Exact lattice chiral symmetry in 2D gauge theory

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We construct symmetry-preserving lattice regularizations of 2D QED with one and two flavors of Dirac fermions, as well as the "3450" chiral gauge theory, by leveraging bosonization and recently proposed modifications of Villain-type lattice actions. The internal global symmetries act just as locally on the lattice as they do in the continuum, the anomalies are reproduced at finite lattice spacing, and in each case we find a sign-problem-free dual formulation.

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I. INTRODUCTION

Numerical Monte Carlo simulations of quantum field theories (QFTs) discretized on Euclidean spacetime lattices are one of the few known nonperturbative techniques to study strongly coupled QFTs. However, it is famously difficult to discretize fermions while preserving all of their symmetries [1]. For example, a free massless Dirac fermion has the internal global symmetry $[(U(1)_V \times U(1)_A)/\mathbb{Z}_2] \rtimes (\mathbb{Z}_2)_C$ for even d. The continuous symmetries have a mixed 't Hooft anomaly, and standard lattice regularizations do not preserve the continuum version of the chiral symmetry at finite lattice spacing a.

If we integrate out a massless Dirac fermion in a Euclidean QFT, we obtain the path integral

$$Z = \int \mathcal{D}\phi \det[\not{D}(\phi)] e^{-S(\phi)}, \tag{1}$$

where $\not{D}(\phi) = \gamma^\mu D_\mu(\phi)$ is the Dirac operator and ϕ stands for an appropriate set of bosonic fields with path integral measure $\mathcal{D}\phi$ and Euclidean action $S(\phi)$. The starting point for lattice Monte Carlo studies is a discretization of Z that preserves as much of the internal symmetry of the QFT as possible.

Replacing the massless continuum Dirac operator by a simple lattice difference operator on a (hyper)cubic lattice does not give the desired symmetries and anomalies. Instead, it yields 2^d massless Dirac fermions in the continuum limit with the symmetry charges of the "doubler" fermions such that the chiral anomaly cancels [2,3]. The Nielsen-Ninomiya theorem [4–7] states that in fact there is no lattice Dirac operator which is simultaneously local, has the desired continuum limit with just one massless Dirac fermion, and is consistent with a locally acting chiral symmetry $\{\Gamma, \not{D}\} = 0$ where $\{\Gamma, \gamma^\mu\} = 0$.

The standard ways around this "fermion doubling problem" all give up some desirable features of the continuum theory. Wilson fermions remove the doublers but explicitly break chiral symmetry [2,3,8]. Staggered fermions [9–13] do not remove all the doublers.[1] Domain-wall and overlap fermions [16–24], which satisfy[2] the Ginsparg-Wilson relation $\{\Gamma, \not{D}\} = a\not{D}\Gamma\not{D}$ [25], remove all of the undesired doubler modes at the cost of making both chiral symmetry transformations and the Dirac operator itself nonlocal at finite lattice spacing [26].

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[1]However, when the continuum theory of interest has the same number of fermions as produced via doubling, one can use staggered fermions (or the closely related Kahler-Dirac fermions) to reproduce the anomalies of the continuum theory; see e.g., [14,15].

[2]In the case of domain-wall fermions the Ginsparg-Wilson relation is satisfied in the limit where the extra dimension is infinitely large.

This was historically viewed as an unavoidable consequence of anomalies, which in popular textbook presentations are characterized as solely arising from subtleties in regularizing fermions. Relatedly, there is a belief that 't Hooft anomalies are necessarily absent in lattice theories with locally acting symmetries [1,2,4,5,25], so that the overlap formulation is the best one can do [16–24].

However, anomalies are not restricted to fermionic systems, and it has recently become appreciated that there exist lattice discretizations in which anomalies of locally acting symmetries can appear even at finite lattice spacing [27–32]. We show that these results straightforwardly lead to lattice discretizations of Dirac fermions coupled to Abelian gauge fields in $d=2$ which preserve the internal symmetries and anomalies *exactly*, with chiral symmetries acting locally even at finite lattice spacing. Our approach is to first apply Abelian bosonization to N_f Dirac fermions and then discretize the resulting bosonic theory using an appropriate modified Villain action.[3] We discuss how this works in 2D QED with $N_f=1$ and $N_f=2$ charge Q fermions and in the "3450" Abelian chiral gauge theory. We also discuss a related spatial lattice Hamiltonian for $N_f=1$ QED in Appendix A.

A. Bosonization

Consider the charge Q Schwinger model: 2D QED with a massless Dirac fermion coupled to a $U(1)$ gauge field a_μ with electric charge $Q\in\mathbb{Z}$ [36–50]. We normalize a_μ such that $\frac{1}{4\pi}\int_M d^2x\epsilon^{\mu\nu}f_{\mu\nu}\in\mathbb{Z}$, where $f_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu$, and write the action as

$$S=\int d^2x\left[\frac{1}{4e^2}f_{\mu\nu}f^{\mu\nu}+\bar{\psi}\gamma^\mu(\partial_\mu-iQa_\mu)\psi\right]. \quad (2)$$

The Nielsen-Ninomiya theorem constrains discretizations of $\not{D}$ but does not directly constrain $\det\not{D}$. We thus aim to circumvent this theorem by discretizing $\det\not{D}$ directly, by using the fact that in $d=2$ [38,39,51,52]

$$\det(\not{D}(a_\mu))=\int\mathcal{D}\varphi\exp\left[-\int d^2x\left(\frac{1}{8\pi}\partial_\mu\varphi\partial^\mu\varphi+\frac{iQ}{2\pi}\epsilon^{\mu\nu}a_\mu\partial_\nu\varphi\right)\right]. \quad (3)$$

In this "bosonized" action φ is a compact real scalar field $\varphi\equiv\varphi+2\pi$ and the mapping of the $U(1)_V$ and $U(1)_A$ currents is $\bar{\psi}\gamma^\mu\psi\leftrightarrow-\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\varphi$ and $\bar{\psi}\gamma^\mu\gamma^5\psi\leftrightarrow\frac{i}{4\pi}\partial_\mu\varphi$. We hasten to emphasize that the existence of a map to bosonic variables (3) does *not* mean that the fermion discretization problem is trivially solvable. Such a solution requires exhibiting a lattice action in which all the desired symmetries and anomalies are preserved.

The Adler-Bell-Jackiw (ABJ) anomaly is encoded at tree level in (3), where it is clear that the 0-form symmetry counting chiral charges of local operators is $(\mathbb{Z}_Q)_A$, acting as $\varphi\to\varphi+2\pi k/Q$, rather than $U(1)_A$. There is also a 1-form [53] "electric" symmetry $(\mathbb{Z}_Q)_\mathrm{e}$ which counts the charges of Wilson loops modulo Q, as well as a mixed 't Hooft anomaly between $(\mathbb{Z}_Q)_A$ and $(\mathbb{Z}_Q)_\mathrm{e}$ which is matched by the spontaneous breaking of *both* symmetries, with the walls separating chiral vacua carrying electric charge [42–44,46–49]. The spectrum in each degenerate discrete chiral vacuum consists of a single free massive scalar field with mass $m_\gamma=eQ/\pi$, often called the Schwinger boson.

II. MODIFIED VILLAIN DISCRETIZATION

We will work with an $N\times N$ periodic Euclidean spacetime lattice with spacing $a=1$, with sites s, links ℓ, and plaquettes p. The corresponding simplices on the dual lattice are denoted by $\tilde{s}$, $\tilde{\ell}$, and $\tilde{p}$. Following Villain [54], we represent the continuum $U(1)$ gauge field a_μ by a pair of lattice fields $\{a_\ell\in\mathbb{R},r_p\in\mathbb{Z}\}$ and the compact scalar field by the pair $\{\varphi_{\tilde{s}}\in\mathbb{R},n_{\tilde{\ell}}\in\mathbb{Z}\}$ on the dual lattice. We adopt the modified [27,28] Villain formulation and also introduce an auxiliary field $\chi_s\in\mathbb{R}$ which can be viewed as the T-dual of $\varphi_{\tilde{s}}$. See Fig. 1 for an illustration.

The action for our discretization of $N_f=1$ QED is

$$S_{N_f=1}=\frac{\beta}{2}[(da)_p-2\pi r_p]^2+\frac{\kappa}{2}[(d\varphi)_{\tilde{\ell}}-2\pi n_{\tilde{\ell}}]^2-i\chi_s(dn)_{\star s}+\frac{iQ}{2\pi}\varphi_{\star p}[(da)_p-2\pi r_p]-iQa_\ell n_{\star\ell}, \quad (4)$$

where repeated indices are summed and d is the lattice exterior derivative $(d\omega)_{c^{r+1}}=\sum_{c^r\in\partial c^{r+1}}\omega_{c^r}$, where c^r is an r cell, so that, for example, $(d\chi)_\ell=\chi_{s+\hat{\ell}}-\chi_s$, and $d^2=0$.

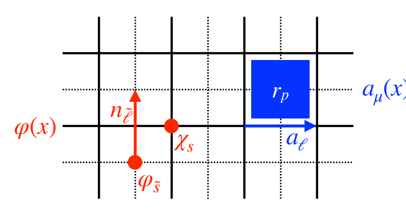

FIG. 1. The setting for the field content of our lattice action (4). The solid grid is the primary lattice with sites s, links ℓ, and plaquettes p. The dotted grid is the dual lattice with sites $\tilde{s}$, links $\tilde{\ell}$, and plaquettes $\tilde{p}$. The three red fields φ, χ, and n are associated with the continuum φ. The two blue fields a and r correspond to the continuum a_μ.

[3] A more conventional discretization of the bosonized Schwinger model was studied in Refs. [33,34]. Here our main focus is on the symmetries and global aspects of the model, and our analysis leverages a number of special features of the modified Villain formalism. An alternative approach to discretization of bosonized 2D gauge theories that shares some (but not all) features of the modified Villain construction was recently discussed in Ref. [35].

