The derivative of $\mathcal{G}$ with respect to z_0 is given by

$$\partial_{z_0}\mathcal{G} = -s_{01}(m_1^2 - m_2^2 + s_{11} + z_1 - z_2) - 2s_{11}z_0. \tag{4.12}$$

Substituting this expression into (3.4), we obtain

$$\mathsf{A}_{2,r}I_n^{(r)} + \mathsf{B}_{2,r}I_2^{(r-1)} + \mathsf{C}_{2,r}I_2^{(r-2)} + \mathsf{B}_{2;\hat{i}}^{(r-1)}I_{2;\hat{i}}^{(r-1)} + \mathsf{C}_{2,r;\hat{i}}I_{2;\hat{i}}^{(r-2)} + \mathsf{C}_{2,r;\widehat{ij}}I_{2;\widehat{ij}}^{(r-2)} = 0. \tag{4.13}$$

To derive expressions for $I_{2;\widehat{11}}^{(r-2)}$ and $I_{2;\widehat{22}}^{(r-2)}$, we utilize (3.6) to write z_1 and z_2 in the following form:

$$z_1 = m_2^2 - m_1^2 - s_{11} + 2\ell \cdot q_2 + z_2 = z_2 + m_2^2 - m_1^2 - s_{11} + q_2 \cdot \partial_R z_0, \tag{4.14}$$

$$z_2 = m_1^2 - m_2^2 + s_{11} - 2\ell \cdot q_2 + z_1 = z_1 + m_1^2 - m_2^2 + s_{11} - q_2 \cdot \partial_R z_0. \tag{4.15}$$

Substituting these equations in, we can express the terms $I_{2;\widehat{11}}^{(r-2)}$ and $I_{2;\widehat{22}}^{(r-2)}$ as follows:

$$I_{2;\widehat{11}}^{(r-2)} = (m_2^2 - m_1^2 - s_{11})I_{2;\hat{1}}^{(r-2)} + \frac{q_2 \cdot \partial_R}{(r-1)}I_{2;\hat{1}}^{(r-1)} + I_{2;\widehat{12}}^{(r-2)}, \tag{4.16}$$

$$I_{2;\widehat{22}}^{(r-2)} = -(m_2^2 - m_1^2 - s_{11})I_{2;\hat{2}}^{(r-2)} - \frac{q_2 \cdot \partial_R}{(r-1)}I_{2;\hat{2}}^{(r-1)} + I_{2;\widehat{12}}^{(r-2)}. \tag{4.17}$$

Then we can get the bubble version of (3.12),

$$I_2^{(r)} = \frac{-1}{A_{2,r}}(A_{2,r}^{-}I_2^{(r-1)} + A_{2,r}^{--}I_2^{(r-2)} + \tilde{A}_{2,r;\hat{i}}^{-}I_{2;\hat{i}}^{(r-1)} + \tilde{A}_{n,r;\hat{i}}^{--}I_{2;\hat{i}}^{(r-2)}). \tag{4.18}$$

Next, we provide a detailed expression for (3.3)

$$\mathcal{G}(\{z\})_{\cancel{13};\cancel{13}} = \det(\, s_{11}\,), \qquad \mathcal{G}(\{z\} = 0)_{\cancel{3};\cancel{1}} = \det\begin{pmatrix} -\frac{1}{2}(m_1^2 + m_2^2 - s_{11}) & 0 \\ s_{11} & s_{01} \end{pmatrix},$$

$$\mathcal{G}(\{z\} = 0) = \det\begin{pmatrix} m_1^2 & -(m_1^2 - m_2^2 + s_{11})/2 & 0 \\ -(m_1^2 - m_2^2 + s_{11})/2 & s_{11} & s_{01} \\ 0 & s_{01} & s_{00} \end{pmatrix}. \tag{4.19}$$

The specific components of $\mathcal{G}$ are given by

$$\mathcal{G}_{00} = -\frac{1}{4}s_{11}, \tag{4.20}$$

$$\mathcal{G}_0 = \frac{1}{2}(m_1^2 - m_2^2 + s_{11})s_{01}, \tag{4.21}$$

$$\mathcal{G}_c = -\frac{1}{4}((m_1^2 - m_2^2)^2 s_{00} - (2m_1^2 + 2m_2^2 - s_{11})s_{11}s_{00} + 4m_1^2 s_{01}^2), \tag{4.22}$$

$$\mathcal{G}_{01} = \frac{1}{2}s_{01}, \qquad \mathcal{G}_{02} = -\frac{1}{2}s_{01}, \tag{4.23}$$

$$\mathcal{G}_1 = -\frac{1}{2}((m_1^2 - m_2^2 - s_{11})s_{00} + 2s_{01}^2), \qquad \mathcal{G}_2 = \frac{1}{2}(m_1^2 - m_2^2 + s_{11})s_{00}, \tag{4.24}$$

$$\mathcal{G}_{11} = -\frac{1}{4}s_{00}, \qquad \mathcal{G}_{12} = \frac{1}{2}s_{00}, \qquad \mathcal{G}_{22} = -\frac{1}{4}s_{00}. \tag{4.25}$$

Then we give the explicit expression of (4.18),

$$\begin{aligned} I_2^{(r)} = {} & \frac{(d+2r-4)(m_1^2 - m_2^2 + s_{11})s_{01}}{(d+r-3)s_{11}} I_2^{(r-1)} + \frac{r-1}{(d+r-3)s_{11}}[((2m_1^2 + 2m_2^2 - s_{11})s_{11}s_{00} - (m_1^2 - m_2^2)^2 s_{00} - 4m_1^2 s_{01}^2)I_2^{(r-2)} \\ & + (m_1^2 - m_2^2 + s_{11})s_{00}I_{2;\hat{2}}^{(r-2)} + ((m_2^2 - m_1^2 + 3s_{11})s_{00} - 4s_{01}^2)I_{2;\hat{1}}^{(r-2)}] \\ & + \frac{(d+2r-4)s_{01} + s_{00}q_2 \cdot \partial_R}{(d+r-3)s_{11}}(I_{2;\hat{1}}^{(r-1)} - I_{2;\hat{2}}^{(r-1)}). \end{aligned} \tag{4.26}$$

Clearly, we can reduce $I_2^{(r)}$ to I_2 and $I_{2;\hat{i}}$ through repeatedly applying (4.26),

$$I_2^{(r)} = C_2^{(r)} I_2 + C_{2;\hat{2}}^{(r)} I_{2;\hat{2}} + C_{2;\hat{1}}^{(r)} I_{2;\hat{1}}. \tag{4.27}$$

Here are the expressions for C_2, $C_{2;\hat{2}}$, and $C_{2;\hat{1}}$ for different values of r:

(i) $r = 1$

$$C_2^{(1)} = \frac{s_{01}(m_1^2 - m_2^2 + s_{11})}{s_{11}}, \tag{4.28}$$

$$C_{2;\hat{1}}^{(1)} = \frac{s_{01}}{s_{11}}, \qquad C_{2;\hat{2}}^{(1)} = -\frac{s_{01}}{s_{11}}. \tag{4.29}$$

(ii) $r = 2$

$$C_2^{(2)} = \frac{((m_1^2 - m_2^2)^2 - s_{11}(2m_1^2 + 2m_2^2 - s_{11}))(s_{11}s_{00} - ds_{01}^2)}{(1-d)s_{11}^2} + \frac{4m_1^2 s_{01}^2}{s_{11}}, \tag{4.30}$$

$$C_{2;\hat{2}}^{(2)} = \frac{(m_1^2 - m_2^2 + s_{11})(s_{11}s_{00} - ds_{01}^2)}{(d-1)s_{11}^2}, \tag{4.31}$$

$$C_{2;\hat{1}}^{(2)} = \frac{(-m_1^2 + m_2^2 + s_{11})(s_{11}s_{00} - ds_{01}^2) + 4(d-1)s_{11}s_{01}^2}{(d-1)s_{11}^2}. \tag{4.32}$$

