Flavon vacuum alignment beyond SUSY

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In flavor models the vacuum alignment of flavons is typically achieved via the F-terms of certain fields in the supersymmetric limit. We propose a method for preserving such alignments, up to a rescaling of the vacuum expectation values, even after softly breaking supersymmetry (and the flavor symmetry). This facilitates the vacuum alignment in models which are nonsupersymmetric at low energies. Examples of models with different flavor groups, namely, A_4, T_7, S_4, and $\Delta(27)$, are discussed.

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I. INTRODUCTION

The origin of the curious triplication of the fermionic representations of the Standard Model (SM) and of their patterns of masses and mixing remains an open question. As symmetries have proven crucial in understanding and organizing the gauge sector, a well-studied approach to this problem has been to employ symmetries acting on flavor space in the quark and lepton sectors [1–3]; for reviews see, e.g., Refs. [4–9].

In many models with flavor symmetries, these are broken spontaneously, and a peculiar form of the vacuum is required in order to correctly describe the fermion masses and/or mixing.[1] For this reason, supersymmetric (SUSY) extensions of the SM are often considered in which gauge singlets, flavons, are responsible for this breaking. If the latter occurs while SUSY is still intact, the vacuum can be aligned via F-terms (of further fields, often called driving fields); see, e.g., Refs. [11,12].[2]

An important point is the sufficient segregation of different symmetry breaking sectors such that the vacuum of flavons contributing to, e.g., the neutrino and the charged lepton sectors, respectively, can be independently (and in different directions) aligned. In general, further symmetries, also called shaping symmetries, have to be invoked in order to achieve this aim. However, such a procedure usually cannot be applied in non-SUSY models in which, e.g., quartic interactions involving two different fields and their complex conjugates are invariant.[3] Thus, it is often assumed that certain couplings are absent (or highly suppressed), although their expected size is of order one.[4] A further possibility that has been explored in the literature is to considerably enlarge the flavor symmetry of the flavon potential, restricting the allowed scalar couplings, without affecting the structure of the Yukawa couplings [22–24].

In this paper, we study how vacuum alignments that are achieved via F-terms in the SUSY limit can be realized, up to a rescaling of the vacuum expectation values (VEVs), in non-SUSY models. For this, we include certain soft SUSY (and potentially also flavor symmetry) breaking terms in the potential.[5] We compute the expected size of the rescaling factor. Furthermore, we identify conditions which

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[1]See, e.g., Ref. [10] for models in which the flavor symmetry is broken at the boundaries of an extra dimension.

[2]If the employed fields transform under a new gauge symmetry, D-terms can also be relevant for the vacuum alignment; see, e.g., Refs. [13–15].

[3]The study of the orbit space of N-Higgs doublet potentials (see, e.g., Refs. [16–20]) allows us to analyze especially highly symmetric potentials and their minima.

[4]In extra-dimensional models, the flavons belonging to different symmetry breaking sectors can be separated via their localization in the extra dimension, and consequently, couplings between these are suppressed (see, e.g., Ref. [21]).

[5]In [25], soft SUSY breaking terms have been used in order to align the vacuum of flavons. In this work, however, we do not use these terms for the alignment but instead study how they can impact the vacuum alignment achieved in the SUSY limit.

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flavor symmetry breaking soft SUSY breaking terms must fulfill in order to maintain the direction of the aligned vacuum. These conditions are similar to those obtained in [26], where the authors have explored the vacuum alignment in multi-Higgs doublet potentials with a softly broken discrete symmetry. In particular, they have shown that the vacuum alignment, obtained in the symmetric potential, remains preserved, up to a rescaling of the VEVs, as long as the soft breaking terms have the aligned vacuum as an eigenvector. In this study, we focus on flavons which are triplets of a (discrete) flavor symmetry and analyze certain potentials in general. Furthermore, we present concrete examples with the flavor symmetries A_4, T_7, S_4, and $\Delta(27)$. These have been widely used in the literature; see, e.g., Refs. [11,21,27–41].

The paper is organized as follows. In Sec. II we determine the conditions under which a vacuum alignment found for a SUSY potential remains preserved even after introducing certain soft SUSY (and flavor symmetry) breaking terms for both flavons in real and complex representations. In Sec. III we exemplify these results in an A_4 model with one or two flavons. Section IV contains further examples in which flavons transform as (complex) triplets of T_7 and $\Delta(27)$ as well as a case with general soft masses for the group S_4. We summarize in Sec. V. A summary of the relevant properties of the different flavor groups considered, details of the minimization, and the effects of higher-order terms in the A_4 model are relegated to the appendices.

II. VACUUM ALIGNMENT FROM SUSY TO NON-SUSY POTENTIALS

In this section, we present the general idea and show that it is indeed possible to maintain the vacuum alignment achieved in the SUSY limit upon including soft SUSY breaking terms. We consider only cases with isolated minima and assume that the effects of the soft SUSY breaking terms are small, given that the flavor symmetry is usually spontaneously broken at a high energy scale where SUSY is still intact, while the scale of the soft SUSY breaking terms is taken to be of the order of a (few) TeV.

The aim is to obtain the same vacuum alignment from the non-SUSY potential (with certain soft SUSY breaking effects) as in the SUSY limit, up to a real rescaling factor ζ that should be close to one; see Eq. (14).[6] In the first part of this section, we focus on the size of ζ and the necessary conditions on the soft SUSY breaking terms in order to maintain the direction of the aligned vacuum, while in the latter part we comment on more general changes in the vacuum alignment.

[6]The following considerations can be generalized for complex ζ.

A. Employed superpotentials

We begin by specifying the framework that we consider in the following. First of all, we assume that the fields responsible for the breaking of the flavor symmetry are gauge singlets, commonly denoted as flavons. We call these ϕ throughout, potentially with subscripts that refer to a certain flavon multiplet and its components. As is well-known [11,12], the achievement of a certain vacuum alignment can be facilitated by the introduction of a continuous R-symmetry $U(1)_R$ and a further set of gauge singlet fields, called driving fields. These are denoted by Φ and also potentially have subscripts.

Assuming that the driving fields carry R-charge 2, while the flavons have no R-charge, the superpotential W is at most linear in the driving fields. In fact, the terms relevant for the derivation of the scalar potential are all linear in the driving fields, while terms representing Yukawa-type interactions do not contain these fields since supermultiplets containing SM fermions are assigned R-charge 1. In the current study, only the former part of W is of interest. Furthermore, we make the simplifying assumption that it is enough to consider only renormalizable terms of W.[7]

In order to present the idea, it is sufficient to focus on the situation of one driving field and one flavon. Since we imagine that the flavon is responsible for the generation of a certain flavor pattern (e.g., fermion mixing), this field is supposed to transform in a nontrivial representation of the (discrete) flavor symmetry G_f, usually as a two- or three-dimensional irreducible representation. For concreteness, we take $\phi \sim \mathbf{3}$, where this representation can be either real or complex.[8] For the driving field, we have, in general, two options: it can be either a (trivial) singlet of G_f or, like the flavon, in a nontrivial representation. In order to determine its assignment, we consider separately the case in which ϕ is in a real representation and ϕ is complex with respect to G_f. In either case, we want to ensure the appearance of an explicit mass scale in the superpotential, either in the form of a mass term $M^2\Phi$ or in the form of a dimensionful coupling $M\Phi\phi$,[9] such that the scale of the VEV of the flavon ϕ is determined by M.[10]

[7]The impact of nonrenormalizable operators is expected to be small. If the vacuum alignment necessitates such terms, one can always imagine introducing further fields, both flavons and driving fields, such that all nonrenormalizable terms can originate from renormalizable ones in a certain ultraviolet completion.

[8]In the case of even-dimensional irreducible representations, these can also be pseudoreal, and such instances can be discussed analogously.

[9]As we see in the example given in Sec. III, M might also originate from the spontaneous breaking of a further symmetry and thus be set by another VEV.

[10]In particular, we would like to avoid the VEV of any of the flavons being related to a flat direction of the potential and, consequently, its value being fixed only once additional terms, e.g., soft SUSY breaking terms, are taken into account as well. At the same time, the inclusion of a dimensionful parameter should ensure that none of the components of the flavons and driving fields remains massless.

For ϕ being a real triplet, $\phi \sim \mathbf{3}$, the relevant terms in the superpotential are

$$W_s = M^2\Phi + \lambda\Phi\phi^2 \tag{1}$$

with the driving field $\Phi \sim \mathbf{1}$; i.e., Φ transforms as a trivial singlet of the flavor symmetry G_f. These two couplings necessarily exist since ϕ is a real triplet. Further types of terms cannot exist, unless more fields are included in the considerations. A typical example of such a potential can be found in Sec. III for the flavor symmetry A_4.

The corresponding F-term (of the driving field Φ) reads[11]

$$F_s \equiv -F_\Phi^\star = M^2 + \lambda c_{ij}\phi_i\phi_j, \tag{2}$$

with the coefficients c_{ij} taking into account the possible nontrivial contraction of the indices of two triplets in order to obtain the trivial singlet. We can simplify this expression by choosing a basis in which the latter has the form of an ordinary scalar product, ϕ_i^2. Thus, we use

$$F_s = M^2 + \lambda\phi_i^2. \tag{3}$$

Since we assume that the vacuum alignment of the flavons occurs in the SUSY limit, the vanishing of this F-term partially aligns the flavon VEVs,

$$F_s|_{V_S} = M^2 + \lambda\phi_i^2|_{V_S} = 0, \quad \text{i.e.,} \quad \phi_i^2|_{V_S} = -\frac{M^2}{\lambda}. \tag{4}$$

Given that Φ is a singlet, only one constraint can be obtained on the flavon VEVs, and consequently, further driving fields (and flavons) are necessary in order to fix the VEVs of all components of the flavon ϕ.[12] We note that both parameters M and λ can be made real by a suitable choice of the phases of the fields Φ and ϕ. Thus, the form of the potential V_S itself is given by

$$V_S = F_sF_s^\star = M^4 + M^2\lambda(\phi_i^2 + (\phi_i^\star)^2) + V_4, \tag{5}$$

with V_4 containing the quartic terms. In the SUSY limit, consequently, we find for the first derivative of V_S with respect to $\phi_j^\star$,

$$\left.\frac{\partial V_S}{\partial\phi_j^\star}\right|_{V_S} = 0 = 2M^2\lambda\phi_j^\star|_{V_S} + \left.\frac{\partial V_4}{\partial\phi_j^\star}\right|_{V_S}. \tag{6}$$

For ϕ being a real triplet, another possible form of the superpotential is

$$W_t = M\Phi\phi + \lambda\Phi\phi^2, \tag{7}$$

with the driving field Φ transforming in the same way as ϕ, namely, $\Phi \sim \mathbf{3}$. While the first coupling is guaranteed to exist, the second one requires that the product of two triplets contains the triplet itself in its symmetric part. Note that we assume here, for simplicity, that the multiplicity of the triplet in the product of two triplets is one (which is true for many discrete groups). One such example is the group S_4 (see [7], Sec. IV C, as well as Appendix A 4).

