QCD with an infrared fixed point and a dilaton

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Following previous work we further explore the possibility that the chirally broken phase of gauge theories admits an infrared fixed point interpretation. The slope of the β function, β'_*, is found to vanish at the infrared fixed point which has several attractive features such as logarithmic running. We provide a more in-depth analysis of our previous result that the mass anomalous assumes $\gamma_* = 1$ at the fixed point. The results are found to be consistent with $\mathcal{N} = 1$ supersymmetric gauge theories. In a second part the specific properties of a dilaton, the (pseudo-) Goldstone, due to spontaneous symmetry breaking are investigated. Dilaton soft theorems indicate that a soft dilaton mass can only originate from an operator of scaling dimension two. In the gauge theory this role is taken on by the $\bar{q}q$-operator. The quantum chromodynamics (QCD) dilaton candidate, the $\sigma = f_0(500)$ meson is investigated and the singlet-octet mixing is found to be relevant. We briefly discuss the dilaton as a candidate for the Higgs boson, which relies on the ratio of dilaton to pion decay constant being close to unity. In QCD this is approximately satisfied but it is remains unclear if this is accidental or whether there is yet to be uncovered principle behind it.

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I. INTRODUCTION

That the strong interaction could be described by an infrared (IR) fixed point is an idea [1–6] that predates quantum chromodynamics (QCD) itself. Applied in our previous work [7], it was found in three different ways, that pion physics can be reproduced when the quark mass anomalous dimension at the IR fixed point assumes $\gamma_* = 1$. The methods included the hyperscaling relation for the pion mass, compatibility of the Feynman-Hellmann theorem with the trace of the energy momentum tensor (TEMT) and matching a long distance correlator. The main idea that underlies the scenario, is that the gauge theory flows into an IR fixed point and that it is the quark condensate $\langle \bar{q}q \rangle \neq 0$ that breaks *both* the chiral and the conformal symmetry spontaneously.[1] This leads, besides the pion, to an additional (pseudo-) Goldstone, the dilaton. The fact that the conformal symmetry is only emergent and that the dilaton has vacuum quantum numbers complicates matters. Let us turn to the assumptions. The minimal implementation of IR conformality, adopted in [7], is that the TEMT on single particle IR-states φ, which include the vacuum, the pion and possibly the dilaton

$$\langle \varphi'(p') | T^\rho{}_\rho | \varphi(p) \rangle_{q=0} = 0, \tag{1.1}$$

vanishes for zero momentum transfer $q = p - p'$. In a true CFT (1.1) holds on any physical state and momentum transfer and can be seen as its definition.[2] We make the reasonable assumption that if (1.1) holds true that there exists a scheme with vanishing β function in the IR: $\beta_* \equiv \beta|_{\mu=0} = 0$.[3] Additionally, we assume that correlators obey conformal field theory (CFT) scaling in the deep-IR

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[1]Throughout this paper we do not distinguish conformal and scale symmetry. In $d = 4$ it is believed that these are equivalent for nontrivial unitary theories (cf. the review [8] or [7] for more comments and references).

[2]In the presence of a massless dilaton (1.1) holds for the LO EFT: $T^\rho{}_\rho|_{\rm LO} = 0$ [9]; and extends to the case of single nucleon states $\langle N(p')|T^\rho{}_\rho|N(p)\rangle_{q=0} = 0$ despite $m_N \neq 0$ [9,10].

[3]Note that $\beta(g^*) = 0$ is invariant under analytic redefinitions of g [11] but not necessarily when nonanalytic (e.g., canonical versus holomorphic coupling in $\mathcal{N} = 1$ supersymmetric QCD [12–14]). In such exotic schemes the physics is hidden away in the field strength since it is the product $T^\rho{}_\rho = \frac{\beta(\mu)}{2g(\mu)} G^2(\mu)$ which is physical and not the β function itself. The EFT formulation is valid in schemes where $\beta(g^*) = 0$ only.

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \propto \frac{1}{(x^2)^{\Delta_{\mathcal{O}}}} + \text{GB-corrections}, \quad (1.2)$$

corrected by terms due to Goldstone boson (GB). Above $\Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}}$ is the scaling dimension, defined as the sum of the engineering and the anomalous dimension. We will see that (1.2) essentially emerges by combining RG and EFT reasoning. Following the terminology in [10] we will refer to this scenario as conformal dilaton (CD) QCD.

It is widely believed that the dilaton retains a *chiral mass* (no explicit symmetry breaking)

$$\bar{m}_D \equiv m_D|_{m_q=0}, \quad (1.3)$$

in the case of an IR-emergent conformal symmetry.[4] To the best of our knowledge this issue has never been fully settled.[5] We take a pragmatic attitude by studying both cases separately. (We stress that all other hadrons remain massive in the chiral limit, primarily since there is no dynamical reason for them to be massless cf. footnote 2.) For $\bar{m}_D = 0$ we use dilaton-χPT ($d\chi$PT) and for $\bar{m}_D \neq 0$ we use standard chiral perturbation theory (χPT) since the dilaton can be integrated out. Phenomenologically a dilaton approach might still be of if $\bar{m}_D \ll \Lambda_{\text{hadron}} = \mathcal{O}(m_N)$ which is what most approaches have in mind. This includes lattice Monte Carlo investigation of gauge theories [15–29], EFT descriptions thereof [30–33]; the dilaton as a Higgs boson within [34–36] and investigations not necessarily linked to gauge theories [37–41]. Or the dilaton as a driving force of inflation [42], in dense nuclear interactions [43–45], in cosmology [46–48] and the σ-meson in QCD [49,50]. The name dilaton is also used in composite models for a light 0^{++}-partner of pseudo-Goldstones [51].