(iii) $r = 3$

$$\begin{aligned} C_2^{(3)} = {} & \frac{(-(m_2^2 - s_{11})^2 - m_1^4 + 2m_2^2 m_1^2)(m_1^2 - m_2^2 + s_{11})s_{01}(3s_{11}s_{00} - (d+2)s_{01}^2)}{(d-1)s_{11}^3} \\ & + \frac{2m_1^2(m_1^2 - m_2^2 + s_{11})s_{01}((d-4)s_{01}^2 + 3s_{11}s_{00})}{(d-1)s_{11}^2}, \end{aligned} \tag{4.33}$$

$$C^{(3)}_{2;\hat{2}} = \frac{((m_2^2 - s_{11})^2 + m_1^4 - 2m_2^2 m_1^2) s_{01}(3s_{11}s_{00} - (d+2)s_{01}^2)}{(d-1)s_{11}^3} - \frac{2m_1^2 s_{01}((d^2 - 2d + 4)s_{01}^2 + 3(d-2)s_{11}s_{00})}{(d-1)ds_{11}^2}, \tag{4.34}$$

$$C^{(3)}_{2;\hat{1}} = \frac{s_{01}(-4(d-1)m_2^2 s_{11} + dm_1^4 - 2dm_2^2 m_1^2 + dm_2^4)((d+2)s_{01}^2 - 3s_{11}s_{00})}{(d-1)ds_{11}^3} + \frac{s_{01}(4(d-1)dm_1^2 s_{11}s_{01}^2 + ds_{11}^2((7d-10)s_{01}^2 + 3s_{11}s_{00}))}{(d-1)ds_{11}^3}. \tag{4.35}$$

1. $\mathcal{G}_{00} = 0$

In this subsection, it is important to note that the $I_2^{(r)}$ integrals discussed here are modified versions that have been adjusted to the same limit. As we can see, when $\mathcal{G}_{00} = 0$, i,e., $\mathsf{A}_{2,r} = 0$, Eq. (4.18) is no longer applicable. We can divide both sides of Eq. (4.13) by $\mathsf{B}_{2,r} = 0$ and shift $r-1$ to r, resulting in the following expression:

$$\begin{aligned} I_2^{(r)} &= \frac{-1}{\mathsf{B}_{2,r+1}}\left(\mathsf{C}_{2,r+1} I_2^{(r-1)} + \mathsf{B}_{2,r+1;\hat{i}} I_{2;\hat{i}}^{(r)} + \mathsf{C}_{n,r+1;\hat{i}} I_{2;\hat{i}}^{(r-1)}\right) \\ &= \frac{r}{(d+2r-2)(m_1^2 - m_2^2)s_{01}}\left[((m_1^2 - m_2^2)^2 s_{00} + 4m_1^2 s_{01}^2) I_2^{(r-1)} - (m_1^2 - m_2^2)s_{00} I_{2;\hat{2}}^{(r-1)} - ((m_2^2 - m_1^2)s_{00} - 4s_{01}^2) I_{2;\hat{1}}^{(r-1)}\right] \\ &\quad - \frac{(d+2r-2)s_{01} + s_{00} q_2 \cdot \partial_R}{(d+2r-2)(m_1^2 - m_2^2)s_{01}}\left(I_{2;\hat{1}}^{(r)} - I_{2;\hat{2}}^{(r)}\right). \end{aligned} \tag{4.36}$$

It is evident that $r = 0$ can be computed, which implies that I_2 can be represented in terms of $I_{2;\hat{2}}$ and $I_{2;\hat{1}}$. Consequently, I_2 does not appear in the final result, and we have

$$I_2^{(r)} = C^{(r)}_{2;\hat{2}} I_{2;\hat{2}} + C^{(r)}_{2;\hat{1}} I_{2;\hat{1}}. \tag{4.37}$$

Next, we provide the specific expressions for $C_{2;\hat{1}}$ and $C_{2;\hat{2}}$ for different values of r:

(i) $r = 0$

$$C_{2;\hat{1}} = -\frac{1}{m_1^2 - m_2^2}, \qquad C_{2;\hat{2}} = \frac{1}{m_1^2 - m_2^2}, \tag{4.38}$$

(ii) $r = 1$

$$C^{(1)}_{2;\hat{1}} = \frac{2s_{01}((d-2)m_2^2 - dm_1^2)}{d(m_1^2 - m_2^2)^2}, \qquad C^{(1)}_{2;\hat{2}} = \frac{4s_{01}m_1^2}{d(m_1^2 - m_2^2)^2}, \tag{4.39}$$

(iii) $r = 2$

$$\begin{aligned} C^{(2)}_{2;\hat{1}} &= -\frac{4(-2(d^2-4)m_2^2 m_1^2 + d(d+2)m_1^4 + (d-2)dm_2^4)s_{01}^2}{d(d+2)(m_1^2 - m_2^2)^3} - \frac{4m_2^2 s_{00}}{dm_1^2 - dm_2^2}, \\ C^{(2)}_{2;\hat{2}} &= \frac{32m_1^4 s_{01}^2}{d(d+2)(m_1^2 - m_2^2)^3} + \frac{4m_1^2 s_{00}}{dm_1^2 - dm_2^2}. \end{aligned} \tag{4.40}$$

2. $\mathcal{G}_{00} = 0$ and $\mathcal{G}_0 = 0$

There is still another pole $m_1 = m_2 = m$. In this case, $\mathsf{B}_{2,r} = 0$. By dividing both sides of (4.13) by $\mathsf{C}_{2,r} = 0$ and shifting $r-2$ to r, we obtain the following expression:

$$I_2^{(r)} = \frac{-1}{\mathsf{C}_{2,r+2}}(\mathsf{B}_{2,r+2;\hat{i}} I_{2;\hat{i}}^{(r+1)} + \mathsf{C}_{n,r+2;\hat{i}} I_{2;\hat{i}}^{(r)}) = \frac{ds_{01} + s_{00} q_2 \cdot \partial_R}{4m^2 s_{01}^2}(I_{2;\hat{1}}^{(r+1)} - I_{2;\hat{2}}^{(r+1)}) - \frac{1}{m^2} I_{2;\hat{1}}^{(r)}. \quad (4.41)$$

In this scenario, $I_{2;\hat{1}}$ is equal to $I_{2;\hat{2}}$. Therefore, the final result can be expressed as

$$I_2^{(r)} = C_{2;\hat{1}}^{(r)} I_{2;\hat{1}}. \quad (4.42)$$

Now we provide the specific expressions for $C_{2;\hat{1}}$ for different values of r:

(i) $r = 0$

$$C_{2;\hat{1}} = \frac{d-2}{2m^2}, \quad (4.43)$$

(ii) $r = 1$

$$C_{2;\hat{1}}^{(1)} = \frac{(d-2)s_{01}}{2m^2}, \quad (4.44)$$

(iii) $r = 2$

$$C_{2;\hat{1}}^{(2)} = \frac{2}{3}\left(\frac{(d-2)s_{01}^2}{m^2} + 3s_{00}\right). \quad (4.45)$$