Likewise, we can study the case of ϕ being a complex triplet. Then, the driving field necessarily also has to be a triplet. For $\phi \sim \mathbf{3}$, Φ has to transform as $\bar{\mathbf{3}}$ such that we can write down the first term in W_t in Eq. (7). As is common, we use a basis in which the representation matrices of the complex conjugate representation are the complex conjugate of those of the representation. The second term in Eq. (7) requires that the product of two triplets contains a triplet in its symmetric part. Again, we assume that its multiplicity is one, such that a unique term of the form $\Phi\phi^2$ exists. The group T_7 is an example of a flavor symmetry that leads to a superpotential which is of the form of W_t, and we discuss it explicitly in Sec. IV A. The case of having two independent cubic terms, $\Phi\phi^2$, is obtained with $\Delta(27)$, for which we also present an example in Sec. IV B. We note that a driving field that is a singlet usually cannot be employed, unless further flavons are present in the setup, since the product of a complex triplet with itself does not contain the trivial singlet $\mathbf{1}$.[13]

In this case, the F-terms, one for each component of the driving field Φ, read

$$F_{t,k} \equiv -F_{\Phi_k}^\star = M\phi_k + \lambda c_{k,ij}\phi_i\phi_j, \tag{8}$$

with the coefficients $c_{k,ij}$ representing the relevant combination of components of the fields Φ and ϕ leading to a trivial singlet.[14] Note that the coefficients $c_{k,ij}$ are symmetric in the second and third indices i and j, i.e., $c_{k,ij} = c_{k,ji}$. Again, the vacuum of the flavons is aligned in the SUSY limit; i.e., the F-terms have to vanish,

$$F_{t,k}|_{V_S} = M\phi_k|_{V_S} + \lambda c_{k,ij}\phi_i|_{V_S}\phi_j|_{V_S} = 0. \tag{9}$$

We note that it is, in general, possible to make the parameters M and λ real. The corresponding part of the potential reads

[11]If not stated otherwise, the repeated appearance of an index indicates that we sum over this index.

[12]In the more general case in which flavons and driving fields can also be charged under a new gauge symmetry, the vanishing of D-terms aligns the vacuum as well.

[13]The reader may wonder whether one could use a driving field transforming as a nontrivial singlet, but then the first term in Eq. (1) becomes forbidden, as long as it is assumed that the flavor symmetry is unbroken at the level of the superpotential. A further option is to introduce flavons in reducible representations of G_f that are composed of a triplet and its complex conjugate so that the product of two of these reducible representations contains a trivial singlet.

[14]We separate the index k from i and j since the former refers to the index of the driving field, while the latter denote the index of the flavons.

$$V_S = F_{t,k} F^{\star}_{t,k}. \quad (10)$$

We remark that while we present the idea assuming one pair of driving field Φ and flavon ϕ at a time, some of the considered examples (see Secs. III and IV) contain more driving fields in order to ensure that the VEVs of all components of the flavon can be aligned, as well as more flavons, because in realistic models, at least two of these are usually present. Additionally, the consideration of more than one flavon permits us to study further constraints arising from the requirement that the VEVs in the non-SUSY minimum should be rescaled by the same factor ζ. The situation of several flavons may also be reduced to the instance with one flavon only when it is possible to integrate out all except one.

Finally, we comment on the driving fields. Their vacuum is aligned with the help of the F-terms of the flavons, and due to the linearity of the superpotentials W_s and W_t in the driving fields, the F-terms of the flavons can always be set to zero by assuming a vanishing VEV for the driving fields. We choose such a vacuum in the following. Furthermore, we consider all components of the driving fields to be very heavy and, thus, irrelevant at low energies, allowing us to restrict our focus to the potential with flavons.

B. Adding soft SUSY breaking terms

In order to break SUSY, we add certain soft SUSY breaking terms. We remain agnostic about the origin of these terms and assume that these can have the structure we invoke in the following in order to preserve the vacuum alignment obtained in the SUSY limit. In the case of several flavons, they should also lead to the same rescaling for all flavon VEVs. For this reason, we do not add all possible types of soft SUSY breaking terms (e.g., A-terms and B-terms) that are compatible with the flavor symmetry but only those that are necessary in order to ensure that the masses of the corresponding SUSY partners can be lifted. Consequently, we add only a universal soft mass term for the flavon,[15]

$$V_{\mathrm{soft},G_f} = m^2 \phi^{\star}_k \phi_k. \quad (11)$$

We note that this term is also always invariant under G_f. Furthermore, the soft mass parameter m^2 is expected to be of the order of a (few) TeV and thus, in general, much smaller than M, $m^2 \ll M^2$.

In certain instances, it might be advantageous if the soft SUSY breaking terms, in our case the soft masses, explicitly, but softly, break the flavor symmetry. Then, the most general form is

$$V_{\mathrm{soft},gen} = m^2_{kl} \phi^{\star}_k \phi_l, \quad (12)$$

with m^2_{kl} being a Hermitian matrix, $m^2_{kl} = (m^2_{lk})^{\star}$, such that $V_{\mathrm{soft},gen}$ itself is real. In general, such arbitrary soft masses do not preserve the vacuum alignment achieved in the SUSY limit but have to fulfill certain conditions, similar to those found in the context of soft breaking mass terms in multi-Higgs doublet potentials [26], as we detail below.

While we also present the idea for general soft masses, we mainly concentrate on the case of flavor-symmetry preserving soft masses in the examples found in Secs. III and IV.

C. Non-SUSY potentials and their vacuum

With the information given in the preceding subsections, we can write down the potential V at low energies,

$$V = V_S + V_{\mathrm{soft}}, \quad (13)$$

where the potential V_S can be found in Eqs. (5) and (10), respectively, and V_{soft} is either of the form as in Eq. (11) or (12); we then study its vacuum. In particular, we can show that the vacuum alignment obtained in the SUSY limit, called $\phi|_{V_S}$, is not altered, when taking into account the soft SUSY masses, up to a possible rescaling with a real factor ζ close to one; i.e., the VEV of the non-SUSY potential, denoted by $\phi|_V$, is given by

$$\phi|_V = \zeta \phi|_{V_S} \quad \text{with} \quad \zeta \approx 1. \quad (14)$$

Furthermore, for general soft masses, we derive constraints on their form, which are needed to preserve Eq. (14).

In the case of the driving field Φ being a singlet, we note that for the vacuum aligned in the SUSY limit, the following holds,

$$\lambda \phi^2_i|_{V_S} = -M^2, \quad (15)$$

because of Eq. (4). Similarly, if Φ is a triplet, we find that the vacuum aligned in the SUSY limit has to fulfill

$$\lambda c_{k,ij} \phi_i|_{V_S} \phi_j|_{V_S} = -M \phi_k|_{V_S} \quad (16)$$

arising from Eq. (9).

1. Case 1: Singlet driving field, universal soft mass m^2

In this case, the form of the potential V is given by

$$V = V_S + V_{\mathrm{soft},G_f}, \quad (17)$$

[15] We can also introduce soft SUSY breaking terms for the driving fields. If the soft mass parameters are positive, the corresponding vacuum remains zero. In this case, driving fields do not play a role in the current study.

with V_S as in Eq. (5). We begin by supposing that the vacuum of V is related to the vacuum in the SUSY limit by a simple rescaling ζ; see Eq. (14). The F-term F_s then takes the form

$$F_s|_V = M^2 + \lambda\phi_i^2|_V = M^2 + \lambda\zeta^2\phi_i^2|_{V_S} = M^2(1-\zeta^2) \tag{18}$$

using Eqs. (4) and (14). Treating the soft mass term in an analogous way, we can write the potential V in its presumed minimum as a function of ζ,

$$V(\zeta) = V(\phi|_V) = M^4(1-\zeta^2)^2 + m^2\zeta^2\phi_k^\star|_{V_S}\phi_k|_{V_S}. \tag{19}$$

Computing the first derivative, we see that ζ can take three (approximate) values (for $m^2 \ll M^2$),

$$\zeta = 0, \qquad \zeta \approx -1 + \frac{m^2}{4M^4}\phi_k^\star|_{V_S}\phi_k|_{V_S},$$
$$\zeta \approx 1 - \frac{m^2}{4M^4}\phi_k^\star|_{V_S}\phi_k|_{V_S}. \tag{20}$$

Plugging these possible values for the minimum into the potential, we see that, while $V(\zeta = 0) = M^4$, the other two potential extrema lead to

$$V(\zeta) \approx m^2\phi_k^\star|_{V_S}\phi_k|_{V_S}. \tag{21}$$

As we consider only situations in which the minima are isolated and $m^2 \ll M^2$, the only plausible solution is, indeed, $\zeta \approx 1$. The other solution, $\zeta \approx -1$, is a consequence of the symmetry of the potential in the flavon ϕ. We can also analyze the first derivative of the potential with respect to the flavon $\phi_a^\star$, which can be written as

$$\frac{\partial V}{\partial\phi_a^\star} = 2M^2\lambda\phi_a^\star + \frac{\partial V_4}{\partial\phi_a^\star} + m^2\phi_a \tag{22}$$

using the form in Eq. (5). From

$$2M^2\lambda\phi_a^\star|_V + \left.\frac{\partial V_4}{\partial\phi_a^\star}\right|_V + m^2\phi_a|_V = 0 \tag{23}$$

and supposing that the rescaling of the vacuum holds [see Eq. (14)], the vacuum is real, and using Eq. (6), we have

$$\zeta(2(1-\zeta^2)M^2\lambda + m^2)\phi_a|_{V_S} = 0. \tag{24}$$

This leads to the same result for ζ as in Eq. (20), if we take into account the fact that the vacuum in the SUSY limit is constrained by Eq. (15).