The Hodge star $\star$ maps an r cell c^r on the lattice to the $(d-r)$ cell $(\star c)^{d-r}$ on the dual lattice which pierces c^r.[4] The partition function is

$$Z = \prod_{s,\tilde{s}} \int_{\mathbb{R}} \mathcal{D}\chi_s \mathcal{D}\varphi_{\tilde{s}} \prod_{\ell} \int_{\mathbb{R}} \mathcal{D}a_\ell \sum_{n_\ell \in \mathbb{Z}} \prod_p \sum_{r_p \in \mathbb{Z}} e^{-S_{N_f=1}}, \quad (5)$$

where the products are over all sites, links, and dual sites of our periodic square lattice with $N \times N$ sites. Similar expressions can be written for the other gauge theories considered in this paper. The $\mathbb{Z}$ gauge redundancy on each site makes Z formally infinite, and readers who find this uncomfortable can work in the "Villain" gauge where integrals over $\mathbb{R}$ are replaced by integrals over the interval $(-\pi, \pi]$; see e.g., Refs. [27,55] for discussions of various possible gauge choices. However, Z is in any case not an observable, and all physical observables are necessarily finite even without this gauge choice.

The gauge redundancies of the lattice action (4) are

$$a_\ell \to a_\ell + (d\lambda)_\ell + 2\pi m_\ell, \qquad r_p \to r_p + (dm)_p, \quad (6a)$$

$$\varphi_{\tilde{s}} \to \varphi_{\tilde{s}} + 2\pi k_{\tilde{s}}, \qquad n_{\tilde{\ell}} \to n_{\tilde{\ell}} + (dk)_{\tilde{\ell}}, \quad (6b)$$

$$\chi_s \to \chi_s + Q\lambda_s + 2\pi h_s, \quad (6c)$$

where $\{\lambda_s \in \mathbb{R}, m_\ell, k_{\tilde{s}}, h_s \in \mathbb{Z}\}$ are gauge parameters. They ensure that $\{a, r\}$ and $\{\chi, \varphi, n\}$ describe a $U(1)$ gauge field and a 2π-periodic boson with a conserved winding charge, with the topological properties one expects in the continuum. For example, the instanton number on the spacetime torus $I = -\frac{1}{2\pi}\sum_p [(da)_p - 2\pi r_p] = \sum_p r_p$ is an integer. The path integral over χ_s implies that $(dn)_{\tilde{p}} = 0$ on shell, the $\frac{iQ}{2\pi}$ term in the lattice action (4) is the analog of the continuum $\frac{iQ}{2\pi}$ term (3), and the last contribution to the lattice action (4) is necessary to maintain gauge invariance [28].

Given that $\beta = 1/(2e^2a^2)$, to get a continuum limit with fixed Le, where L is the physical box size $L = Na$, we should take $N \to \infty$ with β/N^2 fixed. While naively one should also set $\kappa = 1/(4\pi)$ to reach the continuum (3), this parameter value is not protected by any symmetries of the lattice theory and can receive some finite renormalization [54,56,57]. Varying κ amounts to varying the coefficient of the marginal Thirring term $(\bar{\psi}\gamma^\mu\psi)^2$.

The lattice action (4) has precisely the desired global symmetries of the continuum (2). There is no continuous $U(1)_A$ symmetry, but thanks to the quantization of instanton number, there is a remnant $(\mathbb{Z}_Q)_A$ symmetry that acts as $\varphi_{\tilde{s}} \to \varphi_{\tilde{s}} + 2\pi q/Q$ with $q \in \mathbb{Z}$. This reproduces the expected ABJ chiral anomaly. The $(\mathbb{Z}_Q)_{\mathrm{e}}$ symmetry acts as $a_\ell \to a_\ell + \frac{2\pi}{Q} v_\ell$ with $v \in \mathbb{Z}$ and $dv = 0$, also matching the continuum.

To see the 't Hooft anomaly between $(\mathbb{Z}_Q)_{\mathrm{e}}$ and $(\mathbb{Z}_Q)_A$ on the lattice, it is easiest to linearize the quadratic terms in the action by integrating in auxiliary fields $\zeta_\ell, \xi_{\tilde{s}} \in \mathbb{R}$ and summing by parts, turning the original action (4) into[5]

$$\begin{aligned} S'_{N_f=1} = &\left(\frac{1}{2\kappa}\zeta_\ell^2 + i\zeta_\ell[(d\varphi)_{\star\ell} - 2\pi n_{\star\ell}]\right) \\ &+ \left(\frac{1}{2\beta}\xi_{\star p}^2 + i\xi_{\star p}[(da)_p - 2\pi r_p]\right) \\ &+ \frac{iQ}{2\pi}\varphi_{\star p}[(da)_p - 2\pi r_p] - in_{\star\ell}[Qa_\ell - (d\chi)_\ell]. \end{aligned} \quad (7)$$

The generators of the axial $(\mathbb{Z}_Q)_A$ and electric $(\mathbb{Z}_Q)_{\mathrm{e}}$ symmetries are topological line and local operators on the lattice and dual lattice, respectively:

$$(U_A)[C] = e^{i\sum_{\ell \in C}(a_\ell + \frac{2\pi}{Q}\zeta_\ell)}, \qquad (U_{\mathrm{e}})_{\tilde{s}} = e^{i(\varphi_{\tilde{s}} + \frac{2\pi}{Q}\xi_{\tilde{s}})}, \quad (8)$$

where C is a closed curve. The fact that $(U_A)[C]$ is charged under $(\mathbb{Z}_Q)_{\mathrm{e}}$ and $(U_{\mathrm{e}})_p$ is charged under $(\mathbb{Z}_Q)_A$ encodes the mixed 't Hooft anomaly of these symmetries, just as in the continuum.

III. ABSENCE OF A SIGN PROBLEM

The discretizations provided above, and indeed those presented below, provide symmetry-preserving nonperturbative definitions of their respective models. While this is interesting in its own right, it is natural to ask whether these definitions have some practical value. Can we learn something new about the physics of these models just from defining them nonperturbatively, either numerically or analytically? On the analytic side, we will see in Sec. V that putting a simple chiral gauge theory (the 3450 model) on the lattice reveals the presence of an exotic symmetry of the model that is not at all obvious from continuum analyses. On the numerical side, one might initially worry that the constructions described in this paper are completely useless, because direct numerical Monte Carlo with the complex lattice actions (4) and (7) would face a severe sign problem. We now show that this issue can be eliminated by a change of variables, so that the discretizations we give here can be explored using numerical Monte Carlo simulations.

Summing (i.e., path integrating) over $n_{\tilde{\ell}}$, r_p in the auxiliary-field action (7) yields constraints that can be

[4]Differential forms on the lattice are reviewed in Appendix A of Ref. [27]. Two useful facts are that $\star^2 = (-1)^{r(d-r)}$ on an r cell and the identity $\sum_{c^{r+1}}(dA)_{c^{r+1}}B_{\star c^{r+1}} = (-1)^{r+1}\sum_{c^r} A_{c^r}(dB)_{\star c^r}$.

[5]Throughout the paper we ignore any overall constant factors in the partition function which appear in "dualization" procedures.

solved by setting

$$\zeta_\ell = \frac{1}{2\pi}(d\chi)_\ell - \frac{Q}{2\pi}a_\ell - u_\ell, \qquad \xi_{\tilde{s}} = -\frac{Q}{2\pi}\varphi_{\tilde{s}} + t_{\tilde{s}} \tag{9}$$

with $u_\ell, t_{\tilde{s}} \in \mathbb{Z}$ which transform as $t_{\tilde{s}} \to t_{\tilde{s}} + Qk_{\tilde{s}}$ and $u_\ell \to u_\ell + (dh)_\ell - Qm_\ell$ under discrete gauge transformations. The field u also transforms under $(\mathbb{Z}_Q)_\mathrm{e}$ transformations $u_\ell \to u_\ell - 2\pi v_\ell/Q$, while t transforms under $(\mathbb{Z}_Q)_A$ as $t_{\tilde{s}} \to t_{\tilde{s}} + q$. Plugging the constraints (9) into the action (7) and dropping total derivatives and integer multiples of $2\pi i$, we obtain an action

$$\frac{1}{2\kappa}\left[\frac{1}{2\pi}(d\chi)_\ell - \frac{Q}{2\pi}a_\ell - u_\ell\right]^2 + \frac{Q^2}{2\beta(2\pi)^2}\left[\varphi_{\tilde{s}} - \frac{2\pi t_{\tilde{s}}}{Q}\right]^2 - \frac{i}{2\pi}(Qa_\ell + 2\pi u_\ell)(d\varphi)_{\star\ell} + it_{\star p}(da)_p. \tag{10}$$

Shifting $a \to a + \frac{1}{Q}d\chi - \frac{2\pi}{Q}u$ and dropping a total derivative gives

$$\frac{1}{2\kappa}\left(\frac{Q}{2\pi}a_\ell\right)^2 + \frac{Q^2}{2\beta(2\pi)^2}\left[\varphi_{\tilde{s}} - \frac{2\pi t_{\tilde{s}}}{Q}\right]^2 - \frac{i}{2\pi}Qa_\ell(d\varphi)_{\star\ell} + it_{\star p}\left[(da)_p - \frac{2\pi}{Q}(du)_p\right], \tag{11}$$

and subsequently integrating over a yields

$$S_{N_{f=1,\mathrm{dual}}} = \frac{\kappa}{2}\left[(d\varphi)_{\tilde{\ell}} - \frac{2\pi}{Q}(dt)_{\tilde{\ell}}\right]^2 + \frac{1}{2\beta}\left(\frac{Q}{2\pi}\right)^2 \times \left(\varphi_{\tilde{s}} - \frac{2\pi}{Q}t_{\tilde{s}}\right)^2 - \frac{2\pi i}{Q}t_{\star p}(du)_p. \tag{12}$$

This action describes Q copies ("universes" [42,46–48,58–63]) of a free massive scalar particle, as expected from continuum arguments. Adding a fermion mass term in the original action (2) corresponds to adding $\sum_{\tilde{s}}\cos(\varphi_{\tilde{s}})$ to the dual action (12), which would lead to strong coupling in general.

