Extremal Higgs couplings

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We critically assess to what extent it makes sense to bound the Wilson coefficients of dimension-six operators. In the context of Higgs physics, we establish that a closely related observable, c_H, is well defined and satisfies a two-sided bound. c_H is derived from the low momentum expansion of the scattering amplitude, or the derivative of the amplitude at the origin with respect to the Mandelstam variable s, expressed as $M(H_iH_i \to H_jH_j) = c_H s + O(g_{\rm SM}, s^{-2})$ where $g_{\rm SM}$ represents all Standard Model couplings. This observable is *nondispersive* and, as a result, not sign-definite. We also determine the conditions under which the bound on c_H is equivalent to a bound on the dimension-six operator $O_H = \partial|H|^2\partial|H|^2$.

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I. INTRODUCTION AND CONCLUSIONS

The Higgs particle stands out as one of the most enigmatic particles discovered thus far. Examining it from every conceivable perspective is a crucial endeavor. In this paper, we initiate the theoretical study of the maximal Higgs coupling strengths. Present-day measurements of Standard Model (SM) Higgs couplings exhibit a good level of agreement with the SM theory. Nevertheless, the possibility of new physics emerging at the few TeVs scale remains a compelling avenue for beyond the Standard Model physics (BSM). Despite the potential need for some degree of fine-tuning, this avenue holds the promise of shedding light on the mechanism that governs electroweak symmetry breaking.

Precise measurements of Higgs couplings are of particular significance in scenarios where the Higgs is a light composite boson. New physics with strong couplings often involves heavy and broad resonances that may prove challenging to directly observe at the LHC. Nevertheless, these resonances can leave their mark as deviations in the SM Higgs couplings.

The deviations are largest in UV completions of the SM featuring strong coupling dynamics. In this context, the strongly interacting light Higgs [1] (SILH) effective field theory (EFT) provides valuable insights. It offers power counting rules for the Higgs EFT, where the Higgs emerges as a light pseudo-Goldstone boson of a strongly interacting sector. The Higgs becomes massless in the limit $g_{\rm SM} \to 0$ (where $g_{\rm SM}$ collectively denotes the SM couplings) and acquires a small mass through radiative corrections for $g_{\rm SM} \neq 0$.

We make the following simplifying assumption: we consider the UV BSM couplings to be significantly larger than the SM ones, allowing us to treat the latter as small perturbations. We should further assume that the composite sector enjoys a custodial global symmetry $SO(4) \simeq SU(2)_L \otimes SU(2)_R$. Under these assumptions, the Higgs sector of SILH simply reads $\mathcal{L}_{\rm H}^{\rm int} = \frac{g_H}{2f^2}\partial_\mu|H|^2\partial^\mu|H|^2 + \mathcal{O}(f^{-4})$, where we are neglecting operators of dimension (dim.) eight and higher. This effective description can break down either because of strong coupling dynamics at energies above the scale f^2, or because of the need to incorporate new resonances in the perturbative regime of the effective description.

The dim.-six operator $\partial_\mu|H|^2\partial^\mu|H|^2 = |H|^2|\partial H|^2 +$ e.o.m. (equations of motion) is interesting because, after accounting for the Higgs vacuum expectation value, it leads to a wave function renormalization of the Higgs which in turn results in a universal shift of all SM Higgs couplings.

We emphasise why our chosen simplification is interesting in the context of bounding Higgs coupling deviations: it retains the key complexities of the real problem and allows gradual step-by-step relaxation of these assumptions, paving the way for a more realistic model.

Causality and unitarity constraints of the two-to-two scattering matrix element imply sharp bounds on the $\mathcal{L}_H$ EFT. The two-to-two scattering of two $SO(4)$ vectors can proceed through three different channels,

$$\mathbf{M} = M(\bar{s}|\bar{t},\bar{u})\delta_{ab}\delta^{cd} + M(\bar{t}|\bar{u},\bar{s})\delta_a^c\delta_b^d + M(\bar{u}|\bar{s},\bar{t})\delta_a^d\delta_b^c, \quad (1)$$

annihilation, reflection, and transmission of the vector indices $\{a,b,c,d\}$ which run from 1 to $n=4$. We often use shifted Mandelstam variables $(\bar{s},\bar{t},\bar{u}) \equiv (s,t,u) - 4m^2/3(1,1,1)$. Recall that momentum conservation implies $s+t+u=4m^2$. Crossing symmetry dictates that the physical process (1) can be described as the boundary value of a single analytic function with the symmetry $M(\bar{s}|\bar{t},\bar{u}) = M(\bar{s}|\bar{u},\bar{t})$.

The unitarity and crossing symmetry imply a cut that extends from $(s,t,u)=4m^2$ to infinity. Therefore, the point $(\bar{s},\bar{t},\bar{u})=0$ is analytic, and the amplitude $M(\bar{s}|\bar{t},\bar{u})$ can be characterized by its series around the origin,

$$M/(4\pi)^2 = c_\lambda + c_H\bar{s} + c_2(\bar{t}^2+\bar{u}^2) + c_2'\bar{s}^2 + \mathcal{O}(\bar{s},\bar{t},\bar{u})^3, \quad (2)$$

where the coefficients c_λ, c_H, c_2, c_2' are real valued, and $(4\pi)^2$ is a convenient normalization. The Wilson coefficient g_H describes the single dim.-six operator contribution to (2) at tree level: $c_\lambda = \mathcal{O}(\frac{m^2}{f^2})$, $c_H = \frac{g_H}{(4\pi^2)f^2} + \mathcal{O}(\frac{m^2}{f^4})$.

The main result of this work is a bound on the parameter c_H in (2). We discuss two extreme *single energy scale scenarios* and argue that interesting physics lies in the interpolation of the two.

In the first scenario, Sec. III A, we look for the extremal values of c_H by making no assumption of weak coupling. We are led to the rigorous bound,[1]

$$-0.46 < c_H \times m^2 < 1.07. \quad (3)$$

The single scale in the problem is the mass m^2; thus, we can set units $m^2 = 1$. This bound is saturated by amplitudes that are maximally strongly coupled all the way down to the IR $s \gtrsim 4m^2$. These amplitudes do not feature an energy scale separation between m^2 and a putative scale of new physics. Therefore, a simple EFT interpretation of (3) in terms of operators is hardly possible. Although this is not a useful bound for Higgs physics, it is nevertheless an interesting *proof of principle* for the existence of a universal bound: any theory with the same symmetries must take values within (3). In Sec. III B, we discuss how to isolate weakly coupled amplitudes within the space of nonperturbative $O(4)$ theories.

The second scenario, Sec. III C, is complementary and assumes that physics below a new energy scale Λ^2 is much weaker than new physics above Λ^2 and a large scale separation $\Lambda^2 \gg m^2$. In this limit, we are left with a single scale Λ^2, and we find

$$-0.31 < c_H \times \Lambda^2 < 0.35. \quad (4)$$

The bound (4) is saturated by amplitudes that on one hand are maximally strongly coupled above the cutoff scale Λ, but on the other hand are very weakly coupled below. This limiting case has been dubbed *UV dominated EFTs* [2]. In this single scale problem, we can set units $\Lambda^2 = 1$ and interpret (4) as a universal bound on the space of UV dominated EFTs. We argue that, under certain specific conditions, this bound can be interpreted in terms of $\mathcal{L}_H$ and identify $\Lambda^2 = f^2$ and $c_H \times f^2 = g_H/(4\pi)^2$. In Sec. III C, we also explain how to smoothly interpolate the bounds of the two limiting scenarios (3) and (4), see Fig. 3. In Sec. III D, we show how to incorporate IR EFT corrections in order to obtain a more refined bound.

We end this paper with a discussion on the interpretation of the bounds in terms of dim.-six operators, Sec. IV, and with a final discussion about future directions, Sec. V.

We have included a number of appendices with details on the calculations.

II. DUAL BOOTSTRAP FOR $O(n)$ THEORIES

A. Setup and constraints

We begin this section by discussing the constraints on the amplitude (1) that we use. The amplitude satisfies the double-subtracted dispersion relation,

$$\begin{aligned}\mathcal{A}^I(s,t) \equiv M^I(s,t) - \mathbb{C}^I(s,t) \\ - \int_4^\infty dz[\mathbb{K}^{IJ}(z;s,t)M_z^J(z,t) \\ + \mathbb{L}^{IJ}(z;s,t)M_z^J(z,z_0)] = 0, \quad (5)\end{aligned}$$

where $\vec{\mathbb{C}}(s,t) = c_\lambda(n+2,2,0) + c_H(n-1,-1,1) \quad \bar{s} + c_H(0,0,2)\bar{t}$ and is decomposed into irreducible representation channels $M^I \equiv (M^{\text{sing}}, M^{\text{sym}}, M^{\text{anti}})$. The Kernels $\mathbb{K}$ and $\mathbb{L}$ are simple rational functions of its arguments; the derivation of (5) is given in Appendix A. In (5) and in the

[1] We worked with high precision numerical integrals up to $O(10^{-80})$ to avoid the rounding errors and kept a duality gap of $O(10^{-15})$ when solving the optimization problem. The final source of systematic errors is due to the number of negativity constraints imposed (C4), under the limit $L_c \to \infty$ as explained in Appendix C 2. We set the spin cutoff to $L_c = 20$, beyond which bounds (3) and (4) change only by $O(10^{-3})$.

rest of this section, we set $m^2 = 1$. This is a technical section that some readers may prefer to skip on their first reading and proceed directly to Sec. III for a discussion of the physical results.

By taking derivatives of (5), one can express any low energy coefficient c_i of (2) in terms of a sum rule involving integrals over the amplitude's discontinuity. If the definition of c_i involves more than two derivatives of the amplitude with respect to "s", the subtraction terms are not present in the sum rule because $\partial_s^2 \vec{\mathbb{C}}(s,t)$ in (5) vanishes. In this case, the c_i's may enjoy positivity properties that follow from $\mathrm{Im}M^J \geq 0$. Instead, if the definition of c_i involves less than two derivatives of the amplitude, the subtraction terms are present, and thus, the positivity of the sum rule is typically spoiled. This is the case of c_H,

$$c_H \frac{\pi}{3}(s-4) = \mathrm{Re} f_1^{(3)}(s) - \int_4^\infty dv k_{1,\ell}^{(3,J)}(s,v) \mathrm{Im} f_\ell^{(J)}(v), \quad (6)$$

where repeated indices ℓ and J are summed over, and the Kernel $k_{1,\ell}^{(3,J)}$ is the partial wave projection of $\mathbb{K}$ and $\mathbb{L}$, its exact form is given in Appendix A, and $f_j^{(I)}$ are the partial wave projections of the amplitude.

Because of the presence of subtraction terms in the sum rule (6), the value of c_H is not sign definite. Previous works analyzed the sign constraints of c_H by means of unsubtracted dispersion relations [3,4] and positivity constraints $\mathrm{Im} f_j^{(I)} > 0$.

Positivity constraints follow from the unitary inequality,

$$\mathcal{U}_\ell^{(I)} \equiv 2\mathrm{Im} f_\ell^{(I)}(s) - \rho(s) |f_\ell^{(I)}(s)|^2 \geq 0, \quad (7)$$

where $\rho(s) = \sqrt{(s-4)/4}$. Unitarity constraints bound both the real $\mathrm{Re} f_j^{(I)}$ and imaginary $\mathrm{Im} f_j^{(I)}$ parts of the amplitude. Therefore, by using the unitarity constraints (7) (instead of positivity constraints only), we may hope to be able to bound the minimal and maximal value that c_H in (6) can attain. Establishing the existence of this bound is nontrivial, as it involves an infinite sum over partial waves on the right-hand side. Nevertheless, we demonstrate in the next section that this hope is indeed realized.