Let us briefly summarize the main findings of our work. By RG methods it is inferred that the slope of the β function vanishes at the fixed point: $\beta'_* = 0$. A result with many attractive features, as it is compatible with $\mathcal{N} = 1$ supersymmetric gauge theories, makes a light dilaton more likely and implies logarithmic running near the fixed point as one would expect from the EFT itself. A more thorough justification for $\gamma_* = 1$ in the context of matching correlation function is given combining RG and EFT methods. Applying a double-soft dilaton theorem, it is found that an operator $\mathcal{O}$ giving a mass to the dilaton must be of scaling dimension $\Delta_{\mathcal{O}} = 2$. The second part of the paper is more qualitative and contains a discussion on whether a dilaton can be massless, applies dχPT to the σ-meson in QCD, and concerns the dilaton as a Higgs boson. The main outstanding questions are: (i) the size of the chiral mass $\bar{m}_D$ (as the distance from the conformal window) and (ii) the ratio of decay constants $r_{N_f} = F_\pi/F_D$ as a function of the number of flavors N_f. If $r_2 \approx 1$ were true, then this would provide the rationale for coupling the dilaton like a Higgs and quite possibly make it compatible with large hadron collider (LHC) constraints.

The paper is organized as follows. In Sec. II the gauge theory and dχPT are defined including a review of the conformal window. In Sec. III QCD correlators matched to the EFTs in the deep-IR and consistency with $\mathcal{N} = 1$ supersymmetric gauge theories is discussed in Sec. IV. In Sec. V soft and double-soft dilaton theorems are exploited Sec. V. The remaining part consists of a discussion of mass or no mass for the dilaton.

Arguments in favor and disfavor of a massless dilaton are addressed in Sec. VI. The dilaton as the σ-meson in QCD and as the Higgs boson are discussed in Secs. VII and VIII respectively. The paper ends with summary and conclusions in Sec. IX. Conventions, more on soft theorems and details on mixing are deferred to Appendixes A, B and C respectively.

II. THE GAUGE THEORY AND ITS LOW ENERGY EFFECTIVE THEORY

The core ingredients to this work are the gauge theory, its RG quantities and the low energy EFT which corresponds to χPT and dχPT when a dilaton is added and are introduced in minimal form below.

A. The gauge theory and its renormalization group quantities

The gauge theory Lagrangian is given by

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}G^2 + \sum_{q=1}^{N_f} \bar{q}(i\not{D} - m_q)q, \quad (2.1)$$

where $G^2 = G^a_{\mu\nu}G^{a\mu\nu}$ is the field strength tensor with a denoting the adjoint index of the gauge group, $D_\mu = (\partial + igA)_\mu$ is the gauge-covariant derivative and the quarks are in some unspecified representation of the gauge group. For the more formal part of the paper the N_f flavors are assumed to be degenerate. There are two parameters in the Lagrangian, the gauge coupling g and the quark mass m_q whose RG-behavior we must follow. The gauge coupling is relevant and irrelevant in the UV and IR respectively whereas for the quark mass it is the other way around. Global flavor symmetries are discussed in the next section.

The anomalous dimensions of the parameters and their conjugate operators are defined from the renormalized quantities as follows

[4]Throughout this work we refer to the dilaton as D when generic matters are discussed and to $\sigma(\leftrightarrow f_0(500))$ when referring to the dilaton candidate in QCD and to h in the context of the Higgs boson.

[5]In the context of gauge theories, it is generally believed that $\bar{m}_D$ becomes larger with respect to the other hadronic scales as one moves away from the conformal window by lowering the number of flavors.

$$\beta \equiv \frac{d}{d\ln\mu} g, \qquad \gamma_{G^2} \equiv -\frac{d}{d\ln\mu} \ln G^2,$$
$$\gamma_m \equiv -\frac{d}{d\ln\mu} \ln m_q, \qquad \gamma_{\bar{q}q} \equiv -\frac{d}{d\ln\mu} \ln \bar{q}q. \qquad (2.2)$$

A central quantity to this work is the TEMT [52–58]

$$T^\rho{}_\rho|_{\text{phys}} = \frac{\beta}{2g} G^2 + \sum_q m_q (1+\gamma_m) \bar{q}q, \qquad (2.3)$$

since its vanishing on physical states signals conformality. The subscript "phys" indicates that we have omitted terms proportional to the equation of motion (and BRST exact terms which arise upon gauge fixing) which nota bene vanish on physical states [56–58]. It is very useful to note that (2.3) consists of two separately RG-invariant terms $T^\rho{}_\rho = \mathcal{O}_1 + \mathcal{O}_2$

$$\mathcal{O}_1 = \frac{\beta}{2g} G^2 + \sum_q \gamma_m m_q \bar{q}q, \qquad \mathcal{O}_2 = \sum_q m_q \bar{q}q. \qquad (2.4)$$

Imposing $\frac{d}{d\ln\mu}\mathcal{O}_{1,2} = 0$ results in

$$\gamma_{G^2} = \beta' - \frac{\beta}{g}, \qquad \gamma_{\bar{q}q} = -\gamma_m, \qquad (2.5)$$

if one considers a quark mass independent scheme for the β function. Quantities at the IR fixed point are denoted by a star

$$(\gamma_{G^2})_* = \beta'_*, \qquad \gamma_* \equiv (\gamma_m)_*, \qquad (2.6)$$

where $\beta_* = 0$ has been assumed as stated earlier. In the context of conformal field theories (CFT), e.g., [59–64], the important quantities are the operator scaling dimensions $\Delta_{\mathcal{O}} \equiv d_{\mathcal{O}} + \gamma_{\mathcal{O}}$ (sum of engineering the anomalous dimension)

$$\Delta_{G^2} = d + \beta'_* \quad \to \quad 4 + \beta'_*,$$
$$\Delta_{\bar{q}q} = (d-1) - \gamma_* \to \quad 3 - \gamma_*. \qquad (2.7)$$

The d-dimensional expressions have been given for later use.