2. Case 2: Singlet driving field, general soft masses m_{kl}^2

With general soft masses, the potential V reads

$$V = V_S + V_{\text{soft},gen}, \tag{25}$$

where V_S is given in Eq. (5). We proceed in a similar way as for case 1. First, we assume again that the vacuum is only rescaled with a real parameter ζ; see Eq. (14). We consider the first derivative of the potential in Eq. (25) with respect to $\phi_a^\star$,

$$\frac{\partial V}{\partial\phi_a^\star} = 2M^2\lambda\phi_a^\star + \frac{\partial V_4}{\partial\phi_a^\star} + m_{ai}^2\phi_i. \tag{26}$$

Using Eqs. (6) and (14) and ζ real, we obtain the following equation:

$$\zeta(2(1-\zeta^2)M^2\lambda\phi_a^\star|_{V_S} + m_{ai}^2\phi_i|_{V_S}) = 0. \tag{27}$$

With this we can not only compute the size of the rescaling ζ, but also determine the form of the general soft masses that is compatible with this minimum in the non-SUSY case. The latter becomes more obvious by noticing that for a real vacuum, $\phi_i|_{V_S} = \phi_i^\star|_{V_S}$, the equation can be written as (discarding the solution $\zeta = 0$)

$$m_{ai}^2\phi_i|_{V_S} = -2(1-\zeta^2)M^2\lambda\phi_a|_{V_S}, \tag{28}$$

meaning that the vacuum in the SUSY limit should be an eigenvector of the soft mass matrix m_{kl}^2 with the eigenvalue $-2(1-\zeta^2)M^2\lambda$ in order to be compatible with the rescaling of the vacuum. This is similar to the findings of [26] obtained for multi-Higgs doublet potentials.

3. Case 3: Triplet driving field, universal soft mass m^2

Here, we have as potential V

$$V = V_S + V_{\text{soft},G_f}, \tag{29}$$

with V_S found in Eq. (10). We also consider in this case the value of the potential as a function of the parameter ζ, assuming that Eq. (14) holds. We find

$$\begin{aligned} V(\zeta) = V(\phi|_V) &= V_S|_V + V_{\text{soft},G_f}|_V \\ &= \zeta^2((1-\zeta)^2M^2 + m^2)\phi_k^\star|_{V_S}\phi_k|_{V_S} \end{aligned} \tag{30}$$

since

$$\begin{aligned} F_{t,k}|_V &= M\phi_k|_V + \lambda c_{k,ij}\phi_i|_V\phi_j|_V \\ &= \zeta M\phi_k|_{V_S} + \lambda\zeta^2 c_{k,ij}\phi_i|_{V_S}\phi_j|_{V_S} \\ &= \zeta(1-\zeta)M\phi_k|_{V_S}; \end{aligned} \tag{31}$$

see Eqs. (8) and (16). Extremizing V in Eq. (30), we obtain as (approximate) solutions (for $m^2 \ll M^2$)

$$\zeta = 0, \qquad \zeta \approx \frac{1}{2} + \frac{m^2}{M^2}, \qquad \zeta \approx 1 - \frac{m^2}{M^2}. \tag{32}$$

The values of the potential at these points are $V(\zeta = 0) = 0$ and

$$V\left(\zeta \approx \frac{1}{2} + \frac{m^2}{M^2}\right) \approx \frac{1}{16} M^2 \phi_k^\star|_{V_S} \phi_k|_{V_S},$$
$$V\left(\zeta \approx 1 - \frac{m^2}{M^2}\right) \approx m^2 \phi_k^\star|_{V_S} \phi_k|_{V_S}. \tag{33}$$

Inspecting these, only the last one can correspond to a minimum that arises from a small perturbation of the one obtained in the SUSY limit.

The first derivative of V with respect to the field $\phi_a^\star$ at the minimum of this potential is given by

$$\left.\frac{\partial V}{\partial \phi_a^\star}\right|_V = 0 = F_{t,k}|_V \left.\frac{\partial F_{t,k}^\star}{\partial \phi_a^\star}\right|_V + m^2 \phi_a|_V. \tag{34}$$

Expanding the F-terms around the minimum obtained in the SUSY limit, i.e.,

$$F_{t,k}|_V \approx \left.\frac{\partial F_{t,k}}{\partial \phi_b}\right|_{V_S} (\phi_b|_V - \phi_b|_{V_S}), \qquad \left.\frac{\partial F_{t,k}^\star}{\partial \phi_a^\star}\right|_V \approx \left.\frac{\partial F_{t,k}^\star}{\partial \phi_a^\star}\right|_{V_S}, \tag{35}$$

and assuming the rescaling [see Eq. (14)], we obtain

$$(\zeta - 1) \left.\frac{\partial F_{t,k}^\star}{\partial \phi_a^\star}\right|_{V_S} \left.\frac{\partial F_{t,k}}{\partial \phi_b}\right|_{V_S} \phi_b|_{V_S} + \zeta m^2 \phi_a|_{V_S} \approx 0. \tag{36}$$

We identify the first expression as the mass matrix of the flavons in the SUSY limit

$$\mathcal{M}_{ab}^2 \equiv \left.\frac{\partial F_{t,k}^\star}{\partial \phi_a^\star}\right|_{V_S} \left.\frac{\partial F_{t,k}}{\partial \phi_b}\right|_{V_S}. \tag{37}$$

Using the expectation that ζ is close to one, up to corrections of order m^2/M^2, we arrive at

$$(1 - \zeta) \mathcal{M}_{ab}^2 \phi_b|_{V_S} \approx m^2 \phi_a|_{V_S}. \tag{38}$$

The solution to this equation is, indeed, $(1 - \zeta)$ of order m^2/M^2 since the eigenvalues of the mass matrix $\mathcal{M}_{ab}^2$ are of order M^2.

4. Case 4: Triplet driving field, general soft masses m_{kl}^2

For the last case, we begin with the potential V, being of the form

$$V = V_S + V_{\mathrm{soft},gen}, \tag{39}$$

with V_S taken from Eq. (10). The first derivative of V with respect to the field $\phi_a^\star$ at the minimum of this potential is of the form

$$\left.\frac{\partial V}{\partial \phi_a^\star}\right|_V = 0 = F_{t,k}|_V \left.\frac{\partial F_{t,k}^\star}{\partial \phi_a^\star}\right|_V + m_{al}^2 \phi_l|_V. \tag{40}$$

Expanding the F-terms around the minimum obtained in the SUSY limit as in Eq. (35) and employing the rescaling [see Eq. (14)], we have

$$(\zeta - 1) \mathcal{M}_{ab}^2 \phi_b|_{V_S} + \zeta m_{al}^2 \phi_l|_{V_S} \approx 0 \tag{41}$$

with the mass matrix $\mathcal{M}_{ab}^2$ given in Eq. (37). Neglecting higher orders in m^2/M^2, we find

$$(1 - \zeta) \mathcal{M}_{ab}^2 \phi_b|_{V_S} \approx m_{al}^2 \phi_l|_{V_S}. \tag{42}$$

Now, if the vacuum $\phi_b|_{V_S}$ is an eigenvector of the mass matrix $\mathcal{M}_{ab}^2$, we see that this condition simplifies, and the form of the general soft masses is constrained to also have this vacuum as an eigenvector with a certain eigenvalue. This result is equivalent to the one obtained in Eq. (28).

If the vacuum $\phi_b|_{V_S}$ and the coefficients $c_{k,ij}$ are real and the latter fulfill $c_{k,ij} = c_{i,kj} = c_{j,ik}$, it is, indeed, straightforward to show that the vacuum $\phi_b|_{V_S}$ is an eigenvector of the mass matrix $\mathcal{M}_{ab}^2$. Consider $\frac{\partial F_{t,k}}{\partial \phi_b}$, which can be computed from Eq. (8) as

$$\frac{\partial F_{t,k}}{\partial \phi_b} = M \delta_{kb} + 2\lambda c_{k,ib} \phi_i \tag{43}$$

using the symmetry of the coefficients $c_{k,ij}$. Thus, we have in the vacuum in the SUSY limit

$$\left.\frac{\partial F_{t,k}}{\partial \phi_b}\right|_{V_S} \phi_b|_{V_S} = M\phi_k|_{V_S} + 2(-M\phi_k|_{V_S}) = -M\phi_k|_{V_S}, \tag{44}$$

taking into account Eq. (16). For real $\phi_b|_{V_S}$ and $c_{k,ij}$ and remembering that M and λ can be made real without loss of generality, we have

$$\mathcal{M}_{ab}^2 \phi_b|_{V_S} = \left.\frac{\partial F_{t,k}^\star}{\partial \phi_a^\star}\right|_{V_S} (-M\phi_k|_{V_S})$$
$$= -M^2 \phi_a|_{V_S} - 2M\lambda c_{k,ia} \phi_i|_{V_S} \phi_k|_{V_S}. \tag{45}$$

This can be further simplified, if $c_{k,ij} = c_{i,kj} = c_{j,ik}$ holds, and we again use Eq. (16),

$$\mathcal{M}_{ab}^2 \phi_b|_{V_S} = -M^2 \phi_a|_{V_S} - 2M(-M\phi_a|_{V_S}) = M^2 \phi_a|_{V_S}. \tag{46}$$

In this case, the condition in Eq. (42) reduces to

$$(1-\zeta)M^2\phi_a|_{V_S} \approx m^2_{al}\phi_l|_{V_S}, \tag{47}$$

meaning the general soft mass matrix should have $\phi|_{V_S}$ as an eigenvector with the eigenvalue $(1-\zeta)M^2$.

D. Comment on other corrections to vacuum alignment

In the preceding part, we have focused on corrections to the vacuum alignment in the form of a real rescaling ζ. One may wonder whether corrections orthogonal to the direction of the vacuum alignment, obtained in the SUSY limit, could lead to a deeper minimum of the non-SUSY potential than the one due to the rescaled vacuum. We can write the vacuum in the form

$$\phi|_V = \zeta\phi|_{V_S} + \alpha\phi_{\perp,1} + \beta\phi_{\perp,2} \tag{48}$$

with α and β being (real) coefficients and $\phi_{\perp,1}$ and $\phi_{\perp,2}$ being vectors orthogonal to the vacuum $\phi|_{V_S}$ and orthogonal to each other. Plugging this form of $\phi|_V$ into the F-terms of the driving fields leads to a positive contribution to the potential V for α and/or β nonzero. Furthermore, for universal soft masses, this form of $\phi|_V$ leads for nonzero α and/or β to a larger value than for $\alpha=\beta=0$ (corresponding to the rescaled vacuum). Thus, if any of the parameters α and β are nonzero, this does not lead to smaller values of the potential than the rescaled vacuum. For general soft masses, one can require these to be of the form

$$m^2_{kl} = c_0\phi|_{V_S}\phi|^\dagger_{V_S} + c_1\phi_{\perp,1}\phi^\dagger_{\perp,1} + c_2\phi_{\perp,2}\phi^\dagger_{\perp,2} \tag{49}$$

such that $\phi|_{V_S}$, $\phi_{\perp,1}$, and $\phi_{\perp,2}$ are eigenvectors of the general soft masses with (real) eigenvalues proportional to c_0, c_1, and c_2, respectively. These can be chosen appropriately such that the rescaled vacuum leads to the smallest value of the potential V.