Interestingly, high-pole tadpole integrals can be found using the $r = 0$ integral. Specifically, when the momentum $q_2 \to 0$ and mass $m_2 \to m_1$, the second bubble propagator z_2 degenerates into z_1. In other words, the bubble integral transforms into a higher-pole tadpole integral, i.e., $I_{\{1,1\}} \to I_{\{2\}}$. Applying (4.43) with $q_2 = 0$ then yields a recurrence relation directly connecting the higher-pole bubble integral to the tadpole base case

$$I_{\{2\}} = \frac{d-2}{2m_1^2} I_{\{1\}}. \quad (4.46)$$

When r is odd, $I_{\{a_1\}}^{(r)} = 0$, which can follow from (4.7). Specifically,

$$I_{\{2\}}^{(1)} = 0. \quad (4.47)$$

When r is even,

$$I_{\{1\}}^{(r)} = \frac{2^r (r-1)!! m_1^r R^r}{(d+r-2)!!/(d-2)!!} I_{\{1\}}. \quad (4.48)$$

Arbitrary powers of the tadpole propagator z_1 can be generated by repeatedly applying the differential operator $\partial_{m_1^2}$. For example,

$$I_{\{2\}}^{(r)} = \frac{2^{r-1}(r-1)!! m_1^{r-2} R^r}{(d+r-2)!!/(d-2)!!}(r I_{\{1\}} + 2m_1^2 I_{\{2\}}). \quad (4.49)$$

Substituting (4.46) into it gives

$$I_{\{2\}}^{(r)} = \frac{2^{r-1}(r-1)!! m_1^{r-2} R^r}{(d+r-2)!!/(d-2)!!}(r+d-2) I_{\{1\}}. \quad (4.50)$$

3. Det $Q = 0$

As stated in [33], $\det Q = 0$ is also a degenerate case where $Q_{ij} = (m_i^2 + m_j^2 - (q_i - q_j^2))/2)$. But it cannot be well handled in our algorithm; we can use their scalar reduction result as an input to get the tensor reduction. In a bubble integral $\det Q = 0$ gives $s_{11} = (m_2 \pm m_1)^2$. Take the $s_{11} = (m_2 + m_1)^2$ as an example [Eq. (4.21) in [33]],

$$I_2 = \frac{d-2}{2(d-3)m_2(m_1+m_2)} I_{2;\hat{1}} + \frac{d-2}{2(d-3)m_1(m_1+m_2)} I_{2;\hat{2}}. \quad (4.51)$$

Substitute the above equation into (4.28) and (4.29),

$$I_2^{(1)} = \frac{((d-2)m_1 + (d-3)m_2)s_{01}}{(d-3)m_2(m_1+m_2)^2} I_{2;\hat{1}} + \frac{s_{01}}{(d-3)(m_1+m_2)^2} I_{2;\hat{2}}, \quad (4.52)$$

which is consistent with [33]. Similarly,

$$I_2^{(2)} = \frac{2}{(d-3)(d-1)m_2(m_1+m_2)^3}(N_{2;\hat{1}}^{(1)} I_{2;\hat{1}} + N_{2;\hat{2}}^{(1)} I_{2;\hat{2}}), \quad (4.53)$$

where

$$N_{2;\hat{1}}^{(1)} = (d-2)s_{01}^2[(d-1)m_1^2 + (d-3)m_2^2] + (d-3)m_2[2(d-1)m_1 s_{01}^2 + m_2 s_{00}(m_1+m_2)^2],$$
$$N_{2;\hat{2}}^{(1)} = m_1 m_2[(d-3)s_{00}(m_1+m_2)^2 + 2s_{01}^2]. \quad (4.54)$$

4. $I_{\{2,1\}}$ and $I_{\{2,1\}}^{(1)}$

For clarity, we calculate $I_{\{2,1\}}$ using (3.20),

$$\mathsf{H}_2^{1+} I_{\{2,1\}} + \mathsf{H}_2 I_{\{1,1\}} + \mathsf{H}_{2;\hat{1}} I_{\{0,1\}} + \Big[(H_{2;\hat{2}} + \mathsf{H}^{1+}_{2;\widehat{22}}) I_{\{1,0\}} + ((m_2^2 - m_1^2 + q_2^2)\mathsf{H}^{1+}_{2;\widehat{22}} + \mathsf{H}^{1+}_{2;\hat{2}}) I_{\{2,0\}} - \mathsf{H}^{1+}_{2;\widehat{22}} q_2 \cdot \partial_R I_{\{2,0\}}^{(1)}\Big] = 0. \quad (4.55)$$

Based on the results derived in the preceding section,

$$I_{\{2,0\}} = \frac{d-2}{2m_1^2} I_{\{1,0\}}, \qquad I_{\{2,0\}}^{(1)} = 0. \quad (4.56)$$

We can get the final result of $I_{\{2,1\}}$,

$$I_{\{2,1\}} = -\frac{\mathsf{H}_2}{\mathsf{H}_2^{1+}} I_{\{1,1\}} - \frac{\mathsf{H}_{2;\hat{1}}}{\mathsf{H}_2^{1+}} I_{\{0,1\}} - \left(\frac{\mathsf{H}_{2;\hat{2}} + \mathsf{H}^{1+}_{2;\widehat{22}}}{\mathsf{H}_2^{1+}} + \frac{(m_1^2 - m_2^2 + s_{11})\mathsf{H}^{1+}_{2;\widehat{22}} + \mathsf{H}^{1+}_{2;\hat{2}}}{\mathsf{H}_2^{1+}} \frac{d-2}{2m_1^2} \right) I_{\{1,0\}}$$
$$= \frac{(d-3)(m_1^2 - m_2^2 - s_{11})}{4\mathsf{H}_2^{1+}} I_{\{1,1\}} + \frac{d-2}{4\mathsf{H}_2^{1+}} I_{\{0,1\}} + \frac{(2-d)(m_1^2 + m_2^2 - s_{11})}{8m_1^2 \mathsf{H}_2^{1+}} I_{\{1,0\}}, \tag{4.57}$$

where the second equality holds since we have used the coefficient expression

$$\mathsf{H}_2^{1+} = \frac{1}{4}(-2m_1^2(m_2^2 + s_{11}) + (m_2^2 - s_{11})^2 + m_1^4),$$
$$\mathsf{H}_2 = -\frac{1}{4}(d-3)(m_1^2 - m_2^2 - s_{11}), \qquad \mathsf{H}_{2;\hat{1}} = \frac{2-d}{4},$$
$$\mathsf{H}^{1+}_{2;\hat{2}} = \frac{1}{2}(-m_1^2 + m_2^2 - s_{11}), \qquad \mathsf{H}_{2;\hat{2}} = \frac{d-3}{4}, \qquad \mathsf{H}^{1+}_{2;\widehat{22}} = \frac{1}{4}. \tag{4.58}$$

The results of applying $\partial_{m_1^2}$ on $I^{(1)}_{\{1,1\}}$ can be derived using Eqs. (4.27)–(4.29),

$$I^{(1)}_{\{2,1\}} = \frac{s_{01}}{s_{11}} I_{\{1,1\}} + \frac{s_{01}(m_1^2 - m_2^2 + s_{11})}{s_{11}} I_{\{2,1\}} - \frac{s_{01}}{s_{11}} \frac{d-2}{2m_1^2} I_{\{1,0\}}. \tag{4.59}$$

Plugging (4.57) into it,

$$I^{(1)}_{\{2,1\}} = \frac{s_{01}[s_{11}((d-4)(m_2^2 - s_{11}) - 2m_1^2) + (d-2)((m_1^2 - m_2^2)^2 - m_2^2 s_{11})]}{4s_{11}\mathsf{H}_2^{1+}} I_{\{1,1\}}$$
$$+ \frac{(d-2)s_{01}(m_1^2 - m_2^2 + s_{11})}{4s_{11}\mathsf{H}_2^{1+}} I_{\{0,1\}} + \frac{(2-d)s_{01}(m_1^2 - m_2^2 - s_{11})}{4s_{11}\mathsf{H}_2^{1+}} I_{\{1,0\}}. \tag{4.60}$$