B. Synopsis of the conformal window

For the reader not acquainted with the basic notation of the conformal window we briefly summarize its minimal content (the $\mathcal{N} = 1$ supersymmetric case is described in Sec. IV). The conformal window is the study of different phases of gauge theories, cf. Fig. 1, as a function of the gauge group, e.g., $SU(N_c)$ and the quark representation of N_f massless fermions. The conformal window has first been intensely studied within technicolor model building

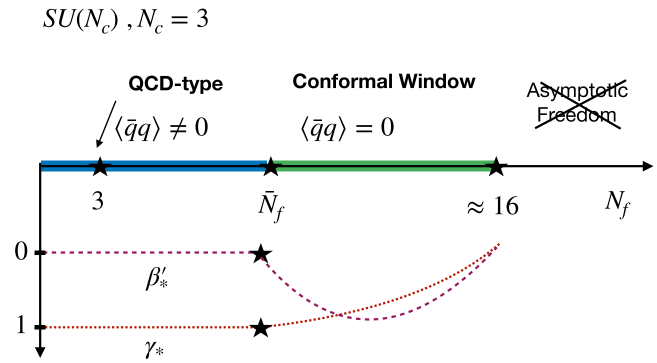

FIG. 1. Sketch of the conformal window for $G = SU(N_c)$ with $N_c = 3$. For $N_f > 16$ asymptotic freedom is lost and those theories are not considered. Below there is the perturbative Caswell-Banks-Zaks IR fixed point. As N_f is lowered the IR fixed point coupling becomes stronger until chiral symmetry breaking sets in (for some unknown critical $\bar{N}_f$). In this paper we explore the possibility that the theory admits an IR fixed point interpretation in parts or all of the broken phase as well. The evolution of the IR fixed point quantities γ_* and β'_*, to be deduced in Sec. III, are shown as a function of N_f. We stress that the evolution between $\bar{N}_f$ and 16 is only schematic (the question of continuity of the transition at $\bar{N}_f$ cannot be assessed since N_f is a discrete number) and most importantly that the evolution below $\bar{N}_f$ is non-standard (not certain) but shown to be consistent with the IRFP assumption. In Sec. IV arguments are given in favor of this picture for $\mathcal{N} = 1$ supersymmetric gauge theories.

(reviewed in [65–67]). The current understanding is in large based on the $\mathcal{N} = 1$ supersymmetric case and nonsupersymmetric lattice studies. The main point is that in the (N_f, N_c)-plane, in the domain of asymptotic freedom, there are two phases: one where the theory flows into an IR fixed point and the QCD-phase. In the latter there is confinement and chiral symmetry is spontaneously broken by the quark condensate $\langle \bar{q}q \rangle \neq 0$ (the global flavor symmetry breaks from $SU(N_f)_L \otimes SU(N_f)_R \to SU(N_f)_I$ to the diagonal subgroup accompanied by $N_f^2 - 1$ massless pions). If we fix $N_c = 3$ then we know that around $N_f \approx 16$ the theory admits a perturbative (Caswell-Banks-Zaks [68,69]) IR fixed point and that somewhere in between $N_f = 16$ and $N_f = 3$ (QCD) there is the transition into the QCD phase. For what critical $\bar{N}_f$ this happens is a matter of intense debate, but known exactly in the $\mathcal{N} = 1$ case.

C. Low energy effective theory—dilaton-χPT at LO

In this section the dilaton EFT is discussed, its starting point is the well-understood χPT [70–73] which is the EFT of $N_f^2 - 1$ pions describing a wealth of low-energy data. Based on this successful framework the dilaton, the Goldstone due to the scale symmetry breaking, is added. The Goldstone bosons are parametrized in exponential form

$$U = e^{i2\pi^a T^a/F_\pi}, \qquad \chi \equiv F_D e^{-D/F_D}, \tag{2.8}$$

where the quantities $F_{\pi,D}$ are fundamental in that they are the order parameters of the symmetry breaking

$$\langle \pi^b(q)|J^a_{5\mu}(x)|0\rangle = iF_\pi q_\mu \delta^{ab} e^{iqx},$$
$$\langle D(q)|J^D_\mu(x)|0\rangle = iF_D q_\mu e^{iqx}, \tag{2.9}$$

defined as the matrix elements of the global symmetry currents

$$J^a_{5\mu}(x) = \bar{q}(x)T^a\gamma_\mu\gamma_5 q(x), \qquad J^D_\mu(x) = x^\nu T_{\mu\nu}(x), \tag{2.10}$$

between the vacuum and the corresponding Goldstone. It is noted that the equation for the dilaton in (2.9) is equivalent to

$$\langle D(q)|T_{\mu\nu}|0\rangle = \frac{F_D}{d-1}(m_D^2\eta_{\mu\nu} - q_\mu q_\nu), \tag{2.11}$$

when taking into account that $x^\nu \to -i\partial_{q_\nu}$. From the zeroth component of the currents one can see heuristically that the charge of the symmetry does not annihilate the vacuum and thus signals SSB. The symmetries of the Goldstones are said to be realized nonlinearly

$$U \to LUR^\dagger, \qquad \chi \to \chi e^{\alpha(x)}, \tag{2.12}$$

where $(L,R) \in SU(N_f)_L \otimes SU(N_f)_R$ and $\alpha(x) \in \mathbb{R}$ parametrize the standard chiral and the Weyl transformation respectively. The latter can be interpreted as the implementation of scale transformations on the metric $g_{\mu\nu}$, rather than the coordinates, and the fields φ

$$g_{\mu\nu} \to e^{-2\alpha(x)} g_{\mu\nu}, \qquad \varphi \to e^{s_\varphi \alpha(x)}\varphi, \tag{2.13}$$

where s_φ is called the Weyl weight which is not to be confused with the engineering dimension d_φ of the field. The Weyl weight of the metric is $s_{g_{\mu\nu}} = -2$ since it naturally contracts two coordinate vectors $x^2 = g_{\mu\nu}x^\mu x^\nu$ which would transform as $x^\mu \to e^{\alpha(x)}x^\mu$. Local Weyl invariance,

$$\sqrt{-g}\mathcal{L} \to (e^{-\alpha d}\sqrt{-g})(e^{\alpha d}\mathcal{L}) = \sqrt{-g}\mathcal{L}, \tag{2.14}$$

is the guiding principle for the low energy EFT. The α space-dependence has and will be suppressed occasionally hereafter. In the case of explicit and anomalous symmetry breaking the spurion technique is adapted.