III. VACUUM ALIGNMENT IN THE A_4 MODEL

In this section, we discuss in detail the case of flavons that transform as (real) triplets under the flavor symmetry A_4. We first present the example of a single flavon and then generalize to the case of more flavons. Since the main purpose of these examples is to illustrate the procedure of Sec. II, they should be understood as toy models.

A. One flavon triplet

Suppose $\phi_a^T \equiv (a_1, a_2, a_3)$ transforms as a triplet of A_4 (see Table I) and that, in the SUSY limit, only one of the components of ϕ_a attains a nonzero VEV. This alignment is achieved with the help of the driving fields Φ_a and Φ_d, $\Phi_a \sim \mathbf{1}$ and $\Phi_d \sim \mathbf{3}$ under A_4. The relevant superpotential is of the form

TABLE I. Charge assignment for the flavon and driving fields of the A_4 model with a single flavon.

Fields	ϕ_a	Φ_a	Φ_d
A_4	**3**	**1**	**3**
$U(1)_R$	0	2	2
Z_N	1	$N-2$	$N-2$

$$W_a = \lambda_a\Phi_a(\phi_a^2 - x^2) + \lambda_d\Phi_d\phi_a^2. \tag{50}$$

We note that in addition to the flavor symmetry A_4 and the R-symmetry $U(1)_R$, we use a Z_N shaping symmetry in order to restrict the number of allowed terms. The first term of the superpotential is $SO(3)$-invariant. The term $\lambda_a\Phi_a x^2$, which is not invariant under the Z_N shaping symmetry, is assumed to be generated by the interaction with other fields that obtain a VEV at a higher scale. We also assume that the couplings λ_a and λ_d as well as the parameter x are real.

1. Vacuum alignment in the SUSY limit

In the SUSY limit, the vacuum alignment of the flavon ϕ_a is determined by the vanishing of the F-terms of the driving fields. These are given by

$$\frac{\partial W_a}{\partial\Phi_a} = -F^\star_{a^0} = \lambda_a(a_1^2 + a_2^2 + a_3^2 - x^2), \tag{51}$$

$$\frac{\partial W_a}{\partial\Phi_{d_i}} = -F^\star_{d_i^0} = \lambda_d a_{i+1}a_{i+2}. \tag{52}$$

In these equations, the subscripts related to the components of the triplets are understood to be cyclic, with values from 1 to 3.

Setting Eqs. (51) and (52) to zero fixes both the direction and size of the VEV of ϕ_a. In particular, Eq. (52) establishes that $\langle\phi_a\rangle$ must have only one component different from zero, and Eq. (51) sets its size. Without loss of generality, we choose

$$\langle\phi_a\rangle^T = (x, 0, 0). \tag{53}$$

The freedom to choose the position of the nonzero component reflects the fact that the minimum of the potential is symmetric under A_4; i.e., the solution shown in Eq. (53) can be transformed by any element of the group (which changes the position of the nonzero component) and still be a solution. This is also true for the sign of the flavon VEV.

The corresponding potential V_S can be written as

$$V_S = \lambda_a^2|a_1^2 + a_2^2 + a_3^2 - x^2|^2 + \lambda_d^2(|a_1|^2|a_2|^2 + |a_2|^2|a_3|^2 + |a_3|^2|a_1|^2). \tag{54}$$

In general, the extrema of the potential are derived from the vanishing of the first derivatives with respect to the components a_i of the flavon ϕ_a,

$$\left.\frac{\partial V_S}{\partial a_i}\right|_{\rm ext} = 2\lambda_a^2 a_i((a_1^\star)^2 + (a_2^\star)^2 + (a_3^\star)^2 - x^2) + \lambda_d^2 a_i^\star(|a_{i+1}|^2 + |a_{i+2}|^2) = 0. \quad (55)$$

It is easy to check that these conditions are consistent with the vacuum alignment in Eq. (53) for the minimum derived from the vanishing of the F-terms. The value of the potential V_S is zero at this minimum, as expected.

The next step is to break SUSY but maintain this vacuum alignment.

2. Vacuum alignment including soft SUSY breaking

As discussed in Sec. II, the vacuum alignment derived in the SUSY limit can be preserved, if the soft mass matrix fulfills certain conditions. In the A_4 model, we introduce a universal soft mass for the flavon ϕ_a so that the potential V becomes

$$V = V_S + \mu_a^2 a_i^\star a_i, \quad (56)$$

with V_S given in Eq. (54). Since the added soft mass is universal, we expect that the alignment achieved in the SUSY limit remains preserved.

Applying the results of Sec. II, we assume that the aligned vacuum is only rescaled by a real parameter ζ and can compute the value of the potential V as a function of ζ,

$$V(\zeta) = V(\phi_a|_V) = \lambda_a^2 x^4 (1-\zeta^2)^2 + \zeta^2\mu_a^2 x^2. \quad (57)$$

Minimizing the potential with respect to ζ, we obtain

$$0 = 2\zeta(2(\zeta^2-1)\lambda_a^2 x^4 + \mu_a^2 x^2). \quad (58)$$

Therefore, we have (discarding $\zeta = 0$)

$$\zeta^2 = 1 - \frac{\mu_a^2}{2\lambda_a^2 x^2}, \quad (59)$$

which is consistent with the expectation that ζ is of order one up to corrections suppressed by m^2/M^2.

This result can be compared to an explicit calculation which shows that the addition of the universal soft mass to the potential indeed preserves the vacuum alignment of the flavon. The minimization conditions are similar to the previous ones in Eq. (55) except for the new contribution proportional to the soft mass parameter μ_a^2,

$$\left.\frac{\partial V}{\partial a_i}\right|_{\rm ext} = \left.\frac{\partial V_S}{\partial a_i}\right|_{\rm ext} + \mu_a^2 a_i^\star = 0. \quad (60)$$

TABLE II. Charge assignment for the flavons and driving fields of the A_4 model with two flavons.

Fields	ϕ_a	ϕ_b	Φ_a	Φ_b	Φ_c	Φ_d	Φ_e
A_4	**3**	**3**	**1**	**1**	**1**	**3**	**3**
$U(1)_R$	0	0	2	2	2	2	2
Z_N	1	0	$N-2$	0	$N-1$	$N-2$	0
Z_M	0	1	0	$M-2$	$M-1$	0	$M-2$

As we have the freedom to choose the direction of the VEV, setting $a_2 = a_3 = 0$, the solution of Eq. (60) yields as global minimum of the potential,

$$a_1^2 = x^2\left(1 - \frac{\mu_a^2}{2\lambda_a^2 x^2}\right), \qquad a_2 = a_3 = 0; \quad (61)$$

see the detailed discussion in Appendix B. The alignment of the flavon remains the same, but its magnitude is rescaled by an amount proportional to the soft mass parameter μ_a^2. This is consistent with Eq. (59).

B. Two flavon triplets

We now present the case of an A_4 model with two flavon triplets whose vacua are aligned orthogonally in the SUSY limit.[16] In addition to the flavon and driving fields discussed in this section so far, we introduce a second flavon $\phi_b^T \equiv (b_1, b_2, b_3)$ and driving fields Φ_b, Φ_c, and Φ_e that allow us to fix the vacuum alignment of the flavons. We use the following superpotential:

$$W_{ab} = \lambda_a\Phi_a(\phi_a^2 - x^2) + \lambda_b\Phi_b(\phi_b^2 - y^2) + \lambda_c\Phi_c\phi_a\phi_b + \lambda_d\Phi_d\phi_a^2 + \lambda_e\Phi_e\phi_b^2. \quad (62)$$

The charge assignment of the flavons and driving fields is listed in Table II. Note that we have to use a further shaping symmetry, Z_M. The term $\lambda_b\Phi_b y^2$ is not Z_M-invariant and is assumed to arise in a similar way as the term $\lambda_a\Phi_a x^2$. Like the couplings λ_a and λ_d as well as the parameter x, we take λ_b, λ_c, λ_e, and y to be real.

The vacuum alignment in the SUSY limit is obtained from the vanishing of the F-terms. In addition to Eqs. (51) and (52), we have, for the F-terms,

$$\frac{\partial W_{ab}}{\partial \Phi_b} = -F_{b^0}^\star = \lambda_b(b_1^2 + b_2^2 + b_3^2 - y^2), \quad (63)$$

$$\frac{\partial W_{ab}}{\partial \Phi_c} = -F_{c^0}^\star = \lambda_c(a_1 b_1 + a_2 b_2 + a_3 b_3), \quad (64)$$

[16] For examples of A_4 models in which flavon triplets with orthogonal VEVs are employed, see, e.g., Refs. [15,42].

$$\frac{\partial W_{ab}}{\partial \Phi_{e_i}} = -F^{\star}_{e_i^0} = \lambda_e b_{i+1} b_{i+2}, \tag{65}$$

with cyclic indices assumed. As before, Eqs. (63)–(65) determine the size of the VEV of ϕ_b and imply that only one of the components of this flavon can obtain a nonzero VEV. In particular, Eq. (64) is used to ensure the orthogonality of the vacua of ϕ_a and ϕ_b. Keeping the alignment of the VEV of ϕ_a as given in Eq. (53), we choose

$$\langle \phi_b \rangle^T = (0, y, 0). \tag{66}$$

The corresponding potential reads

$$\begin{aligned} V_S = {} & \lambda_a^2 |a_1^2 + a_2^2 + a_3^2 - x^2|^2 + \lambda_b^2 |b_1^2 + b_2^2 + b_3^2 - y^2|^2 \\ & + \lambda_c^2 |a_1 b_1 + a_2 b_2 + a_3 b_3|^2 \\ & + \lambda_d^2 (|a_1|^2 |a_2|^2 + |a_2|^2 |a_3|^2 + |a_3|^2 |a_1|^2) \\ & + \lambda_e^2 (|b_1|^2 |b_2|^2 + |b_2|^2 |b_3|^2 + |b_3|^2 |b_1|^2). \end{aligned} \tag{67}$$