C. Triangles

The preceding subsections provided detailed and specific computations for the tadpole and the bubble. Consequently, in what follows, the step-by-step calculations will be omitted for brevity. Unlike tadpoles and bubbles, triangles involve integrals where two propagators have been removed. Using Eq. (3.8), we can get the triangle equation,

$$\mathsf{A}_{3,r} I_n^{(r)} + \mathsf{B}_{3,r} I_3^{(r-1)} + \mathsf{C}_{3,r} I_3^{(r-2)} + \mathsf{B}_{3,r;\hat{i}} I_{3;\hat{i}}^{(r-1)} + \mathsf{C}_{n,r;\hat{i}} I_{3;\hat{i}}^{(r-2)} + \sum_{i<j} \mathsf{C}_{3,r;\widehat{ij}} I^{(r-2)}_{3;\widehat{ij}} = 0. \tag{4.61}$$

Since only $\mathcal{G}_{00}$, $\mathcal{G}_0$, and $\mathcal{G}_c$ are associated with singularities, we will focus exclusively on presenting their specific results. The other coefficients in the reduction formula do not contribute to singular configurations and hence will be omitted for brevity,

$$\mathcal{G}_{00} = \frac{1}{4}(s_{11}s_{22} - s_{12}^2), \tag{4.62}$$

$$\mathcal{G}_0 = \frac{1}{2}(m_1^2(s_{12} - s_{22}) - m_3^2 s_{12} + s_{22}(m_2^2 - s_{11} + s_{12}))s_{01}$$
$$+ \frac{1}{2}(m_1^2(s_{12} - s_{11}) + m_3^2 s_{11} - m_2^2 s_{12} + s_{11}s_{12} - s_{11}s_{22})s_{02}, \tag{4.63}$$

$$
\begin{aligned}
\mathcal{G}_c = & -m_1^2 s_{12}^2 s_{00} + \frac{1}{4}(s_{22}^2 s_{01}^2 + (m_1^2(s_{01}-s_{02}) - m_3^2 s_{01} + m_2^2 s_{02})^2 + s_{11}^2(s_{02}^2 - s_{22}s_{00})) \\
& - \frac{s_{22}}{4}(m_1^4 s_{00} + m_1^2(2s_{01}(s_{01}+s_{02}) - 2m_2^2 s_{00}) + 2m_3^2 s_{01}^2 + m_2^4 s_{00} - 2m_2^2 s_{01}s_{02}) \\
& + \frac{s_{12}}{2}((m_1^2 - m_2^2)s_{22}s_{00} + (m_1^4 s_{00} - m_1^2(m_2^2 s_{00} + m_3^2 s_{00} - 4s_{01}s_{02}) + m_2^2 m_3^2 s_{00})) \\
& + s_{11}\left(\frac{1}{2}s_{22}(m_2^2 s_{00} + m_3^2 s_{00} - s_{01}s_{02}) + \frac{s_{12}}{2}((m_1^2 - m_3^2)s_{00} + s_{22}s_{00}) - \frac{1}{4}s_{22}^2 s_{00}\right) \\
& - \frac{s_{11}}{4}(m_1^4 s_{00} + m_1^2(2s_{02}(s_{01}+s_{02}) - 2m_3^2 s_{00}) + 2m_2^2 s_{02}^2 + m_3^4 s_{00} - 2m_3^2 s_{01}s_{02}). \qquad (4.64)
\end{aligned}
$$

First, the case of $\mathcal{G}_{00} \neq 0$ will be considered. Following a procedure analogous to that applied to the bubble, the final result for the triangle can be derived as

$$
I_3^{(r)} = C_3^{(r)} I_3 + \sum_{i=1,2,3} C_{3;\hat{i}}^{(r)} I_{3;\hat{i}} + \sum_{1\leq i<j\leq 3} C_{3;\widehat{ij}}^{(r)} I_{3;\widehat{ij}}. \qquad (4.65)
$$

The specific coefficients for different values of r are as follows:

(i) $r = 1$

$$
\begin{aligned}
& C_3^{(1)} = -\frac{\mathcal{G}_0}{2\mathcal{G}_{00}}, \\
& C_{3;\hat{3}}^{(1)} = \frac{s_{12}s_{01} - s_{11}s_{02}}{4\mathcal{G}_{00}}, \qquad C_{3;\hat{2}}^{(1)} = \frac{s_{12}s_{02} - s_{22}s_{01}}{4\mathcal{G}_{00}}, \\
& C_{3;\hat{1}}^{(1)} = \frac{(s_{22} - s_{12})s_{01} - (s_{12} - s_{11})s_{02}}{4\mathcal{G}_{00}}, \qquad (4.66)
\end{aligned}
$$

(ii) $r = 2$

In this case, the analytic expressions are unwieldy, so a numerical solution will be adopted. To facilitate a comparison with prior results from [33], the parameters are set as $d = 4$, $\{m_1^2, m_2^2, m_3^2\} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}$, $\{s_{12}, s_{11}, s_{22}\} = \{\frac{7}{13}, \frac{5}{7}, \frac{7t}{5} + \frac{343}{845}\}$, where $t = 4\mathcal{G}_{00}$.

$$
\begin{aligned}
C_3^{(2)} = & \frac{-1573040 s_{00} - 476472423 s_{01}^2 + 1084364190 s_{02}s_{01} - 596126375 s_{02}^2}{2519080200t} \\
& + \frac{5618(49s_{01} - 65s_{02})^2}{17738523075t^2} + \mathcal{O}(t^0), \qquad (4.67)
\end{aligned}
$$

$$
C_{3;\hat{3}}^{(2)} = \frac{530s_{00} + 24787s_{01}^2 - 5200s_{02}s_{01} - 38025s_{02}^2}{35490t} - \frac{53(49s_{01} - 65s_{02})^2}{6997445t^2} + \mathcal{O}(t^0), \qquad (4.68)
$$

$$
C_{3;\widehat{23}}^{(2)} = \frac{-2401s_{01}^2 + 6370s_{02}s_{01} - 4225s_{02}^2}{3185t} + \mathcal{O}(t^0). \qquad (4.69)
$$

1. $\mathcal{G}_{00} = 0$

There are many viable options to satisfy the condition $\mathcal{G}_{00} = 0$; without loss of generality, we impose the constraint $s_{11} = s_{12}^2/s_{22}$. As with the bubbles, in this subsection $I_3^{(r)}$ represents the modified version of the integral. Using (3.13), the final result for $I_3^{(r)}$ is

$$
I_3^{(r)} = \sum_{i=1,2,3} C_{3;\hat{i}}^{(r)} I_{3;\hat{i}} + \sum_{1\leq i<j\leq 3} C_{3;\widehat{ij}}^{(r)} I_{3;\widehat{ij}}. \qquad (4.70)
$$

The coefficients for different values of r are as follows:

(i) $r=0$

$$C_{3;\hat{3}}=\frac{s_{12}}{m_1^2(s_{12}-s_{22})-m_3^2s_{12}+m_2^2s_{22}-s_{12}^2+s_{12}s_{22}}, \tag{4.71}$$

$$C_{3;\hat{2}}=\frac{s_{22}}{m_1^2(s_{12}-s_{22})-m_3^2s_{12}+m_2^2s_{22}-s_{12}^2+s_{12}s_{22}}, \tag{4.72}$$

$$C_{3;\hat{1}}=\frac{s_{22}-s_{12}}{m_1^2(s_{12}-s_{22})-m_3^2s_{12}+m_2^2s_{22}-s_{12}^2+s_{12}s_{22}}, \tag{4.73}$$

(ii) $r=1$

$$C^{(1)}_{3;\hat{3}}=-\frac{169(568463s_{01}-972595s_{02})}{943824},\qquad C^{(1)}_{3;\widehat{23}}=-\frac{169}{371}(49s_{01}-65s_{02}), \tag{4.74}$$

where we use the same parameter sets as those in the $r=2$ case discussed in the Sec. IV C.