The leading order (LO) Lagrangian consists of

$$\mathcal{L}_{\rm LO} = \mathcal{L}^{(\pi)}_{\rm kin} + \mathcal{L}^{(D)}_{\rm kin} + \mathcal{L}^R_d + \mathcal{L}_{m_q} + \mathcal{L}_{\rm anom} - V(\chi), \tag{2.15}$$

two kinetic-, an improvement-, a quark mass-, an anomaly- and a dilaton potential term. The kinetic terms read

$$\mathcal{L}^{(\pi)}_{\rm kin} = \frac{F_\pi^2}{4}\hat{\chi}^{d-2}{\rm Tr}[\partial^\mu U \partial_\mu U^\dagger], \quad \mathcal{L}^{(D)}_{\rm kin} = \frac{1}{2}\hat{\chi}^{d-4}(\partial\chi)^2, \tag{2.16}$$

where hatted quantities are, hereafter, understood to be dimensionless divided by the appropriate power of F_D, e.g., $\hat{\chi} \equiv \chi/F_D$. The pion kinetic term is standard and the prefactor χ^2 indicates that the pion has zero Weyl weight which can be deduced directly from the conformal algebra [3]. In Ref. [9] the following coupling of the dilaton to the Ricci tensor was introduced

$$\mathcal{L}^R_d = \frac{1}{2(d-2)(d-1)}R\chi^{d-2} \to \frac{1}{12}R\chi^2. \tag{2.17}$$

It renders the dilaton kinetic term locally Weyl-invariant which has many advantages.[6] The quark mass term reads $[\mathcal{M} \equiv {\rm diag}(m_{q_1}, \ldots, m_{q_{N_f}})]$

$$\mathcal{L}_{m_q} = \frac{B_0 F_\pi^2}{2}{\rm Tr}[\mathcal{M}U + U^\dagger\mathcal{M}^\dagger]\hat{\chi}^{\Delta_{\bar{q}q}}, \qquad B_0 \equiv -\frac{\langle\bar{q}q\rangle}{F_\pi^2}, \tag{2.18}$$

where B_0 assures that the Gell-Mann–Oakes–Renner (GMOR)-relation $m_\pi^2 F_\pi^2 = -2m_q\langle\bar{q}q\rangle$ [73,75] is reproduced. The chiral symmetry is formally restored by assigning the spurious transformation rule $\mathcal{M} \to L^\dagger \mathcal{M} R$ which in turn dictates the form in (2.18). The quantities B_0 and F_π are the two (independent) low energy constants of LO χPT. The quark mass corresponds to explicit Weyl symmetry breaking but the latter can be restored by assigning the spurious transformation $m_q \to e^{(1+\gamma_m)\alpha}m_q$ and the extra factor $\hat{\chi}^{\Delta_{\bar{q}q}}$ is then required to render $\sqrt{-g}\mathcal{L}_{m_q}$ Weyl invariant. Alternatively, we may regard m_q as the source for $\bar{q}q$ with the factor $\hat{\chi}^{\Delta_{\bar{q}q}}$ capturing its scaling. The remaining two terms are more delicate and need more elaboration.

The anomalous term reads $T^\rho{}_\rho = \frac{\beta}{2g}G^2$ in the $m_q \to 0$ limit. It is not well understood whether a term needs to be added for the trace anomaly in the EFT in analogy with the WZW-term for the chiral anomaly in χPT (e.g., [71]). Clearly, the leading effect would be captured by β'_* and is sometimes parametrized in the EFT as β'_* times operators in the LO TEMT cf. [36,50] (and also [39]). Using RG methods we will establish $\beta'_* = 0$ in Sec. III which implies that $\delta g = g - g_*$ runs logarithmically instead of powerlike close the fixed point. We would therefore think that the EFT can capture these effects through its own loops and matching of NLO low energy constants with QCD. Hence, we will drop $\mathcal{L}_{\rm anom}$ from the LO Lagrangian.

[6]The term (2.17) is the adaption of the improved EMT [74] to the Goldstone case for which the presence of the dilaton is needed to improve the pion. It leads to $T^\rho{}_\rho|^{\rm d\chi PT}_{\rm LO} = 0$ for $m_q = 0$. It improves flow theorems [9].