Before we proceed, there is one remaining class of constraints to address. We encoded analyticity in the fixed-t dispersion relation (6), which is $s \leftrightarrow u$ symmetric, but lacks $t \leftrightarrow u$ crossing symmetry constraints,

$$\vec{\mathcal{F}}(s,t) = \vec{M}(s,t) - C_{tu}.\vec{M}(s, 4-s-t) = 0. \quad (8)$$

In order to extract a discrete number of constraints from the last equation, we plug the dispersion relation (5) for M^I in the crossing Eq. (8), next expand into partial waves, and finally take a number of derivatives $\partial_s^n \partial_t^m$ at $s,t = 4/3$. We are left with

$$\mathcal{F}_{n,m}^{(I)} \equiv \int_4^\infty dv \sum_{\ell,J}^\infty F_{n,m;\ell}^{(IJ)}(v) n_\ell \mathrm{Im} f_\ell^{(J)}(v) = 0, \quad (9)$$

with $F_{n,m;\ell}^{(IJ)}(v) = \partial_t^n \partial_s^m F_\ell^{(IJ)}(v,s,t)|_{s,t=\frac{4}{3}}$, $n_\ell = 16\pi(2\ell+1)$. The exact form of $F_\ell^{(IJ)}$ follows from projecting the Kernels in (5) into partial waves. On a first reading of this paper, its exact details are not too important to follow the logic flow. We note that the lowest nontrivial constraint is for $(n,m) = (1,3)$ derivatives. For instance, for the $(I,J) = (3,1)$ channel, we have $F_{1,3;\ell}^{(31)}(v) = \frac{(l+1)(x-1)^3[([l+2]x^2 - l)P_\ell(x) - 2xP_{\ell+1}(x)]}{x^3(x+1)^2}$, with $x = \frac{4-3v}{12-3v}$, and P_ℓ are Legendre polynomials. Similarly, higher order derivatives give rise to functions $F_{n,m;\ell}^{(IJ)}(v)$ consisting of linear combinations of Legendre polynomials times rational functions of v and ℓ. The constraints in (9) are equivalent to the null constraints [5,6].

B. Rigorous dual bounds

Our task now is to find the extremal values of c_H under the constraints of unitarity, analyticity, and crossing symmetry. We adapt to our needs the rigorous setup developed in [7].[2] With all the constraints laid down, an optimization problem is best summarized by means of a Lagrangian,

$$L_\pm(\{P\},\{D\}) = \pm c_H + \underbrace{\sum_{(n,m)=(1,3)}^{(n_c,m_c)} \nu_{n,m}^{(I)} \mathcal{F}_{n,m}^{(I)}}_{s\leftrightarrow t\,\text{crossing}} + \underbrace{\int_4^\infty dv \lambda_\ell^{(I)}(v) \mathcal{U}_\ell{}^{(I)}(v)}_{\text{unitarity}} + \underbrace{\int_4^{\mu_c} dv \sum_{j=0}^{J_c} \omega_j^{(I)}(v) a_j^{(I)}(v)}_{\text{analiticity}+s\leftrightarrow u\,\text{crossing}}, \quad (10)$$

where repeated indices I and ℓ are summed over, and $a_j^{(I)}(v)$ is the jth partial wave projection of $\mathcal{A}^I(s,t)$.

The first term in (10) is the objective to optimize. The next terms encode the crossing symmetry (9), unitarity (8), and analiticity (5) constraints by means of the Lagrange multipliers $\{\nu_{n,m}^{(I)}, w_J^{(I)}, \lambda_\ell^{(I)} \geq 0\}$. Collectively, all the Lagrange multipliers are denoted as *dual variables* $\{D\}$. The primal variables $\{P\}$ are given by $\{\mathrm{Re} f_\ell^{(I)}(s), \mathrm{Im} f_\ell^{(I)}(s), c_H, c_\lambda\}$.

[2]The dual approach to the bootstrap was first revisited in two dimensions [8] and later generalized to scattering of several species [9] and flux tubes [10]. In higher dimensions, dual bounds were studied already long ago in [11–15]. A different approach based on the Mandelstam representation was developed in [16].

We keep a finite number of crossing $\mathcal{F}_{m,n}$'s and spin projections $a_j(s)$'s constraints—the maximal number is labeled by (n_c, m_c) and J_c, respectively. Similarly, we keep a maximal value μ_c in the evaluation of the projected dispersion relation $\mathcal{A}$. Even though this is enough for the derivation of a rigorous bound, we are not leveraging the full power of all the constraints that we know of. However, we argue that the observable that we are studying converges rapidly in (n_c, m_c), J_c, and μ_c. Therefore, despite of truncating the number of constraints, our bounds will be close to optimality.

Say we are interested in maximizing c_H in (10) (i.e., we take $+$ in the first term). The *weak duality theorem* [17] states that

$$c_H \leq d_+(\{D\}) \equiv \max_{\{P\}} L_+(\{P\}, \{D\}). \tag{11}$$

The maximization over the primal variables is straightforward since the Lagrangian (10) is a quadratic function. Below. we summarize the features of the dual problem; further details can be found in Appendix B.

First, we discuss the maximization of L with respect to c_λ and c_H. Since they enter linearly into the Lagrangian as αc_i, by taking derivatives we get that α must vanish, yielding the two normalization conditions,

$$c_\lambda\colon\ 0 = \int_4^{\mu_c} \frac{dv}{16\pi} \vec{n}_1 \cdot \vec{N}(v), \tag{12}$$

$$c_H\colon 1 = \int_4^{\mu_c} \frac{dv}{16\pi} \left[\left(v - \frac{4}{3}\right)\vec{n}_2 + (v-4)\vec{n}_3\right] \cdot \vec{N}(v), \tag{13}$$

where $\vec{n}_1 = (n+2, 2, 0)$, $\vec{n}_2 = (n-1, -1, 0)$, $\vec{n}_3 = (0, 0, 1/3)$, and $\vec{N} = (\omega_0^1, \omega_0^2, \omega_1^3)$. Maximizing the Lagrangian (10) with respect to $\{\mathrm{Re}f_l^{(I)}, \mathrm{Im}f_\ell^{(I)}\}$ and substituting the equations of motion leads to

$$d_+(\nu, \lambda, \omega) = \int_4^\infty \frac{dv}{\lambda_\ell \rho} \left(\frac{n_\ell \mu_\ell}{2} + \lambda_\ell\right)^2 + \int_4^{\mu_c} \frac{dv}{4\lambda_j \rho} \omega_j^2, \tag{14}$$

where we left implicit a sum on the channels I, and repeated indices are summed over according to (10),

$$\mu_\ell^{(I)} = \nu_{n,m}^{(K)} \cdot F_{n,m;\ell}^{(KI)}(v) - \text{p.v.} \int_4^{\mu_c} w_j^{(K)} k_{j,\ell}^{(KJ)}(s, v), \tag{15}$$

where repeated indices $\{K, n, m, j\}$ are summed over, the j and (n, m) sums are cut according to (10), "p.v." denotes the Cauchy principal value, and $\ell = 0, 1, \ldots\infty$. The kernel $k_{j,\ell}^{(KJ)}$ is given in Appendix A.

For any value of the multipliers, the inequality holds $d_+(\nu, \lambda, \omega) \geq c_H$, and we obtain a bound on c_H. Thus, to obtain the best bound, we should minimize d_+ over the Lagrange multipliers. In practice, it is hard to perform such minimization analytically. Nevertheless, an efficient numerical algorithm to search for the minimal value of d_+ in the ν, λ, ω space was developed in [7]. The generalization to our problem is explained in Appendix C.

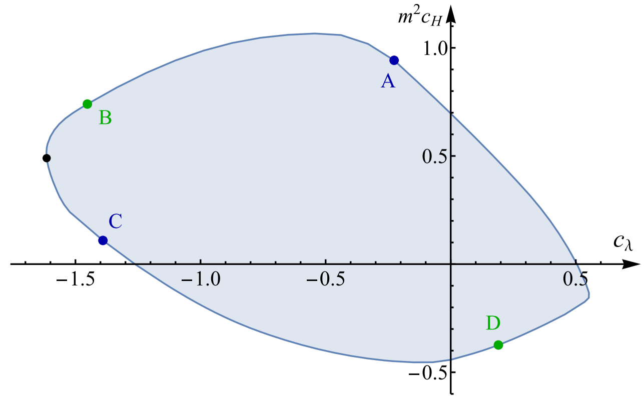

FIG. 1. All $O(4)$ theories must lie inside the colored region.

III. THE SPACE OF $O(4)$ THEORIES

A. The $O(4)$ nonperturbative island

Our first goal is to determine universal bounds on c_λ and c_H defined in (2). By universal, we mean that we are not making any assumption beyond the rigorous analyticity, crossing, and unitarity properties [18]. For concreteness, we set $c_\lambda = R\cos\theta$, $c_H = R\sin\theta$, and for each fixed θ, we bound the maximum value of R. Our numerical results are shown in Fig. 1—see Appendix C for detailed explanations on the numerics. Everything except for the blue region is rigorously excluded: all $O(4)$ theories must take values inside the blue "$O(4)$ island" in Fig. 1.

The boundary of the island is determined by the extremal values that c_λ and c_H can take. As we are not making any assumption, it is natural to expect that our bounds will be saturated by strongly coupled amplitudes all the way to the IR. A signature of strongly coupled IR dynamics is the presence of bound states or resonances. We experimentally observe the presence of scalar threshold bound states in the spin zero singlet and/or in the symmetric channel. Using this knowledge, we can define four distinct regions on the boundary of the island, whose properties are summarized in Table I.

For instance, in the region DA, we have both the singlet and the symmetric threshold bound states. On the other hand, in region BC, there are none. However, even without threshold bound states, there are other strong coupling

TABLE I. Threshold singularities along the boundary.

	A-B	B-C	C-D	D-A
Singlet	✓	✗	✗	✓
Symmetric	✗	✗	✓	✓

phenomena happening. Between the point B and the point with the minimum value of c_λ denoted by a black dot, although the value of c_H is positive, we measure a negative spin one scattering length in the antisymmetric channel. This change of sign cannot be realized with a weakly coupled field theory description. On the other hand, between the black dot and point C, we find a spin one resonance at low energies. In Appendix F, we have included a number of plots showing the phase shifts of the amplitude around the boundary of Fig. 1.

B. Perturbative boundary regions

There are two linear combinations of dimension-eight operators of the $O(n)$ theory (2) that are positive [19],

$$c_2 = \frac{1}{\pi}\int_4^\infty \frac{dv}{\bar{v}^3}\mathrm{Im}\vec{M}(v)\cdot\left(0,\frac{1}{2},\frac{1}{2}\right) \geq 0, \tag{16}$$

$$2c_2 + c_2' = \frac{1}{\pi}\int_4^\infty \frac{dv}{\bar{v}^3}\mathrm{Im}\vec{M}(v)\cdot\left(\frac{1}{n},\frac{n-1}{n},0\right) \geq 0. \tag{17}$$

Both inequalities are saturated when the theory is free. Therefore, scanning the values of c_λ and c_H in the region where these two linear combinations are small, we single out weakly coupled extremal amplitudes at the boundary of the allowed region. An example of this, is shown in Fig. 2. The red line is analytically computed by performing a one-loop computation in $\lambda|\vec{\phi}|^4$, choosing the scheme in which $\lambda = c_\lambda$, and plotting the parametric curve $\{c_H(c_\lambda), c_2(c_\lambda)\}$—in Appendix D 1, these functions are given. The red line agrees with the boundary of the allowed region for small c_λ. Interestingly, we do not have a perturbative description of the whole region around the origin. The boundary is expected to be saturated by amplitudes obtained from integrating out strongly coupled UV dynamics.[3]

C. EFT bounds

A key property of EFTs is the scale separation between the mass of the scattered particle and the scale of "new physics" $\Lambda^2 \gg m^2$. To incorporate the separation of scales nonperturbatively, it is useful to introduce the concept of *UV/IR domination* of the sum rules. Consider the dispersive representation of c_2 split into two pieces,

$$c_2^{\mathrm{IR}} = \int_4^{\Lambda^2} \frac{dv}{\pi\bar{v}^3}\mathrm{Im}\vec{M}(v)\cdot\left(0,\frac{1}{2},\frac{1}{2}\right), \qquad c_2^{\mathrm{UV}} = c_2 - c_2^{\mathrm{IR}}. \tag{18}$$

[3]It is possible to ask several other variations of the questions that we have asked so far; e.g., one could min./max. c_H a a function of $\{\alpha,\beta\}$, with $\alpha \equiv \max(c_2)$ and $\beta \equiv \max(2c_2 + c_2')$. Small values of $\{\alpha,\beta\}$ isolate perturbative amplitudes.