2. $\mathcal{G}_{00}=0$ and $\mathcal{G}_0=0$

Under the imposed constraint $s_{11}=s_{12}^2/s_{22}$, $\mathcal{G}_0$ and $\mathcal{G}_c$ take the forms

$$\mathcal{G}_0=\frac{(m_1^2(s_{12}-s_{22})-m_3^2s_{12}+m_2^2s_{22}-s_{12}^2+s_{12}s_{22})(s_{12}s_{02}-s_{22}s_{01})}{2s_{22}}, \tag{4.75}$$

$$\mathcal{G}_c=\frac{\mathsf{N}_c}{4(-m_1^2+m_2^2+s_{12})^2(-m_1^2+m_3^2+s_{12})^2}, \tag{4.76}$$

where

$$\begin{aligned}\mathsf{N}_c=&((m_1^2-m_3^2-s_{12})s_{01}+(m_2^2-m_1^2+s_{12})s_{02})^2((m_1+m_2)(m_1-m_3)-s_{12})\\&\times((m_1-m_2)(m_1+m_3)-s_{12})((m_1-m_2)(m_1-m_3)-s_{12})((m_1+m_2)(m_1+m_3)-s_{12}).\end{aligned} \tag{4.77}$$

To satisfy $\mathcal{G}_0=0$, we set

$$s_{22}=-\frac{m_3^2s_{12}-m_1^2s_{12}+s_{12}^2}{m_1^2-m_2^2-s_{12}}. \tag{4.78}$$

Clearly, utilizing alternative constraints is also valid, as the underlying algorithm remains unchanged. The final result for $I_3^{(r)}$ can be expressed as

$$I_3^{(r)}=\sum_{i=1,2,3}C^{(r)}_{3;\hat{i}}I_{3;\hat{i}}+\sum_{1\le i<j\le 3}C^{(r)}_{3;\widehat{ij}}I_{3;\widehat{ij}}. \tag{4.79}$$

The coefficients for different values of r:

(i) $r=0$

$$\begin{aligned}C_{3;\hat{3}}&=\frac{(3-d)(m_1^2-m_2^2-s_{12})(m_1^2-m_3^2-s_{12})((m_3^2-m_1^2)(m_2^2-m_1^2)-s_{12}^2)s_{01}}{\mathsf{D}_{3;\hat{3}}},\\C_{3;\widehat{23}}&=\frac{(d-2)(m_1^2-m_2^2-s_{12})(m_1^2-m_3^2-s_{12})}{\mathsf{D}_{3;\widehat{23}}},\end{aligned} \tag{4.80}$$

where

$$\begin{aligned}\mathsf{D}_{3;\hat{3}} &= ((m_1^2 - m_3^2 - s_{12})s_{01} + (m_2^2 - m_1^2 + s_{12})s_{02})((m_1+m_2)(m_1-m_3) - s_{12})\\ &\quad \times ((m_1-m_2)(m_1+m_3) - s_{12})((m_1-m_2)(m_1-m_3) - s_{12})((m_1+m_2)(m_1+m_3) - s_{12}),\\ \mathsf{D}_{3;\hat{2}\hat{3}} &= ((m_1+m_2)(m_1-m_3) - s_{12})((m_1-m_2)(m_1+m_3) - s_{12})\\ &\quad \times ((m_1-m_2)(m_1-m_3) - s_{12})((m_1+m_2)(m_1+m_3) - s_{12}). \end{aligned} \tag{4.81}$$

Specific results for rank $r \geq 1$ are omitted here in the interest of brevity, but the computational procedure remains unchanged. As is evident from (4.62) and (4.63), setting $q_3 = 0$ results in both $\mathcal{G}_{00}$ and $\mathcal{G}_0$ vanishing. In this case, the coefficients for different values of r are as follows:

(i) $r = 0$

$$\begin{aligned} &C_{3;\hat{1}} = -\frac{1}{m_1^2 - m_3^2}, \qquad C_{3;\hat{3}} = \frac{1}{m_1^2 - m_3^2}, \qquad C_{3;\hat{2}} = 0,\\ &C_{3;\hat{2}\hat{3}} = 0, \qquad C_{3;\hat{1}\hat{3}} = 0, \qquad C_{3;\hat{1}\hat{2}} = 0, \end{aligned} \tag{4.82}$$

(ii) $r = 1$

$$\begin{aligned} C^{(1)}_{3;\hat{3}} &= \frac{(m_1^2 - m_2^2 + s_{11})(2m_1^2 - m_2^2 - m_3^2 + s_{11})s_{01}}{(m_1^2 - m_3^2)^2 s_{11}}, \qquad C^{(1)}_{3;\hat{2}} = 0,\\ C^{(1)}_{3;\hat{1}} &= -\frac{(m_1^2 + m_2^2 - 2m_3^2 - s_{11})(-m_2^2 + m_3^2 + s_{11})s_{01}}{(m_1^2 - m_3^2)^2 s_{11}},\\ C^{(1)}_{3;\hat{2}\hat{3}} &= \frac{(-2m_1^2 + m_2^2 + m_3^2 - s_{11})s_{01}}{(m_1^2 - m_3^2)^2 s_{11}}, \qquad C^{(1)}_{3;\hat{1}\hat{3}} = \frac{(m_1^2 - 2m_2^2 + m_3^2 + 2s_{11})s_{01}}{(m_1^2 - m_3^2)^2 s_{11}},\\ C^{(1)}_{3;\hat{1}\hat{2}} &= \frac{(m_1^2 + m_2^2 - 2m_3^2 - s_{11})s_{01}}{(m_1^2 - m_3^2)^2 s_{11}}. \end{aligned} \tag{4.83}$$

The results of integrals with a tensor structure in the examples examined herein match those derived in [33].

3. Det $Q = 0$

Analogous to the bubble, using the scalar result in [33], we set the parameters as $\{m_1^2, m_2^2, m_3^2\} = \{\frac{1}{2}, \frac{1}{3}, \frac{5}{338}\}$, $\{s_{11}, s_{12}, s_{22}\} = \{\frac{5}{7}, \frac{7}{13}, \frac{3552}{5915}\}$,

$$\begin{aligned} I_3 = \frac{21}{1151(d-4)}[&4225(d-2)I_{3;\hat{1}\hat{2}} - 780(d-2)I_{3;\hat{1}\hat{3}} + 455(d-2)I_{3;\hat{2}\hat{3}}\\ &+ 402(d-3)I_{3;\hat{1}} - 592(d-3)I_{3;\hat{2}} + 130(d-3)I_{3;\hat{3}}]. \end{aligned} \tag{4.84}$$

Substitute the above equation into (4.66),

$$\begin{aligned} I_3^{(1)} = \frac{7}{11510(d-4)}[&-4225(d-2)(12s_{01} - 65s_{02})I_{3;\hat{1}\hat{2}} + 780(d-2)(12s_{01} - 65s_{02})I_{3;\hat{1}\hat{3}}\\ &- 455(d-2)(12s_{01} - 65s_{02})I_{3;\hat{2}\hat{3}} + 130((37d-160)s_{01} + 65s_{02})I_{3;\hat{3}}\\ &+ 2(65(692 - 247d)s_{02} + 3552s_{01})I_{3;\hat{2}} - 2((2045d - 5768)s_{01} + 65(667 - 217d)s_{02})I_{3;\hat{1}}]. \end{aligned} \tag{4.85}$$