Extremizing the potential with respect to the components of the flavons, we find

$$\begin{aligned} \left.\frac{\partial V_S}{\partial a_i}\right|_{\text{ext}} = {} & 2\lambda_a^2 a_i ((a_1^\star)^2 + (a_2^\star)^2 + (a_3^\star)^2 - x^2) \\ & + \lambda_c^2 b_i (a_1^\star b_1^\star + a_2^\star b_2^\star + a_3^\star b_3^\star) \\ & + \lambda_d^2 a_i^\star (|a_{i+1}|^2 + |a_{i+2}|^2) = 0, \end{aligned} \tag{68}$$

$$\begin{aligned} \left.\frac{\partial V_S}{\partial b_i}\right|_{\text{ext}} = {} & 2\lambda_b^2 b_i ((b_1^\star)^2 + (b_2^\star)^2 + (b_3^\star)^2 - y^2) \\ & + \lambda_c^2 a_i (a_1^\star b_1^\star + a_2^\star b_2^\star + a_3^\star b_3^\star) \\ & + \lambda_e^2 b_i^\star (|b_{i+1}|^2 + |b_{i+2}|^2) = 0. \end{aligned} \tag{69}$$

These constraints are satisfied by the vacuum alignment in Eqs. (53) and (66).

As before, we use universal soft mass terms for the flavons. These are of the form $\mu_a^2 a_i^\star a_i$ and $\mu_b^2 b_i^\star b_i$. As a consequence, the vacuum alignment obtained in the SUSY limit remains preserved. In the following, we assume a single real rescaling ζ for both vacua and calculate the condition that results for the soft mass parameters μ_a^2 and μ_b^2.

The relevant potential V as a function of ζ is given by

$$\begin{aligned} V(\zeta) & = V(\phi_a, \phi_b|_V) \\ & = \lambda_a^2 x^4 (1-\zeta^2)^2 + \lambda_b^2 y^4 (1-\zeta^2)^2 \\ & \quad + \zeta^2 \mu_a^2 x^2 + \zeta^2 \mu_b^2 y^2. \end{aligned} \tag{70}$$

Minimizing the potential and discarding $\zeta = 0$, we obtain

$$\zeta^2 = 1 - \frac{\mu_a^2 x^2 + \mu_b^2 y^2}{2(\lambda_a^2 x^4 + \lambda_b^2 y^4)}. \tag{71}$$

Nevertheless, we can proceed as in the case of one flavon only and extremize the potential V with respect to the flavons

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\text{ext}} = \left.\frac{\partial V_S}{\partial \phi_i}\right|_{\text{ext}} + \mu_i^2 \phi_i^\star = 0, \tag{72}$$

for $i = a$, b. As before, we keep $a_2 = a_3 = 0$. Then, the solution of Eq. (72) yields, as a global minimum of the potential,

$$a_1^2 = x^2 \left(1 - \frac{\mu_a^2}{2\lambda_a^2 x^2}\right), \qquad a_2 = a_3 = 0, \tag{73}$$

$$b_1 = 0, \qquad b_2^2 = y^2 \left(1 - \frac{\mu_b^2}{2\lambda_b^2 y^2}\right), \qquad b_3 = 0, \tag{74}$$

as shown in Appendix B. Thus, we choose the free soft mass parameters to fulfill $\mu_a^2/(\lambda_a^2 x^2) = \mu_b^2/(\lambda_b^2 y^2)$ such that

$$\zeta^2 = 1 - \frac{\mu_a^2}{2\lambda_a^2 x^2} = 1 - \frac{\mu_b^2}{2\lambda_b^2 y^2}. \tag{75}$$

This result is consistent with the one in Eq. (71) for the assumed relation among the soft mass parameters.

C. Effect of higher-order terms on SUSY potential

The purpose of the Z_n shaping symmetries is not only to constrain the number of terms in the superpotential at the renormalizable level but also to control the effect of higher-order terms on the vacuum alignment achieved with the help of the F-terms of the driving fields. These higher-order terms contain, in general, more than two flavons (and always one driving field in order to keep the R-symmetry intact) and are thus nonrenormalizable. They are taken to be suppressed by the cutoff scale Λ, which is assumed to be larger than all scales present in the superpotential, especially larger than the scales x and y.

In order to show that the effect of higher-order terms can be kept well under control, we study the flavon combinations ϕ_a^n, ϕ_b^m, and $\phi_a^n \phi_b^m$ for n and m integers and $n + m$ larger than two, their VEVs following from the VEVs of the flavons ϕ_a and ϕ_b [see Eqs. (53) and (66)] and how they couple to the different driving fields $\Phi_a, ..., \Phi_e$. Indeed, we can see that only small shifts in the size of the VEVs of ϕ_a and ϕ_b, x and y, are induced, as long as we assume that the indices of the Z_n shaping symmetries, N and M, are both even. The smallest possible viable choice is $N = 4$ and $M = 4$ since, in the case of N and/or M being two, the driving fields Φ_a and Φ_d and/or Φ_b and Φ_e would turn out to be uncharged under the shaping symmetries (see Table II), and thus, further terms would become allowed. Choosing values for N and/or M larger than four is also possible and leads to a larger suppression of the shift in the size of the VEVs of the flavons. We note that N and M may also take on distinct values.

A detailed discussion of the flavon combinations, their VEVs, and their impact on the F-terms of the different driving fields can be found in Appendix C.

D. Comment on generalization to three flavon triplets

The example of two flavons whose VEVs are aligned orthogonally can be generalized to three flavon triplets with orthogonal alignment by adding another flavon, suitably chosen driving fields, and a further shaping symmetry Z_P. All statements made regarding the vacuum alignment in the SUSY limit and including soft SUSY breaking terms, as well as those regarding the impact of higher-order terms, can be adapted straightforwardly to the case of three flavon triplets.

IV. VACUUM ALIGNMENT FOR OTHER GROUPS

In the previous example for the group A_4, it has been shown that the explicit minimization of the scalar potential renders the same results as those derived in section II.

In the following, further examples are presented based on the groups T_7, $\Delta(27)$, and S_4. These cover different situations including the case of real and complex triplet representations as well as universal and general soft masses.

A. T_7 model

We first consider the simple case with two fields, a flavon and a driving field. Since the group T_7 only contains complex triplet representations, the flavon, $\phi^T \equiv (a_1, a_2, a_3)$, and the driving field, $\Phi^T \equiv (a_1^0, a_2^0, a_3^0)$, transform as a triplet and an antitriplet, respectively. More information about the group T_7 can be found in Appendix A 2 and references therein.

For our discussion, the relevant terms of the superpotential are

$$\begin{aligned} W &= M\Phi\phi + \lambda\Phi\phi^2 \\ &= M(a_1^0 a_1 + a_2^0 a_2 + a_3^0 a_3) \\ &\quad + \lambda(a_1^0 a_3^2 + a_2^0 a_1^2 + a_3^0 a_2^2). \end{aligned} \tag{76}$$

Differentiating with respect to the components of the driving field, the vanishing of their F-terms gives

$$\frac{\partial W}{\partial a_i^0} = -F_i^\star = M a_i + \lambda a_{i+2}^2 = 0, \tag{77}$$

where, again, cyclic indices are understood. Apart from the trivial solution, the alignment arising from Eq. (77) is

$$\phi|_{V_S} = -\frac{M}{\lambda}\begin{pmatrix} \omega_7^n \\ \omega_7^{2n} \\ \omega_7^{4n} \end{pmatrix} \quad \text{with} \quad \omega_7 = e^{\frac{2\pi i}{7}}, \quad n = 0, \ldots, 6. \tag{78}$$

Including universal soft masses [see Eq. (11)] and taking the vacuum of the non-SUSY potential V to be rescaled by ζ compared to the vacuum in Eq. (78), V as a function of ζ reads

$$V(\zeta) = V(\phi|_V) = \zeta^2((1-\zeta)^2 M^2 + m^2)\frac{3M^2}{\lambda^2}, \tag{79}$$

compare Eq. (30). Assuming $m^2 \ll M^2$, the possible extrema correspond to

$$\zeta_0 = 0, \qquad \zeta_1 \approx \frac{1}{2} + \frac{m^2}{M^2}, \qquad \zeta_2 \approx 1 - \frac{m^2}{M^2}, \tag{80}$$

where ζ_0 is the trivial minimum, ζ_1 corresponds to a local maximum, and ζ_2 is the shifted minimum, consistent with Eq. (32).

B. $\Delta(27)$ model

This example aims to show that the results of Sec. II can also be applied to the case of more than one flavon. This model has two pairs of flavons and driving fields. The flavons, $\phi_a^T \equiv (a_1, a_2, a_3)$ and $\phi_b^T \equiv (b_1, b_2, b_3)$, transform as a triplet and an antitriplet of $\Delta(27)$, respectively. The corresponding driving fields, $\Phi_a^T \equiv (a_1^0, a_2^0, a_3^0)$ and $\Phi_b^T \equiv (b_1^0, b_2^0, b_3^0)$, are instead $\Phi_a \sim \bar{\mathbf{3}}$ and $\Phi_b \sim \mathbf{3}$.