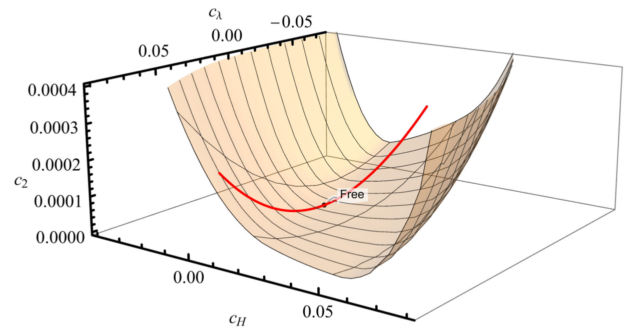

FIG. 2. A portion in the boundary of $O(4)$ theories that can be described by perturbative quantum field theory.

If $c_2^{\mathrm{IR}} \gg c_2^{\mathrm{UV}}$, then the sum rule is *IR dominated*. Conversely, if $c_2^{\mathrm{IR}} \ll c_2^{\mathrm{UV}}$, the dispersive integral receives the largest contribution from values at $s \gtrsim \Lambda^2$, in which case we say it is *UV dominated*. In the case of c_H, this separation is less universal because of the explicit subtraction term. However, being IR or UV dominated is a physical property of the amplitude, not just of the sum rule. In the case of c_H, we apply this definition to the dispersive part of the sum rule.

The bounds discussed in Secs. III and III A are IR dominated either because the amplitude is strongly coupled at values of $s \gtrsim m^2$ (Fig. 1) or because the amplitude is weakly coupled at all energies with no significant resonance behavior (Fig. 2). In either case, there is no effective separation of scales between m^2 and a putative "new physics" scale Λ^2.

Next, we are interested in the other limit, i.e., theories that are fully UV dominated. These are EFTs that are so weakly coupled in the IR $s \leq \Lambda^2$ that the IR contribution to the dispersive integrals is negligible with respect to the UV contribution.[4] In this scenario, the sum rule (6) gets replaced by

$$\frac{\pi}{3}c_H s = \mathrm{Re}f_1^{(3)}(s) - \int_{\Lambda^2}^\infty dv k_{1,\ell}^{(3,J)}(s,v)\mathrm{Im}f_\ell^{(J)}(v), \tag{19}$$

where ignore powers of m^2 because we are considering $\Lambda^2 \gg m^2$. In [2], it was shown that the bounds obtained in this regime are valid even in presence of a small physical IR imaginary part, which can be incorporated into a systematic error on the bound itself. The smaller the IR physics is, the better is this approximation.

The UV contribution is not necessarily strongly coupled for UV domination to hold. For instance, the whole amplitude may be well approximated at tree level at all energies, but the exchange of a tree-level resonance localizes

[4]Many recent interesting developments [20] exploiting positivity constraints apply in this regime.

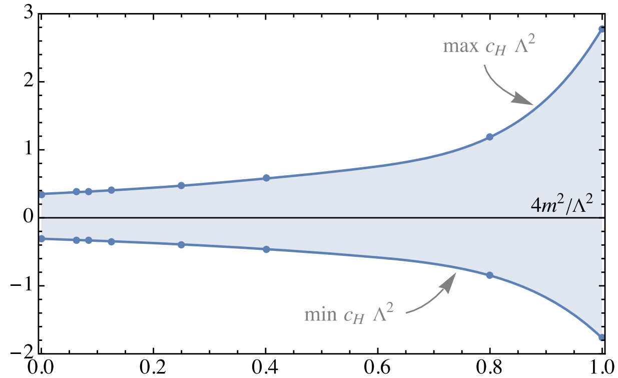

FIG. 3. Allowed value of $c_H \times \Lambda^2$ as a function of $4m^2/\Lambda^2$.

with a delta function $[\mathrm{Im}(s-\Lambda^2+i\epsilon)^{-1} \propto \delta(s-\Lambda)]$ the UV integral at $s=\Lambda^2$.[5]

Next, we find the min/max values of c_H. We do so by neglecting the imaginary part of the amplitude at values $s<\Lambda^2$ and by taking the massless limit $m\to 0$. The procedure is a simple modification of what we described in Sec. II, and thus, details are relegated to Appendix B. There is a single scale in the problem Λ^2, and therefore, bounds on c_H are naturally expressed by normalizing with respect to Λ^2. We are interested in looking for a field theory interpretation of the bound; therefore, we set $c_\lambda \ll 1$. We call the theories showing UV dominated sum rules *UV dominated EFTs*.

All in all, we find the result in (4). This is a universal bound to all UV dominated theories: as long as $m^2 \ll \Lambda^2$ and the dispersive part of $M(s|t,u)$ is negligible for $s<\Lambda^2$, any such theory should satisfy the bound.

For completeness, it is interesting to interpolate between the UV and IR domination regimes. We do so by min/max c_H defined through $\frac{\pi}{3}c_H(s-4m^2) = \mathrm{Re}f_1^{(3)}(s) - \int_{\Lambda^2}^{\infty} dv k_{1,\ell}^{(3,J)}(s,v)\mathrm{Im}f_\ell^{(J)}(v)$ neglecting the imaginary part of the amplitude in $[4m^2,\Lambda^2]$ and varying Λ within $[4m^2,\infty)$. The result of this exercise is given in Fig. 3. The rightmost points correspond to the min/max values of c_H in the $\frac{4m^2}{\Lambda^2}\to 1$ limit. Those points agree with Fig. 1, at $c_\lambda=0$, after accounting for the $\frac{4m^2}{\Lambda^2}$ normalization factor, $4\times[-0.44,0.70]=[-1.76,2.8]$. The leftmost points are instead the UV domination limit $\frac{m^2}{\Lambda^2}\to 0$, in agreement with (4). Note that for $\frac{m}{\Lambda}\lesssim\frac{1}{8}$, the bound is close to the asymptotic bound $\frac{m^2}{\Lambda^2}\to 0$ and shows variation only below the percent level.

One might ponder how the just-derived boundary will be influenced by the introduction of a small nonvanishing discontinuity in the IR. This matter is discussed in the next section.

[5]This is often the case for large N QCD-like theories [21,22].

D. Rigorous bounds assuming an IR model

In this section, we introduce a small IR imaginary part and study its effect on our dual bounds. We address this question using the following model. We imagine that somebody gives us a functional fit of the imaginary part of the amplitude for all spins and irreducible representations below a certain energy scale Λ,

$$\mathrm{Im}f_\ell^{(J)}(s) \equiv g_\ell^{(J)}(s), \qquad s<\Lambda^2. \tag{20}$$

We can now extremize the values of c_H combining the set of constraints in (10) with the new conditions (20). The generalization of the dual problem to include this additional constraint is straightforward and is discussed in Appendix E. The final dual bounds take the form

$$D_-^{\mathrm{UV}} + D_-^{\mathrm{IR}}[g_\ell^{(I)}] \le c_H \le D_+^{\mathrm{UV}} + D_+^{\mathrm{IR}}[g_\ell^{(I)}], \tag{21}$$

where D^{UV} is the functional used to obtain the EFT bounds shown in Fig. 3, and $D^{\mathrm{IR}}[g_\ell^{(I)}]$ is the IR contribution which vanishes if $g_\ell^{(I)}=0$. For concreteness, if we take $\Lambda=8m$, and choose

$$g_0^{(I)} = \frac{\lambda^2/2}{(16\pi)^2}\sqrt{\frac{s-4}{s}}((n+2)^2,4,0), \qquad g_{\ell>0}^{(I)}=0, \tag{22}$$

with $\lambda=0.1$, we obtain

$$-0.33 < c_H \times 8^2 < 0.38. \tag{23}$$

The difference between this bound and the one obtained by neglecting the imaginary part in Fig. 3 is of order 10^{-5} for this value of the quartic coupling.

In this analysis, we have not included nonlinear unitarity in the IR. If included, it would be interesting to compare with the bounds in [2], which were obtained by solving the primal problem. We leave this exploration to a future work.

IV. DIMENSION-SIX OPERATORS

The bounds on c_H that we have presented thus far are sharp and rigorous. Moving forward, next we aim to interpret them through an effective field theory Lagrangian. While this will necessitate making additional assumptions, it will also enable us to make further predictions. Once we establish a match between the amplitude's coefficient, denoted as c_H, and the effective operator $|\partial H|^2|H|^2$, it opens up new opportunities to test the constraints on c_H. Besides altering high energy $2\to 2$ scattering of the Higgs particles or longitudinal electroweak gauge bosons, this operator universally modifies all Higgs couplings. Indeed, after accounting for the Higgs vacuum expectation value, it leads to a Higgs wave function renormalization. Thus, the interpretation of our bound on

c_H in terms of the field theory operator allows us to determine the maximal deviation on Higgs couplings due to the O_H operator. For instance, $\Gamma(h \to VV)_{\rm SILH}/\Gamma(h \to VV)_{\rm SM} = 1 - (v/f)^2 c_H + \cdots$, where $\cdots$ denote other Wilson coefficients [1]. Thus, even though the dimension-six operator is not a clean observable, establishing a connection with the bound on c_H is a worthwhile exercise due to the physics motivations just explained.

A. Maximally UV dominated EFTs

Consider the field theory given by

$$\mathcal{L}_H = \mathcal{L}_{\rm Free} - \epsilon |H|^4 - \frac{g_H}{f^2} |\partial H|^2 |H|^2 + \mathcal{O}(f^{-4}), \qquad (24)$$

with $\epsilon = m^2/f^2 \ll 1$. A simple calculation of the amplitude (2) reveals $c_\lambda = O(\epsilon)$ and $c_H \times \Lambda^2 = \frac{g_H}{(4\pi)^2} \frac{\Lambda^2}{f^2} + \mathcal{O}(\epsilon)$. Higher order coefficients do receive possibly large corrections from g_H, e.g., $c_2 \times \Lambda^4 = g_H^2 \frac{\Lambda^4}{f^4} + \cdots$. Note also that ${\rm Im} M = \mathcal{O}(\epsilon^2, \epsilon g_H) + g_H^2 \times \mathcal{O}(s^2/f^4)$, and therefore, in the perturbative computation of the two-to-two scattering amplitude, it is justified to neglect the imaginary part of the amplitude provided that s is well below the value of f^2/g_H.