4. $I_{\{2,1,1\}}$

The integral $I_{\{2,1,1\}}$ is evaluated using (3.20)

$$\mathsf{H}_3^{1+}I_{\{2,1,1\}}+\mathsf{H}_3 I_{\{1,1,1\}}+\mathsf{H}_{3;\hat{1}}I_{\{0,1,1\}}+\sum_{j=2,3}\Big[\Big(\mathsf{H}_{3;\hat{j}}+\mathsf{H}^{1+}_{3;\widehat{jj}}\Big)I_{\{a_j=0\}} + \Big((m_1^2-m_j^2+q_j^2)\mathsf{H}^{1+}_{3;\widehat{jj}}+\mathsf{H}^{1+}_{3;\hat{j}}\Big)I_{\{a_1=2,a_j=0\}}-\mathsf{H}^{1+}_{3;\widehat{jj}}q_j\cdot\partial_R I^{(1)}_{\{a_1=2,a_j=0\}}\Big]=0. \tag{4.86}$$

Noting that the specific integrals $I_{\{a_1=2,a_j=0\}}$ and $I^{(1)}_{\{a_1=2,a_j=0\}}$ follow from the corresponding bubbles section,

$$I_{\{2,1,1\}}=C^{1+}_{3\to3}I_{\{1,1,1\}}+\sum_{i=1,2,3}C^{1+}_{3\to3;\hat{i}}I_{\{a_i=0\}}+\sum_{i\neq j}C^{1+}_{3\to3;\widehat{ij}}I_{\{a_i=0,a_j=0\}}, \tag{4.87}$$

where the coefficients are

$$\begin{aligned}
C^{1+}_{3\to3}&=\frac{(d-4)[(m_1^2-s_{12})(s_{11}-2s_{12}+s_{22})+m_3^2(s_{12}-s_{11})+m_2^2(s_{12}-s_{22})]}{4B_3^{1+}},\\
C^{1+}_{3\to3;\hat{1}}&=\frac{(d-3)(s_{11}-2s_{12}+s_{22})}{4B_3^{1+}},\\
C^{1+}_{3\to3;\hat{2}}&=\frac{\mathsf{N}_{3\to3;\hat{2}}}{4(m_1^4-2m_1^2(m_3^2+s_{22})+(m_3^2-s_{22})^2)B_3^{1+}},\\
C^{1+}_{3\to3;\hat{3}}&=\frac{\mathsf{N}_{3\to3;\hat{3}}}{4(m_1^4-2m_1^2(m_2^2+s_{11})+(m_2^2-s_{11})^2)B_3^{1+}},\\
C^{1+}_{3\to3;\widehat{12}}&=\frac{(d-2)(m_1^2(s_{22}-s_{12})+m_3^2s_{12}-s_{22}(m_2^2-s_{11}+s_{12}))}{4(m_1^4-2m_1^2(m_3^2+s_{22})+(m_3^2-s_{22})^2)B_3^{1+}},\\
C^{1+}_{3\to3;\widehat{13}}&=\frac{(d-2)(m_1^2(s_{11}-s_{12})+m_2^2s_{12}-s_{11}(m_3^2-s_{22}+s_{12}))}{4(m_1^4-2m_1^2(m_2^2+s_{11})+(m_2^2-s_{11})^2)B_3^{1+}},\\
C^{1+}_{3\to3;\widehat{23}}&=\frac{\mathsf{N}_{3\to3;\widehat{23}}}{\mathsf{D}_{3\to3;\widehat{23}}},
\end{aligned} \tag{4.88}$$

where

$$\begin{aligned}
B_3^{1+}&=\frac{1}{2}(s_{22}-s_{12})(-m_3^2s_{11}-m_1^2m_2^2)+\frac{1}{4}(s_{11}-2s_{12}+s_{22})(-2m_1^2s_{12}+m_1^4+s_{11}s_{22})\\
&\quad+\frac{1}{4}(m_3^4s_{11}-2m_2^2m_3^2s_{12}+2m_1^2m_3^2(s_{12}-s_{11})+m_2^2s_{22}(m_2^2-2s_{11}+2s_{12})),\\
\mathsf{N}_{3\to3;\hat{2}}&=(3-d)s_{22}[(m_3^2-m_1^2)(-m_2^2+m_3^2+s_{11})+2s_{12}(m_1^2+m_3^2-s_{22})+s_{22}(-m_1^2-m_2^2-2m_3^2+s_{11}+s_{22})],\\
\mathsf{N}_{3\to3;\hat{3}}&=(3-d)s_{11}[(m_1^2-m_2^2)(-m_2^2+m_3^2-s_{22})+2s_{12}(m_1^2+m_2^2-s_{11})+s_{11}(-m_1^2-2m_2^2-m_3^2+s_{11}+s_{22})],\\
\mathsf{N}_{3\to3;\widehat{23}}&=(d-2)\{[(m_1^2-m_2^2)^3+s_{11}^3](s_{22}-m_1^2-m_3^2)s_{22}+2s_{12}(m_1^2-m_2^2)[m_1^6+m_2^2(m_3^2-s_{22})^2\\
&\quad-m_1^2(m_1^2+m_2^2)(m_3^2+s_{22})]+s_{11}^2[2m_3^2(m_1^2-m_3^2)s_{12}+3(m_2^2+m_3^2)s_{22}(m_3^2-s_{22})\\
&\quad+s_{22}(m_1^2(3m_2^2-m_3^2+2s_{12}-2s_{22})+4m_3^2s_{12}+s_{22}(s_{22}-2s_{12}))+(m_1^2-m_3^2)^3]\\
&\quad+s_{11}[(3m_2^2(m_2^2+m_3^2)-m_1^2(m_2^2-3m_3^2))s_{22}^2-(m_1^2+m_2^2)((m_1^2-m_3^2)^3+s_{22}^3)\\
&\quad+(2m_1^4-3m_2^4-3m_3^4+4m_2^2m_3^2)s_{22}m_1^2+3s_{22}(m_2^2+m_3^2)(m_1^4-m_2^2m_3^2)\\
&\quad+s_{12}(2m_1^2((m_3^2-s_{22})^2-2m_2^2(m_3^2+s_{22}))+4m_2^2(m_3^2-s_{22})^2-2m_1^6)]\},\\
\mathsf{D}_{3\to3;\widehat{23}}&=8m_1^2((m_1-m_2)^2-s_{11})((m_1+m_2)^2-s_{11})((m_1-m_3)^2-s_{22})((m_1+m_3)^2-s_{22})B_3^{1+}.
\end{aligned} \tag{4.89}$$

The integrals with higher poles presented in the tadpoles, bubbles, and triangles sections are consistent with [36].

V. SUMMARY AND OUTLOOK

This work has demonstrated a unifying framework that synergizes the Baikov representation and IBP relations to uniformly reduce one-loop integrals with arbitrary tensor structures and high poles. Although our recursion formula includes a term with ∂_R, this poses little difficulty given the simplicity it provides in avoiding tedious algebraic manipulations. Most importantly, one can easily and consistently treat various degenerate cases appearing in our method. The degeneracy of $\det Q = 0$ in [33] may not be immediately apparent using our method. However, it is worth noting that our degenerate origin, represented by $\mathcal{G}_{00}$ and $\mathcal{G}_0$, does not vanish, and our recursion relation remains valid. Although their tensor reduction cannot be effectively handled by our algorithm, we can utilize their scalar reduction result as an input to obtain the tensor reduction.