We turn to the discussion of the potential. It has been recognized by Zumino a long time ago [76] that, at least in the absence of anomalous breaking, the only permissible term in the potential is $V_Z(\chi) = \lambda\chi^d$. Such a term on its own is troublesome, cf. Ref. [36] for detailed discussion, as it generates a linear as well as a mass term. Hence it was concluded that λ must be a function of a symmetry breaking parameter. A simple and frequently used form of a potential is given by adding another power χ^Δ which can be written as[7]

$$V_\Delta(\hat{\chi}) = \frac{m_D^2 F_D^2}{\Delta - d}\left(\frac{1}{\Delta}\hat{\chi}^\Delta - \frac{1}{d}\hat{\chi}^d\right) = \text{const} + m_D^2 F_D^2\left(\frac{1}{2}\hat{D}^2 - \frac{d+\Delta}{3!}\hat{D}^3 + \frac{(d+\Delta)^2}{4!}\hat{D}^4 + \mathcal{O}(D^5)\right), \tag{2.19}$$

with a well-defined minimum ($V'(1) = 0$ and $V''(1) = m_D^2$). The extra term is usually associated with the presence of an operator of scaling dimension Δ which breaks conformal invariance. In Sec. V we will see that in our framework the soft theorem indicate that $\Delta = d - 2$. Indeed the quark mass term in $\mathcal{L}_{m_q}|_{\pi=0}$ (2.18) corresponds to the $\Delta = \Delta_{\bar{q}q} = d - 2$ term for which the Zumino-term (2.20) assumes the form

$$V_Z(\hat{\chi}) = \frac{\Delta_{\bar{q}q}}{d}\sum_{q=1}^{N_f} m_q\langle qq\rangle\hat{\chi}^d, \tag{2.20}$$

where $\lambda \propto m_q$ is a special case of a symmetry-breaking parameter discussed above. It assures positivity of the dilaton mass

$$F_D^2 m_D^2 = (d - \Delta_{\bar{q}q})\Delta_{\bar{q}q}\frac{N_f}{2}F_\pi^2 m_\pi^2|_{d=4,\Delta_{\bar{q}q}=2} = 2N_f F_\pi^2 m_\pi^2, \tag{2.21}$$

as it contributes the $d\Delta_{\bar{q}q}$-term. In analogy to the pion case we will refer to (2.21) as the *dilaton GMOR-relation*. It will be derived from a double-soft theorem in Sec. V. To this end let us summarize the LO dχPT Lagrangian considered for $d = 4$ ($\Delta_{\bar{q}q} = 2$)

$$\mathcal{L}_{\text{LO}}^{\text{d}\chi\text{PT}} = \frac{F_\pi^2}{4}\hat{\chi}^2\text{Tr}[\partial^\mu U\partial_\mu U^\dagger] + \frac{1}{2}(\partial\chi)^2 + \frac{1}{12}\chi^2 R + \frac{B_0 F_\pi^2}{2}(\text{Tr}[\mathcal{M}U^\dagger + U\mathcal{M}^\dagger]\hat{\chi}^2 - \text{Tr}[\mathcal{M} + \mathcal{M}^\dagger]\hat{\chi}^4), \tag{2.22}$$

[7] In the literature this potential is often discussed without specific reference to the $\bar{q}q$-operator. For $\Delta \to 4$ it assumes the famous logarithmic form used as an ansatz for an EFT of pure Yang-Mills [77].

where the anomalous part has been dropped as per above. We refrain from parametrizing further potential terms at LO, as they are not required for our work and neither is it clear what form they would take. If we want to derive the insertion of the $\bar{q}q$-operator then it is important to realize that its source $s(x)$ is only to be substituted in the term containing U where it acts as a true spurion, $\mathcal{M}U^\dagger \to (\mathcal{M} + s)U^\dagger$, and not in the Zumino-term. We further note that similar Lagrangians have frequently appeared in the literature [30,32,34,50,78]. The novelty in our case is the justification for the absence of the anomaly term, the $\Delta_{\bar{q}q} = 3 - \gamma_* = 2$ requirement and with respect to many references the inclusion of the $R\chi^2$-term.

As we sometimes consider the χPT Lagrangian on its own, we quote its LO version

$$\mathcal{L}_{\text{LO}}^{\chi\text{PT}} = \frac{F_\pi^2}{4}\text{Tr}[\partial^\mu U\partial_\mu U^\dagger] + \frac{B_0 F_\pi^2}{2}\text{Tr}[\mathcal{M}U^\dagger + U\mathcal{M}^\dagger], \tag{2.23}$$

which follows from (2.22) by setting $\chi \to 1$ and dropping constant terms.

III. SCALAR CORRELATORS WITH GOLDSTONES IN THE DEEP-IR ($m_q = 0$)

In CFTs correlation functions are determined by a minimal amount of information. For example, two-point functions are characterized by a single parameter

$$\langle\mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle_{\text{CFT}} \propto \frac{1}{(x^2)^{\Delta_\mathcal{O}}}, \tag{3.1}$$

the scaling dimension $\Delta_\mathcal{O}$, e.g., (2.7) ($\langle\ldots\rangle$ stands for the vacuum expectation value (VEV) hereafter). For theories which flow into an IR fixed point (i.e., conformal window cf. Fig. 1), Eq. (3.1) represents the leading behavior in the deep-IR, $x^2 \to \infty$. The aim of this section is to investigate how this picture is affected in the presence of Goldstone bosons (due to scale and chiral symmetry breaking). The basic reasoning is that since the EFT and QCD describe the same IR-physics, the following must hold

$$\langle\mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle_{\text{CDQCD}} = \langle\mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle_{\text{d}\chi\text{PT}}, \quad \text{for } x^2 \to \infty, \tag{3.2}$$

for the correlation functions as they represent physical observables. We will first analyze the two-point functions from the RG viewpoint combining it with some knowledge from the EFT described in Sec. II C. This will provide a more profound understanding of the analysis in our previous work [7] and the reason why the scalar adjoint correlation function was singled out.