The sole imaginary term in the dual action (12) involves u, but summing over u just gives the constraint $(dt)_{\tilde{\ell}} = 0 \bmod Q$. One can thus avoid the sign problem entirely by proposing updates for $t_{\tilde{s}}$ that satisfy $(dt)_{\tilde{\ell}} = 0 \bmod Q$ in a Monte Carlo calculation.

Procedures to generate field configurations which satisfy constraints such as $(dt)_{\tilde{\ell}} = 0 \bmod Q$, and hence avoid apparent sign problems, are well known; see, for example, Ref. [64]. Nevertheless, to make our presentation self-contained, we now give a brief discussion of how simple constraints such as the one above—and indeed others that we encounter later in this paper—can be taken into account in Monte Carlo calculations without sign problems.

Consider a lattice field theory with an action S where the only term where the field u appears is

$$S \ni \frac{2\pi i}{Q}u_\ell(dt)_\ell, \tag{13}$$

where $Q \in \mathbb{N}$, $u_\ell \in \mathbb{Z}$, and $t_s \in \mathbb{Z}$. The dual one-flavor (12), two-flavor (20), and 3450 (29) models all have this character.

Because it is purely imaginary, direct Monte Carlo evaluation of the QFT path integral based on importance sampling with this term as part of the action suffers from a sign problem. However, the path integral over u can be done analytically and yields a delta function setting

$$(dt)_\ell = 0 \bmod Q. \tag{14}$$

If we can make proposals that maintain this constraint but are otherwise ergodic, we will consider all supported configurations of t_s and avoid the sign problem caused by the phase (13).

There is a simple solution to the constraint (14) on a spacetime torus, where we can write[6]

$$t_s = x + Qy_s, \tag{15}$$

$x, y_s \in \mathbb{Z}$ and x is a constant so that $(dx)_\ell = 0$. This decomposition is not unique, since (for example) we can shift x by Q and all y's by -1 without changing t. But the key point is that any constraint-satisfying t_s can be written in the form above. This helps us define two kinds of proposals which together reach all constraint-satisfying configurations.

The first proposal is a global update of x. We randomly pick a site-independent integer $\Delta x \in [-X, +X]$ with $X \in \mathbb{Z}$ and Metropolis test $t_s \to t_s + \Delta x$ for all sites s simultaneously.

The second is a local update of y which we can sweep across the lattice. On a particular site s we pick an integer $\Delta y_s \in [-Y, +Y]$ with $Y \in \mathbb{Z}$ and test $t_s \to t_s + Q\Delta y_s$.

An ergodic algorithm should offer proposals of both kinds, and their relative frequency may be adjusted to control autocorrelation times. This algorithm also manifestly satisfies detailed balance thanks to the Metropolis tests described above. The algorithm parameters X and Y, or more generally the distributions for Δx and Δy_s, may be adjusted to optimize acceptance and thermalization.

Similarly, suppose the action includes a term

$$S \ni i\eta_p(dn)_p, \tag{16}$$

where $\eta_p \in \mathbb{R}$, $n_\ell \in \mathbb{Z}$, and η does not appear in any other terms. The dual 3450 action (29) has a term of this character. Integrating out η yields the constraint $(dn)_p = 0$.

[6]Formally, this very simple solution is possible because all of the integer cohomology groups of a torus are torsion-free.

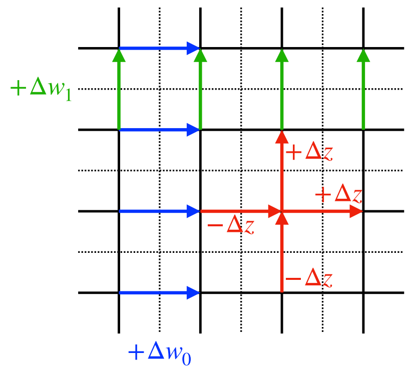

FIG. 2. Two kinds of proposals for n_ℓ which satisfy $(dn)_p = 0$. In red is a local update which is an exact form, i.e., the exterior derivative dz of a single-site 0-form. The blue and green updates consist of noncontractible strips of links and are closed because opposite edges of a plaquette contribute inversely to $(dn)_p$.

The field n_ℓ may be split into a closed 1-form $w_\ell \in \mathbb{Z}$ and an exact 1-form $(dz)_\ell \in \mathbb{Z}$:

$$n_\ell = w_\ell + (dz)_\ell. \tag{17}$$

As we will see below, updates of z are local, while updates of w "wrap" around cycles of the torus. Again, for any given field configuration n_ℓ, the decomposition above is not unique, but the key point is that any constraint-satisfying n_ℓ can be written in the form above.

As before, we offer two kinds of proposals, depicted in Fig. 2, which together reach all constraint-satisfying configurations of n.

We first an offer update of the closed 1-form w. We update a torus-wrapping strip of parallel links at once, as shown by the blue and green updates in Fig. 2. We pick a single integer $\Delta w \in [-W, +W]$ with $W \in \mathbb{Z}$ and Metropolis test $n_\ell \to n_\ell + \Delta w$ for all the links on the strip. We can sweep this update across all the strips of the lattice in either orientation.

The second proposal offers a local update to the exact 1-form dz. We pick $\Delta z \in [-Z, +Z]$ with $Z \in \mathbb{Z}$ and a site s and build a 0-form z that vanishes everywhere except at s where it is $-\Delta z$. We propose $n_\ell \to n_\ell + (dz)_\ell$, which amounts to the simultaneous proposal

$$\begin{aligned} n_{s,\hat{0}} &\to n_{s,\hat{0}} + \Delta z, & n_{s-\hat{0},\hat{0}} &\to n_{s-\hat{0},\hat{0}} - \Delta z, \\ n_{s,\hat{1}} &\to n_{s,\hat{1}} + \Delta z, & n_{s-\hat{1},\hat{1}} &\to n_{s-\hat{1},\hat{1}} - \Delta z, \end{aligned} \tag{18}$$

where $\hat{0}$ and $\hat{1}$ are unit vectors in the positive space and time directions.

An ergodic algorithm should offer proposals of both kinds, and as before their relative frequency may be adjusted to control autocorrelation times. The distributions of Δw and Δz may be adjusted to optimize performance, for example by changing W and Z.

The constrained update algorithms presented here are simple examples of field update methods which evade the sign problem. We again emphasize that the problem of constructing ergodic detailed-balanced algorithms to sample discrete gauge fields satisfying flatness constraints has long been solved in the literature, and we have only described the algorithms above to make our paper self-contained. The specific algorithms we have presented might have long autocorrelation times in some parameter regimes, especially given that some of the proposals touch a number of variables growing with the lattice size N and may be rejected often. In practice one may want to construct more efficient constraint-satisfying field update algorithms. For example, worm algorithms [65,66] can quickly decorrelate worldline formulations which have closed-loop constraints; it would be interesting to try and adapt these powerful tools to our actions.

IV. 2D QED WITH $N_f = 2$

Let us now consider 2D QED with two flavors of massless Dirac fermions ψ and $\hat{\psi}$ with a common charge Q. The global flavor symmetry is

$$G_{N_f=2} = \frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2} \times (\mathbb{Z}_2)_G \times (\mathbb{Z}_Q)_A,$$

where $SU(2)_{L,R}$ act on the left and right-handed components of ψ and $\hat{\psi}$, the quotient is by the gauge transformation $\psi, \hat{\psi} \to -\psi, -\hat{\psi}$, $(\mathbb{Z}_2)_G$ is G parity [67], and the discrete axial symmetry $(\mathbb{Z}_Q)_A$ is the same as before. There is also a $(\mathbb{Z}_Q)_e$ 1-form symmetry. This model is believed to be equivalent to a self-dual $c = 1$ compact boson conformal field theory (CFT) plus a decoupled massive Schwinger boson [38,39,68,69]. Mass terms and other perturbations can make this model strongly coupled, and so this field theory has been a popular testing ground for analytic and numeric approaches to confining gauge theories [33,42–44,70–90].

Abelian bosonization maps $\psi, \hat{\psi}$ to a pair of 2π-periodic compact bosons $\varphi, \hat{\varphi}$, so we can discretize it in a parallel way to the one-flavor case (4):

$$\begin{aligned} S_{N_f=2} = \frac{\beta}{2}&[(da)_p - 2\pi r_p]^2 \\ &+ \frac{\kappa}{2}([(d\varphi)_{\star\ell} - 2\pi n_{\star\ell}]^2 + [(d\hat{\varphi})_{\star\ell} - 2\pi \hat{n}_{\star\ell}]^2) \\ &+ \frac{iQ}{2\pi}(\varphi_{\star p} + \hat{\varphi}_{\star p})[(da)_p - 2\pi r_p] \\ &- iQ(n_{\star\ell} + \hat{n}_{\star\ell})a_\ell + i n_{\star\ell}(d\chi)_\ell + i\hat{n}_{\star\ell}(d\hat{\chi})_\ell, \end{aligned} \tag{19}$$

where $a, \varphi, \hat{\varphi}, \chi, \hat{\chi} \in \mathbb{R}$, $n, \hat{n}, r \in \mathbb{Z}$, and gauge transformations act as in the one-flavor case (6) plus the analogous shifts of $\hat{\varphi}$, $\hat{n}$, and $\hat{\chi}$ with $\hat{k}, \hat{h} \in \mathbb{Z}$.