The superpotential at the renormalizable level is given by

$$\begin{aligned} W = {} & M_a\Phi_a\phi_a + M_b\Phi_b\phi_b + \Phi_a(\lambda_1\phi_b^2|_{\mathbf{3}_1} + \lambda_2\phi_b^2|_{\mathbf{3}_2}) \\ & + \Phi_b(\lambda_3\phi_a^2|_{\bar{\mathbf{3}}_1} + \lambda_4\phi_a^2|_{\bar{\mathbf{3}}_2}). \end{aligned} \tag{81}$$

We note that there are two independent cubic terms of the form $\Phi_a\phi_b^2$ and $\Phi_b\phi_a^2$, respectively. The explicit form of the resulting triplets (antitriplets) from the contractions ϕ_b^2 (ϕ_a^2) can be found in Appendix A 3. It can be checked that the conditions arising from the vanishing of the F-terms of the driving fields are

$$\frac{\partial W}{\partial a_i^0} = -F_{a_i^0}^\star = M_a a_i + \lambda_1 b_i^2 + \lambda_2 b_{i+1}b_{i+2} = 0, \tag{82}$$

$$\frac{\partial W}{\partial b_i^0} = -F_{b_i^0}^\star = M_b b_i + \lambda_3 a_i^2 + \lambda_4 a_{i+1}a_{i+2} = 0, \tag{83}$$

where again cyclic indices are understood. Two types of alignment are

$$\begin{aligned} &\phi_a, \phi_b|_{V_{S_1}} \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{with} \quad \langle a_1\rangle = -\omega^n \frac{M_a^{1/3}M_b^{2/3}}{\lambda_1^{1/3}\lambda_3^{2/3}}, \\ &\qquad \langle b_1\rangle = \frac{M_a M_b}{\lambda_1\lambda_3}\langle a_1\rangle^{-1} \end{aligned} \tag{84}$$

and

$$\phi_a, \phi_b|_{V_{S_2}} \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{with}$$

$$\langle a_1 \rangle = -\omega^n \frac{M_a^{1/3} M_b^{2/3}}{(\lambda_1+\lambda_2)^{1/3}(\lambda_3+\lambda_4)^{2/3}}, \qquad (85)$$

$$\langle b_1 \rangle = \frac{M_a M_b}{(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)} \langle a_1 \rangle^{-1}, \qquad (86)$$

where $n = 0, 1, 2$ and $\omega = e^{\frac{2\pi i}{3}}$. Including universal soft masses [see Eq. (11)], we can analyze the minima of the non-SUSY potential V. As in the case of the A_4 model with two flavon triplets (compare Sec. III B), the soft mass parameters are chosen such that the vacuum of the flavons ϕ_a and ϕ_b is rescaled by the same factor ζ. If we take the first type of alignment as an example, this requires, at first order in m^2/M^2,

$$\frac{m_b^2}{m_a^2} = \left(\frac{M_b \lambda_1}{M_a \lambda_3}\right)^{2/3} \frac{2M_a^{4/3}\lambda_1^{2/3} - M_b^{4/3}\lambda_3^{2/3}}{2M_b^{4/3}\lambda_3^{2/3} - M_a^{4/3}\lambda_1^{2/3}} + \mathcal{O}\left(\frac{m_a^2}{M_{a,b}^2}\right) \qquad (87)$$

for $M_a^{4/3}\lambda_1^{2/3} < 2M_b^{4/3}\lambda_3^{2/3}$. The rescaling ζ then reads

$$\zeta = 1 - \frac{\lambda_1^{2/3} m_a^2}{2M_a^{2/3}M_b^{4/3}\lambda_3^{2/3} - M_a^2\lambda_1^{2/3}} + \mathcal{O}\left(\frac{m_a^4}{M_{a,b}^4}\right). \qquad (88)$$

Furthermore, the value of the potential is given by

$$V(\phi_a, \phi_b|_{V_1}) = \frac{M_a^{2/3} M_b^{4/3} m_a^2}{\lambda_3^{4/3}\lambda_1^{2/3}} \frac{M_a^{4/3}\lambda_1^{2/3} + M_b^{4/3}\lambda_3^{2/3}}{2M_b^{4/3}\lambda_3^{2/3} - M_a^{4/3}\lambda_1^{2/3}} + \mathcal{O}(m_a^4). \qquad (89)$$

An analogous analysis can be performed for the second type of alignment in order to obtain the relation among the soft mass parameters, the shifted minimum, and the value of the potential V.

C. S_4 model

In this example, we study the constraints on general soft masses arising from requiring that these do not alter the vacuum alignment. The group S_4 contains two real triplets, **3** and **3**′. Further details and the relevant multiplication rule can be found in Appendix A 4. We consider one flavon, $\phi^T \equiv (a_1, a_2, a_3)$, and one driving field, $\Phi^T \equiv (a_1^0, a_2^0, a_3^0)$, both transforming as **3**.

The renormalizable terms in the superpotential are

$$\begin{aligned} W &= M\Phi\phi + \frac{\lambda}{2}\Phi\phi^2 \\ &= M(a_1^0 a_1 + a_2^0 a_2 + a_3^0 a_3) \\ &\quad + \lambda(a_1^0 a_2 a_3 + a_2^0 a_1 a_3 + a_3^0 a_1 a_2). \end{aligned} \qquad (90)$$

Setting the F-terms of the components of the driving field to zero leads to

$$\frac{\partial W}{\partial a_i^0} = -F_i^\star = M a_i + \lambda a_{i+1} a_{i+2} = 0, \qquad (91)$$

with cyclic indices being understood. Equation (91) is compatible with the following alignments (discarding the trivial vacuum):

$$\phi|_{V_S} = -\frac{M}{\lambda} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}. \qquad (92)$$

For universal soft masses [see Eq. (11)] and rescaling the SUSY vacuum by ζ, we obtain the potential V as a function of ζ,

$$V(\zeta) = V(\phi|_V) = \zeta^2((1-\zeta)^2 M^2 + m^2)\frac{3M^2}{\lambda^2}, \qquad (93)$$

compare Eq. (30). As expected, the solutions for ζ are those in Eq. (32).

For general soft masses [see Eq. (12)], we instead have to check Eq. (42) in order to determine the form compatible with a rescaling only of the vacuum as well as the value of ζ. We exemplify this for the first alignment mentioned in Eq. (92) and compute the form of the mass matrix $\mathcal{M}_{ab}^2$ [see Eq. (37)],

$$\mathcal{M}^2 = M^2 \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}. \qquad (94)$$

As we can see, the first alignment in Eq. (92) is, indeed, an eigenvector of this matrix with the eigenvalue M^2. Following from Eq. (42), this alignment also has to correspond to an eigenvector of the general soft mass matrix m_{kl}^2. Its form (assuming, for simplicity, that m_{kl}^2 is real) is then given by

$$m^2 = \begin{pmatrix} m_{11}^2 & m_{12}^2 & m_{12}^2 + m_{22}^2 - m_{33}^2 \\ m_{12}^2 & m_{22}^2 & m_{11}^2 + m_{12}^2 - m_{33}^2 \\ m_{12}^2 + m_{22}^2 - m_{33}^2 & m_{11}^2 + m_{12}^2 - m_{33}^2 & m_{33}^2 \end{pmatrix}. \qquad (95)$$

For ζ, we obtain, at first order in m^2/M^2,

$$\zeta \approx 1 - \frac{m_{11}^2}{M^2} - 2\frac{m_{12}^2}{M^2} - \frac{m_{22}^2}{M^2} + \frac{m_{33}^2}{M^2}. \quad (96)$$

Similar results are obtained for the other three alignments which are mentioned in Eq. (92).

V. SUMMARY

In this work, we have advised a way to implement vacuum alignments, obtained in SUSY theories, in non-SUSY models with (discrete) flavor symmetries. As is well-studied, the vacua of gauge singlets, flavons, can be aligned in specific directions in SUSY theories with the help of the F-terms of driving fields. It is desirable to apply such an alignment mechanism also in non-SUSY models in which it is notoriously difficult to obtain the correct vacuum alignment without suppressing some couplings by hand. This can be achieved by adding certain soft SUSY (and potentially flavor symmetry) breaking masses to the SUSY potential. The only effect of these terms is to rescale the aligned vacuum by a factor close to one, up to corrections of the order of the soft mass parameters which are small compared to the mass scales in the superpotential. In the case of general soft masses, we have identified conditions that must be fulfilled in order to maintain the alignment, up to rescaling. These are similar to those found for mass terms softly breaking the flavor symmetry in multi-Higgs doublet potentials [26]. Beyond the general case, we have discussed examples with the well-known flavor symmetries A_4, T_7, $\Delta(27)$, and S_4. For concreteness, we have assumed the flavons to be triplets of the flavor symmetry. In Sec. III, we have investigated the vacuum alignment in A_4 models with one or two flavons and shown that the alignment realized in the SUSY limit is only rescaled after the inclusion of universal soft masses. Furthermore, we have presented examples with T_7, $\Delta(27)$, and S_4 in order to study concrete cases with flavons in complex three-dimensional representations, as well as with a potential with general soft masses, in Sec. IV.

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APPENDIX A: GROUP THEORY DETAILS

In this appendix, we summarize information about the employed flavor symmetries A_4, T_7, $\Delta(27)$, and S_4.

1. Group theory of A_4

The group A_4 has twelve distinct elements and four irreducible representations. Apart from the trivial singlet **1**, it has two complex conjugated singlets, **1′** and **1″**, and one real triplet **3**. This group can be generated by two generators, s and t, that satisfy $s^2 = (st)^3 = t^3 = e$, where e is the neutral element. For the three-dimensional representation **3**, these elements can be chosen as the 3×3 matrices

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (A1)$$

The multiplication rule of two triplets is

$$\begin{aligned}(\mathbf{3}_a \otimes \mathbf{3}_b) = {} & (a_1 b_1 + a_2 b_2 + a_3 b_3)_{\mathbf{1}} \\ & + (a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3)_{\mathbf{1}'} \\ & + (a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3)_{\mathbf{1}''} \\ & + (a_2 b_3, a_3 b_1, a_1 b_2)_{\mathbf{3}} \\ & + (a_3 b_2, a_1 b_3, a_2 b_1)_{\mathbf{3}} \end{aligned} \quad (A2)$$

with $\omega = e^{\frac{2\pi i}{3}}$. Further details can be found in, e.g., Ref. [27].

2. Group theory of T_7

The group T_7 has 21 distinct elements. It contains five irreducible representations: three singlets, **1** and the complex conjugated pair **1′** and **1″**, and two complex conjugated triplets **3** and $\bar{\mathbf{3}}$. This group can be generated by two generators, s and t, which fulfill $s^7 = t^3 = e$ and $st = ts^4$. For the three-dimensional representation **3**, the generators can be chosen as

$$S = \begin{pmatrix} \omega_7 & 0 & 0 \\ 0 & \omega_7^2 & 0 \\ 0 & 0 & \omega_7^4 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (A3)$$

where $\omega_7 = e^{\frac{2\pi i}{7}}$. The relevant product rules for T_7 are

$$(\mathbf{3}_a \otimes \mathbf{3}_b) = (a_3b_3, a_1b_1, a_2b_2)_{\mathbf{3}} + (a_2b_3, a_3b_1, a_1b_2)_{\bar{\mathbf{3}}} + (a_3b_2, a_1b_3, a_2b_1)_{\bar{\mathbf{3}}},$$

$$(\mathbf{3}_a \otimes \bar{\mathbf{3}}_b) = (a_1b_1 + a_2b_2 + a_3b_3)_{\mathbf{1}} + (a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3)_{\mathbf{1}'} + (a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3)_{\mathbf{1}''} + (a_2b_1, a_3b_2, a_1b_3)_{\mathbf{3}} + (a_1b_2, a_2b_3, a_3b_1)_{\bar{\mathbf{3}}}, \tag{A4}$$

with $\omega = e^{\frac{2\pi i}{3}}$. Further information can be found in Ref. [30], for example.