Let us introduce operators $\mathcal{O}$, split into ones

$$S = \bar{q}q, \qquad P^a = \bar{q}i\gamma_5 T^a q, \tag{3.3}$$

which couple to single Goldstones and those

$$S^a = \bar{q}T^a q, \qquad T^{\rho}{}_{\rho} = \frac{\beta}{2g}G^2, \tag{3.4}$$

that do not. Their quantum numbers and scaling dimensions are

$$J^{PC}(S) = 0^{++}, \qquad J^P(P^a) = 0^-,$$
$$J^P(S^a) = 0^+, \qquad J^{PC}(T^{\rho}{}_{\rho}) = 0^{++}, \tag{3.5}$$

and

$$\Delta_{P^a} = \Delta_S = \Delta_{S^a} = 3 - \gamma_*, \qquad \Delta_{T^{\rho}{}_{\rho}} = 4 + \beta'_*, \tag{3.6}$$

respectively. The reason of why (3.6) holds true for the currents can be found in [7], and for $T^{\rho}{}_{\rho}$ it is given in Appendix A. It is noted that the scaling dimension of G^2 and $T^{\rho}{}_{\rho}$ are the same, $\Delta_{T^{\rho}{}_{\rho}} = \Delta_{G^2}$, since multiplying by the β function which is a scalar does not alter the long-distance behavior of the operator. Two-point functions will sometimes be abbreviated as

$$\Gamma_{\mathcal{O}}(x^2) \equiv \langle \mathcal{O}(x)\mathcal{O}(0)\rangle, \tag{3.7}$$

and refer to Euclidean space unless otherwise stated.

A. Renormalization group analysis of correlators

Since Goldstones are massless it is to be expected that they will affect the IR-behavior. From the formal RG viewpoint the main change is the presence of an additional scale [10]. There are in fact two, the pion and the dilaton decay constant but as they are of the same order we may group them into one single quantity $F = F_{D,\pi}$. Under these circumstances the RG equation for the correlators of the type (3.7) assume the following form, e.g., [79],

$$(2\partial_{\ln x^2} - y_{F^2}\partial_{\ln F^2} + \Delta_{\mathcal{O}})\Gamma_{\mathcal{O}}(x^2, F^2) = 0, \tag{3.8}$$

where we have neglected the $\beta\partial_{\ln g}$-term since its effect is subleading as will become clear later on. Note that $y_{F^2} = d_{F^2} + \gamma_{F^2} = 2$ since F has vanishing anomalous dimension. The solution of this equation reads

$$\Gamma_{\mathcal{O}}(x^2, F^2) \propto \frac{1}{(x^2)^{\Delta_{\mathcal{O}}}} h_{\mathcal{O}}(x^2F^2), \tag{3.9}$$

where in general $h_{\mathcal{O}}$ is arbitrary such that predictiveness is essentially lost. However, we can improve this situation by matching to (d)χPT, as in (3.2), taking advantage of the explicit Lagrangian (2.22). We make the following observations:

(a) We will argue that the EFT cannot produce any noninteger powers of $1/x^2$. Whereas the EFT expansion is in powers of $1/(x^2F^2)$ it could be that $\ln x^2$-corrections, related to the neglect of the β function, resum to noninteger powers (n integer and η not)

$$\frac{1}{(x^2)^{(n+\eta)}} = \frac{1}{x^{2n}} e^{-\eta \ln x^2}$$
$$= \frac{1}{x^{2n}}\left(1 - \eta \ln x^2 + \frac{\eta^2}{2}\ln x^2 + \ldots\right). \tag{3.10}$$

The answer is however negative since η itself must be proportional to inverse powers of F, but then there is no other scale in the chiral limit to make η dimensionless. Hence we conclude that $\ln x^2$-terms can only appear in subleading terms which is fairly natural from an EFT perspective (cf. Sec. III B 6 for a more explicit discussion of this aspect). Thus we may write

$$h_{\mathcal{O}}(z) = a_{k_{\mathcal{O}}} z^{k_{\mathcal{O}}} + \cdots + a_0 + a_{-1}\frac{1}{z}$$
$$+ \mathcal{O}(z^{-2}) + \ln\text{-terms}, \tag{3.11}$$

where $k_{\mathcal{O}}$ is some positive integer and subleading terms contain $\ln x^2$-corrections which we have not indicated explicitly.

(b) Having learned that the overall powers are integers, there is still an ambiguity left and that is the interpretation of the $\Delta_{\mathcal{O}}$ coefficient (or the actual number $k_{\mathcal{O}}$). If the operator $\mathcal{O}$ shares the quantum numbers with the Goldstone boson π and $\langle \pi|\mathcal{O}|0\rangle \neq 0$, then $\Gamma_{\mathcal{O}}(x^2)$ scales as

$$\Gamma_{\mathcal{O}}(x^2) \propto \frac{1}{x^2} + \mathcal{O}(x^{-4}), \tag{3.12}$$

as a consequence of the spectral representation. Are we to interpret $\Delta_{\mathcal{O}}$ with this contribution, that is $\Delta_{\mathcal{O}} = 1$? The analysis in terms of the spectral function, further below, suggests otherwise since these contributions are discontinuous with respect to the conformal window phase. The RG analysis in Sec. III B 7 provides a further tool. Since the single [two] Goldstone cases are of $\mathcal{O}(F^2)$ [$\mathcal{O}(B_0)$] with $\Delta_{F^2} = 2$ and $\Delta_{B_0} = 0$ it is the latter which agrees with the pure CFT scaling. And thus one is to discard the single Goldstone case. Another way to look at it is to the pure CFT case for which the operator product expansion (OPE) provide a strong tool. These corrections can then be seen as emerging due to VEVs which is a more formal way to see the discontinuity with the conformal window.

(c) There is a second exception. It could be that in the leading term $\Gamma_{\mathcal{O}}(x^2) = c/(x^2)^{\Delta_{\mathcal{O}}}$, $c = 0$ due to some symmetry. The identification of $\Delta_{\mathcal{O}}$ would then proceed through a next-leading correction which complicates matters as is the case for the $T^{\rho}{}_{\rho}$-correlator for dχPT.

Finally, we conclude that it is the leading multiparticle state contribution which is to be identified with $\Delta_{\mathcal{O}}$ (provided the exceptional case under the last item does not occur).

