Schwarzschild deformed supergravity background: Possible geometry origin of fermion generations and mass hierarchy

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The problem of fermion masses hierarchy in the Standard Model is considered on a toy model of a 10-dimensional space-time with a IIA supergravity type background. The Dirac equation written on this background, after compactification of extra four- and one-dimensional subspaces, gives the spectrum of Fermi fields which profiles in five dimensions and corresponding Higgs generated masses in four dimensions depend on the eigenvalues of Dirac operator on the named compact subspaces. Schwarzschild Euclidean deformation of the supergravity throat with the "apple-shaped" conical singularity permits to leave only three nondivergent angular modes interpreted as three generations of the down-type quarks. The resulting expressions for the quark masses are a geometry version of the Froggatt-Nielsen mechanism.

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I. INTRODUCTION

Models in warped extra dimensions are among the popular trends aimed at resolving the Standar Model (SM) fermions mass hierarchy enigma. In the works of this approach, based upon the five-dimensional (5D) Randall-Sundrum (RS) model [1], chiral SM fermions "live" in the bulk, and their small masses m_f in four dimensions result from the overlaps of the fermionic bulk wave functions and the Higgs field profile confined to the IR end of the slice of the RS AdS_5 background; see reviews [2–4] and references therein. Then, the calculated values of masses of quarks or leptons strongly depend on the ratios $c_f = M_f/k$ of bulk masses M_f of corresponding Fermi fields to the curvature k of AdS_5: $m_f \sim \epsilon^{2c_f - 1}$, where $\epsilon = 10^{-16}$ is the Planck-TeV hierarchy parameter. Thus, to get the observed masses m_f, the special choice of values of parameters c_f in the vicinity of $1/2$ for every SM fermion is demanded. This fine-tuning of fermion bulk masses is an essential drawback of models [2–4]. We also pay attention to Refs. [5] and [6], in which each Standard Model family lives in its own slice of the 5D bulk separated by additional 3-branes and fermion masses in four dimensions essentially depend on the choice of parameters of the model, and to Ref. [7], in which fermions bulk masses are dynamically generated through fermions interaction with the bulk scalar field, but again bulk Yukawa coupling constants of this interaction are fine tuned for every SM fermion.

In the present paper, the Dirac equation with zero bulk mass on a 10-dimensional background is considered where profiles of Fermi fields in the 5D space-time remaining after compactification of $(4+1)$ extra dimensions are determined by eigenvalues $K_l^{(4)}$ of Dirac operator on compact subspace $K^{(4)}$ and by the angular momentum q with respect to the warped S^1 dimension. In this way, the present paper generalizes the approach of Refs. [8–11], in which the named angular momentum numbers the fermion generations and three fermion families arise from a single Fermi field in six dimensions. In the 6D model of Ref. [10], three fermion generations originate thanks to the introduced codimension-2 brane and corresponding "apple-shaped" conical singularity of the extra two-dimensional sphere. We use the same trick at the point of the Schwarzschild Euclidean "horizon" of the supergravity throat.

Beginning from Witten's seminal work [12], it is known that the chiral fermions of SM cannot be obtained from Dirac equation on D-dimensional space $M_4 \times \Sigma_{D-4}$ purely from the metric and spin connection on compact extra space Σ_{D-4}. For the Dirac operator on extra space to have physically necessary zero modes the topologically nontrivial background gauge fields must be introduced, as it is done, for example, in Refs. [13–15]. However, Witten's no-go theorem was formulated for the direct product of $M_4(x)$ and compact extra space $\Sigma_{D-4}(y)$, whereas we are considering what is typical for the RS-type models nonfactorizable warped geometries of type $A(y)M_4(x) \times \Sigma_{D-4}(y)$ where the nonzero eigenvalues of the extra space

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Dirac operator play the role similar to the bulk fermion masses in lower dimensions; that is, they, as it will be shown below, determine the bulk profiles of Fermi fields and do not prevent the appearance of chiral fermions in four dimensions.

The picture considered in this paper is a toy model that ignores the $SU(3)\times SU(2)\times U(1)$ group nature of the Standard Model and the left-right asymmetry of its Weyl fermions (see the discussion in the Conclusion). We also consider only the down-type quarks (d, s, b) that differ in $U(1)$ charge associated with the rotational symmetry around the brane. Thus, the results of this paper are in a sense complementary to approach of Ref. [13], in which only a single generation of the Standard Model spectrum is considered in a high-dimensional model incorporating the full group nature of Standard Model and four types of the SM fermions.