3. Group theory of $\Delta(27)$

The group $\Delta(27)$ contains 27 different elements. It has nine irreducible singlets, $\mathbf{1_1}, \ldots, \mathbf{1_9}$, that correspond to the trivial singlet $\mathbf{1_1}$ and four pairs of complex conjugated singlets, as well as two complex conjugated triplets, $\mathbf{3}$ and $\bar{\mathbf{3}}$. This group can be described in terms of two generators, s and t, that satisfy $s^3 = t^3 = (st)^3 = (s^2t)^3 = e$. In the representation $\mathbf{3}$, these generators are represented by the matrices

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{A5}$$

with $\omega = e^{\frac{2\pi i}{3}}$. The relevant product rule for $\Delta(27)$ reads

$$(\mathbf{3}_a \otimes \mathbf{3}_b) = (a_1b_1, a_2b_2, a_3b_3)_{\bar{\mathbf{3}}_1} + (a_2b_3, a_3b_1, a_1b_2)_{\bar{\mathbf{3}}_2} + (a_3b_2, a_1b_3, a_2b_1)_{\bar{\mathbf{3}}_3}. \tag{A6}$$

For more information, see, e.g., Ref. [43].

4. Group theory of S_4

The group S_4 has 24 different elements and five real irreducible representations: two singlets, $\mathbf{1}$ and $\mathbf{1}'$; one doublet, $\mathbf{2}$; and two triplets, $\mathbf{3}$ and $\mathbf{3}'$. The group can be defined by two generators, a and b, fulfilling $a^4 = e = b^3$, $ab^2a = b$, and $aba = ba^2b$. In particular, for the representation $\mathbf{3}$, the following pair of real matrices can be chosen:

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{A7}$$

The relevant product rule for the triplet $\mathbf{3}$ is

$$(\mathbf{3}_a \otimes \mathbf{3}_b) = (a_1b_1 + a_2b_2 + a_3b_3)_{\mathbf{1}} + \left(\frac{a_2b_2 - a_3b_3}{\sqrt{2}}, \frac{-2a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{6}}\right)_{\mathbf{2}} + (a_2b_3 + a_3b_2, a_1b_3 + a_3b_1, a_1b_2 + a_2b_1)_{\mathbf{3}} + (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2)_{\mathbf{3}'}. \tag{A8}$$

For more information on S_4, see, e.g., Ref. [34].

APPENDIX B: MINIMUM OF A_4 POTENTIAL INCLUDING SOFT SUSY BREAKING

1. One flavon triplet

First, we analyze the extremization conditions in Eq. (60) for the A_4 potential including soft SUSY breaking masses and derive the form of the shifted minimum. Writing Eq. (60) in terms of the components of ϕ_a, we obtain

$$2\lambda_a^2 a_1((a_1^*)^2 - x^2) + \mu_a^2 a_1^* = 0, \tag{B1}$$

which yields $a_1^* = a_1(2\lambda_a^2 x^2)/(2\lambda_a^2|a_1|^2 + \mu_a^2)$, implying that a_1 and a_1^* have the same phase and must be real. In that case,

$$a_1^2 = x^2 - \frac{\mu_a^2}{2\lambda_a^2}, \qquad a_2 = a_3 = 0. \tag{B2}$$

We can check that this shifted vacuum leads to a minimum of the potential by examining its Hessian, expressed as a 3×3 symmetric matrix $\mathcal{H} = \partial^2 V/\partial a_i \partial a_j$. The principal minors $\mathcal{H}_i$ of this Hessian are defined as the determinants of the $i \times i$ upper-left submatrices. According to Sylvester's criterion, all principal minors of $\mathcal{H}$ must be positive at the point where the potential has a local minimum [44]. Explicitly calculating them, we obtain

$$\mathcal{H}_1 = 8\lambda_a^2\left(x^2 - \frac{\mu_a^2}{2\lambda_a^2}\right), \qquad \mathcal{H}_2 = 2\lambda_a^2\mathcal{H}_1\left(x^2 - \frac{\mu_a^2}{2\lambda_a^2}\right),$$
$$\mathcal{H}_3 = 2\lambda_d^2\mathcal{H}_2\left(x^2 - \frac{\mu_a^2}{2\lambda_a^2}\right). \tag{B3}$$

These are positive, implying that the vacuum in Eq. (B2) yields a local minimum of the potential.

It remains to verify that this minimum is also a global minimum of the potential including soft SUSY breaking masses. The value of the potential at the local minimum is given by

$$V_{\min} = \mu_a^2\left(x^2 - \frac{\mu_a^2}{4\lambda_a^2}\right). \tag{B4}$$

We would like to check whether or not a different choice of shifted vacuum could yield a value of the potential smaller than this. Suppose the addition of soft SUSY breaking masses causes a (real) shift $x^2 \to x^2 + \delta_a$. The value of the potential then reads

$$V_{\min}(\delta_a) = \mu_a^2\left(x^2 + \delta_a + \frac{\lambda_a^2\delta_a^2}{\mu_a^2}\right) \tag{B5}$$

with δ_a unknown. Extremizing the potential with respect to this quantity, $\partial V_{\min}/\partial\delta_a = 0$, yields $\delta_a = -\mu_a^2/(2\lambda_a^2)$. Hence, the minimum in Eq. (B2) is the global one of the potential.

2. Two flavon triplets

We move on to the case of two flavon triplets. Writing the extremization conditions in Eq. (72) in terms of the components of ϕ_a and ϕ_b, we find, for $a_2 = a_3 = 0$,

$$2\lambda_a^2 a_1((a_1^*)^2 - x^2) + \lambda_c^2 a_1^*|b_1|^2 + \mu_a^2 a_1^* = 0, \tag{B6}$$

$$\lambda_c^2 a_1^* b_1^* b_2 = 0, \tag{B7}$$

$$\lambda_c^2 a_1^* b_1^* b_3 = 0, \tag{B8}$$

$$2\lambda_b^2 b_1((b_1^*)^2 + (b_2^*)^2 + (b_3^*)^2 - y^2) + \lambda_c^2|a_1|^2 b_1^* + \lambda_e^2 b_1^*(|b_2|^2 + |b_3|^2) + \mu_b^2 b_1^* = 0, \tag{B9}$$

$$2\lambda_b^2 b_2((b_1^*)^2 + (b_2^*)^2 + (b_3^*)^2 - y^2) + \lambda_e^2 b_2^*(|b_1|^2 + |b_3|^2) + \mu_b^2 b_2^* = 0, \tag{B10}$$

$$2\lambda_b^2 b_3((b_1^*)^2 + (b_2^*)^2 + (b_3^*)^2 - y^2) + \lambda_e^2 b_3^*(|b_1|^2 + |b_2|^2) + \mu_b^2 b_3^* = 0. \tag{B11}$$

From Eqs. (B7) and (B8), we obtain two cases, (I) $b_1 = 0$ and (II) $b_2 = b_3 = 0$.

a. Case (I) Equation (B6) leads to $a_1^* = a_1(2\lambda_a^2 x^2)/(2\lambda_a^2|a_1|^2 + \mu_a^2)$, which implies that a_1 must be real and is given by

$$a_1^2 = x^2 - \frac{\mu_a^2}{2\lambda_a^2}. \tag{B12}$$

Furthermore, setting $b_1 = 0$, Eqs. (B10) and (B11) are solved for $b_2 = 0$ and $b_3 = 0$, respectively. However, we are not interested in the trivial solution, $b_1 = b_2 = b_3 = 0$. We, therefore, consider the following two cases.

Case (I.a) If we assume $b_3 = 0$, Eq. (B10) implies that b_2 must be real, and we arrive at

$$b_1 = 0, \qquad b_2^2 = y^2 - \frac{\mu_b^2}{2\lambda_b^2}, \qquad b_3 = 0. \tag{B13}$$

We note that an analogous result can be obtained for $b_2 = 0$ instead.

Case (I.b) If $b_2 \neq 0$ and $b_3 \neq 0$, Eqs. (B10) and (B11) yield

$$(b_2^*)^2 + (b_3^*)^2 - y^2 = -\frac{b_2^*}{b_2}\left(\frac{\lambda_e^2|b_3|^2 + \mu_b^2}{2\lambda_b^2}\right) = -\frac{b_3^*}{b_3}\left(\frac{\lambda_e^2|b_2|^2 + \mu_b^2}{2\lambda_b^2}\right), \tag{B14}$$

implying that b_2 and b_3 must have the same phase. Then, each term on the left-hand side of Eq. (B10) or (B11) has to have the phase of b_2^*, telling us that b_2 and b_3 are real. Consequently, Eq. (B14) leads to $b_1 = 0, b_2^2 = b_3^2 = (2\lambda_b^2 y^2 - \mu_b^2)/(4\lambda_b^2 + \lambda_e^2)$.