In the extreme UV domination limit, and when the gap is large $\epsilon^{-1} \gg 1$, Eq. (4) implies the bound on the dimension-six operator coefficient,

$$-0.31 < \frac{g_H}{(4\pi)^2} \times \frac{\Lambda^2}{f^2} < 0.35. \qquad (25)$$

What is the appropriate value of Λ^2/f^2? The scale Λ^2 was introduced to ensure that the IR contribution to the c_H sum rule (6) is small with respect to the UV contribution. Thus, the derivation of (25) is valid as long as

$$\int_0^{\Lambda^2} dv \kappa_\ell^{(J)}(v) {\rm Im} f_\ell^{(J)}(v) \ll \int_{\Lambda^2}^{\infty} dv \kappa_\ell^{(J)}(v) {\rm Im} f_\ell^{(J)}(v), \qquad (26)$$

which follows from (6) in the limit $\epsilon \to 0$, and we have defined $\kappa_\ell^{(J)}(v) \equiv k_{1,\ell}^{(3,J)}(0, v)$. Instead, the scale f^2 does not have an intrinsic definition within the EFT.[6] Thus we define f^2 as the scale at which (26) is satisfied for the largest[7] possible Λ^2. That is we take $f^2 = \Lambda^2$.

[6]Given the value of g_H, f^2 is often associated to the lowest energy scale at which the perturbative calculation of $2 \to 2$ breaks down. However, this definition would be somewhat circular and not useful to us. The other standard choice is to associate f^2 to the scale of new physics, where the EFT breaks down. This definition is not useful for our purposes either.

[7]Equation (26) is trivially satisfied if Λ^2 is taken arbitrarily small.

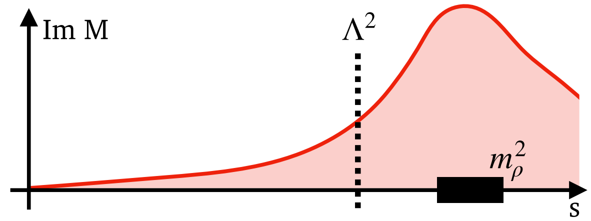

FIG. 4. Representation of strongly coupled and UV dominated amplitudes. The amplitude is strongly coupled at energies $s \gtrsim \Lambda^2$; nevertheless, (26) is satisfied; i.e., the c_H sum rule is UV dominated.

We remark that we are not claiming a regime such that the imaginary part of the amplitude is necessarily negligible at energies $s \lesssim f^2$. We are instead arguing for the existence of a regime such that the IR contribution to the sum rule is subdominant with respect to the UV contribution (26). After identifying $f^2 = \Lambda^2$, the remaining question is for what type of theories the condition being assumed (26) is less constraining than the actual result (25). While we do not know the answer to this question in its most general terms, next we provide two sources of intuition.

The first one comes from simple perturbative models. As argued above, the effect of exchanging heavy weakly coupled resonances on the dispersion relation is to localize the dispersive integrals at the heavy particle threshold. Thus, if the IR couplings are parametrically smaller than the UV couplings to heavy states, then UV domination (26) follows. As the UV couplings becomes stronger, the condition (26) still holds if the IR couplings are hold weaker. A simple perturbative example fulfilling this behavior of UV/IR domination is worked out in Appendix D2.

Even if the scattering amplitude cannot be computed in perturbation theories for energies $s \leq \Lambda^2$, the condition (26) may still be satisfied. For an intuitive picture see Fig. 4. In our previous work [2], we constructed scattering amplitudes meeting this trait, as well as amplitudes that smoothly interpolate between the UV and IR domination regimes—providing our second source of intuition. These theories show broad resonance behavior for $s > \Lambda^2$, with large values of the imaginary part for $s > f^2$. For energies below the resonances, the amplitude decays in powers of energy over the scale of the new resonances.

Accidentally, Eq. (25) with $\Lambda^2 = f^2$ agrees with the rough "loop-democracy" estimate—often called naive dimensional analysis (NDA) [23–25]. Indeed, using (24) to compare tree vs one-loop corrections to the four-point function gives $g_H \gtrsim g_H^2/(4\pi)^2 s/f^2$. We are conservatively arguing for $s \sim \Lambda^2 = f^2$ and are led to the bound $|g_H| \leq \mathcal{O}(1) \times (4\pi)^2$ in (25). What we have achieved here is to turn the NDA estimate into a precise "theorem" by determining the order one factors. The bound we have uncovered shows the symmetry $|g_H^{\max}| \approx |g_H^{\min}|$, a trait that was hardly predictable prior to the calculations presented here.[8]

Various composite Higgs models, where the light Higgs is a pseudo-Goldstone boson, have been shown to fall in within the SILH power counting [1]. For instance, the holographic minimal composite Higgs model [26] gives $g_H = 1$. These type of models are well within (25). Our bound could be made more stringent with further understanding or assumptions about the extent to which UV domination (26) holds for these particular class of models. For instance, one could argue to improve the bound by pushing Λ^2 to larger values by setting $\Lambda^2 \equiv f^2 g_\rho^2 \equiv m_\rho^2$. A more interesting possibility is to improve our bound by further modeling of the IR, in the spirit of Sec. III D. Besides of requiring a separation of scales with weak coupling in the IR (26), models where the Higgs arises as a pseudo-Goldstone boson can be further characterized by imposing chiral zeros on the scattering amplitude, very much like in pion physics [27]. We leave this intriguing possibility to improve the bound for these class of theories to future investigations.

Defining f^2 in terms of Λ^2 appears to us as the only logical possibility for establishing rigorous bounds on dimension-six operators. Given that the SM is very weakly coupled at TeV energies, it is reasonable to associate Λ ($\equiv f$) with the largest energy scale for which new physics, or new resonances, have been excluded. Namely, to the extent that no new physics contributes to the left-hand side of (26) up to an energy scale Λ, it is safe to neglect it.[9] Therefore, if no new physics appears up to the scale $f^2 \equiv \Lambda^2$, then (25) is the maximal value of g_H that one can hope of measuring. The current fit to the LHC data reveals $|g_H| \leq 1 \times \Lambda^2/(1\ \text{TeV})^2$ [28]. Our bound is universal in units of the scale Λ^2 (recall that as long as $\Lambda \gtrsim 8m$, we are on the asymptotic left region of Fig. 1). Thus, if we assume no new physics enters on the lhs of (26) up to an energy scale 5 TeV, then our bound reads $-0.31 \times (4\pi)^2/5^2 < g_H < 0.35 \times (4\pi)^2/5^2$, i.e., $-2.0 < g_H < 2.1$, which is comparable to the current experimental bound. Our construction can thus be used to figure out the precision needed on Higgs coupling measurements given an exclusion bound on the energy scale of new physics.

V. FUTURE DIRECTIONS

This program is in its early stages. We presented the answer to a very specific problem, what is the maximal/minimal value of c_H and where do SM-like EFTs fit within this bound. We are led to many more questions that would be interesting to investigate, next we present few of them:

(i) An interesting aspect of the starting point we took is that it can be extended to other theories and make it more realistic. Further modeling of the IR amplitudes will allow for more refined bounds. This modeling biased on the particular IR physics one is interested in probing, which is why in this work we have restricted ourselves to fairly simple choices.

(ii) It would be interesting to constrain the dimension-six operators involving two H's and two electroweak gauge bosons.[10] These operators, together with c_H, control the deviation of $h \to W^+W^-/ZZ$ within the SILH framework.

(iii) Our bound (25) is fairly symmetric, $|g_H^{\max}| \approx |g_H^{\min}|$. It would be phenomenologically interesting to know whether other dimensions-six operators enjoy instead very asymmetric bounds.

(iv) Related to the previous point, we remark that our bound is not optimal. It is nevertheless sufficiently stringent to provide a rigorous constraint within the relevant experimental ballpark. The primal S-matrix bootstrap [29] is well suited to study in detail the physical properties of the UV completed amplitudes saturating the bounds studied. It would be interesting to close the gap between the UV dominated primal amplitudes constructed in [2,30] and the dual approach here presented in Sec. III D by including also nonlinear unitarity in the IR.

(v) As a final remark, there are many generalizations of our approach that could be done. From the dual perspective, it might be interesting to extend the dual problem including crossing symmetric dispersion relations, recently reviewed in [31], see also [32,33], and check whether this helps reconstructing the phase shifts in a larger domain. Another direction is to study the scattering of massless particles, assuming an IR EFT input, and compare, for instance, with the results obtained in [34]. Finally, it would be also interesting to use the recursive approach of [36] and reconstruct the full amplitude starting from an IR EFT input.

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[8]Indeed, such symmetry is absent for the closely related bound (3).

[9]For UV completions above Λ that are parametrically weaker than IR physics, the lhs is not negligible. However, bounds for these theories are of little use, and we do not consider them further.

[10]The are five operators $D_\mu W^{a\mu\nu} H^\dagger \sigma^a D_\nu H$, $D_\mu B^{\mu\nu} H^\dagger D_\nu H$, $H^\dagger \sigma^i H W^{i\mu\nu} B_{\mu\nu}$, $|H|^2 W_{\mu\nu} W^{\mu\nu}$, $|H|^2 B_{\mu\nu} B^{\mu\nu}$.

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APPENDIX A: $O(n)$ AMPLITUDES AND DISPERSION RELATIONS

The two-to-two scattering amplitude of scalars transforming as vectors under an $O(n)$ internal symmetry, $v_a + v_b \to v_c + v_d$, can proceed through three different channels, as shown in (1). In this theory, crossing symmetry amounts to the fact that the function M is symmetric under the exchange of its last two variables $M(\bar{s}|\bar{t},\bar{u}) = M(\bar{s}|\bar{u},\bar{t})$.

Unitary acts diagonally when expressed in terms of the three irreducible representations of $v_a \otimes v_b$, respectively, the singlet, symmetric, and antisymmetric irreducible representations,

$$M^{(\text{sing})} = nM(\bar{s}|\bar{t},\bar{u}) + M(\bar{t}|\bar{u},\bar{s}) + M(\bar{u}|\bar{s},\bar{t}), \tag{A1}$$

$$M^{(\text{sym})} = M(\bar{t}|\bar{u},\bar{s}) + M(\bar{u}|\bar{s},\bar{t}), \tag{A2}$$

$$M^{(\text{anti})} = M(\bar{t}|\bar{u},\bar{s}) - M(\bar{u}|\bar{s},\bar{t}). \tag{A3}$$

We often group the irreducible representations into the vector $\vec{M} \equiv (M^{(\text{sing})}, M^{(\text{sym})}, M^{(\text{anti})})$. Then, the action of crossing symmetry on the irreducible representations follows from their definition and the symmetry properties of M. It turns out that

$$\vec{M}(\bar{s},\bar{t},\bar{u}) = C_{st}.\vec{M}(\bar{t},\bar{s},\bar{u}), \qquad \vec{M}(\bar{s},\bar{t},\bar{u}) = C_{su}.\vec{M}(\bar{u},\bar{t},\bar{s}),$$
$$\vec{M}(\bar{s},\bar{t},\bar{u}) = C_{tu}.\vec{M}(\bar{s},\bar{u},\bar{t}), \tag{A4}$$

where the crossing matrices satisfy $C_{st}.C_{su} = C_{tu}.C_{st} = C_{su}.C_{tu}$, $C_{tu} = \text{diag}(1,1,-1)$,

$$C_{st} = \begin{pmatrix} \frac{1}{n} & \frac{1}{2} - \frac{1}{n} + \frac{n}{2} & \frac{n}{2} - \frac{1}{2} \\ \frac{1}{n} & \frac{1}{2} - \frac{1}{n} & -\frac{1}{2} \\ \frac{1}{n} & -\frac{1}{2} - \frac{1}{n} & \frac{1}{2} \end{pmatrix},$$
$$C_{su} = \begin{pmatrix} \frac{1}{n} & \frac{1}{2} - \frac{1}{n} + \frac{n}{2} & \frac{1}{2} - \frac{n}{2} \\ \frac{1}{n} & \frac{1}{2} - \frac{1}{n} & \frac{1}{2} \\ -\frac{1}{n} & \frac{1}{2} + \frac{1}{n} & \frac{1}{2} \end{pmatrix}, \tag{A5}$$

and $C_{st}^2 = C_{su}^2 = C_{tu}^2 = \mathbb{1}$.