II. TEN-DIMENSIONAL BACKGROUND

Let us begin with the action $S^{(D)}$ in D dimensions for gravity described by metric g_{AB} ($A, B = 0, 1, 2\ldots(D-1)$), for scalar field ϕ, n-form $F_{(n)i_1 i_2\ldots in}$ and with Λ term, $F_{(n)}$ and Λ interacting with ϕ:

$$S^{(D)} = M_{(D)}^{D-2}\int\left\{R^{(D)} - \frac{1}{2}(\nabla\phi)^2 - \frac{e^{\alpha\phi}}{2n!}F_{(n)}^2 - 2\Lambda e^{-\frac{\alpha}{n-1}\phi}\right\}\times\sqrt{-g^{(D)}}d^D x, \tag{1}$$

where $M_{(D)}$ and $R^{(D)}$ are Planck mass and scalar curvature in D dimensions. Following Ref. [16], we postulate the invariance of the action (1) under the simultaneous scale transformation $g_{AB}\to e^{2\lambda}g_{AB}$, $\phi\to\phi + (2(n-1)/\alpha)\lambda$, $M_{(D)}\to e^{-\lambda}M_{(D)}$ ($\lambda = \mathrm{const}$); this gives the coupling constant of ϕ in the Λ term in (1). This action is a modified and reduced version of the written in Einstein frame bosonic part of the action of IIA supergravity (where $D = 10$, $n = 4$, $\alpha = 1/2$, $\Lambda = 0$, but plus to 4-form there are nonzero 2- and 3-forms and Chern-Simons term L_{SC} (see Ref. [17]):

$$S_{IIA}^{(10)} = M_{(10)}^8\int\left\{R^{(8)} - \frac{1}{2}(\nabla\phi)^2 - \frac{e^{\phi/2}}{24!}F_{(4)}^2 - \frac{e^{-\phi}}{23!}F_{(3)}^2 - \frac{e^{3\phi/2}}{22!}F_{(2)}^2 + L_{\mathrm{CS}}\right\}\sqrt{-g^{(10)}}d^{10}x. \tag{2}$$

And in general, exponent α in (1) is not arbitrary in the supergravity and string theories but depends on the numbers of compactified and noncompactified dimensionalities and order of the n-form; see the recent review [18]. The dynamical equations following from action (1) admit the long-time-known [16,19–24] p-brane (fluxbrane) throat-like solution which also permits the Schwarzschild-type Euclidean modification. This solution looks deep into the throat as

$$ds_{(D)}^2 = \left(\frac{r}{L}\right)^{2\beta(n-1)}\left[\eta_{\mu\nu}dx^\mu dx^\nu + U(r)\left(\frac{T_\theta}{2\pi}\right)^2 d\theta^2\right] + \left(\frac{L}{r}\right)^{2\xi(n-1)}\left[\frac{dr^2}{U(r)} + \delta^2 r^2 d\Omega_{(n)}^2\right], \tag{3}$$

$$e^{\phi} = e^{\phi_0}\left(\frac{r}{L}\right)^{\frac{2\alpha(n-1)}{\Delta}}, \qquad F_{(n)} = Q_{(n)}dy_1\wedge\ldots\wedge dy_n,$$
$$U(r) = 1 - \left(\frac{r_{\mathrm{Sch}}}{r}\right)^{n-1}, \tag{4}$$

where $\eta_{\mu\nu}$ is the most plus metric of the p-dimensional Minkowski space-time $M_{(p)}$; $\mu, \nu = 0, 1, \ldots(p-1)$; T_θ is the period of compact coordinate S^1 ($0<\theta<2\pi$); r is the isotropic coordinate along the throat; $d\Omega_{(n)}^2$ is the volume element on sphere S^n of unit radius, y_i are angles of this sphere ($i = 1, 2, \ldots n$); and length L and dimensionless constant δ are expressed through $Q_{(n)}$, e^{ϕ_0}, Λ; $D = p + n + 2$,

$$\beta = \frac{2(n-1)}{(p+n)\Delta}, \qquad \xi = \frac{2(p+1)}{(p+n)\Delta},$$
$$\Delta = \alpha^2 + \frac{2(p+1)(n-1)}{p+n}. \tag{5}$$

In metric (3) $\delta = 1$, if $\Lambda = 0$ in action (1), nonzero Λ-term in (1) results in $\delta\neq 1$; thus, δ is a free parameter of the considered model, and its physical meaning is clarified below.

Introduction of the imaginary periodic "time" (reverse temperature) is a standard tool when temperature effects in gravity are studied. The Schwarzschild type Euclidean deformation of the 4-brane metric ("4-soliton") was first introduced in IIA supergravity in Refs. [25] and [26] and is used in particular in the Witten-Sakai-Sugimoto model of holographic QCD; see, e.g., Refs. [27] and [28], in which metric (3) [with account of (4) and (5)] is written down in the string frame for the IIA supergravity values of parameters given in action (2).