As before, the lattice parameter $\beta = 1/(2e^2a^2)$ and the continuum limit requires the same scaling as in $N_f = 1$ QED. However, we will see below that now $\kappa = 1/4\pi$ *is* associated with an enhanced symmetry of the action (19), and thus we can set $\kappa = 1/4\pi$ on the lattice and be sure that the lattice theory will flow precisely to $N_f = 2$ charge Q massless QED in the continuum limit without any Thirring terms.

The Abelian subgroup of $G_{N_f=2}$ is manifestly respected by the two-flavor action (19), and we will argue below that the theory flows to a continuum limit where all of $G_{N_f=2}$ is preserved. Following a similar dualization procedure to the $N_f = 1$ case (see Appendix B) we reach

$$\begin{aligned} S_{N_f=2,\mathrm{dual}} = \frac{1}{4\kappa(2\pi)^2}[(d\sigma)_\ell - 2\pi u_\ell]^2 + i\phi_{\star p}(du)_p \\ + \frac{\kappa}{4}\left[(d\eta)_{\tilde{\ell}} - \frac{2\pi}{Q}(dt)_{\tilde{\ell}}\right]^2 - \frac{2\pi i}{Q}\hat{u}_\ell(dt)_{\star\ell} \\ + \frac{1}{2\beta}\left(\frac{Q}{2\pi}\right)^2\left(\eta_{\tilde{s}} - \frac{2\pi}{Q}t_{\tilde{s}}\right)^2, \end{aligned} \tag{20}$$

where the fields $u, \hat{u}, t \in \mathbb{Z}$ emerge during the dualization process and

$$\sigma = \chi - \hat{\chi}, \qquad \eta = \varphi + \hat{\varphi}, \qquad \phi = \frac{\hat{\varphi}}{2} - \frac{\varphi}{2} - \frac{\pi}{Q}t$$

are real and invariant under $U(1)$ gauge transformations. The remaining gauge redundancies are

$$\sigma_s \to \sigma_s + 2\pi h_s, \qquad u_\ell \to u_\ell + (dh)_\ell, \tag{21a}$$

$$\eta_{\tilde{s}} \to \eta_{\tilde{s}} + 2\pi k_{\tilde{s}}, \qquad t_{\tilde{s}} \to t_{\tilde{s}} + Qk_{\tilde{s}}, \tag{21b}$$

$$\phi_{\tilde{s}} \to \phi_{\tilde{s}} + 2\pi\hat{h}_{\tilde{s}}, \qquad \hat{u}_\ell \to \hat{u}_\ell + (dw)_\ell + Qg_\ell, \tag{21c}$$

with all gauge parameters taking values in $\mathbb{Z}$. Finally, the terms with factors of i simply impose constraints $(du)_p = 0$ and $(dt)_{\tilde{\ell}} = 0 \bmod Q$, and solving these constraints when generating field configurations in a Monte Carlo calculation avoids the sign problem, as discussed in the preceding section.

The dual formulation (20) shows that in the massless limit our lattice theory decomposes into two decoupled sectors. The top line of (20) is simply the modified Villain discretization of the compact boson σ. But when we set $\kappa = 1/(4\pi)$, the effective radius of σ makes the theory self-dual under Poisson resummation on u_ℓ, which implements T-duality on the lattice [28]. This implies the existence of a topological line operator which is absent for generic κ [91,92], so that the $\kappa = 1/(4\pi)$ point is protected by an enhanced symmetry against quantum corrections. The continuum limit is thus guaranteed to be the self-dual $c = 1$ compact boson CFT with non-Abelian $[SU(2) \times SU(2)]/\mathbb{Z}_2$ global symmetry. The decoupled "Schwinger boson" QFT in the lower lines of the dual action (20) matches the remaining symmetries:

$$\begin{aligned} (\mathbb{Z}_2)_G &: \eta_{\tilde{s}} \to -\eta_{\tilde{s}}, \quad \phi_{\tilde{s}} \to \phi_{\tilde{s}} + \frac{2\pi}{Q}t, \\ &\quad \hat{u}_\ell \to -\hat{u}_\ell + u_\ell, \quad t_{\tilde{s}} \to -t_{\tilde{s}}, \\ (\mathbb{Z}_Q)_A &: \eta_{\tilde{s}} \to \eta_{\tilde{s}} + \frac{2\pi q}{Q}, \quad t_{\tilde{s}} \to t_{\tilde{s}} + q_{\tilde{s}}, \quad q \in \mathbb{Z}, \\ (\mathbb{Z}_Q)_e &: \hat{u}_\ell \to \hat{u}_\ell + \frac{2\pi}{Q}v_\ell, \quad v_\ell \in \mathbb{Z}, \quad (dv)_p = 0. \end{aligned} \tag{22}$$

V. CHIRAL GAUGE THEORY

We now turn to a popular example [32,93–110] of a 2D Abelian chiral gauge theory, namely the 3450 model, which has two left-handed Weyl fermions ψ_L and $\hat{\psi}_L$ coupled to a $U(1)$ gauge field with charges 3 and 4 as well as two right-handed Weyl fermions ψ_R and $\hat{\psi}_R$ with charges 5 and 0.[7] This QFT satisfies the gauge anomaly cancellation condition $(Q_L)^2 + (\hat{Q}_L)^2 = (Q_R)^2 + (\hat{Q}_R)^2$ as well as the gravitational 't Hooft anomaly cancellation condition on the left and right central charges $c_L = c_R$. After repackaging the matter into two Dirac fermions $\psi = (\psi_R, \psi_L)^\top$ and $\hat{\psi} = (\hat{\psi}_R, \hat{\psi}_L)^\top$, the gauge field couples to the vector and axial currents of ψ and $\hat{\psi}$ with charges $Q_V = 8$, $Q_A = -2$, $\hat{Q}_V = 4$, and $\hat{Q}_A = 4$, and the gauge anomaly cancellation condition becomes $Q_V Q_A + \hat{Q}_V \hat{Q}_A = 0$.

We will study the variant of the 3450 model with a gauged $(-1)^F$ symmetry to avoid dealing with the Arf invariant [111,112]. Our discretization takes the form

$$\begin{aligned} S_{3450} = \frac{\beta}{2}[(da)_p - 2\pi r_p]^2 + \frac{\kappa}{2}([(d\varphi)_{\tilde{\ell}} - Q_A a_{f(\tilde{\ell})} - 2\pi n_{\tilde{\ell}}]^2 \\ + [(d\hat{\varphi})_{\tilde{\ell}} - \hat{Q}_A a_{f(\tilde{\ell})} - 2\pi\hat{n}_{\tilde{\ell}}]^2) \\ + \frac{i}{2\pi}(Q_V\varphi_{\star p} + \hat{Q}_V\hat{\varphi}_{\star p})[(da)_p - 2\pi r_p] \\ - i(Q_V n_{\star\ell} + \hat{Q}_V\hat{n}_{\star\ell})a_\ell + in_{\star\ell}(d\chi)_\ell + i\hat{n}_{\star\ell}(d\hat{\chi})_\ell \\ - ir_{f(\star s)}(Q_A\chi_s + \hat{Q}_A\hat{\chi}_s), \end{aligned} \tag{23}$$

where $f\colon\ s \to s + \frac{1}{2}(\hat{x} + \hat{y})$ shifts cells from the lattice to the dual lattice, and the gauge redundancies are

$$\begin{aligned} &a_\ell \to a_\ell + (d\lambda)_\ell + 2\pi m_\ell, \quad r_p \to r_p + (dm)_p, \\ &\varphi_{\tilde{s}} \to \varphi_{\tilde{s}} + Q_A\lambda_{f(\tilde{s})} + 2\pi k_{\tilde{s}}, \quad \hat{\varphi}_{\tilde{s}} \to \hat{\varphi}_{\tilde{s}} + \hat{Q}_A\lambda_{f(\tilde{s})} + 2\pi\hat{k}_{\tilde{s}}, \\ &n_{\tilde{\ell}} \to n_{\tilde{\ell}} + (dk)_{\tilde{\ell}} - Q_A m_{f(\tilde{\ell})}, \quad \hat{n}_{\tilde{\ell}} \to \hat{n}_{\tilde{\ell}} + (d\hat{k})_{\tilde{\ell}} - \hat{Q}_A m_{f(\tilde{\ell})}, \\ &\chi_s \to \chi_s + Q_V\lambda_s + 2\pi h_s, \quad \hat{\chi}_s \to \hat{\chi}_s + \hat{Q}_V\lambda_s + 2\pi\hat{h}_s. \end{aligned} \tag{24}$$

[7] In particular, [32] mentioned that a Villain Hamiltonian [29,31] formulation of the 3450 model should exist.

Modulo $2\pi i$ and total derivative terms, the gauge variation of S_{3450} is

$$\Delta S_{3450} = i(Q_V Q_A + \hat{Q}_V \hat{Q}_A)\Big[m_\ell a_{f^{-1}(\star\ell)} + \lambda_s\Big(\frac{1}{2\pi}(da)_{f^{-1}(\star s)} - r_{f^{-1}(\star s)} - r_{f(\star s)} - (dm)_{f(\star s)}\Big)\Big], \tag{25}$$

which vanishes precisely when the charges satisfy the anomaly cancellation condition.