To restore the general tensor structure in the tensor reduction of L-loop integrals, it is necessary to introduce L auxiliary vectors. In contrast to the one-loop case, the inclusion of ISPs such as $\ell_i \cdot R_j$ and $\ell_i \cdot p_j$ becomes necessary. We refer to the ISP $\ell_i \cdot R_i$ as R-ISP as they emerge in the momentum representation. In general, we can derive L recursion relations by considering differentiation with respect to the L R-ISPs.[1] However, these relations alone are insufficient due to the presence of ISPs in $\partial_{z_{0i}}\mathcal{G}(z)$ and $\mathcal{G}(z)$, where $z_{0i} = \ell_i \cdot R_i$. Unfortunately, there is no established method for handling these terms effectively. One approach is to translate these terms into differential operators acting on the auxiliary vectors. Consequently, unlike in the one-loop case, we obtain differential equations instead of pure recursion relations. To solve the tensor reduction problem, one can consider expanding the reduction coefficient based on its tensor structure, which leads to L recursion relations. Next, we introduce a linear combination of propagators in the numerator by applying differential operators. This gives rise to N recursion relations, where N is the number of propagators. Assuming that there are E independent external momenta, if $L + N < \frac{L(L+1)}{2} + EL$, we can consider applying $\partial_{z_{\mathrm{ISP}}}$ to generate additional required recursion relations. In this way, we can ultimately solve the tensor reduction problem. However, it should be noted that redundancy may arise when the number of ISPs is large compared to the number of extra recursion equations needed.

To illustrate this process, let us consider the sunset diagram as an example. We introduce two auxiliary vectors, R_1 and R_2, which results in more ISPs involved in the Baikov representation, namely, $\ell_1 \cdot p$, $\ell_2 \cdot p$, $\ell_1 \cdot R_2$, and $\ell_2 \cdot R_1$. There are five tensor structures involving R_1 and R_2, i.e., $R_1^2, R_2^2, R_1 \cdot R_2, R_1 \cdot p$, and $R_2 \cdot p$. The sunset has exactly two R-ISPs and three propagators. We are fortunate enough to solve the reduction problem by utilizing a set of $L + N = 5$ recursion relations which are derived through the loop-by-loop reduction and constructing propagators [31]. Here, we can simply make use of the IBP relations generated by $\partial_{z_{0i}}$:

$$\int \partial_{z_{0i}} \left[\frac{\mathcal{G}(z)^\gamma z_{0i}^{r_i}}{\prod_j z_j} \right] = 0. \tag{5.1}$$

During the reduction process, one may come across terms in the numerator that involve z_{ISP} and z_j. The presence of terms involving z_j allows for a reduction of the integral to a known sector with a lower topology. On the other hand, handling the terms containing z_{ISP} is relatively straightforward, as they can be readily translated into differential operators acting on R_1 and R_2. So finally we obtain two partial differentials of the standard integral $I_{1,1,1}^{(r_1,r_2)}$. The remaining three recursion relations are obtained by constructing propagators in the numerator through the application of three differential operators:

$$\partial_{R_1} \cdot \partial_{R_1}, \qquad \partial_{R_2} \cdot \partial_{R_2}, \qquad \partial_{R_1} \cdot \partial_{R_2}. \tag{5.2}$$

It is easy to find

$$\begin{aligned}
\partial_{R_1} \cdot \partial_{R_1} I_{1,1,1}^{(r_1,r_2)} &= 4r_1(r_1-1)\left[m_1^2 I_{1,1,1}^{(r_1-2,r_2)} + I_{0,1,1}^{(r_1-2,r_2)}\right], \\
\partial_{R_2} \cdot \partial_{R_2} I_{1,1,1}^{(r_1,r_2)} &= 4r_2(r_2-1)\left[m_2^2 I_{1,1,1}^{(r_1,r_2-2)} + I_{1,0,1}^{(r_1,r_2-2)}\right], \\
\partial_{R_1} \cdot \partial_{R_2} I_{1,1,1}^{(r_1,r_2)} &= 2r_1 r_2\Big[I_{1,1,1;\hat{3}-\hat{1}-\hat{2}}^{(r_1-1,r_2-1)} - (p^2 + m_1^2 + m_2^2 - m_3^2) \\
&\quad \times I_{1,1,1}^{(r_1-1,r_2-1)}\Big] + 2r_2 p \cdot \partial_{R_1} I_{1,1,1}^{(r_1,r_2-1)} \\
&\quad + 2r_1 p \cdot \partial_{R_2} I_{1,1,1}^{(r_1-1,r_2)}.
\end{aligned} \tag{5.3}$$

To convert the aforementioned differential equations into recursion relations, one can expand the reduction coefficient $C_a^{(r_1,r_2)}$ in $I_{1,1,1}^{(r_1,r_2)} = \sum_{a=1}^7 C_a^{(r_1,r_2)} I_a$ based on the tensor structure of R_1 and R_2,

$$\begin{aligned}
C_a^{(r_1,r_2)} = \sum_{\{v\}} & c_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a;(r_1,r_2)} (R_1^2)^{\nu_1} (R_2^2)^{\nu_2} (R_1 \cdot R_2)^{\nu_3} \\
& \times (R_1 \cdot p)^{\nu_4} (R_2 \cdot p)^{\nu_5},
\end{aligned} \tag{5.4}$$

with

$$2\nu_1 + \nu_3 + \nu_4 = r_1, \qquad 2\nu_2 + \nu_3 + \nu_5 = r_2. \tag{5.5}$$

The five recursion relations of the expansion coefficients $c_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a;(r_1,r_2)}$, as explicitly discussed in [31], provide a complete solution for the tensor reduction of integrals with the sunset topology. For higher-loop tensor integrals with $L + N < \frac{L(L+1)}{2} + EL$, one can initially generate $L + N$ recursion relations using a similar approach. Subsequently, the remaining recursion relations needed for these cases can

[1] In the subsequent discussion, unless explicitly indicated, the term “ISP” refers to ISPs other than R-ISPs.

be obtained from IBPs generated by taking derivatives with respect to other ISP parameters. As can be anticipated, higher loops will inevitably result in high-degree polynomials. As previously discussed, it is necessary to convert all ISPs in the numerator (excluding the R-ISPs) into differential operators. This transformation can lead to a series of complex partial differential equations with high-order derivatives. In principle, these intricate calculations can be delegated to a computer. However, we must actually resolve the linear equations for the expansion coefficients. Indeed, the method encounters challenges as the total rank $r_1 + r_2$ (in the case of two loops) and the number of external momenta increases. This leads to a rapid proliferation in the number of linear equations involved in the reduction process. Improvement for this method for higher loops is left for future research and exploration.

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APPENDIX: PENTAGON

In this Appendix, we will provide an additional illustrative example of a pentagon. For simplicity, we will give the numerical result. Let us begin by setting up the numerical framework: $s_{11} = \frac{1}{13}, s_{12} = \frac{1}{17}, s_{13} = \frac{1}{19}$, $s_{14} = \frac{1}{23}, s_{22} = \frac{2}{29}, s_{23} = \frac{2}{31}, s_{24} = \frac{2}{37}, s_{33} = \frac{3}{41}, s_{34} = \frac{3}{43}$, and $s_{44} = \frac{3433409242718675-830060336174604588086 8t}{48244934730591561}$, with $t = \mathcal{G}_{00}$. The definition of s_{ij} remains consistent with the previous formulation:

(i) $r = 1$

$$C_5^{(1)} = -\frac{\mathsf{N}_5^{(1)}(48324393052 s_{01} - 863984088446 s_{02} + 1984130693427 s_{03} - 1318419772377 s_{04})}{109436795947849827995337870191832 36t} + \mathcal{O}(t^0), \quad \text{(A1)}$$

$$C_{5;\hat{5}}^{(1)} = \frac{-48324393052 s_{01} + 863984088446 s_{02} - 1984130693427 s_{03} + 1318419772377 s_{04}}{226835825478808676t} + \mathcal{O}(t^0), \quad \text{(A2)}$$

where

$$\mathsf{N}_5^{(1)} = 165198902691154 + 5487075499569992 m_1^2 + 1768334514951836 m_2^2 - 31615769748504478 m_3^2 + 72605294464574211 m_4^2 - 48244934730591561 m_5^2. \quad \text{(A3)}$$