The partial wave decomposition can be carried independently on each of the three different channels,

$$M^{(I)}(s,t) = \sum_{\ell,I}^{\infty} n_\ell f_\ell^{(I)}(s) P_\ell\left(1 + \frac{2t}{s-4}\right), \tag{A6}$$

where $I = \text{sing}, \text{sym}, \text{anti}$, or equivalently $I = 1, 2, 3$. As usual,

$$f_\ell^{(I)} = \frac{1}{32\pi}\int_{-1}^{1} dz P_\ell(z) M^{(I)}(s,t(z)), \tag{A7}$$

with $t(z) = -1/2(s-4)(1-z)$ and $n_\ell \equiv 16\pi(2\ell+1)$. Singlet and symmetric channels are symmetric under the exchange of $t \leftrightarrow u$; therefore, their odd spin partial wave projections vanish. Similarly, even spin partial wave projection of the antisymmetric channel amplitude are zero. The sum over the indices $\{\ell, I\}$ in (A6) means $\{\ell\,\text{even}, I = (\text{sing},\text{sym})\}$ and $\{\ell\,\text{odd}, I = (\text{anti})\}$.

1. Deriving the $O(n)$ dispersion relation

In this section, we outline double subtracted dispersion relations for the $O(n)$ amplitude inspired by the original derivation of Roy in [36]. Roy equations are often used to analyze pion scattering amplitudes (see, [37] for example), and the main difference of the treatment here is that our subtractions points are the first and second derivatives of $\vec{M}(s,t)$ at $(s = 4/3, t = 4/3)$ (namely, c_λ and c_H), instead of the scattering lengths as in [37]. Let us start by writing a contour integral counterclockwise around an analytic point $z = s$ for fixed-t,

$$\vec{M}(s,t) = \frac{1}{2\pi i}\oint dz \frac{1}{z^2}\frac{s^2}{z-s}\vec{M}(z,t). \tag{A8}$$

Then, we blow up the contour to infinity, and by using $\vec{M}(s,t) = \frac{1}{2}\vec{M}(s,t) + \frac{1}{2}C_{su}.\vec{M}(u,t)$, we get

$$\vec{M}(s,t) = \vec{f}(t;s,u) + \frac{1}{\pi}\int_4^\infty dz K(z;s,u).\vec{M}_z(z,t), \tag{A9}$$

where the kernel K is a matrix given by

$$K(z;s,u) = \frac{1}{z^2}\left(\frac{s^2}{z-s}\mathbb{1} + \frac{u^2}{z-u}C_{su}\right), \tag{A10}$$

satisfying $K(z;s,u) = C_{su}.K(z;u,s)$, and $\vec{M}_z \equiv \text{Disc}_z \times \vec{M}(z,t)/2i$, and $\vec{f}$ is a function that contains residues at $z = 0$ and integrals along the left-hand cuts. Notice that the $s \leftrightarrow u$ symmetry of the amplitude M and the kernel K force $\vec{f}$ to satisfy $\vec{f}(z;s,u) = C_{su}.\vec{f}(z;u,s)$. Below, we replaced $\vec{f}$ with a simple ansatz with the desired $s \leftrightarrow u$ symmetry property and obtained the following equations:

$$\vec{M}(s,t) = C_{st}.[\vec{c}(t) + (s-u)\vec{d}(t)] + \frac{1}{\pi}\int_4^\infty dz K(z;s,u).\vec{M}_z(z,t), \tag{A11}$$

$$\partial_s \vec{M}(s,t) = C_{st}.[2\vec{d}(t)] + \frac{1}{\pi}\int_4^\infty dz \partial_s[K(z;s,u)].\vec{M}_z(z,t), \tag{A12}$$

where $\vec{c}(t) = (c^{\text{sing}}(t), c^{\text{sym}}(t), 0)^T$ and $\vec{d}(t) = (0,0,d^{\text{anti}}(t))^T$. Evaluating (A11) and (A12) at the crossing symmetric point, we can derive the following:

$$\vec{c}\left(\frac{4}{3}\right) = (n+2,2,0)^T c_\lambda - \frac{1}{\pi}\int_4^\infty C_{st}.K\left(\frac{4}{3},\frac{4}{3}\right).\vec{M}_z\left(z,\frac{4}{3}\right), \tag{A13}$$

$$2\vec{d}\left(\frac{4}{3}\right) = (0,0,2)^T c_H - \frac{1}{\pi}\int_4^\infty C_{st}.\partial_s K\left(s,\frac{8}{3}-s\right)\Big|_{s=4/3}.\vec{M}_z\left(z,\frac{4}{3}\right). \tag{A14}$$

Then, using the equation $\vec{M}(4/3,t) = C_{st}.\vec{M}(t,4/3)$, we can reexpress t-dependent subtraction constants with the ones above,

$$C_{st}.[\vec{c}(t) + (s-u)\vec{d}(t)] = \vec{c}\left(\frac{4}{3}\right) + 2\left(t-\frac{4}{3}\right)\vec{d}\left(\frac{4}{3}\right) + 2\left(s-\frac{4}{3}\right)C_{st}.\vec{d}(t) + \int absorptive, \tag{A15}$$

as well as

$$2\vec{d}(t) = 2\vec{d}\left(\frac{4}{3}\right) + \frac{(\mathbb{1}-C_{tu})}{(t-4/3)}\frac{1}{\pi}\int_4^\infty dz\left[K\left(t,\frac{8}{3}-t\right) - K\left(\frac{4}{3},\frac{4}{3}\right)\right].\vec{M}_z\left(z,\frac{4}{3}\right) - C_{st}.K\left(\frac{4}{3},\frac{8}{3}-t\right).\vec{M}_z(z,t).$$

The integrand in the second term goes like $O((t-4/3)^2)$, so it is regular when $t \to 4/3$. Finally, plugging everything back into (A11) yields

$$M^{(I)}(s,t) = \mathbb{C}^{(I)}(s,t) + \int_4^\infty dz[\mathbb{K}^{(IJ)}(z;s,t)\text{Im}M^{(J)}(z,t) + \mathbb{L}^{(IJ)}(z;s,t)\text{Im}M^{(J)}(z,4/3)], \tag{A16}$$

where

$$\vec{\mathbb{C}}(s,t) = c_\lambda\begin{pmatrix} n+2 \\ 2 \\ 0 \end{pmatrix} + c_H\begin{pmatrix} n-1 \\ -1 \\ 1 \end{pmatrix}(s-4/3) + c_H\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}(t-4/3). \tag{A17}$$

Notice that the absorptive pieces vanish at $(s,t) = (4/3,4/3)$. The explicit form of the kernels is given by

$$\mathbb{K}(z;s,t) = \frac{(3s-4)(3s+3t-8)}{\pi n(3z-4)(3t+3z-8)(s+t+z-4)} \times \begin{bmatrix} 1-\frac{n(s+t+z-4)}{s-z} & \frac{1}{2}(n^2+n-2) & -\frac{1}{2}(n-1)n \\ 1 & -\frac{n(s+2t+3z-8)}{2(s-z)} - 1 & \frac{n}{2} \\ -1 & \frac{n+2}{2} & -\frac{n(s+2t+3z-8)}{2(s-z)} \end{bmatrix}, \tag{A18}$$

$$\mathbb{L}(z;s,t) = \frac{3(n-1)(3t-4)}{\pi n(3z-4)(3t+3z-8)} \times \begin{bmatrix} \frac{2(3t-4)}{3(n-1)(z-t)} + \frac{4-3s}{3z-4} & \frac{3ns-4n+6s-8}{2(3z-4)} + \frac{(n+2)(3t-4)}{3(z-t)} & \frac{n(2s+t-4)}{z-t} - \frac{n(3s-4)}{2(3z-4)} \\ \frac{3s-4}{(n-1)(3z-4)} + \frac{2(3t-4)}{3(n-1)(z-t)} & \frac{(n-2)(3t-4)}{3(n-1)(z-t)} - \frac{(n+2)(3s-4)}{2(n-1)(3z-4)} & \frac{-2ns-nt+4n}{(n-1)(z-t)} + \frac{3ns-4n}{2(n-1)(3z-4)} \\ \frac{2(3t-4)}{3(n-1)(z-t)} - \frac{3(s+2t-4)}{(n-1)(3z-4)} & \frac{3(ns+2nt-4n+2s+4t-8)}{2(n-1)(3z-4)} - \frac{(n+2)(3t-4)}{3(n-1)(z-t)} & \frac{2ns+nt-4n}{(n-1)(z-t)} - \frac{3(ns+2nt-4n)}{2(n-1)(3z-4)} \end{bmatrix}. \tag{A19}$$

The jth partial wave projection of (A16) gives rise to the Roy equations [36],

$$a_j^{(I)}(s) = \mathrm{Re}f_j^{(I)}(s) - \frac{\mathcal{C}_0^{(I)}}{16\pi}\delta_{0,j} - \frac{\mathcal{C}_1^{(I)}}{16\pi}\delta_{1,j} - \mathrm{P.V.}\int_4^\infty dv \sum_{\ell,J}^{\infty} k_{j,\ell}^{(IJ)}(s,v) n_\ell \mathrm{Im}f_\ell^{(J)}(v) = 0. \tag{A20}$$

where the subtraction constants are given by

$$\vec{\mathcal{C}}_0 = \begin{pmatrix} c_\lambda(n+2) + c_H(n-1)(s-4/3) \\ 2c_\lambda - c_H(s-4/3) \\ 0 \end{pmatrix}, \qquad \vec{\mathcal{C}}_1 = \begin{pmatrix} 0 \\ 0 \\ (c_H/3)(s-4), \end{pmatrix}, \tag{A21}$$

and the kernels by

$$k_{j,\ell}^{(IJ)}(s,v) = \frac{2}{32\pi}\int_0^1 dz P_j(z)\left[\mathbb{K}^{(IJ)}(v;s,t) P_\ell\left(1 + \frac{2t(z)}{v-4}\right) + \mathbb{L}^{(IJ)}(v;s,t) P_\ell\left(1 + \frac{8/3}{v-4}\right)\right]. \tag{A22}$$