But we shall use solution (3) in another way. Our goal is to study Dirac equation on the background (3). Let us perform the coordinate transformation

$$r\sim z^{-q},$$
$$q = \frac{\Delta}{2(n-1)-\Delta} = \frac{2(n-1)(p+1)+(p+n)\alpha^2}{2(n-1)^2-(p+n)\alpha^2}, \tag{6}$$

which takes metric (3) to the Poincare-like form convenient for writing down the Dirac equation on this background,

$$ds^2_{(D)} = \frac{1}{(kz)^{2s}}\left[\eta_{\mu\nu}dx^\mu dx^\nu + U(z)\left(\frac{T_\theta}{2\pi}\right)^2 d\theta^2 + \frac{dz^2}{U(z)} + \kappa^2 z^2 d\Omega^2_{(n)}\right], \tag{7}$$

where

$$U(z) = 1 - \left(\frac{z}{z_{\rm IR}}\right)^\gamma, \qquad s = \frac{2(n-1)^2}{2(n-1)^2 - (p+n)\alpha^2}, \qquad \gamma = (n-1)q, \qquad \kappa = \frac{\delta}{q}, \tag{8}$$

and q is given in (6). We note that for $\alpha = 0$ in action (1) metric (7) describes the Schwarzschild deformed $\mathrm{AdS}_{D-n} \times S^n$ space-time where κ is the ratio of radius of S^n to radius k^{-1} of AdS_{D-n}.

k and $z_{\rm IR}$ in (7) and (8) are simply expressed through L and $r_{\rm Sch}$ in (3) and (4), and there is no need to put down these expressions since in what follows only background (7) and (8) will be used.

To determine finally the background, it is necessary to match $z_{\rm IR}$ and T_θ. Following Refs. [19–24], we change coordinate z in (7) and (8) to τ,

$$z = z_{\rm IR}\left[1 - \frac{\gamma}{4}\left(\frac{\tau}{z_{\rm IR}}\right)^2\right]. \tag{9}$$

This coordinate transformation gives in vicinity of $z_{\rm IR}$, $(z_{\rm IR} - z) \ll z_{\rm IR}$, the following expressions for metric (7) and for U (8),

$$ds^2_{(D)} = \frac{1}{(kz_{\rm IR})^{2s}}\left[\eta_{\mu\nu}dx^\mu dx^\nu + \eta^2\tau^2 d\theta^2 + d\tau^2 + \kappa^2 z^2_{\rm IR} d\Omega^2_{(n)}\right], \tag{10}$$

where

$$\eta = \gamma \cdot \frac{T_\theta}{4\pi z_{\rm IR}}, \qquad U = \frac{\gamma^2}{4}\left(\frac{\tau}{z_{\rm IR}}\right)^2. \tag{11}$$

For $\eta = 1$, we have the smooth IR end of the IR throat, like was supposed, for example, in Refs. [25–28]. For $\eta \neq 1$, there is conical singularity with the codimension-2 IR plane at this point. As was shown in Ref. [10] in the 6D model, in case $2 < \eta < 4$, there are three fermion generations corresponding to three permitted quantum numbers along the angular θ coordinate. The same will happen in our model; see Sec. IV.

Throat (7) is terminated at its IR end at $z = z_{\rm IR}$. According to the conventional Randall-Sundrum approach, we limit the throat (7) also from below with the codimension-1 Planck UV brane located at $z = z_{\rm UV} = 1/k$ so that

$$z_{\rm UV} = \frac{1}{k} \leq z \leq z_{\rm IR}. \tag{12}$$

Again, according to the familiar Randall-Sundrum approach, the Planck-TeV hierarchy ($\epsilon = 10^{-16}$) is equal to the ratio of warp factors at the UV and IR ends of the throat; for metric (7), this means

$$\epsilon = 10^{-16} = \left(\frac{z_{\rm UV}}{z_{\rm IR}}\right)^s = (kz_{\rm IR})^{-s}. \tag{13}$$

To conclude this section, a few remarks are in order about UV and IR branes named above. At the UV end components of the extrinsic curvature (equal to the logarithmic derivatives over z of scale factors of subspaces $M_{(p)}$, S^1 and S^n of metric (7)), and also derivative of scalar field $\phi(z)$ are nonzero. Hence, Z_2 symmetry and the corresponding Israel junction conditions on the codimension-1 Planck brane $M_{(p)} \otimes S^1 \otimes S^n$ demand anisotropy of the UV-brane energy-momentum tensor. It is always possible to introduce the local surface action of the UV-brane possessing the necessary anisotropy of the UV-brane energy-momentum tensor. A more detailed discussion of this topic is beyond the scope of the present paper.

The case of the codimension-2 brane $M_{(p)} \otimes S^n$ at the IR end of the throat (7) is theoretically less trivial [29–33], while in our case, it is technically more simple since there are no discontinuities on the brane in the extrinsic curvatures of brane's subspaces and in the derivative of scalar field, which is evident if metric (7) is considered near $z = z_{\rm IR}$ in the form (10). This, in turn, means that the action of the codimension-2 IR brane cannot depend on the field ϕ and is of the isotropic Nambu-Goto type $S^{br}_{\rm IR} = -M^{D-2}_{(D)} \oint \sigma\sqrt{-h} d^{D-2}x$, where h is determinant of the induced metric and σ is the dimensionless brane's tension. Then, nonzero components of the brane's energy-momentum tensor are equal to $T^{br\nu}_\mu \sim \sigma\delta^\nu_\mu$, $T^{bri}_k \sim \sigma\delta^i_k$ and $T^{br} \sim (D-2)\sigma$ (T^{br} is the trace of the brane's energy-momentum tensor). Although components T^{br}_{AB} are proportional to delta function fixing the position of IR brane $[\delta(\tau)/[2\pi\sqrt{g_{\tau\tau}g_{\theta\theta}}]$ for metric (10)], the components $(\mu\nu)$ and (ik) of the rhs of the Einstein equations written in the form $M^{D-2}_{(D)} R^{(D)}_{AB} = T^{br}_{AB} - \frac{1}{D-2} g_{AB} T^{br}$ identically vanish for the codimension-2 brane. Hence, there is no delta function in the components R^μ_ν and R^i_k of the Ricci tensor and, as expected, no discontinuity in corresponding extrinsic curvatures.