(ii) $r = 2$

In this case, we additionally assume that all m_i are equal, denoted as $m_i = m$, and we take $d = 6$ to avoid excessive complexity,

$$C_5^{(2)} = \frac{6822669362590342075877462929\,\mathsf{N}_5^{(2)}}{19960687178885534221068998904386539063038990526693942059576757238616t^2} + \mathcal{O}(t^{-1}), \quad \text{(A4)}$$

$$C_{5;\hat{5}}^{(2)} = -\frac{82599451345577\,\mathsf{N}_5^{(2)}}{827472864886215336096756770932755381383603943518512t^2} + \mathcal{O}(t^{-1}), \quad \text{(A5)}$$

$$C_{5;\hat{4}\hat{5}}^{(2)} = \frac{67\,\mathsf{N}_5^{(2)}}{380016026464966029473624813316t} + \mathcal{O}(t^0), \quad \text{(A6)}$$

where

$$\mathsf{N}_5^{(2)} = (48324393052 s_{01} - 863984088446 s_{02} + 1984130693427 s_{03} - 1318419772377 s_{04})^2. \quad \text{(A7)}$$

The time required to obtain the results for the tensor rank of a pentagon is presented in Table I. The parameter setting remains the same as for the case of $r = 1$, without any constraints on the values of m_i and d. Our method is a simple *Mathematica* program, while the FIRE6 algorithm is executed in parallel using a total of ten computing threads. It is evident that our method offers significantly faster computational efficiency than directly solving the IBP relations does.

We can also give the symbolic result for the pentagon using our algorithm. Setting all of the mass equal to zero, we find that the times for $r = 1$ and 2 are about 6 and 15 s, respectively,

$$C_5^{(1)} = \frac{\mathsf{N}_5^{(1)}}{\mathsf{D}_5^{(1)}}, \qquad C_{5;\hat{5}}^{(1)} = \frac{\mathsf{N}_{5;\hat{5}}^{(1)}}{\mathsf{D}_5^{(1)}}, \tag{A8}$$

where

TABLE I. The consuming time for different ranks of our method vs FIRE6. To expedite the reduction process, we made the decision to set $m_1 = m_2 = m_3$ and $m_4 = m_5$ during the reduction process in FIRE6.

	$r = 1$	$r = 2$	$r = 3$
Our method	6 s	16 s	39 s
FIRE6	60 s	180 s	557 s

$$\begin{aligned}
\mathsf{N}_5^{(1)} = {} & s_{04}[s_{13}(s_{11}(s_{23}s_{24} - s_{22}s_{34}) + s_{12}s_{22}s_{34} - 2s_{12}s_{23}s_{44} + s_{12}s_{24}s_{33} + s_{14}s_{22}(s_{23} - s_{33})) \\
& + s_{11}(s_{12}(s_{23}s_{34} - s_{24}s_{33}) + s_{14}(s_{22}s_{33} - s_{23}^2) + s_{22}(s_{24}s_{33} - s_{23}s_{34} + s_{33}s_{34} - s_{33}s_{44}) \\
& + s_{23}^2 s_{44} - s_{23}s_{24}s_{33}) + s_{12}s_{33}(-s_{12}s_{34} + s_{12}s_{44} - s_{14}s_{22} + s_{14}s_{23}) + s_{13}^2 s_{22}(s_{44} - s_{24})] \\
& + s_{02}[s_{14}(s_{11}(s_{23}s_{34} - s_{24}s_{33}) + s_{12}s_{33}(s_{34} - s_{44}) + s_{13}(-2s_{22}s_{34} + s_{23}s_{44} + s_{24}s_{33})) \\
& + s_{13}s_{44}(s_{12}(s_{34} - s_{33}) + s_{13}(s_{22} - s_{24})) + s_{11}s_{34}(s_{34}(s_{22} - s_{12}) + s_{24}(s_{13} - s_{33})) \\
& + s_{11}s_{44}(s_{33}(s_{12} - s_{22} + s_{23} + s_{24}) - s_{23}(s_{13} + s_{34})) + s_{14}^2 s_{33}(s_{22} - s_{23})] \\
& + s_{03}[s_{14}(s_{11}(s_{23}s_{24} - s_{22}s_{34}) + s_{12}(s_{22}s_{34} + s_{23}s_{44} - 2s_{24}s_{33}) + s_{13}s_{22}(s_{24} - s_{44})) \\
& + s_{11}s_{44}(-s_{12}s_{23} + s_{22}(s_{13} + s_{23} - s_{33} + s_{34}) - s_{23}s_{24}) + s_{14}^2 s_{22}(s_{33} - s_{23}) \\
& + s_{11}s_{24}(s_{34}(s_{12} - s_{22}) - s_{13}s_{24} + s_{24}s_{33}) + s_{12}s_{44}(s_{12}(s_{33} - s_{34}) + s_{13}(s_{24} - s_{22}))] \\
& + s_{01}[s_{14}(s_{22}s_{23}s_{34} - s_{22}s_{33}(s_{24} + s_{34} - s_{44}) + s_{23}(s_{24}s_{33} - s_{23}s_{44})) \\
& + s_{44}(s_{12}s_{33}(s_{22} - s_{23} - s_{24}) + s_{12}s_{23}s_{34} - s_{13}s_{22}(s_{23} - s_{33} + s_{34}) + s_{13}s_{23}s_{24}) \\
& + s_{11}(s_{22}(s_{34}^2 - s_{33}s_{44}) + s_{23}(s_{23}s_{44} - 2s_{24}s_{34}) + s_{24}^2 s_{33}) \\
& - (s_{13}s_{24} - s_{12}s_{34})(s_{24}s_{33} - s_{22}s_{34})], \\
\mathsf{N}_{5;\hat{5}}^{(1)} = {} & -s_{04}[s_{11}(s_{23}^2 - s_{22}s_{33}) + s_{12}^2 s_{33} - 2s_{12}s_{13}s_{23} + s_{13}^2 s_{22}] + s_{03}[s_{11}s_{23}s_{24} - s_{11}s_{22}s_{34} \\
& + s_{12}^2 s_{34} - s_{12}s_{13}s_{24} - s_{12}s_{14}s_{23} + s_{13}s_{14}s_{22}] + s_{02}[s_{11}s_{23}s_{34} - s_{11}s_{24}s_{33} + s_{12}s_{14}s_{33} \\
& - s_{13}(s_{12}s_{34} + s_{14}s_{23}) + s_{13}^2 s_{24}] + s_{01}[s_{12}s_{24}s_{33} - s_{12}s_{23}s_{34} + s_{13}s_{22}s_{34} - s_{13}s_{23}s_{24} \\
& + s_{14}(s_{23}^2 - s_{22}s_{33})], \\
\mathsf{D}_5^{(1)} = {} & s_{44}(-s_{11}s_{22}s_{33} + s_{11}s_{23}^2 + s_{12}^2 s_{33} - 2s_{12}s_{13}s_{23} + s_{13}^2 s_{22}) + s_{11}s_{22}s_{34}^2 - 2s_{11}s_{23}s_{24}s_{34} \\
& + s_{11}s_{24}^2 s_{33} - s_{12}^2 s_{34}^2 + 2s_{14}(s_{12}(s_{23}s_{34} - s_{24}s_{33}) + s_{13}(s_{23}s_{24} - s_{22}s_{34})) \\
& + 2s_{12}s_{13}s_{24}s_{34} - s_{13}^2 s_{24}^2 + s_{14}^2(s_{22}s_{33} - s_{23}^2).
\end{aligned} \tag{A9}$$

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