APPENDIX B: DUAL FUNCTIONAL

Let us find $d_\pm$ defined in Eq. (11), corresponding to the problem of maximizing or minimizing c_H using only rigorous analyticity, crossing, and unitarity. First, maximizing $L_\pm$ with respect to c_λ and c_H yields the following normalization constraints:

$$\begin{aligned} &\frac{\partial L_\pm}{\partial c_\lambda} = 0\colon\ 0 = \int_{\Lambda^2}^{\Lambda_c^2} \frac{dv}{16\pi}(n+2,2,0)\cdot\vec{\omega}(v), \\ &\frac{\partial L_\pm}{\partial c_H} = 0\colon\ \pm 1 = \int_{\Lambda^2}^{\Lambda_c^2} \frac{dv}{16\pi}\left[\left(v - \frac{4}{3}\right)(n-1,-1,0) + (v-4)\left(0,0,\frac{1}{3}\right)\right]\cdot\vec{\omega}(v), \end{aligned} \tag{B1}$$

where $\vec{\omega} \equiv (w_0^{(1)}, w_0^{(2)}, w_1^{(3)})$. Next, comes the maximization with respect to $\{\mathrm{Re}f_l^{(I)}, \mathrm{Im}f_\ell^{(I)}\}$. On their equations of motion,

$$\rho\cdot\mathrm{Re}f_{(I\ell)} = \frac{w_{(I\ell)}}{2\lambda_{(I\ell)}}, \tag{B2}$$

$$\rho\cdot\mathrm{Im}f_{(I\ell)} = 1 + n_\ell\frac{\mu_{(I\ell)}}{2\lambda_{(I\ell)}}, \tag{B3}$$

where $\rho = \sqrt{(s-4)/s}$. After plugging in the solution, we get Eq. (14) in full glory

$$\begin{aligned} d_\pm(\mathcal{D}) = &\int_{\Lambda^2}^\infty \frac{dv}{\rho}\cdot\frac{1}{4\lambda_{(I\ell)}}(2\lambda_{(I\ell)} + n_\ell\mu_{(I\ell)})^2 \\ &+ \int_{\Lambda^2}^{\Lambda_c^2}\frac{dv}{\rho}\cdot\frac{w_{(Kj)}^2}{4\lambda_{(Kj)}}, \end{aligned} \tag{B4}$$

where we have grouped isospin and spin indices together in a common parentheses for convenience, and we have defined an auxiliary function,

$$\mu_\ell^{(I)}(v) \equiv \nu_{n,m}^{(K)}\cdot F_{n,m;\ell}^{(KI)}(v) - \mathrm{p.v.}\int_{\Lambda^2}^{\Lambda_c^2} ds w_j^{(K)}(s) k_{j,\ell}^{(KI)}(s,v). \tag{B5}$$

Here, repeated indices are summed over, meaning that (Kj) and (n,m) sums are cut by J_c and (n_c, m_c), and $(I\ell)$ sum goes up to ∞. The functional $d_\pm$ is to be minimized over the set of dual variables $\mathcal{D}$. However, to keep under control the infinite ℓ sum, one needs to make sure the squared expression in the first term of (B4) must be suppressed as $\ell\to\infty$. One way to achieve this is to minimize analytically for $\lambda_{(I\ell)}$, which gives

$$2\lambda_{(I\ell)}/n_\ell = \sqrt{\mu_{(I\ell)}(v)^2 + W_{(I\ell)}(v)^2} \geq 0. \tag{B6}$$

Plugging this expression in (B4) produces

$$\begin{aligned} d_\pm(\{\nu,w\}) = &\int_{\Lambda^2}^\infty \frac{dv}{\rho(v)}\sum_{(I\ell)}^\infty n_\ell\Big[\mu_{(I\ell)}(v) \\ &+ \sqrt{\mu_{(I\ell)}(v)^2 + W_{(I\ell)}(v)^2}\Big], \end{aligned} \tag{B7}$$

where we have defined

$$W_{(I\ell)}(v) \equiv \begin{cases} w_{(I\ell)}(v)/n_\ell & \text{for } \ell \le J_c \text{ and } v \le \Lambda_c^2 \\ 0 & \text{otherwise} \end{cases}, \quad \text{(B8)}$$

and dropped the $\pm$ label on d since its effect only enters into c_H normalization condition, but not in the objective. We stress that the functional $d_\pm$ must be evaluated on the solution of the two normalization conditions (B1).

Now, a final remark is in order: note that in the region where $W_{(I\ell)}$ has no support, the integrand in the dual objective reduces to

$$\mu_{(I\ell)}(v) \cdot \Theta[\mu_{(I\ell)}(v)],$$

where Θ is the Heaviside theta function. This means that higher spins will not contribute to the objective, as long as $\mu_{(I\ell)} \le 0$.

APPENDIX C: NUMERICAL IMPLEMENTATION

The (dual) problem presented before is a mathematically well-defined nonlinear optimization problem, which can be studied by the favorite methods of the reader. For our purposes, we numerically looked for the minimum of (B7), and we chose to work with a linear problem solver named SDPB, mostly used in the conformal bootstrap literature [38,39]. The reason of choice for us was its ability to achieve a reasonable degree of precision in the optimization objective and the possibility to parallelize the computations on a cluster.

To put the problem on a computer, we need to take several steps to transform it into a suitable form. We start with turning the nonlinear problem into a linear one, by the relaxation method.

1. Relaxation

Let us write a new objective function, by extending the set of dual variables to

$$\mathcal{D}^{\text{rel}} = \mathcal{D} \cup \{\mathcal{X}^{\text{IR}}_{(I\ell)}(v), \mathcal{X}^{\text{UV}}_{(I\ell)}(v)\}.$$

Then, the *relaxed* dual objective, in which the new set of dual variables enter linearly,

$$d^{\text{rel}}(\mathcal{D}^{\text{rel}}) = \int_{\Lambda^2}^{\Lambda_c^2} \frac{dv}{\rho^2(v)} \sum_{(I\ell)}^{\infty} n_\ell \mathcal{X}^{\text{IR}}_{(I\ell)}(v) + \int_{\Lambda^2}^{\Lambda_c^2} \frac{dv}{\rho^2(v)} \sum_{(I\ell)}^{\infty} n_\ell \mathcal{X}^{\text{UV}}_{(I\ell)}(v), \quad \text{(C1)}$$

subject to semipositive conditions on the following 2×2 matrices:

$$\begin{pmatrix} \mathcal{X}^{\text{IR}}_{(I\ell)}(v) & W_{(I\ell)}(v) \\ W_{(I\ell)}(v) & \mathcal{X}^{\text{IR}}_{(I\ell)}(v) - 2\mu_{(I\ell)}(v) \end{pmatrix} \succeq 0,$$
$$\begin{pmatrix} \mathcal{X}^{\text{UV}}_{(I\ell)}(v) & 0 \\ 0 & \mathcal{X}^{\text{UV}}_{(I\ell)}(v) - 2\mu_{(I\ell)}(v) \end{pmatrix} \succeq 0. \quad \text{(C2)}$$

When these matrix constraints are saturated, rhs of (C1) reduces to the rhs of (B7), and both dual objectives become equal $d^{\text{rel}} = d$.

In this section, we assume the lower boundary of integration in (C1) to be the generic value Λ^2, instead of the normal threshold $4m^2$. This simple generalization will allow us to describe the setup discussed in Sec. III C, where we impose that $\text{Im}M = 0$ for $s < \Lambda^2$.

Notice that the positivity of the determinants imply $\mathcal{X}^{\text{IR}}_\ell \ge \mu_\ell + \sqrt{\mu_\ell^2 + (w_\ell/n_\ell)^2}$ and $\mathcal{X}^{\text{UV}}_\ell \ge 2\mu_\ell \Theta[\mu_\ell]$. As a result, determinants measure how far away we are from saturating the true inequality $d^{\text{rel}} \ge d$. We call the positive difference $d^{\text{rel}} - d$ measured by the determinant as *the relaxation gap*.

The relaxation gap adds an additional layer of difficulty on the way of achieving the optimal solution to the dual problem. However, it does not compromise the rigor of our approach, since its sole impact is to increase the objective, which still qualifies as a valid dual bound. To reduce the gap, we should increase the number of degrees of freedom in $\mathcal{X}^{\text{IR}}$ and $\mathcal{X}^{\text{UV}}$ variables as much as we can.

2. Spin cutoff

We truncate the spin sum in (B7) to deal with only a finite number of spins. Remember that for $\ell > J_c$ the integrand for each spin becomes

$$\mu_{(I\ell)} \cdot \Theta[\mu_{(I\ell)}],$$

where Θ is the Heaviside theta function. This is interesting, because if we can show that there exists an L_c such that $\mu_{(I\ell)}(v) < 0$ for all $v \in [4, \infty]$ and $\ell > L_c > J_c$, the infinite tail would not contribute to the spin sum, and we can safely truncate the sum at L_c.

It turns out that there is such a corner in dual variables space, providing a feasible solution for our objective. The way we enforce the conditions are twofold: (i) we study large-ℓ expansion of each term in $\mu_{(I\ell)}$ and make sure that it stays negative as $\ell \to \infty$. (ii) for the intermediate spins, we impose by hand the negativity conditions.

$$\text{(i)}: \ \mu_{(I\ell)}(v) \le 0 \quad \text{for } L_c \le \ell, \quad \text{(C3)}$$

$$\text{(ii)}: \ \mu_{(I\ell)}(v) \le 0 \quad \text{for } J_c \le \ell \le L_c. \quad \text{(C4)}$$

By analyzing the asymptotics of (i) at large ℓ, we find out that it is equivalent to the conditions,

$$\begin{aligned} &(-1)^{n_c+m_c}\cdot y^{(\mathrm{sing})}<0 \\ &(-1)^{n_c+m_c}\cdot y^{(\mathrm{sym})}<0 \quad \text{where} \quad \vec{y}\equiv\begin{pmatrix} n+1 & 1 & 1 \\ \frac{2(n^2+n-2)}{n} & \frac{2(3n-2)}{n} & \frac{2(-n-2)}{n} \\ n-1 & -1 & 3 \end{pmatrix}\cdot\vec{\nu}, \\ &(-1)^{n_c+m_c+1}\cdot y^{(\mathrm{anti})}<0 \end{aligned} \tag{C5}$$

and $\vec{v}\equiv(\nu^{(\mathrm{sing})}_{n_c,m_c},\nu^{(\mathrm{sym})}_{n_c,m_c},\nu^{(\mathrm{anti})}_{n_c,m_c})^T$, together with

$$\sum_{j=0}^{J_c} w_j^{(\mathrm{sym})}(\Lambda_c^2)P_j(0)>0 \quad \text{with} \quad \Lambda_c^2\in[v_c,4\Lambda^2-4],$$
$$\sum_{j=0}^{J_c}\left(w_j^{(\mathrm{sing})}(\Lambda_c^2)-\frac{w_j^{(\mathrm{sym})}(\Lambda_c^2)}{n-1}\right)P_j(0)=0. \tag{C6}$$

See the next subsection for a detailed account on how to derive them.

We impose (i) through (C5) and (C6) and (ii) with a fixed L_c on the relaxed problem, and we increase L_c until a convergence in d^{rel} is obtained.

a. Large ℓ asymptotics in more detail

Large ℓ behavior of $\mu_\ell^{(I)}(v)$ is determined by Legendre polynomials in the definition of the kernels (A22). Their argument is either $x(v,t)\equiv 1+2t/(v-4)$ or $x(v,4/3)$. Remember that Legendre polynomials will grow exponentially whenever their argument exceed ±1. It turns out that there is a critical value,

$$v_c=2-4/3+\Lambda_c^2/2, \tag{C7}$$

along the integration range of v, such that

$$|x(v,4/3)|\geq|x(v,t)| \qquad \text{for} \;\; v_c\leq v, \tag{C8}$$

$$|x(v,t)|\geq|x(v,4/3)| \qquad \text{for} \; \Lambda^2\leq v\leq v_c. \tag{C9}$$

We call the two regions the outer and inner region, respectively.

Outer region $v_c<v$. It is easy to see that at large ℓ,

$$\bar{w}_\ell^{(I)}(v)\sim P_\ell(x(v,4/3)). \tag{C10}$$

We have checked the crossing term for all $\mathsf{n}+\mathsf{m}\leq 7$, and the leading contributions go like

$$F^{(KI)}_{\mathsf{n},\mathsf{m};\ell}(v)\sim(-\ell)^{(\mathsf{n}+\mathsf{m}-2)}P_{\ell+\mathsf{n}+\mathsf{m}-2}(x(v,4/3)), \tag{C11}$$

with a positive overall ℓ-independent factor that we omitted. Crossing symmetry contribution clearly wins over the other terms. We choose the set of crossing constraints such that n_c+m_c gives uniquely the highest sum, and (C5) will suffice to enforce $\mu_\ell^{(I)}<0$.