On the other hand, the scaling factor before the S^1 subspace of metric (10) is singular at $\tau = 0$, and its logarithmic derivative is equal to $1/\tau$. The terms of the $\theta\theta$ component of the Einstein equations that include the delta function $\delta(\tau)$ give the known relation between brane tension and angle deficit factor: $\sigma = 2\pi(1-\eta)$ [29–33]. Hence, the choice of arbitrary parameter η (11) of the model is equivalent to the choice of the IR-brane dimensionless tension σ.

As it was noted above and will be proved in Sec. IV to have three fermion generations, we must have $2 < \eta < 4$; that is, tension σ must be negative. A negative-tension IR brane is quite common in the RS-type models. More generally, surfaces with negative tension are called in the literature not p-branes but O-planes or orientifolds in the literature; see Appendix in Ref. [18] or review [34].

III. DIRAC EQUATION

From now on, we will limit ourselves to the dimensionalities of IIA supergravity $D = 10$, $p = 4$, $n = 4$, leaving the α exponent arbitrary for now. The Dirac equation with zero bulk mass in ten dimensions follows from the action

$$S_\Psi = \int \bar{\Psi}_{(32)} \tilde{\Gamma}^A D_A \Psi_{(32)} \sqrt{-g_{(10)}} d^{10}x, \qquad (14)$$

where $\tilde{\Gamma}^A$ are 32×32 gamma matrices in curved 10-dimensional space-time and D_A are covariant derivatives with account of spin connection. We first write down the Dirac equation for the general metric of type (7),

$$ds^2_{(10)} = b^2(z)\eta_{\mu\nu} dx^\mu dx^\nu + P^2(z) d\theta^2 + N^2(z) dz^2 + a^2(z) d\Omega^2_{(4)} \qquad (15)$$

(μ, $\nu = 0$, 1, 2, 3), making the following choice of flat anticommuting Γ^A (cf. Refs. [35] and [36]):

$$\Gamma^\mu = \gamma^\mu \otimes \sigma^1 \otimes I_4; \qquad \Gamma^z = \gamma_5 \otimes \sigma^1 \otimes I_4;$$
$$\Gamma^\theta = I_4 \otimes \sigma^2 \otimes \tau_5; \qquad \Gamma^i = I_4 \otimes \sigma^2 \otimes \tau^i, \qquad (16)$$

where γ^μ are ordinary gamma matrices in Minkowski space-time, τ^i are the same on S^4, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\tau_5 = \tau^1\tau^2\tau^3\tau^4$, $\sigma^{1,2}$ are Pauli matrices, and I_4 is 4×4 unit matrix. The Corresponding Dirac equation on space-time (15) looks as (prime means derivative over z):

$$\left[\frac{1}{b}\Gamma^\mu \frac{\partial}{\partial x^\mu} + \frac{1}{N}\Gamma^z\left(\frac{\partial}{\partial z} + 2\frac{b'}{b} + \frac{1}{2}\frac{P'}{P} + 2\frac{a'}{a}\right) + \frac{1}{P}\Gamma^\theta \frac{\partial}{\partial\theta} + \frac{1}{a}\Gamma^i \nabla_i\right]\Psi_{(32)} = 0. \qquad (17)$$

The 32-component spinor may be presented as a decomposition of 8-component spinors $\Psi^{(l)}_{(8)}$ in six-dimensional space-time $\{x^\mu, \theta, z\}$ and some eigenfunctions $\chi_{(l)}(y^i)$ of Dirac operator $\Gamma^i \nabla_i$ on sphere S^4 of unit radius:

$$\Psi_{(32)} = \Sigma_{(l)} \Psi^{(l)}_{(8)}(x^\mu, \theta, z) \cdot \chi_{(l)}(y^i); \qquad \Gamma^i \nabla_i \chi_{(l)} = i K^{(4)}_l \chi_{(l)}; \qquad K^{(4)}_l = \pm(l+2), \qquad (18)$$

where (l) enumerates main eigenvalues of Dirac operator on S^4 and sets of azimuthal numbers (on sphere S^n, these main eigenvalues are $K^{(n)}_l = \pm(l + n/2)$, $l = 0, 1, 2\ldots$ [37]). The 8-component spinor is, in turn, a couple of 4-component spinors ($\pm$), each of them consisting of right and left Weyl 2-spinors. Thus, for definite eigenvalue of the Dirac operator on S^4 and definite angular mode q on S^1 ($q = 0, 1, 2\ldots$),

$$\Psi^{(l,q)}_{(8)} = \begin{pmatrix} \psi^+_R(x) F^+_R(z) \\ \psi^+_L(x) F^+_L(z) \\ \psi^-_R(x) F^-_R(z) \\ \psi^-_L(x) F^-_L(z) \end{pmatrix} \cdot \frac{e^{iq\theta}}{b^2 P^{\frac{1}{2}} a^2}; \qquad (19)$$

here and below we omit the indices (l, q) of $\psi^\pm_{R,L}$ and profiles $F^\pm_{R,L}$.