The function f was introduced to allow the $U(1)$ gauge field to couple to fields that live on the "primary" lattice as well as on the dual lattice, which is necessary in the present context when trying to couple the gauge field to both vector and axial currents. But the presence of the function f in (23) appears to break $\mathbb{Z}_4$ lattice rotation symmetry. Also, $\mathrm{Im}S_{3450} \neq 0$, leading to an apparent sign problem. However, following the same method as in $N_f = 1$, 2 vectorlike QED, one can derive (see Appendix C) a dual representation which both avoids the sign problem and shows that $\mathbb{Z}_4$ lattice rotation symmetry is actually preserved, since it is manifest in the dual variables. This dual representation can be written as

$$\begin{aligned} S = {} & \frac{\kappa}{2}\frac{1}{5}((d\phi)_{\star\ell} - 2\pi v_{\star\ell})^2 \\ & + \frac{1}{2\kappa}\frac{1}{20(2\pi)^2}(2(d\hat{\psi})_{\tilde{\ell}} - 2\pi((dy)_{\tilde{\ell}} - 4v_{\tilde{\ell}}))^2 \\ & + \frac{1}{2\beta}\frac{1}{(2\pi)^2}(4\phi_{\tilde{s}} + 2\hat{\psi}_{\tilde{s}} - 2\pi y_{\tilde{s}})^2 \\ & + i\sigma_{\star\tilde{p}}(dv)_{\tilde{p}} - i\pi\hat{n}_{\star\tilde{\ell}}(dy)_{\tilde{\ell}} - 2\pi i\frac{4}{5}v_{\star\ell}v_{f(\ell)}. \end{aligned} \tag{26}$$

The fields ϕ and $\hat{\psi}$ are $U(1)$-gauge invariant combinations of the fields in (23):

$$\phi_{\tilde{s}} = 2\varphi_{\tilde{s}} + \hat{\varphi}_{\tilde{s}}, \qquad \hat{\psi}_{\tilde{s}} = 2\hat{\chi}_{f(s)} - \chi_{f(s)}. \tag{27}$$

When $\hat{\psi}_{\tilde{s}} \to \psi_{\tilde{s}} + 2\pi q_{\tilde{s}}$ and $\phi_{\tilde{s}} \to \phi_{\tilde{s}} + 2\pi b_{\tilde{s}}$, the discrete fields shift as $v_{\tilde{\ell}} \to v_{\tilde{\ell}} + (db)_{\tilde{\ell}}$ and $y_{\tilde{s}} \to y_{\tilde{s}} + 4b_{\tilde{s}} + 2q_{\tilde{s}}$.

We can simplify the expression above further by introducing the variable

$$\rho_{\tilde{s}} \equiv \frac{\pi}{2}y_{\tilde{s}} - \frac{1}{2}\hat{\psi}_{\tilde{s}} = \frac{\pi}{2}y_{\tilde{s}} + \frac{1}{2}\chi_{f(s)} - \hat{\chi}_{f(s)}, \tag{28}$$

which shifts as $\rho_{\tilde{s}} \to \rho_{\tilde{s}} + 2\pi b_{\tilde{s}}$ under the discrete gauge transformations described in the preceding paragraph. We can then rewrite the dualized action of the discretized 3450 chiral gauge theory as

$$\begin{aligned} S = {} & \frac{\kappa}{10}[(d\phi)_{\tilde{\ell}} - 2\pi v_{\tilde{\ell}}]^2 + \frac{1}{10\pi^2\kappa}[(d\rho)_{\tilde{\ell}} - 2\pi v_{\tilde{\ell}}]^2 \\ & + \frac{2}{\pi^2\beta}(\phi_{\tilde{s}} - \rho_{\tilde{s}})^2 + i\sigma_{\star\tilde{p}}(dv)_{\tilde{p}} - i\pi\hat{n}_{\star\tilde{\ell}}(dy)_{\tilde{\ell}} \\ & - 2\pi i\frac{4}{5}v_{\star\ell}v_{f(\ell)}. \end{aligned} \tag{29}$$

This action has several interesting physical consequences. First, the $i\pi\hat{n}dy$ term is a decoupled topological $\mathbb{Z}_2$ gauge theory. This topological quantum field theory appears because the charges of the Dirac fermions in the 3450 chiral gauge theory are even with a minimal charge of 2. Therefore the model has a $\mathbb{Z}_2$ 1-form symmetry, and this symmetry is spontaneously broken. Second, we see that one linear combination of ϕ and ρ acquires a Schwinger-type mass. These two features are analogous to the behavior of the charge-Q $N_f = 1$ Schwinger model. Third, another linear combination of ϕ and ρ remains exactly massless, thanks to the presence of the σdv term, which ensures that there are no dynamical v vortices, implying that there cannot be any Berezinskii-Kosterlitz-Thouless transition [113,114] as a function of κ. This gapless mode matches the $U(1) \times U(1)$ 't Hooft anomaly of the 3450 model, which is analogous to what we saw above in the $N_f = 2$ vector-like QED.

The last interesting physical consequence of Eq. (29) we want to highlight is that the model has an extra $\mathbb{Z}_2$ 0-form symmetry that acts by exchanging ϕ and ρ if we set $\kappa = 1/\pi$. Dialing κ maps to dialing the coefficients of the Thirring interaction terms $(\bar{\psi}\gamma^\mu\psi)^2$ and $(\bar{\hat{\psi}}\gamma^\mu\hat{\psi})^2$ in the original fermionic theory. The extra $\mathbb{Z}_2$ symmetry is therefore not present at weak coupling, helping to explain why it is not obvious in the original fermionic description of the model. It is even quite opaque after bosonization and only becomes obvious after finding a particularly simple duality frame. Another reason this symmetry is not obvious from the start is that it involves exchanging ϕ, a six-fermion operator, with ρ, which is an exotic defect operator from the point of view of the original fermionic description.

Before closing this section, we can return to our original motivations for looking for a dual representation of Eq. (23): the sign problem and the lack of a manifest $\mathbb{Z}_4$ rotation symmetry. We have already seen that the latter issue is automatically taken care of by passing to the representation in Eq. (29), so all that remains is understanding why in the end there is no sign problem. The first two terms in the second line of Eq. (29) yield constraints $(dv)_{\tilde{p}} = 0$ and $(dy)_{\tilde{\ell}} = 0$ mod 2. We have already seen that such constraints are easy to enforce when generating field configurations, and consequently these terms are harmless. The remaining term in the bottom of Eq. (29) looks alarming at first glance, but when $dv = 0$, it can be shown to be a total derivative (see Appendix C) and can be dropped, giving a sign-problem-free formulation with a manifest $\mathbb{Z}_4$ rotation symmetry.

VI. OUTLOOK

We leveraged recent advances in the understanding of anomalies on the lattice [27–29,31,32] to construct symmetry-preserving discretizations of $d = 2$ Abelian gauge theories with massless fermions with vectorlike and chiral couplings. These discretizations evade the Nielsen-Ninomiya theorem essentially because they directly discretize the fermion determinant $\det \not{D}$ rather than the fermion matrix $\not{D}$ itself, and $\det \not{D}$ is rewritten in terms of a path integral over bosonic fields with local interactions. We emphasize that in our construction the vectorlike and chiral symmetries act just as locally at finite lattice spacing as they do in the continuum, and all of the ABJ and 't Hooft anomalies are reproduced on the lattice. Finally, while the lattice actions we construct are complex, we have shown that the resulting sign problems can be avoided by judicious choices of dual variables.

Our results open many directions for future work. Numerical lattice calculations using this formalism can be used to explore strongly coupled regions in parameter space. It would be interesting to see if our approach can be generalized to $d > 2$, for example by taking advantage of advances in the understanding of continuum bosonization in $d = 3$ [115–121] and the development of symmetry-preserving discretizations of Chern-Simons terms [122]. It would also be nice to see if generalizations of our construction can preserve non-Abelian chiral symmetries at finite lattice spacing [14,123–130]. Finally, to get inspiration toward constructing more direct symmetry-preserving fermion discretizations, one can compute the discretized Dirac operators corresponding to our representations of the fermion determinant $\det \not{D}$.

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APPENDIX A: HAMILTONIAN FORMULATION

To construct the Hamiltonian for the one-flavor bosonized Schwinger model we follow the discussion of Ref. [29] (see Sec. 3.5 therein for a parallel discussion). We go to Lorentzian signature, take time to be continuous, drop the timelike integer-valued fields in the lattice action (4), and assume that space is discretized on a lattice with periodic boundary conditions. The Lagrangian density becomes

$$\mathcal{L} = \frac{\kappa}{2}\dot{\varphi}_{\tilde{s}}^2 - \frac{\kappa}{2}[(d\varphi)_{\tilde{\ell}} - 2\pi n_{\tilde{\ell}}]^2 + \frac{\beta}{2}\dot{a}_{\ell}^2 - \frac{Q}{2\pi}\varphi_{\star\ell}\dot{a}_{\ell} + \chi_s \dot{n}_{\star s}. \tag{A1}$$

Note that on the 1D lattice $\star\ell = \tilde{s}, \star s = \tilde{\ell}$. The canonical momenta (which live on the duals of the cells of their respective fields) can thus be written as

$$(\Pi_\varphi)_\ell = \kappa\dot{\varphi}_{\star\ell}, \qquad (\Pi_a)_{\tilde{s}} = \beta\dot{a}_\ell - \frac{Q}{2\pi}, \tag{A2}$$

$$(\Pi_n)_s = \chi_s, \qquad (\Pi_\chi)_{\tilde{\ell}} = 0. \tag{A3}$$

The lower two lines are second-class constraints. They can be taken into account using Dirac brackets, which lead us to the quantum Hamiltonian

$$\begin{aligned}\mathbf{H} = {}& \frac{1}{2\kappa}(\mathbf{\Pi}_\varphi)_\ell^2 + \frac{1}{2\beta}\left[(\mathbf{\Pi}_a)_{\tilde{s}} + \frac{Q}{2\pi}\boldsymbol{\varphi}_{\tilde{s}}\right]^2 \\ & + \frac{\kappa}{2}[(d\boldsymbol{\varphi})_{\tilde{\ell}} - 2\pi\mathbf{n}_{\tilde{\ell}}]^2,\end{aligned} \tag{A4}$$

where $\boldsymbol{\varphi}$, $\mathbf{\Pi}_\varphi$, $\mathbf{a}$, $\mathbf{\Pi}_a$, $\mathbf{n}$, and $\boldsymbol{\chi}$ are operators with the commutation relations

$$\begin{aligned}[\boldsymbol{\varphi}_{\tilde{s}}, (\mathbf{\Pi}_\varphi)_\ell] &= i\delta_{\tilde{s},\star\ell}, \\ [\mathbf{a}_\ell, (\mathbf{\Pi}_a)_{\tilde{s}}] &= i\delta_{\ell,\star\tilde{s}}, \\ [\mathbf{n}_{\tilde{\ell}}, \boldsymbol{\chi}_s] &= i\delta_{\star\tilde{\ell},s},\end{aligned} \tag{A5}$$

1. Gauge operators

In the Hamiltonian formalism the gauge redundancies must be imposed as constraints. We have four such redundancies: compactness of $\boldsymbol{\chi}$, compactness of $\boldsymbol{\varphi}$, small gauge transformations of $\mathbf{a}$ and $\boldsymbol{\chi}$, and large gauge transformations that ensure that the gauge group is $U(1)$ and not $\mathbb{R}$. These transformations will be associated with four operators $\mathbf{G}_\chi$, $\mathbf{G}_\varphi$, $\mathbf{G}_{\text{small}}$, and $\mathbf{G}_{\text{large}}$ which have to act like identity operators on all physical states.