Inner region $4\leq v\leq v_c$. The only dominant contribution in this case is

$$\bar{w}_\ell^{(I)}(v)\sim P_\ell(x(v,t)). \tag{C12}$$

$P_\ell(x)$ grows exponentially in $x<-1$; therefore, s- and z-integrations in $\bar{w}_\ell^{(I)}(v)$ are dominated by the minimum of $x(v,t(s,z))$ which occurs at the end points $(s=\Lambda_c^2$, $z=0)$. Approximating the result using the saddle point method around the minimum gives

$$\bar{w}_\ell^{(I)}(v)\approx\frac{1}{\ell^2}\frac{(\Lambda_c^2-v)^2}{\Lambda_c^2-4}\sum_{j,K}^{\mathrm{J_{max}}}w_j^{(K)}(\Lambda_c^2)P_j(0)P_\ell$$
$$\times\left(\frac{v-\Lambda_c^2}{v-4}\right)\mathbb{K}^{(KI)}(v;\Lambda_c^2,2-\Lambda_c^2/2). \tag{C13}$$

Notice that $P_j(z\to0)\approx O(1)$ for j even and $O(z)$ for j odd. As a consequence, saddle point contributions from $w_j^{(\mathrm{anti})}$ are $O(1/\ell)$ suppressed with respect to symmetric and singlet channels. Therefore, inner region constraints will be only on the dual variables $w_j^{(\mathrm{sing})}$ and $w_j^{(\mathrm{sym})}$.

Note further that $P_\ell(\frac{v-\Lambda_c^2}{v-4})>\pm1$ for all v and even/odd ℓ. To enforce $\mu_\ell^{(I)}<0$, we need to combine $\bar{w}_\ell^{(\mathrm{sing,sym})}>0$ and $\bar{w}_\ell^{(\mathrm{anti})}<0$. These three conditions put together eventually imply the ones in (C6).

3. Dual ansatze

Next, we describe how to write an ansatz for the dual variables in terms of a finite basis of functions. We send the interval $x\in[-1,1]$ into $v^{\mathrm{IR}}\in[\Lambda^2,\Lambda_c^2]$ and $v^{\mathrm{UV}}\in[\Lambda_c^2,\infty)$ with the following maps:

$$v^{\rm IR}(x) = \frac{1}{2}(\Lambda_c^2+\Lambda^2)+\frac{x}{2}(\Lambda_c^2-\Lambda^2),$$
$$v^{\rm UV}(x) = \frac{\pi}{3}\Lambda_c^2\tan\left(\frac{\pi}{4}(x+1)\right)\sec^2\left(\frac{\pi}{4}(x+1)\right). \quad \text{(C14)}$$

Then, we parametrize the dual variables $\{\mathcal{X}_\ell^{\rm IR},\mathcal{X}_\ell^{\rm UV},w_\ell\}$ in terms of Chebyshev polynomials T_n,

$$w_\ell^{(I)}(x)=\sum_{n=0}^{N_w}a_{\ell,n}^{(I)}T_n(x),\qquad \mathcal{X}_\ell^{(I),\rm IR}(x)=\sum_{n=0}^{N_{\rm IR}}c_{\ell,n}^{(I)}T_n(x),$$
$$\mathcal{X}_\ell^{(I),\rm UV}(x)=\sum_{n=0}^{N_{\rm UV}}d_{\ell,n}^{(I)}T_n(x), \quad \text{(C15)}$$

such that $dx\mathcal{X}_\ell(x)=dv\mathcal{X}_\ell(v)$, and they give the same result under the integral sign. In our runs, we often take $N_{\max}=N_w=N_{\rm IR}=N_{\rm UV}$, and we sample above functions of x on a Chebyshev grid with 199 points on $[-1,1]$ for both IR and UV sections.

Notice that $dv^{\rm UV}/dx$ has a zero at $x=-1$ which can cause us problems, since

$$\lim_{v\to\Lambda_c^2}\mathcal{X}_\ell^{\rm UV}(v)=\lim_{x\to-1}\mathcal{X}_\ell^{\rm UV}(x)\left[\frac{dv^{\rm UV}}{dx}\right]^{-1}$$

will behave singular, unless $\mathcal{X}_\ell^{\rm UV}(x)\overset{x\to-1}{\longrightarrow}O(x+1)$. To make sure it is regular, we require

$$\mathcal{X}_\ell^{(I),\rm UV}(-1)=\sum_{n=0}^{N_{\rm UV}}d_{\ell,n}^{(I)}T_n(-1)=0. \quad \text{(C16)}$$

All in all, the dual problem to be solved numerically is the following:

$$\min \quad d^{\rm rel}(\mathcal{D}^{\rm rel}) \qquad \text{over}\{\nu_{\rm n,m}^{(I)},a_{\ell,n}^{(I)},c_{\ell,n}^{(I)},d_{\ell,n}^{(I)}\}$$
$$\text{subject to}\ \{(B1),(C2),(C4),(C5),(C6),(C16)\}. \quad \text{(C17)}$$

4. The space of $O(n)$ theories

We have left the parameter $n>1$ to be a generic integer so far. In order to explore the space of nonperturbative islands at various n, we solved the radial optimization problem given in Sec. III A for a couple of values $n=2$, 3, 4. The resulting islands are shown in Fig. 5.

Numerical bounds we obtain in Figs. 1, 3, and 5 depend on the cutoff parameters $\{J_c,L_c,(n_c,m_c),N_{\max}\}$. To obtain well converged bounds, we fixed them to the following values in all of our runs:

$$J_c=5,\quad L_c=25,\quad (n_c,m_c)=(1,6),\quad N_{\max}=8.$$

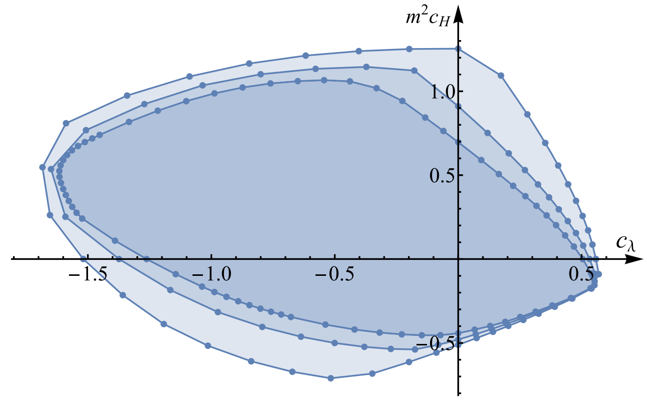


FIG. 5. A family of $O(n)$ dual exclusion plots, for $n=2$, 3, 4, respectively, from outer to innermost boundary. Markers along the boundaries indicate the angular grid chosen in the radial optimization problem described in Sec. III A.

APPENDIX D: PERTURBATIVE COMPUTATIONS

1. Perturbative $\lambda|\vec{\phi}|^4$ low energy constants

Consider the interaction Lagrangian $\mathcal{L}_{\rm int}=-\lambda|\vec{\phi}|^4/8$. The amplitude at tree level is given by $M=-\lambda$. Here, we are interested in computing the leading order contribution to the low energy constants coming from the one-loop interaction. The bare coupling λ diverges at one-loop, so we just redefine the coupling using the physical amplitude $c_\lambda=M(s=t=u=4/3)=-\lambda_R$. The imaginary part at one-loop is simply given by

$$\mathrm{Im}\vec{M}(s\geq 4m^2,t)=\frac{1}{2}\frac{\lambda^2}{16\pi}\sqrt{\frac{s-4m^2}{s}}((n+2)^2,4,0). \quad \text{(D1)}$$

Plugging this expression into the dispersive representations (6), (16), and (17), we obtain

$$c_H=\frac{c_\lambda^2}{256\pi^2}3(n+2)\left(3\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-2\right), \quad \text{(D2)}$$

$$c_2=\frac{c_\lambda^2}{4096\pi^2}9\left(14-15\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right), \quad \text{(D3)}$$

$$2c_2+c_2'=\frac{c_\lambda^2}{8192\pi^2}9(n+8)\left(14-15\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right). \quad \text{(D4)}$$

2. Further comments on (26)

We now present a simple perturbative calculation to gain additional insights into the range of validity of (26). Consider the Lagrangian $\mathcal{L}=(\partial\vec{\phi})^2/2-m^2\vec{\phi}^2/2-\lambda\vec{\phi}^4/8-M^2\Phi^2/2-g\vec{\phi}^2\Phi/2+O(\Phi^3)$, with $M^2\gg m^2$. The imaginary part of the two-to-two scattering amplitude, at lowest nontrivial order, is given by

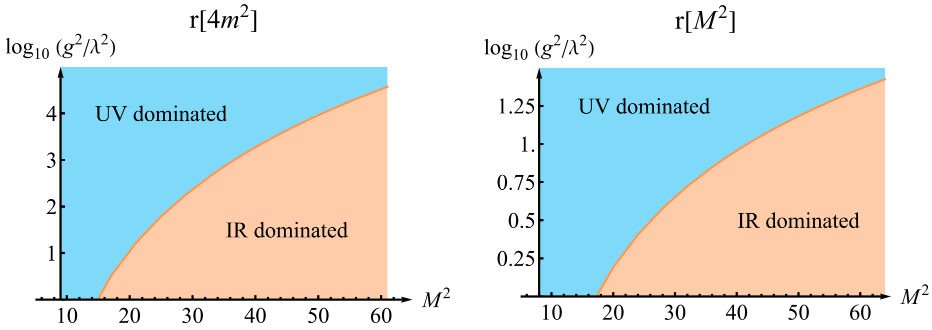
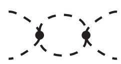
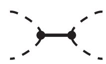


FIG. 6. IR/UV domination regimes of the amplitude (D5). Blue is when $r < 1$, and orange is when $r > 1$.

$$\mathrm{Im}\,\vec{M}(s \geq 4m^2, t) = \quad \text{(diagram)} \quad + \quad \text{(diagram)} \quad = \frac{1}{2}\frac{\lambda^2}{16\pi}\sqrt{\frac{s-4m^2}{s}}\left((n+2)^2, 4, 0\right) + g^2\pi\delta(s-M^2)\left(n, 0, 0\right) \tag{D5}$$





which has nonvanishing partial wave projection to spin-zero only, in both singlet and symmetric isospin channels. Then, the only integrals to evaluate in the sum rule (6) are for $J = 1$, 2 and $\ell = 0$. The corresponding kernels are

$$\kappa_{1,0}^{(3,1)}(s,v) = -\frac{1}{3}\kappa_{1,0}^{(3,2)}(s,v) = \frac{-12(s-4)(v-\frac{4}{3})^2 + 6(v-\frac{4}{3})^2(s+2v-4)\log(\frac{s+v-4}{v}) - (s-4)^3}{192\pi^2(s-4)^2(v-\frac{4}{3})^2}. \tag{D6}$$

Let us choose $\Lambda^2 = M^2$ and study the following IR/UV ratio:

$$r[s] \equiv \frac{c_H^{\mathrm{IR}}(s)}{c_H^{\mathrm{UV}}(s)} \quad \text{with} \quad c_H^{\mathrm{IR}}(s) \equiv \int_4^{\Lambda^2} dv k_{1,\ell}^{(3,J)}(s,v)\mathrm{Im}f_\ell^{(J)}(v), \qquad c_H^{\mathrm{UV}}(s) \equiv \int_{\Lambda^2}^{\infty} dv k_{1,\ell}^{(3,J)}(s,v)\mathrm{Im}f_\ell^{(J)}(v), \tag{D7}$$

to be evaluated on the above amplitude. We fix the subtraction point to two values: $s = 4m^2$ and $s = M^2$.