B. Spectral analysis and matching to (dilaton-)χPT

Here we aim to underlay the previous discussion from the viewpoint of the spectral representation which provides a more intuitive insight. Below we give a brief summary (e.g., [80,81] for further reading). In Minkowski space the spectral density $\rho(s)$[8]

$$\bar{\Gamma}_{\mathcal{O}}(p^2) \equiv i \int d^4x e^{ipx} \langle T\mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle = \int_0^\infty ds \frac{\rho_{\mathcal{O}}(s)}{s - p^2 - i0}, \tag{3.13}$$

of the two-point function is proportional to the imaginary part

$$\rho_{\mathcal{O}}(s) = \frac{1}{\pi} \mathrm{Im} \bar{\Gamma}_{\mathcal{O}}(s). \tag{3.14}$$

The same spectral function enters the Euclidean correlation function

$$\int d^4x e^{ipx} \langle \mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle = \int_0^\infty ds \frac{\rho_{\mathcal{O}}(s)}{s + p^2}. \tag{3.15}$$

It is instructive to first consider an example of a QCD and a CFT spectral function since the CFT-Goldstone case bears aspects of both of them.

1. Spectral function in QCD with a heavy b-quark

In QCD a typical spectral function consists of a few (stable) bound states, characterized by δ-functions, and a continuum, beginning at some branch point s_0,

$$\rho_{\mathcal{O}}^{\mathrm{QCD}}(s) = \sum_n |f_{\mathcal{O}}^{(n)}|^2 \delta(s - m_n^2) + \theta(s - s_0)\sigma_{\mathcal{O}}(s), \tag{3.16}$$

including resonances as well as other multiparticle states. The δ-function prefactor is $f_{\mathcal{O}}^{(n)} \equiv \langle 0|\mathcal{O}|n\rangle$, sometimes referred to as the decay constants since such a quantity governs the pion decay. We consider pure QCD, QED and weak interactions turned off, with a heavy b-quark flavor $\mathcal{O}(x) \to \bar{b} i\gamma_5 q(x)$ for which: $m_1^2 \to m_B^2$ with no further stable states and the continuum threshold is given by $s_0 = (m_B + 2m_\pi)^2$. The x-dependence due to $\sigma_{\mathcal{O}}$ cannot be evaluated without knowing the function but the δ-term part is simply given by the Fourier transform of the propagator

$$\rho_{\mathcal{O}}(s) \propto \delta(s - m^2)$$
$$\Leftrightarrow \quad \langle \mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle \propto \int \frac{d^4x e^{ipx}}{p^2 + m^2}\bigg|_{m^2=0} \propto \frac{1}{x^2}, \tag{3.17}$$

which exhibits the $1/x^2$-scaling stated earlier.

2. Spectral function in a CFT without spontaneous symmetry breaking

It is straightforward to deduce that the following identification holds

$$\rho_{\mathcal{O}}^{\mathrm{CFT}}(s) \propto s^{\Delta_{\mathcal{O}} - \frac{d}{2}} \quad \Leftrightarrow \quad \langle \mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle \propto \frac{1}{(x^2)^{\Delta_{\mathcal{O}}}}, \tag{3.18}$$

either by direct computation [82] or on grounds of dimensional analysis. This function is to be interpreted as belonging to the multiparticle threshold $\sigma_{\mathcal{O}}$. It is noted that the limit $\Delta_{\mathcal{O}} \to 1$ for $d = 4$ is pathological (IR-divergent spectral integral) since the operator really describes a free field rather than a multiparticle spectrum.

3. Spectral function in spontaneous scale symmetry breaking

The case of spontaneous symmetry breaking has elements of both the QCD- and the CFT-case. The stable particle becomes the massless Goldstones and the continuum threshold moves to zero assuming a simple power law. The δ-functions are very singular and have no counterpart in the unbroken CFT-case and it is intuitively clear that they are not to be identified with $\Delta_{\mathcal{O}}$. In Sec. III A this has been argued more formally, that $\Delta_{\mathcal{O}} = 1$ does strictly imply a free field and is therefore not associated with s raised to some power. In conclusion the spectral function, in the case of SSB, generically reads

$$\rho_{\mathcal{O}}^{\mathrm{SSB}} \propto s^{\Delta_{\mathcal{O}} - \frac{d}{2}}(F^2\delta(s) + c\theta(s)) + \delta\rho(s), \tag{3.19}$$

where c is a constant and the δ-term is only present if the Goldstone couples to $\mathcal{O}$. The quantity $\delta\rho(s)$ parametrizes suppressed contributions such as multinucleon thresholds.

4. Multi-Goldstone case: The operator S^a

The case of the operator S^a (3.4) is the simplest as its quantum numbers do not allow for the propagation of a single Goldstone. It was chosen for this reason in our earlier

[8] The representation (3.13) will in general need subtraction terms which are not important since polynomial $(p^2)^n$-terms corresponds to local $\delta^{(n)}(x)$-terms which are irrelevant for the deep-IR.

work [7] which we recapitulate here in the language of the spectral function. Dropping the δ-function piece in (3.19) one gets the standard CFT-scaling

$$\rho_{S^a}(s) \propto s^{\Delta_{S^a}-\frac{d}{2}} \;\Leftrightarrow\; \langle S^a(x)S^a(0)\rangle_{\mathrm{CDQCD}} \propto \frac{1}{(x^2)^{\Delta_{S^a}}}. \tag{3.20}$$

In dχPT these operators are matched at leading order to

$$\begin{aligned} S^a|_{\mathrm{LO}} &= -\frac{F_\pi^2 B_0}{2}\mathrm{Tr}[T^a U^\dagger + UT^a]\hat{\chi}^2 \\ &= \frac{B_0}{2} d^{abc}\pi^b\pi^c + \mathcal{O}(1/F), \end{aligned} \tag{3.21}$$

to two pions using the method of sources [7]. Its evaluation involves the propagation of two free pions $\langle \pi^a(x)\pi^b(0)\rangle = \frac{\delta^{ab}}{4\pi^2 x^2}$ only, as illustrated in Fig. 2, and one gets

$$\langle S^a(x)S^a(0)\rangle_{\chi\mathrm{PT}} \propto \frac{1}{x^4}. \tag{3.22}$$

Matching to (3.2), $\langle S^a(x)S^a(0)\rangle_{\mathrm{CDQCD}} \propto \frac{1}{(x^2)^{\Delta_{S^a}}}$ one finds

$$\Delta_{S^a} = 3 - \gamma_* = 2 \quad \Leftrightarrow \quad \gamma_* = 1, \tag{3.23}$$

that the mass anomalous dimension at the IR fixed point is unity [7].