From the Dirac equations $(\gamma^\mu \partial_\mu - m^\pm)\psi^\pm(x) = 0$ for 4-spinors $\psi^\pm = \psi^\pm_R + \psi^\pm_L$ ($m^\pm$ are the fermion masses in four dimensions), and with account of (18) and (19), Dirac equation (17) comes to four equations for profiles $F^\pm_{R,L}$:

$$\begin{cases} \left(\frac{1}{N}\frac{d}{dz} + \frac{q}{P} + \frac{K^{(4)}_l}{a}\right) F^-_R - \frac{m^-}{b} F^-_L = 0, \\ \left(\frac{1}{N}\frac{d}{dz} - \frac{q}{P} - \frac{K^{(4)}_l}{a}\right) F^-_L + \frac{m^-}{b} F^-_R = 0, \end{cases} \qquad (20)$$

$$\begin{cases} \left(\frac{1}{N}\frac{d}{dz} - \frac{q}{P} - \frac{K^{(4)}_l}{a}\right) F^+_R - \frac{m^+}{b} F^+_L = 0, \\ \left(\frac{1}{N}\frac{d}{dz} + \frac{q}{P} + \frac{K^{(4)}_l}{a}\right) F^+_L + \frac{m^+}{b} F^+_R = 0. \end{cases} \qquad (21)$$

We note that equations for (+) and (−) components of 6D eight-spinor separate thanks to the choice (16) of higher-dimensional gamma matrices.

Finally, taking the scale factors $b(z)$, $P(z)$, $N(z)$, and $a(z)$ from comparison of metrics (15) and (7) (for the specific values of dimensionalities in (6)–(8) named in the beginning of this section), multiplying Eqs. (18) and (19) by N, defining constants c_l which are analogous to $c_f = M^{\rm bulk}_f/k$ in 5D models of Refs. [2–4], and passing to dimensionless quantities t, $\mu^\pm$,

$$t = \frac{2\pi z}{T_\theta}; \qquad \frac{\gamma}{2\eta}\epsilon^{\frac{1}{s}} = t_{\rm UV} < t < t_{\rm IR} = \frac{\gamma}{2\eta};$$
$$\mu^\pm = \frac{T_\theta}{2\pi} \cdot m^\pm; \qquad c_l = \frac{K^{(4)}_l}{\kappa} \qquad (22)$$

[the lower and upper limits for the variable t are determined from (12) and (13), $\epsilon = 10^{-16}$], we obtain from (20) the following system of equations for the profiles $F^-_{R,L}(t)$:

$$\begin{cases} \left(\frac{d}{dt}+\frac{q}{U}+\frac{c_l}{t\sqrt{U}}\right)F^-_R-\frac{\mu^-}{\sqrt{U}}F^-_L=0, & U=1-\left(\frac{2\eta t}{\gamma}\right)^\gamma \\ \left(\frac{d}{dt}-\frac{q}{U}-\frac{c_l}{t\sqrt{U}}\right)F^-_L+\frac{\mu^-}{\sqrt{U}}F^-_R=0, & \gamma=\frac{3(15+4\alpha^2)}{9-4\alpha^2} \end{cases}. \tag{23}$$

Equation (21) for $F^+_{R,L}$ take a similar form with corresponding changes in the signs of various terms. It is important to note that profiles of zero modes of F^-_R, F^+_L and of F^-_L, F^+_R coincide, which is easily seen from (20) and (21) in case $m^\pm=0$; this will be used in Sec. V.

It is known that the Hermiticity of Dirac operators in (20) [or (23)] requires the fulfillment of the boundary conditions (BCs):

$$F^-_L(z_{\rm UV})F^-_R(z_{\rm UV})=F^-_L(z_{\rm IR})F^-_R(z_{\rm IR})=0. \tag{24}$$

System (23), taking into account (22) and BC (24), is our main tool. In Sec. V, we shall analyze its chiral zero modes, when $\mu^-=0$. But it is interesting to note that for $U=1$ in (23) in the general case of nonchiral fermions, $\mu^-\neq 0$, second-order equation that follows from (23) formally coincide with the nonrelativistic Schrodinger equation for the wave function of an electron moving in a Coulomb field, where c_l in (23) play the role of the electron's orbital momentum. Thus, the solutions of these Equations are well known, and their spectra may be easily found. Unfortunately, the physical interpretation of these solutions and spectra, including from the point of view of the Standard Model, and the amazing parallel with nonrelativistic quantum mechanics remain vague. Therefore, we decided not to include these results in this paper.