$r[s]$ then depends on two free parameters of the amplitude (D5): The ratio of couplings g^2/λ^2 and the mass of the heavy particle M^2. Defining $r[s] = 1$ as the transition point between IR/UV domination regimes, we find out the domination regions in the parameter space as shown in Fig. 6.

APPENDIX E: RIGOROUS BOUNDS ASSUMING AN IR MODEL

In this Appendix, we solve the problem discussed in Sec. III D. We assume that

$$\mathrm{Im}f_\ell^{(I)}(s) = g_\ell^{(I)}(s), \qquad s \leq \Lambda^2. \tag{E1}$$

The phenomenological input enters into the choice of Λ and $g_\ell(s)$. With this condition, the Roy equations are given by

$$\mathrm{Re}f_\ell^{(I)}(s) = c_\ell^{(I)}(s) + \frac{1}{\pi}\int_{4m^2}^{\Lambda^2}\sum_{j,K} g_j^{(K)}(v)k_{j\ell}^{(KI)}(v,s)dv + \frac{1}{\pi}\int_{\Lambda^2}^{\infty}\sum_{j,K}\mathrm{Im}f_j^{(K)}(v)k_{j\ell}^{(KI)}(v,s)dv. \tag{E2}$$

We want to solve the problem of maximizing αc_H where $\alpha = \pm 1$ within our model. We write the Lagrangian (we omit the unitarity constraints for simplicity)

$$\begin{aligned}\mathcal{L} = \alpha c_H &+ \sum_{\ell I}^{L}\int_{\Lambda^2}^{\mu^2}\mathrm{Re}f_\ell^{(I)}(s)w_\ell^{(I)}(s)ds + \sum_{\ell I}^{\infty}\int_{\Lambda^2}^{\infty}\mathrm{Im}f_\ell^{(I)}(s)\bar{w}_\ell^{(I)}(s)ds + \sum_{\ell I}^{\infty}\int_{4m^2}^{\Lambda^2} g_\ell^{(I)}(s)\bar{w}_\ell^{(I)}(s)ds \\ &+ \sum_{\ell I}^{L}\int_{\Lambda^2}^{\mu^2} c_\ell^{(I)}(s)w_\ell^{(I)}(s)ds,\end{aligned} \tag{E3}$$

where

$$\bar{w}_j^{(K)}(v) = -\frac{1}{\pi}\sum_{\ell I}^{L}\int_{\Lambda^2}^{\mu^2} w_\ell^{(I)}(s)k_{j\ell}^{(KI)}(v,s)ds. \quad \text{(E4)}$$

Maximizing with respect to the primal variables c_λ, c_H, we obtain the dual constraints

$$\frac{\partial}{\partial c_H}\left(\alpha c_H + \sum_{\ell I}^{L}\int_{\Lambda^2}^{\mu^2} c_\ell^{(I)}(s)w_\ell^{(I)}(s)ds\right) = 0,$$
$$\frac{\partial}{\partial c_\lambda}\left(\alpha c_H + \sum_{\ell I}^{L}\int_{\Lambda^2}^{\mu^2} c_\ell^{(I)}(s)w_\ell^{(I)}(s)ds\right) = 0, \quad \text{(E5)}$$

while first maximizing with respect to the physical partial waves and then minimizing with respect to the unitarity constraints we obtain the inequality

$$c_H \le D = D^{\mathrm{UV}} + D^{\mathrm{IR}}[g_\ell^{(I)}], \quad \text{(E6)}$$

where

$$D^{\mathrm{UV}} = \sum_{\ell I}^{L}\int_{\Lambda^2}^{\mu^2} \frac{\bar{w}_\ell^{(I)}(s) + \sqrt{(\bar{w}_\ell^{(I)}(s))^2 + (w_\ell^{(I)}(s))^2}}{\rho^2(s)}ds, \quad \text{(E7)}$$

and

$$D^{\mathrm{IR}}[g_\ell^{(I)}] = \sum_{\ell I}^{\infty}\int_{4m^2}^{\Lambda^2} g_\ell^{(I)}(s)\bar{w}_\ell^{(I)}(s)ds, \quad \text{(E8)}$$

provided that $\bar{w}_\ell^{(I)}(s) \le 0$ whenever it does not appear in D^{UV} or D^{IR}.

For each given IR model, we can solve a dual problem. Let us discuss now some limiting situations. Imagine $g_\ell^{(I)} = 0$, which gives

$$c_H \le D^{\mathrm{UV}}. \quad \text{(E9)}$$

This approximation can be well justified in two scenarios. One is realized when we have weakly coupled UV complete models such as in gauge theories with large N_c. In this case, we do expect our bound to be extremely loose. The second scenario is realized when, due to nonperturbative effects, there is a cancellation among terms in the low energy expansion, and we can neglect the imaginary parts way beyond the radius of convergence of the EFT. This scenario sometimes is realized on the boundary of the allowed region determined by nonperturbative bootstrap studies [2].

Suppose now that $D^{\mathrm{IR}}[g_\ell^{(I)}] \le 0$. Then, we obtain the chain of inequalities,

$$c_H \le D^{\mathrm{UV}} + D^{\mathrm{IR}}[g_\ell^{(I)}] \le D^{\mathrm{UV}}. \quad \text{(E10)}$$

In it, it could be possible to impose this condition, and the bound should hold for any IR model and therefore contain the $O(4)$ island. In general, however, it is hard to satisfy the inequality $D^{\mathrm{IR}}[g_\ell^{(I)}] \le 0$, and, *a priori*, we cannot rigorously use the D^{UV} functional alone to bound the Wilson coefficients in presence of an IR imaginary part.

However, it is possible to obtain a bound on c_H by solving first the truncated optimization problem

$$d^{\mathrm{UV}} = \min_w D^{\mathrm{UV}}, \quad \text{(E11)}$$

which is attained for some critical w_c, then construct the bound

$$c_H \le D^{\mathrm{IR}}[g_\ell^{(I)}] + d^{\mathrm{UV}}, \quad \text{(E12)}$$

by plugging w_c in $D^{\mathrm{IR}}[g_\ell^{(I)}]$.

So, solving the simple universal problem (E11)—see also Sec. III C—has a conceptual value since it can be used to generate rigorous bounds for any choice of the $g_\ell^{(I)}$. If we commit from the beginning with some IR model, then we can fully optimize $D^{\mathrm{UV}} + D^{\mathrm{IR}}[g_\ell^{(I)}]$, and obtain even stronger bounds.

APPENDIX F: PHASE SHIFTS

We can reconstruct the optimal dual S-matrices from the partial waves $f_\ell^{(I)}(v)$ on the support where we impose the Roy equations. $w_\ell^{(I)}$ exist, because then we can use the fixed-t dispersion relation to reconstruct $\mathrm{Re}f_\ell^{(I)}(v)$, as can be seen in (B3). Then the S-matrix on a single partial wave channel is given by

$$S_\ell^{(I)}(s) = 1 + i\sqrt{\frac{s-4}{s}}f_\ell^{(I)}(s), \quad \text{(F1)}$$

which is a pure complex phase evaluated on the solutions (B3). This allows us then to plot *the phase shift* of the scattered wave as a function of s,

$$\delta_\ell^{(I)}(s) = \frac{1}{2i}\log S_\ell^{(I)}(s). \quad \text{(F2)}$$

Below, in Fig. 7, we give sample dual phase shifts along the four distinct sections of the $O(4)$ nonperturbative island. Note that a threshold singularity puts $\delta_\ell^{(I)}(0) = \pi/2$ which would otherwise be zero.

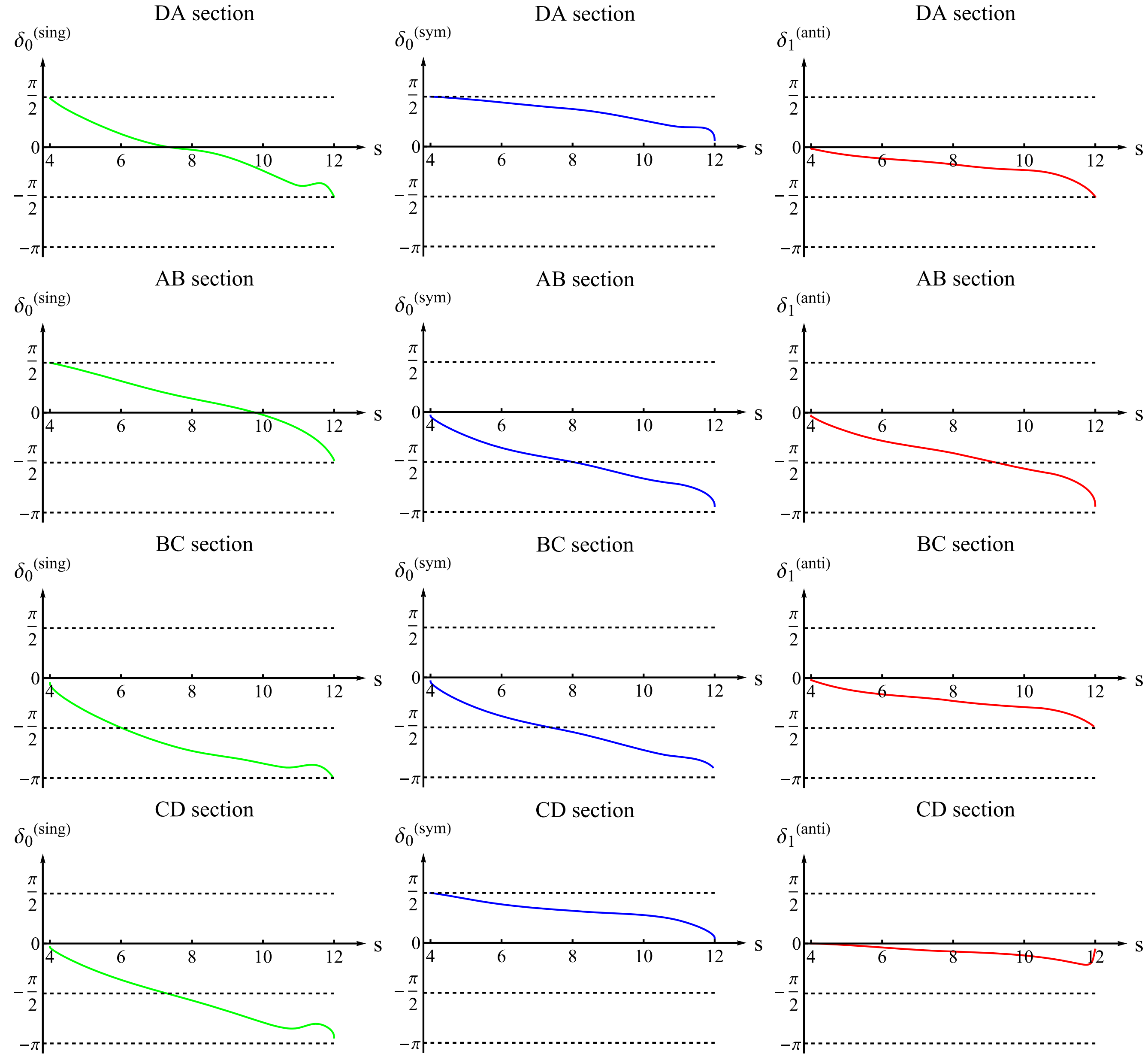


FIG. 7. Sample phase shifts from each of four sections along the boundary of Fig. 1.

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