The 2π shifts of χ are generated by

$$\mathbf{G}_\chi = [\{s_{\tilde{\ell}}\}] = e^{2\pi i \sum_{\tilde{\ell}} s_{\tilde{\ell}}\mathbf{n}_{\tilde{\ell}}}, \tag{A6}$$

where $s_{\tilde{\ell}} \in \mathbb{Z}$. Therefore $\mathbf{n}_{\tilde{\ell}}$ must have an integer spectrum.

The compactness condition for $\boldsymbol{\varphi}$ is associated with the transformation $\boldsymbol{\varphi}_{\tilde{s}} \to \boldsymbol{\varphi}_{\tilde{s}} + 2\pi k_{\tilde{s}}$, $\mathbf{n}_{\tilde{\ell}} \to \mathbf{n}_{\tilde{\ell}} + (dk)_{\tilde{\ell}}$, $(\mathbf{\Pi}_a)_{\tilde{s}} \to (\mathbf{\Pi}_a)_{\tilde{s}} - \frac{Q}{2\pi}k_{\tilde{s}}$ with $k_{\tilde{s}} \in \mathbb{Z}$, which is generated by

$$\mathbf{G}_\varphi[\{k\}] = \exp\left[i\sum_\ell 2\pi k_{\star\ell}\left((\mathbf{\Pi}_\varphi)_\ell - \frac{Q}{2\pi}\mathbf{a}_\ell - \frac{1}{2\pi}(d\chi)_\ell\right)\right]. \tag{A7}$$

Therefore the operator

$$\mathbf{q}_\ell = (\mathbf{\Pi}_\varphi)_\ell + \frac{Q}{2\pi}\mathbf{a}_\ell - \frac{1}{2\pi}(d\boldsymbol{\chi})_\ell \tag{A8}$$

must have an integer spectrum. Its commutation relations are

$$\begin{aligned}[\mathbf{q}_\ell, \boldsymbol{\varphi}_{\tilde{s}}] &= i\delta_{\tilde{s},\star\ell}, \\ [\mathbf{q}_\ell, (\mathbf{\Pi}_a)_{\tilde{s}}] &= \frac{iQ}{2\pi}\delta_{\ell,\star\tilde{s}}, \\ [\mathbf{q}_\ell, \mathbf{n}_{\tilde{\ell}}] &= \frac{i}{2\pi}(\delta_{f(\ell),\tilde{\ell}} - \delta_{f^{-1}(\ell),\tilde{\ell}}),\end{aligned} \tag{A9}$$

where $f(\ell)$ is a positive translation by half a lattice unit, which takes the lattice to the dual lattice.

Continuous ("small") gauge transformations $\mathbf{a} \to \mathbf{a} + d\lambda$ and $\boldsymbol{\chi} \to \boldsymbol{\chi} + Q\lambda$ are generated by

$$\mathbf{G}_{\text{small}}(\{\lambda\}) = \exp\left[i\sum_\ell (d\lambda)_\ell(\mathbf{\Pi}_a)_\ell - iQ\sum_s \lambda_s \mathbf{n}_{\star s}\right], \tag{A10}$$

where $\lambda_s \in \mathbb{R}$. This yields the Gauss law constraint

$$(d\mathbf{\Pi}_a)_{\tilde{\ell}} + Q\mathbf{n}_{\tilde{\ell}} = 0. \tag{A11}$$

Finally, the large gauge transformations $(\mathbf{\Pi}_a)_\ell \to (\mathbf{\Pi}_a)_\ell + 2\pi m_\ell$ are generated by

$$\mathbf{G}_{\text{large}}(\{m\}) = e^{i\sum_\ell 2\pi m_\ell (\mathbf{\Pi}_a)_\ell}, \qquad m_\ell \in \mathbb{Z}, \tag{A12}$$

which implies that $\mathbf{\Pi}_a$ must have an integer spectrum.

2. Symmetry operators

The two internal global symmetries of our Hamiltonian are associated with the following operators. The $\mathbb{Z}_Q$ chiral symmetry is generated by the line operator

$$\begin{aligned}\mathbf{U}_k(L) &= \exp\left[\frac{2\pi i k}{Q}\sum_{\ell\in L}\left((\mathbf{\Pi}_\varphi)_\ell + \frac{Q}{2\pi}(\mathbf{\Pi}_a)_\ell\right)\right] \\ &= e^{\frac{2\pi i k}{Q}\sum_\ell \mathbf{q}_\ell},\end{aligned} \tag{A13}$$

where L is all of space (that is, a time slice). This means that $\mathbf{q}_\ell$ is a charge density operator (which manages to exist in this case despite the fact that chiral symmetry is discrete) and $\mathbf{Q} = \sum_\ell \mathbf{q}_\ell$ is the total charge operator. The equation of motion of $\mathbf{q}$ is $\dot{\mathbf{q}} = i[\mathbf{H}, \mathbf{q}] = 0$, so $\mathbf{U}_k(L)$ is conserved, and the coefficient in front of $\mathbf{Q}$ is quantized thanks to the requirement that $\mathbf{U}_k(L)$ must commute with $\mathbf{G}_{\text{large}}$.

The $\mathbb{Z}_Q$ 1-form symmetry is generated by the local operator

$$\mathbf{V}_w(\tilde{s}) = e^{\frac{2\pi i}{Q}(\mathbf{\Pi}_a)_{\tilde{s}}}. \tag{A14}$$

The coefficient in the exponent must be quantized so that $[\mathbf{V}_w, \mathbf{G}_\varphi] = 0$, and it is topological thanks to the Gauss law.

The 't Hooft anomaly is encoded in the fact that these symmetry operators do not commute:

$$\mathbf{U}_k(L)\mathbf{V}_w(\tilde{s}) = e^{\frac{2\pi i k w}{Q}}\mathbf{V}_w(\tilde{s})\mathbf{U}_k(L). \tag{A15}$$

Therefore (A4) provides a Hamiltonian discretization of the charge Q Schwinger model which encodes all of its continuum internal symmetries and anomalies.

APPENDIX B: DUALIZING $N_f = 2$ QED

We start with the action (19), linearize the gauge and scalar kinetic terms using auxiliary fields, and sum over n, $\hat{n}$, and r to find

$$\begin{aligned}&\frac{1}{2\kappa}\left(\frac{N}{2\pi}\right)^2\left[\left(a_\ell - \frac{1}{N}(d\chi)_\ell - \frac{2\pi}{N}y_\ell\right)^2 \right. \\ &\quad \left. + \left(a_\ell - \frac{1}{N}(d\hat{\chi})_\ell - \frac{2\pi}{N}\hat{u}_\ell\right)^2\right] \\ &\quad + \frac{1}{2\beta}\left(\frac{N}{2\pi}\right)^2\left(\varphi_{\tilde{s}} + \hat{\varphi}_{\tilde{s}} - \frac{2\pi}{N}t_{\tilde{s}}\right)^2 + ia_\ell(dt)_{\star p} \\ &\quad - \frac{i}{2\pi}(Na_\ell + 2\pi y_\ell)(d\varphi)_{\star\ell} - \frac{i}{2\pi}(Na_\ell + 2\pi\hat{u}_\ell)(d\hat{\varphi})_{\star\ell},\end{aligned} \tag{B1}$$

where $y, \hat{u}, t \in \mathbb{Z}$. Doing the Gaussian integral over a gives

$$\begin{aligned}&\frac{1}{4\kappa}\frac{1}{(2\pi)^2}((d\chi)_\ell - (d\hat{\chi})_\ell - 2\pi(y_\ell - \hat{u}_\ell))^2 \\ &\quad + \frac{\kappa}{4}\left((d\varphi)_{\tilde{\ell}} + (d\hat{\varphi})_{\tilde{\ell}} - \frac{2\pi}{N}(dt)_{\tilde{\ell}}\right)^2 \\ &\quad + \frac{1}{2\beta}\left(\frac{N}{2\pi}\right)^2\left(\varphi_{\tilde{s}} + \hat{\varphi}_{\tilde{s}} - \frac{2\pi}{N}t_{\tilde{s}}\right)^2 \\ &\quad - \frac{i}{2}(y_\ell - \hat{u}_\ell)((d\varphi)_{\star\ell} - (d\hat{\varphi})_{\star\ell}) - \frac{i}{2}\frac{2\pi}{N}(y_\ell + \hat{u}_\ell)(dt)_{\star\ell}\end{aligned} \tag{B2}$$

after dropping total derivatives and multiples of $2\pi i$. Now let us define $\sigma = \chi - \hat{\chi}$, $\eta = \varphi + \hat{\varphi}$, $\phi = \frac{\hat{\varphi}}{2} - \frac{\varphi}{2} - \frac{\pi t}{N}$, and $u = y - \hat{u}$:

$$\frac{1}{4\kappa}\frac{1}{(2\pi)^2}((d\sigma)_\ell - 2\pi u_\ell)^2 + \frac{\kappa}{4}\left((d\eta)_{\tilde{\ell}} - \frac{2\pi}{N}(dt)_{\tilde{\ell}}\right)^2 + \frac{1}{2\beta}\left(\frac{N}{2\pi}\right)^2\left(\eta_{\tilde{s}} - \frac{2\pi}{N}t_{\tilde{s}}\right)^2 + iu_\ell(d\phi)_{\star\ell} - \frac{2\pi i}{N}\hat{u}_\ell(dt)_{\star\ell}, \tag{B3}$$

which is the result (20). The first line is the modified Villain formulation of a compact scalar. If we had started with the free-fermion radius $\kappa = \frac{1}{4\pi}$, we end up with an effective radius $\tilde{\kappa} = \frac{1}{2\kappa}\frac{1}{(2\pi)^2} = \frac{1}{2\pi}$, which is the self-dual radius.