IV. THREE GENERATIONS FROM ONE FERMI FIELD

The coefficient in action (14) at the kinetic term in four dimensions $\bar\psi\gamma^\mu\partial_\mu\psi$ must be equal to 1 for each mode q, l, $\pm$ and separately for R and L Weyl components. Thus, with account of $\sqrt{g_{(10)}}=b^4PNa^4$ [we again use generic metric (15)], $\Psi^2_{(8)}\sim(b^4Pa^4)^{-1}$ [see (19)], $\int\bar\chi_l\chi_{l'}d\Omega_{(4)}=\delta_{ll'}$, and $\int e^{i(q-q')\theta}d\theta=2\pi\delta_{qq'}$, the following normalization condition of functions $F^\pm_{R,L}(z)$ must be satisfied:

$$2\pi\int_{z_{\rm UV}}^{z_{\rm IR}}\frac{N}{b}(F^\pm_{R,L})^2dz=2\pi\int_{z_{\rm UV}}^{z_{\rm IR}}\frac{1}{\sqrt{U}}(F^\pm_{R,L})^2dz=1. \tag{25}$$

Transforming in this integral and in Eqs. (20) and (21) coordinating $z\to\tau$ like in (9) and taking into account that near the IR end of the throat $F_{R,L}\sim\tau^{\pm q/\eta}$ [see (20) and (21)] and $dz/\sqrt{U}\sim d\tau$ [the expression for U is given in (11)], we see that finiteness of integral (25) demands

$$\int_0\tau^{\pm\frac{2q}{\eta}}d\tau<\infty. \tag{26}$$

For the smooth IR end ($\eta=1$) of metric (7) or (10), this integral is nondivergent only for one mode, $q=0$. In case

$$2<\eta<4, \tag{27}$$

integral (26) is finite for three modes (three fermion generations) $q=0,\pm1$; hence, for these three modes, kinetic terms in four dimensions may be normalized to 1. Whereas for modes with divergent integral (26) this is impossible, such modes fall out of the spectrum of observable physical fields in Kaluza-Kline–type theories.

Thus, we reproduce here the result of Ref. [10], not for the extra 2-sphere as a background but for the Schwarzschild deformed supergravity background (7) and (8). Below, it is shown that ratios of fermion masses also depend on parameter η. Inequalities (27) mean also that dimensionless tension σ of the codimension-2 O-plane limiting the IR end of the throat (7) or (10) must be in the range $-6\pi<\sigma<-2\pi$; see comments at the end of Sec. II.

V. HIGGS MECHANISM AND ZERO MODES PROFILES

For $m^\pm=0$ in Eqs. (20) and (21), or equivalently $\mu^-=0$ in Eq. (23) for $F^-_{R,L}(t)$ and $\mu^+=0$ in similar equations for $F^+_{R,L}(t)$, and for $U=1$ in these equations, we have four zero modes profiles:

$$F^-_R=C^-_Re^{-qt}t^{-c_l},\qquad F^-_L=C^-_Le^{qt}t^{c_l},$$
$$F^+_R=C^+_Re^{qt}t^{c_l},\qquad F^+_L=C^+_Le^{-qt}t^{-c_l}. \tag{28}$$

Since the Higgs field is located on or near the IR brane, small fermion masses are acquired by the fields which profiles have minimum at the IR end and maximum at the other—UV end of the throat where $t\ll1$ [see definition of t in (22)]. Thus, out of four solutions (28), two proportional to t^{-c_l} must be left (we remind the reader that $c_l>1/2$), while C^-_L and C^+_R are set equal to zero, which, in turn, ensures the fulfillment of the boundary conditions (24).

Nonzero constants C^-_R and C^+_L are equal to each other and are determined from the normalization integral (25), which, being rewritten through variable t (22), for $U=1$, and after substitution nonzero profiles (28), takes the form (C is any of two named constants) $C^2T_\theta\int_{t_{\rm UV}}^{t_{\rm IR}}e^{-2qt}t^{-2c_l}dt=1$. Since the main contribution to this integral comes from small t where the $qt\ll1$ exponent in the integrand can be ignored (approximation $U=1$ is also justified because of it) and neglecting the small contribution from the upper limit of integration, we come to the normalization condition

$$(C_R^-)^2 = (C_L^+)^2 = \frac{2\pi(2c_l - 1)}{T_\theta} t_{\rm UV}^{2c_l-1}, \qquad (29)$$

which dependence on c_l is familiar to the similar conditions in 5D models [2–4].

The conventional way to introduce the SM Yukawa interactions is to add to spinor action (14) the Higgs field mass term,

$$S_H = \int \bar{\Psi}_{(32)} H \Psi_{(32)} \sqrt{-g_{(10)}} d^{10}x, \qquad (30)$$

where the vacuum average of the Higgs field $\langle H \rangle$ lives in the vicinity or at the IR end of the throat. After substitution here, expressions for $\Psi_{(32)}$ from (18) and (19) and integration over extra coordinates with the assumption that Higgs field does not depend on θ and y^i, the nonzero diagonal in the l, q mass terms of the fermion action in four dimensions $S_m^{(l,q)} = \int \bar{\psi}^{(l,q)} m_{l,q} \psi^{(l,q)} d^4x$ is obtained for every component (l, q) of the high-dimensional spinor $\Psi_{(32)}$, where

$$m_{l,q} = 2\pi \int F_R^{-(l,q)} F_L^{+(l,q)} \langle H \rangle N dz, \qquad (31)$$

N is the lapse function in metric (15) or (7).