The potential for cases (I.a) and (I.b) can be evaluated as

$$V_{\mathrm{I.a}} = \mu_a^2\left(x^2 - \frac{\mu_a^2}{4\lambda_a^2}\right) + \mu_b^2\left(y^2 - \frac{\mu_b^2}{4\lambda_b^2}\right),$$
$$V_{\mathrm{I.b}} = V_{\mathrm{I.a}} + \frac{\lambda_e^2(2\lambda_b^2 y^2 - \mu_b^2)^2}{4\lambda_b^2(4\lambda_b^2 + \lambda_e^2)}. \tag{B15}$$

Clearly, case (I.b) cannot yield the global minimum.

b. Case (II) For $b_2 = b_3 = 0$, Eqs. (B6) and (B9) imply that both a_1 and b_1 are real, and we find

$$a_1^2 = \frac{2\lambda_b^2\mu_a^2 - 4\lambda_a^2\lambda_b^2 x^2 + 2\lambda_b^2\lambda_c^2 y^2 - \lambda_c^2\mu_b^2}{\lambda_c^4 - 4\lambda_a^2\lambda_b^2},$$
$$b_1^2 = \frac{2\lambda_a^2\mu_b^2 - 4\lambda_a^2\lambda_b^2 y^2 + 2\lambda_a^2\lambda_c^2 x^2 - \lambda_c^2\mu_a^2}{\lambda_c^4 - 4\lambda_a^2\lambda_b^2}. \tag{B16}$$

The value of the potential at this point is

$$V_{\mathrm{II}} = V_{\mathrm{I.a}} + \lambda_a^2\left[a_1^2 - \left(x^2 - \frac{\mu_a^2}{2\lambda_a^2}\right)\right]^2 + \lambda_b^2\left[b_1^2 - \left(y^2 - \frac{\mu_b^2}{2\lambda_b^2}\right)\right]^2 + \lambda_c^2 a_1^2 b_1^2, \tag{B17}$$

which, again, cannot be the global minimum.

We conclude that the potential is minimized for case (I.a), implying that the universal soft masses do not change the vacuum alignment but simply induce a shift in the vacuum proportional to the soft mass parameters. One may verify that this leads to a local minimum of the potential from the Hessian, which is now a 6×6 symmetric matrix $\mathcal{H} = \partial^2 V/\partial\varphi_i\partial\varphi_j$, where $\varphi^T \equiv (a_1, a_2, a_3, b_1, b_2, b_3)$. The principal minors of the Hessian are given by

$$\begin{aligned}
\mathcal{H}_1 &= 8\lambda_a^2\left(x^2 - \frac{\mu_a^2}{2\lambda_a^2}\right),\\
\mathcal{H}_2 &= 2\mathcal{H}_1\left[\lambda_d^2\left(x^2 - \frac{\mu_a^2}{2\lambda_a^2}\right) + \lambda_c^2\left(y^2 - \frac{\mu_b^2}{2\lambda_b^2}\right)\right],\\
\mathcal{H}_3 &= 2\lambda_d^2\mathcal{H}_2\left(x^2 - \frac{\mu_a^2}{2\lambda_a^2}\right),\\
\mathcal{H}_4 &= \frac{\lambda_d^2}{\lambda_a^2}\mathcal{H}_1^2\left[\lambda_c^2\lambda_d^2\left(x^2 - \frac{\mu_a^2}{2\lambda_a^2}\right)^2\right.\\
&\quad\left. + \lambda_d^2\lambda_e^2\left(x^2 - \frac{\mu_a^2}{2\lambda_a^2}\right)\left(y^2 - \frac{\mu_b^2}{2\lambda_b^2}\right) + \lambda_c^2\lambda_e^2\left(y^2 - \frac{\mu_b^2}{2\lambda_b^2}\right)^2\right],\\
\mathcal{H}_5 &= 8\lambda_b^2\mathcal{H}_4\left(y^2 - \frac{\mu_b^2}{2\lambda_b^2}\right),\\
\mathcal{H}_6 &= 2\lambda_e^2\mathcal{H}_5\left(y^2 - \frac{\mu_b^2}{2\lambda_b^2}\right),
\end{aligned} \tag{B18}$$

which are all positive, confirming that the shifted vacua in Eqs. (B12) and (B13) lead to a local minimum, with the value of the potential given by $V_{\mathrm{I.a}}$ in Eq. (B15). As before, we can further check whether or not it is the global minimum, assuming that $x^2 \to x^2 + \delta_a$ and $y^2 \to y^2 + \delta_b$ could give rise to a value of the potential smaller than $V_{\mathrm{I.a}}$. We find

$$\begin{aligned}
V_{\mathrm{I.a}}(\delta_a, \delta_b) = \mu_a^2\left(x^2 + \delta_a + \frac{\lambda_a^2\delta_a^2}{\mu_a^2}\right)\\
+ \mu_b^2\left(y^2 + \delta_b + \frac{\lambda_b^2\delta_b^2}{\mu_b^2}\right).
\end{aligned} \tag{B19}$$

Extremizing $V_{\mathrm{I.a}}(\delta_a, \delta_b)$, we obtain $\delta_a = -\mu_a^2/(2\lambda_a^2)$ and $\delta_b = -\mu_b^2/(2\lambda_b^2)$, leading to the shifted vacua found in Eqs. (B12) and (B13).

APPENDIX C: HIGHER-ORDER TERMS IN SUSY A_4 POTENTIAL AND SHAPING SYMMETRIES

In this appendix, we present the form and possible impact of the higher-order terms on the SUSY A_4 potential and how the choice of the indices of the Z_n shaping symmetries can reduce their effect to only a (small) shift in the size of the VEVs of the flavons. In the following, we refer to the case of two flavon triplets. Clearly, these results can be directly applied to the case of one flavon triplet only and also generalized to three flavon triplets.

We study the combinations ϕ_a^n, ϕ_b^m, and $\phi_a^n\phi_b^m$ with n and m integers and $n + m$ larger than two.

Assuming the vacuum alignment given in Eq. (53), for the combination ϕ_a^n with n even, only the covariants that transform as singlets have a nonzero VEV, while for n odd, only the covariant that transforms as a triplet has a nonvanishing VEV, whose form is proportional to $\langle\phi_a\rangle^T$. Similarly, we find, for ϕ_b^m, with the vacuum alignment shown in Eq. (66), that the covariants with a nonzero VEV are either singlets for m even or the triplet for m odd, with its VEV being proportional to $\langle\phi_b\rangle^T$.

For the flavon combinations $\phi_a^n\phi_b^m$ with n and m both integers and larger than zero, we can classify the covariants acquiring a nonvanishing VEV for the vacuum alignment in Eqs. (53) and (66) according to whether $n + m$, n, and m are even or odd. In particular, we have, for $n + m$ odd, with n even (odd) and m odd (even), that only the covariant which transforms as a triplet has a nonzero VEV that is proportional to $\langle\phi_b\rangle^T$ ($\langle\phi_a\rangle^T$). For $n + m$ even (and equal or larger than four), with both n and m also even, only the covariants that are singlets acquire a VEV, while for $n + m$ even, with both n and m odd, only the triplet has a nonzero VEV, whose form is proportional to $\langle\phi_a\phi_b\rangle^T$, i.e., proportional to (0,0,1).

In the next step, we check, for each flavon combination, whether or not it can be coupled in a Z_n- and A_4-invariant way to one (or more) of the driving fields $\Phi_a, ..., \Phi_e$ as well as if such a coupling can give rise to a nonzero contribution to the F-terms of the driving fields and, thus, has an impact on the vacuum alignment of the flavons.

Before doing so, we emphasize that the driving fields Φ_a and Φ_b are responsible for the size of the VEVs of the flavons and not for their alignment. Thus, any contribution from higher-order terms to the F-terms of Φ_a and Φ_b can only lead to a shift in the flavon VEVs and, thus, is acceptable, as long as this shift is small compared to the scales x and y. The size of such shifts is determined by the choice of the indices of the Z_n shaping symmetries. On the contrary, the driving fields Φ_c, Φ_d, and Φ_e are responsible for the alignment of the flavon VEVs, and consequently, the impact of higher-order terms on their F-terms should be suppressed or absent due to the choice of the shaping symmetries.

Furthermore, we note that in order to form an invariant under $Z_N \times Z_M$, flavon combinations coupling to Φ_a and Φ_d have to have the charges (2,0), while flavon combinations coupling to Φ_b and Φ_e should carry the charges (0,2). Eventually, those coupling in an invariant manner to the driving field Φ_c must have the Z_n charges (1,1). The invariance under A_4 requires the flavon combinations coupling to Φ_a, Φ_b, and Φ_c to be a trivial singlet, whereas the ones coupling to Φ_d and Φ_e have to transform as a triplet; see Table II.

Clearly, the flavon combinations ϕ_a^n with n an integer cannot couple in an invariant way to the driving fields Φ_b, Φ_c, and Φ_e. An invariant coupling is instead possible to the driving fields Φ_a and Φ_d, in case n equals $2 + \alpha N$ with α an integer larger than zero.[17] As mentioned, for $2 + \alpha N$ being even, the covariants transforming as singlets acquire a nonzero VEV, and thus, a contribution to the F-term of Φ_a, in general, arises. Similarly, for $2 + \alpha N$ odd, the covariant being the triplet obtains a nonvanishing VEV which affects the F-terms of Φ_d and, as a consequence, the vacuum alignment of the flavon ϕ_a. This can be easily avoided, if N is even. Furthermore, N should be larger than two since otherwise the driving fields Φ_a and Φ_d would be neutral and additional couplings become allowed.

Similarly, combinations of the form ϕ_b^m with m an integer cannot form an invariant with the driving fields Φ_a, Φ_c, and Φ_d. For m being $2 + \beta M$ with β an integer and larger than zero, ϕ_b^m can be coupled in an invariant way to Φ_b and Φ_e. With the same arguments as above, it follows that M should also be even and larger than two.

The impact of the flavon combinations $\phi_a^n \phi_b^m$ with n and m integers and $n + m$ larger than two remains to be analyzed. Depending on the values of n and m, they can couple to all driving fields. Indeed, $\phi_a^{2+\alpha N} \phi_b^{\beta M}$ for α and β integers (and at least one of them being larger than zero) can couple to Φ_a and Φ_d. Since N and M are already chosen to be even, both exponents are even and, hence, so is their sum such that the covariants that are singlets get a nonzero VEV, meaning that only a shift in the size of the VEV of the flavon ϕ_a can be induced by these combinations. Likewise, the flavon combinations $\phi_a^{\alpha N} \phi_b^{2+\beta M}$ with α and β integers can couple to Φ_b and Φ_e. Given N and M even, however, these can only impact the size of the VEV of the flavon ϕ_b. Eventually, we see that $\phi_a^{1+\alpha N} \phi_b^{1+\beta M}$ with α and β integers[18] can couple to Φ_c. Nevertheless, these combinations cannot lead to a contribution to the F-term of Φ_c since both exponents are odd, while their sum, $2 + \alpha N + \beta M$, is even for N and M both even; consequently, only the triplet gets a nonvanishing VEV, while Φ_c is a singlet of A_4.

[17]The choice $\alpha = 0$ leads to renormalizable terms that are included in the superpotential.

[18]At least one of these should be nonzero, as otherwise the resulting term is renormalizable and included in the superpotential.

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