For completeness, we note the dual description of the fermion mass terms:

$$\cos(\varphi) \to \cos\left(\frac{\eta}{2} - \phi - \frac{\pi}{N}t\right) \tag{B4}$$

and

$$\cos(\hat{\varphi}) \to \cos\left(\frac{\eta}{2} + \phi + \frac{\pi}{N}t\right). \tag{B5}$$

As a result, a flavor-symmetric mass deformation becomes

$$\cos(\varphi) + \cos(\tilde{\varphi}) \to 2\cos\left(\frac{\eta}{2}\right)\cos\left(\phi + \frac{\pi}{N}t\right). \tag{B6}$$

Note this is respects the 2π periodicity because when $\eta \to \eta + 2\pi k$ and $t \to t + Nk$, both factors pick up a sign $(-1)^k$ which squares away. If we turn on a theta angle, this is modified to $\cos(\frac{\eta}{2} + \frac{\theta}{2})\cos(\phi + \frac{\pi}{N}t)$.

APPENDIX C: DUALIZING THE 3450 MODEL

Let us write the action of the 3450 model using (real) auxiliary fields ζ, $\hat{\zeta}$, and ξ:

$$\begin{aligned} S = {} & \frac{1}{2\kappa}\zeta^2_{\star\tilde{\ell}} + i\zeta_{\star\tilde{\ell}}[(d\varphi)_{\tilde{\ell}} - Q_A a_{f(\tilde{\ell})} - 2\pi n_{\tilde{\ell}}] \\ & + \frac{1}{2\kappa}\hat{\zeta}^2_{\star\tilde{\ell}} + i\hat{\zeta}_{\star\tilde{\ell}}[(d\hat{\varphi})_{\tilde{\ell}} - \hat{Q}_A a_{f(\tilde{\ell})} - 2\pi \hat{n}_{\tilde{\ell}}] \\ & + \frac{1}{2\beta}\xi^2_{\star p} + i\left(\xi_{\star p} + \frac{1}{2\pi}(Q_V\varphi_{\star p} + \hat{Q}_V\hat{\varphi}_{\star p})\right) \\ & \times [(da)_p - 2\pi r_p] - i(Q_V n_{\star\ell} + \hat{Q}_V\hat{n}_{\star\ell})a_\ell \\ & + in_{\star\ell}(d\chi)_\ell + i\hat{n}_{\star\ell}(d\hat{\chi})_\ell \\ & - ir_p(Q_A\chi_{f^{-1}(\star p)} + \hat{Q}_A\hat{\chi}_{f^{-1}(\star p)}). \end{aligned} \tag{C1}$$

Summing over $r_p \in \mathbb{Z}$ sets

$$\begin{aligned} \xi_{\star p} = {} & -\frac{1}{2\pi}(Q_V\varphi_{\star p} + \hat{Q}_V\hat{\varphi}_{\star p} + Q_A\chi_{f^{-1}(\star p)} \\ & + \hat{Q}_A\hat{\chi}_{f^{-1}(\star p)} - 2\pi y_{\star p}), \end{aligned} \tag{C2}$$

where $y_{\tilde{\ell}} \in \mathbb{Z}$. Plugging this back into the action and integrating out the gauge field a_ℓ allows us to solve for ζ:

$$\begin{aligned} \zeta_{\star\tilde{\ell}} = {} & \frac{1}{Q_A}\left[-\hat{Q}_A\hat{\zeta}_{\star\tilde{\ell}} - \frac{1}{2\pi}(Q_A(d\chi)_{\star\tilde{\ell}} + \hat{Q}_A(d\hat{\chi})_{\star\tilde{\ell}}) \right. \\ & \left. + (dy)_{f(\star\tilde{\ell})} - Q_V n_{f(\star\tilde{\ell})} - \hat{Q}_V\hat{n}_{f(\star\tilde{\ell})}\right]. \end{aligned} \tag{C3}$$

If we plug this back into the action and perform a field redefinition $\hat{\zeta}_\ell \to \hat{\zeta}_\ell - \frac{1}{2\pi}(d\hat{\chi})_\ell$, we land on

$$\begin{aligned} S = {} & \frac{1}{2\beta}\frac{1}{(2\pi)^2}\xi^2_{\tilde{s}} + \frac{1}{2\kappa}\left[\hat{\zeta}_\ell - \frac{1}{2\pi}(d\hat{\chi})_\ell\right]^2 \\ & + \frac{1}{2\kappa}\frac{\hat{Q}_A^2}{Q_A^2}\left[\hat{\zeta}_\ell + \frac{Q_A}{\hat{Q}_A}\frac{(d\chi)_\ell}{2\pi} - \frac{(dy)_{f(\ell)} + u_{f(\ell)}}{\hat{Q}_A}\right]^2 \\ & - i\frac{\hat{\zeta}_\ell}{Q_A}\left[Q_A(d\hat{\varphi})_{\star\ell} - \hat{Q}_A(d\varphi)_{\star\ell} + 2\pi\frac{\hat{Q}_A}{Q_V}u_{\star\ell}\right] \\ & + \frac{i}{Q_A}u_{f(\ell)}[(d\varphi)_{\star\ell} - 2\pi n_{\star\ell}] + \frac{2\pi i}{Q_A}(dy)_{f(\ell)}n_{\star\ell}, \end{aligned} \tag{C4}$$

with ξ set to its value in (C2), and for notational convenience we have defined

$$u_{\tilde{\ell}} \equiv Q_V n_{\tilde{\ell}} + \hat{Q}_V\hat{n}_{\tilde{\ell}} = \frac{Q_V}{\hat{Q}_A}(\hat{Q}_A n_{\tilde{\ell}} - Q_A\hat{n}_{\tilde{\ell}}), \tag{C5}$$

where the second equality follows from the anomaly-free condition. Note this is not a $GL(2,\mathbb{Z})$ change of basis, and we have to remember that $u_{\tilde{\ell}} \in Q_V\mathbb{Z} + \hat{Q}_V\mathbb{Z}$. The integral over $\hat{\zeta}$ is Gaussian. To simplify the calculation let us define

$$\phi_{\tilde{s}} = Q_V\varphi_{\tilde{s}} + \hat{Q}_V\hat{\varphi}_{\tilde{s}}, \qquad \psi_s = Q_A\chi_s + \hat{Q}_A\hat{\chi}_s, \tag{C6}$$

$$\eta_s = \hat{Q}_A\chi_s - Q_A\hat{\chi}_s. \tag{C7}$$

Note that ϕ and ψ are (0-form) gauge invariant but η is not. One can check that this defines an invertible change of basis assuming the anomaly-cancelation condition. In terms of these new variables, the result of integrating over $\hat{\zeta}$ is

$$\begin{aligned} S = {} & \frac{\kappa}{2}\frac{\hat{Q}_A^2/Q_V^2}{Q_A^2 + \hat{Q}_A^2}((d\phi)_{\star\ell} - 2\pi u_{\star\ell}))^2 \\ & + \frac{1}{2\kappa}\frac{1}{(2\pi)^2}\frac{1}{Q_A^2 + \hat{Q}_A^2}((d\psi)_\ell - 2\pi((dy)_{f(\ell)} - u_{f(\ell)}))^2 \\ & + \frac{1}{2\beta}\frac{1}{(2\pi)^2}(\phi_{\tilde{s}} + \psi_{f^{-1}(\tilde{s})} - 2\pi y_{\tilde{s}})^2 \\ & + \frac{i}{Q_A}u_{f(\ell)}[(d\varphi)_{\star\ell} - 2\pi n_{\star\ell}] + \frac{2\pi i}{Q_A}(dy)_{f(\ell)}n_{\star\ell} \\ & - \frac{i}{2\pi}\frac{\hat{Q}_A/Q_V}{Q_A^2 + \hat{Q}_A^2}((d\phi)_{\star\ell} - 2\pi u_{\star\ell})) \\ & \times\left[(d\eta)_\ell - 2\pi\frac{\hat{Q}_A}{Q_A}((dy)_{f(\ell)} - u_{f(\ell)})\right]. \end{aligned} \tag{C8}$$

Let us collect the terms linear in du:

$$i(du)_{\tilde{p}}\left[\frac{1}{Q_A}\varphi_{f^{-1}(\star\tilde{p})}-\frac{\hat{Q}_A/Q_V}{Q_A^2+\hat{Q}_A^2}\eta_{\star\tilde{p}}\right.\\\left.-\frac{\hat{Q}_A^2/(Q_AQ_V)}{Q_A^2+\hat{Q}_A^2}(\phi_{f^{-1}(\star\tilde{p})}-2\pi y_{f(\star\tilde{p})})\right]. \tag{C9}$$

Ignoring total derivatives, neither φ nor η appear anywhere else in the action, so we can freely make another change of variables and define a new field $\sigma_{\star\tilde{p}}$ which is equal to the quantity in brackets (note that this quantity is gauge invariant). The role of σ is to set $(du)_{\tilde{p}}=0$.