It makes sense to note that, in contrast to the Dirac equation (17), where due to the choice of gamma matrices (16) the equations for two 4-components of the 8-component spinor (19) are separated [see (20) and (21)], the mass Higgs term of the action (30) entangles these ± 4-spinors, providing the necessary product of right and left Weyl spinors—like in (31). We believe that this is an important advantage of the model under consideration, since, for example, in the 5D models [2–4], right and left spinors are introduced independently and are quite arbitrarily combined into the Higgs terms of the Lagrangian, giving Yukawa interactions. On the emergence of left-right group asymmetry of the Standard Model Weyl fermions in theories with a big fermion multiplets, see the discussion in the Conclusion.

Finally, under the assumption $\langle H \rangle = Y\delta(z - z_{\rm IR})/N$ (Y is a dimensionless Yukawa coupling constant), writing down from (28) and (29) expressions for $F_R^-(t_{\rm IR})$ and $F_L^+(t_{\rm IR})$, taking into account the relationships (11) and (22) between T_θ, $z_{\rm IR}$, and $t_{\rm IR} = \gamma/2\eta$, we get from (31)

$$m_{l,q} = Y \frac{2c_l - 1}{z_{\rm IR}} \left(\frac{z_{\rm UV}}{z_{\rm IR}}\right)^{2c_l-1} e^{-q\frac{\gamma}{\eta}}. \qquad (32)$$

In this paper, three generations of one type of Standard Model fermions are considered. It is worth it to try to identify three masses (32) obeying inequalities $m_{q=1,l=0} < m_{q=0,l=0} < m_{q=-1,l=0}$ with masses of three down-type quarks d, s, b (at scale 2 GeV; errors are shown in brackets): $m_d = 4.67(32)$ MeV $< m_s = 0.093(2)$ GeV $< m(b) = 4.18(2)$ GeV [38]. The observed ratios of these masses are equal to

$$\frac{m_d}{m_s} = 5.0(7) \times 10^{-2}, \qquad \frac{m_s}{m_b} = 2.22(25) \times 10^{-2}, \qquad (33)$$

whereas in the considered model, these ratios, according to (32), are

$$\frac{m_d}{m_s} = \frac{m_s}{m_b} = e^{-\frac{\gamma}{\eta}}. \qquad (34)$$

For $\alpha = 1/2$ like in IIA supergravity (2), and η only slightly above 2, which guarantees three generations, see (27), we have from (6), (8), or (23) $\gamma = 6$; hence, $\gamma/\eta \approx 3$ and $m_d/m_s = e^{-3} = 0.05$, which coincide with its experimental value (33). We may continue to play in different nice values of the model parameters, but this numerology without physical justification is hardly interesting.

In fact, the relation $m_q \sim e^{-q\gamma/\eta}$ (32) is a sort of a geometry form of the Froggatt-Nielsen (FN) mechanism where hierarchical masses of the SM fermions are expressed in a way $m_f \sim \epsilon^{Q_f}$ (ϵ is a universal small constant, and Q_f are fermion's charges of Abelian horizontal symmetry $U(1)$) [39]. In our model, rotation around the S^1 coordinate corresponds to group $U(1)$, angular number q corresponds to Q_f, and parameter ϵ of the FN mechanism is $e^{-\gamma/\eta}$ of our model.

The essential drawback of the considered model is the equality of ratios of masses of lighter and heavier quarks, like in (34), whereas, in practice, they differ; see (33). The possible way to resolve this difficulty is discussed in the Conclusion.

As for the mentioned absolute values of the masses of down quarks, that is, the preexponential factor in (32), we note again that in the 5D approaches to SM parameters c_f, that is, bulk fermion masses, are fine tuned for every fermion mass in four dimensions.

The simple model considered above, where the compact four-dimensional space (the base) of the 10D supergravity (3) or (7) is S^4, is intended to illustrate that bulk fermion masses in lower dimensions can have a geometric origin. And the sensible physical results may be achieved. For example, in addition to the IIA supergravity parameters $\alpha = 1/2$, $D = 10$, $p = 4$, and $n = 4$, let us take free parameter $\kappa = 3$ [κ is introduced in (7) and (10)]; then, in the rhs of (32) $c_0 = 2/3$ [see (18) and (22)] and calculating $z_{\rm UV}/z_{\rm IR}$ from (13) with account of $s = 9/8$ [cf. Eq. (8) for the named values of parameters], for Yukawa coupling constant of order 1 and canonical value $Z_{\rm IR}^{-1} \approx 10$ TeV, expression (32) comes to

$$m_{0,q} \approx m_s e^{-q\frac{\gamma}{\eta}}. \qquad (35)$$

The repetition of similar calculations of $m_{1,q}$ (32) with the same input parameters but for the next ($l = 1$) eigenvalue of Dirac operator on S^4, that is, for $c_1 = 1$ [see (18) and (22)], gives mass scales less than 1 eV. Of course, it would be wonderful if the spectral number l enumerated types of SM fermions (up and down quarks, charged leptons, and neutrinos), while q corresponded to the generations of quarks of the same type. But this can hardly be expected in the framework of the considered extremely simplified model.