The remaining imaginary terms in the action are

$$\frac{2\pi i}{Q_A}((dy)_{f(\ell)}-u_{f(\ell)})n_{\star\ell}+2\pi i\frac{\hat{Q}_A^2/(Q_AQ_V)}{Q_A^2+\hat{Q}_A^2}u_{\star\ell}u_{f(\ell)}. \tag{C10}$$

The last term is trivial when $du=0$. To see this, fix a plaquette $\tilde{p}$ on the dual lattice whose lower left-hand corner is at the site $\tilde{x}$. One can verify the identity[8]

$$\sum_{\tilde{\ell}=(\tilde{x},\mu)}u_{\tilde{\ell}}u_{f(\star\tilde{\ell})}=-\frac{1}{2}(d(u^2))_{\tilde{p}}\\+\frac{1}{2}(du)_{\tilde{p}}(u_{\tilde{x},0}+u_{\tilde{x}+\hat{0},1}+u_{\tilde{x},1}+u_{\tilde{x}+\hat{1},0}), \tag{C11}$$

where the sum is over the two links emanating from $\tilde{x}$. The first term is a total derivative which vanishes when summed over the lattice and the second line vanishes when $du=0$.

Recalling the definition of u, we arrive at the dual formulation (ignoring the term discussed above)

$$\begin{aligned}S=&\frac{\kappa}{2}\frac{\hat{Q}_A^2/Q_V^2}{Q_A^2+\hat{Q}_A^2}((d\phi)_{\star\ell}-2\pi(Q_Vn_{\star\ell}+\hat{Q}_V\hat{n}_{\star\ell}))^2\\&+\frac{1}{2\kappa}\frac{1/(2\pi)^2}{Q_A^2+\hat{Q}_A^2}\\&\times((d\psi)_\ell-2\pi((dy)_{f(\ell)}-Q_Vn_{f(\ell)}-\hat{Q}_V\hat{n}_{f(\ell)}))^2\\&+\frac{1}{2\beta}\frac{1}{(2\pi)^2}(\phi_{\tilde{s}}+\psi_{f^{-1}(\tilde{s})}-2\pi y_{\tilde{s}})^2\\&+i\sigma_{\star\tilde{p}}(Q_V(dn)_{\tilde{p}}+\hat{Q}_V(d\hat{n})_{\tilde{p}})\\&+\frac{2\pi i}{Q_A}((dy)_{f(\ell)}-Q_Vn_{f(\ell)}-\hat{Q}_V\hat{n}_{f(\ell)})n_{\star\ell}.\end{aligned} \tag{C12}$$

[8]This is equivalent to the following identity involving higher cup products: $2u\cup u=-d(u\cup_1 u)+du\cup_1 u-u\cup_1 du$, where u is an arbitrary 1-form. See [122] for explicit formulas for higher cup products on square lattices.

So far we have allowed the charges to be arbitrary but satisfying the anomaly-cancelation condition. Note that there are still imaginary terms which remain, indicating a sign problem. In future work, it would be interesting to determine the set of 2D chiral gauge theories where one can avoid this sign problem, possibly by finding alternatives to (C12) in some cases.

Here we focus on the particular case of the 3450 model, where the sign problem can indeed be removed thanks to the fact that the 3450 charge assignments make the last two terms in (C12) multiples of $2\pi i$. Dropping these terms we find

$$\begin{aligned}S=&\frac{\kappa}{2}\frac{1}{80}((d\phi)_{\star\ell}-2\pi(8n_{\star\ell}+4\hat{n}_{\star\ell}))^2\\&+\frac{1}{2\kappa}\frac{1}{20(2\pi)^2}((d\psi)_\ell-2\pi((dy)_{f(\ell)}-8n_{f(\ell)}-4\hat{n}_{f(\ell)}))^2\\&+\frac{1}{2\beta}\frac{1}{(2\pi)^2}(\phi_{\tilde{s}}+\psi_{f^{-1}(\tilde{s})}-2\pi y_{\tilde{s}})^2\\&+i\sigma_{\star\tilde{p}}(8(dn)_{\tilde{p}}+4(d\hat{n})_{\tilde{p}})-i\pi(dy)_{f(\ell)}n_{\star\ell}\\&-\frac{2\pi i}{20}(8n_{\star\ell}+4\hat{n}_{\star\ell})(8n_{f(\ell)}+4\hat{n}_{f(\ell)}),\end{aligned} \tag{C13}$$

where for completeness we have reinstated the imaginary term we argued was zero above. Now we can perform a $GL(2,\mathbb{Z})$ change of basis to $v_{\tilde{\ell}}=2n_{\tilde{\ell}}+\hat{n}_{\tilde{\ell}}$ and $n_{\tilde{\ell}}$ as the integer degrees of freedom:

$$\begin{aligned}S=&\frac{\kappa}{2}\frac{1}{80}((d\phi)_{\star\ell}-8\pi v_{\star\ell})^2\\&+\frac{1}{2\kappa}\frac{1}{20(2\pi)^2}((d\psi)_\ell-2\pi((dy)_{f(\ell)}-4v_{f(\ell)}))^2\\&+\frac{1}{2\beta}\frac{1}{(2\pi)^2}(\phi_{\tilde{s}}+\psi_{f^{-1}(\tilde{s})}-2\pi y_{\tilde{s}})^2\\&+4i\sigma_{\star\tilde{p}}(dv)_{\tilde{p}}-i\pi(dy)_{f(\ell)}n_{\star\ell}-2\pi i\frac{4}{5}v_{\star\ell}v_{f(\ell)}.\end{aligned} \tag{C14}$$

We now rescale $\phi\to 4\phi$, $\psi\to 2\psi$, and $\sigma\to\frac{1}{4}\sigma$ to reach

$$\begin{aligned}S=&\frac{\kappa}{2}\frac{1}{5}((d\phi)_{\star\ell}-2\pi v_{\star\ell})^2\\&+\frac{1}{2\kappa}\frac{1}{20(2\pi)^2}(2(d\psi)_\ell-2\pi((dy)_{f(\ell)}-4v_{f(\ell)}))^2\\&+\frac{1}{2\beta}\frac{1}{(2\pi)^2}(4\phi_{\tilde{s}}+2\psi_{f^{-1}(\tilde{s})}-2\pi y_{\tilde{s}})^2\\&+i\sigma_{\star\tilde{p}}(dv)_{\tilde{p}}-i\pi(dy)_{f(\ell)}n_{\star\ell}-2\pi i\frac{4}{5}v_{\star\ell}v_{f(\ell)}.\end{aligned} \tag{C15}$$

The $\mathbb{Z}_4$ lattice rotation symmetry is not manifest at this level—for instance the second and third lines involve the shift f which does not commute with rotations. However, the first three lines of the action are invariant if one performs a (counterclockwise) $\pi/2$ rotation together with a shift $\psi_s \to \psi_{s+\hat{0}}$. The last line just encodes the constraints, modulo the last term which we argued is a total derivative. Alternatively, we can make the $\mathbb{Z}_4$ lattice rotation symmetry manifest by simply defining $\hat{\psi}_{\tilde{s}} = \psi_{f^{-1}(\tilde{s})}$ and $\hat{n}_{\tilde{\ell}} = n_{f^{-1}(\star\tilde{\ell})}$, so that

$$\begin{aligned} S = &\frac{\kappa}{2}\frac{1}{5}((d\phi)_{\star\ell} - 2\pi v_{\star\ell})^2 \\ &+ \frac{1}{2\kappa}\frac{1}{20(2\pi)^2}(2(d\hat{\psi})_{f(\ell)} - 2\pi((dy)_{f(\ell)} - 4v_{f(\ell)}))^2 \\ &+ \frac{1}{2\beta}\frac{1}{(2\pi)^2}(4\phi_{\tilde{s}} + 2\hat{\psi}_{\tilde{s}} - 2\pi y_{\tilde{s}})^2 \\ &+ i\sigma_{\star\tilde{p}}(dv)_{\tilde{p}} - i\pi\hat{n}_{\star\tilde{\ell}}(dy)_{\tilde{\ell}} - 2\pi i\frac{4}{5}v_{\star\ell}v_{f(\ell)}. \end{aligned} \tag{C16}$$

The path integrals over σ and $\hat{n}$ serve to impose the constraints $(dv)_{\tilde{p}} = 0, (dy)_{\tilde{\ell}} = 0 \bmod 2$, so that $\sum_\ell v_{\star\ell}v_{f(\ell)} = 0$. Finally, we observe that for any lattice field $g_{\tilde{\ell}}$, $\sum_\ell g^2_{f(\ell)} = \sum_{\tilde{\ell}} g^2_{\tilde{\ell}}$, so that we can rewrite S as

$$\begin{aligned} S = &\frac{\kappa}{2}\frac{1}{5}((d\phi)_{\star\ell} - 2\pi v_{\star\ell})^2 \\ &+ \frac{1}{2\kappa}\frac{1}{20(2\pi)^2}(2(d\hat{\psi})_{\tilde{\ell}} - 2\pi((dy)_{\tilde{\ell}} - 4v_{\tilde{\ell}}))^2 \\ &+ \frac{1}{2\beta}\frac{1}{(2\pi)^2}(4\phi_{\tilde{s}} + 2\hat{\psi}_{\tilde{s}} - 2\pi y_{\tilde{s}})^2 \\ &+ i\sigma_{\star\tilde{p}}(dv)_{\tilde{p}} - i\pi\hat{n}_{\star\tilde{\ell}}(dy)_{\tilde{\ell}} - 2\pi i\frac{4}{5}v_{\star\ell}v_{f(\ell)}. \end{aligned} \tag{C17}$$

This is form of the dual action we will discuss in the main text.

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