Now, a few final remarks about the approximation $U = 1$ in calculation of zero-mode profiles (28) are in order. It is not difficult to write down these profiles for $U \neq 1$ near $z = z_{\rm IR}$, that is, near $\tau = 0$, using the coordinate transformation (9), metric (10), and expression for U (11). Now, in contrast to the situation with $U = 1$ considered above, the zero-modes profiles are singular at $\tau = 0$ (although their norms are integrable for $q = 0, \pm 1$; see Sec. IV). Assumption $\langle H \rangle = Y\delta(z - z_{\rm IR})/N$ does not work now, to get the Higgs generated masses of type (31) and (32), it is necessary to put $\langle H \rangle = {\rm const}$ in some interval in the vicinity of $z = z_{\rm IR}$, like it was done, for example, in Ref. [8]. Estimates show that this way of taking into account $U \neq 1$ slightly changes the result (34) for the quark mass ratios.

VI. CONCLUSION

The first obvious improvement of the considered model would be, following Refs. [8–11], to drop the assumption of independence of Higgs field in (30) on the angular coordinate θ. Then, action (30) will generate mass matrix $m_{q,q'}$ with nonzero nondiagonal elements, $q \neq q'$; in particular, in this case, the ratios of the eigenvalues of this mass matrix should not be equal to each other like in (34). We did not fulfill these calculations because of the arbitrariness of the assumed dependence of the Higgs field on the extra coordinate. In the models incorporating up and down quarks, the knowledge of the $\hat{m}^{\rm up}$ and $\hat{m}^{\rm down}$ mass matrices would allow us to calculate the Cabibbo-Kobayashi-Maskawa matrix and hopefully to find the geometry origin of the so-called flavor puzzle. This surprising correlation between the experimentally observed ratios of quark masses and the Cabibbo-Kobayashi-Maskawa matrix mixing angles was considered in a recent author's paper [40], which is not related to the high-dimensional models.

The model of the present paper does not allow us to carry out the named calculations since it does not incorporate the Standard Model group nature, including the left-right group asymmetry of the SM chiral fermions. In many theories of type [2–8], the restoration of the SM group nature is achieved with the introduction for each SM chiral fermion with its specific group properties of its own bulk fermion field. A different approach, also adopted in this work, is characteristic of the Grand Unified Theories in higher dimensions, when all chiral SM fermions with their left-right group asymmetries and other properties are components of "big" Weyl fermion multiplets and arise as a result of compactification and spontaneous breakdown of a large group. It is also worth noting that the problem of the above-mentioned arbitrariness of the Higgs field can be resolved within the framework of the promising higher-dimensional "gauge-Higgs" unification approach. For this set of issues, see, for example, Refs. [13–15] and [41] or recent works [42,43] and references therein. In many of these theories, the problem of including three generations of fermions is solved by introducing an additional flavor group "horizontal" symmetry. Perhaps the geometry mechanism proposed in the present work can be useful in this context.

Three main results of this paper may be outlined.

It is shown in Sec. IV that Schwarzschild Euclidean deformation of the generalized IIA supergravity background with certain angle deficit factor $-6\pi < 2\pi(1 - \eta) < -2\pi$ of the conical singularity at the horizon leaves nondivergent three fermion angular modes interpreted as three generations of fermions of one and the same type. This result reproduces similar earlier results achieved perhaps in more artificial 6D models.

Proportion $m_q \sim e^{-q\gamma/\eta}$ (32) obtained in Sec. V is a geometry version of the FN mechanism. Also, in IIA supergravity [$\alpha = 1/2$, $D = 10$, $p = 4$, and $n = 4$ in (3)–(8)] and for $\eta = 2 + \epsilon$ in (27) ($\epsilon \ll 1$), the obtained ratio (34) of masses of d and s quarks $m_d/m_s = e^{-3}$ is experimentally viable. However, it is necessary to emphasize that in the unrealistic model under consideration, which ignores the SM group nature, specific numerical predictions are hardly justified.

It is demonstrated that using 10-dimensional supergravity backgrounds the usually arbitrarily selected in the models of 5D warped compactifications fermions' bulk masses may be identified with the eigenvalues of the Dirac operator on a compact 4D subspace $K^{(4)}$. This is possible because, as explained in the Introduction, Witten's no-go theorem [12] is not applicable in the models with warped compactifications. This theorem imposes serious restrictions in unified theories, such as Refs. [13–15], since to obtain chiral fermions in four dimensions it requires having chiral eigenvectors of the Dirac operator on a compactified subspace. Our result suggests that this requirement is not necessary to obtain chiral SM fermions in four dimensions.

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