5. *Single-Goldstone case: The operators P^a and S*

The cases of the operators P^a and S (3.3) differ in that they couple to a single Goldstone, the pion and the dilaton respectively, and thus both terms in (3.19) are present. Omitting the truly subleading terms we have

$$\rho_{P^a}(s) \propto s^{\Delta_{P^a}-\frac{d}{2}}(F_\pi^2\delta(s) + c), \tag{3.24}$$

implying

$$\langle P^a(x)P^a(0)\rangle_{\mathrm{CDQCD}} \propto \frac{F_\pi^2}{(x^2)^{\Delta_{P^a}-1}} + \frac{c'}{(x^2)^{\Delta_{P^a}}}, \tag{3.25}$$

where c and c' are constants. The case of S is analogous with $F_\pi \to F_D$, cf. Fig. 2. In dχPT the operators are given by

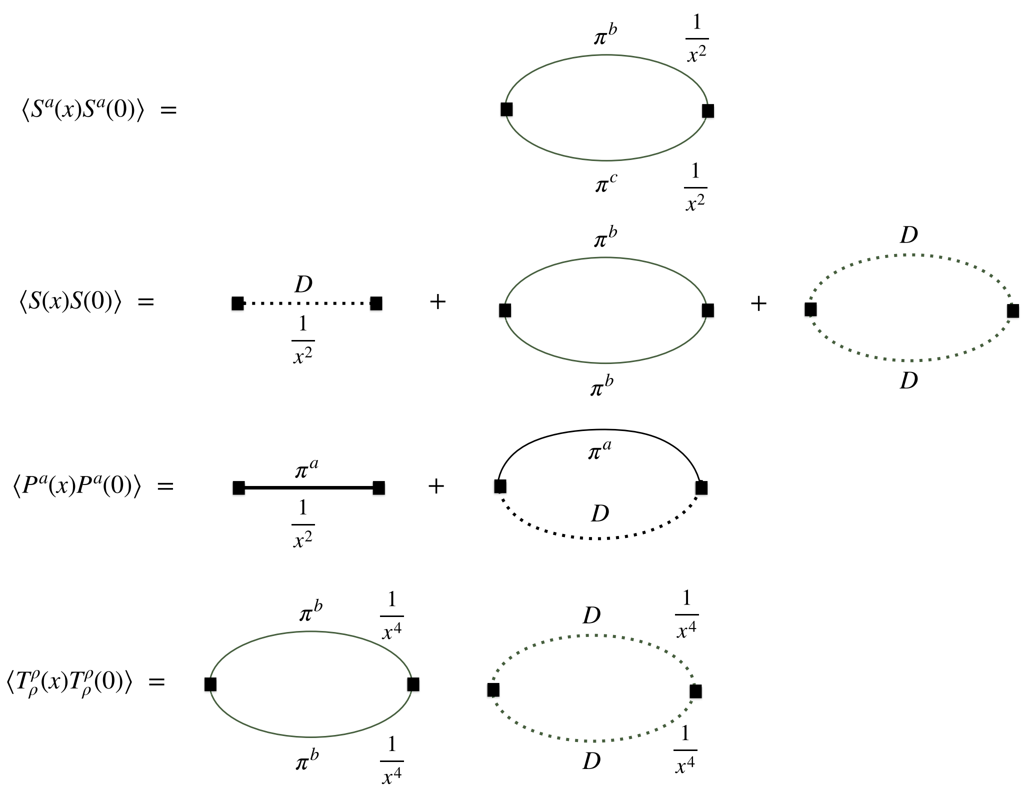

FIG. 2. Two-point functions of operators defined in (3.3), (3.4) evaluated in the deep-IR. Top: scalar adjoint correlator with no coupling to Goldstones scaling as $1/x^4$. Center: scalar and adjoint pseudoscalar correlators with a single and a two dilaton intermediate state. Bottom: $T^\rho{}_\rho$-correlator in χPT (the one for dχPT is zero at LO).

$$
\begin{aligned}
S &= -\frac{F_\pi^2 B_0}{2}\mathrm{Tr}[U^\dagger + U]\hat{\chi}^2 \\
&= 2B_0 N_f F_\pi^2 \hat{D} + \frac{B_0}{4}(\pi^a\pi^a - N_f F_\pi^2 \hat{D}^2) + \mathcal{O}(1/F), \\
P^a &= -\frac{F_\pi^2 B_0}{2} i\mathrm{Tr}[T^a U^\dagger - U T^a]\hat{\chi}^2 \\
&= -B_0 F_\pi \pi^a (1 - 2\hat{D}) + \mathcal{O}(1/F).
\end{aligned} \tag{3.26}
$$

From these expressions the following LO correlation functions result

$$
\begin{aligned}
\langle S(x)S(0)\rangle_{\mathrm{d}\chi\mathrm{PT}} &= \frac{c_S^{(2)}}{x^2} + \frac{c_S^{(4)}}{x^4}, \\
\langle P^a(x)P^a(0)\rangle_{\mathrm{d}\chi\mathrm{PT}} &= \frac{c_{P^a}^{(2)}}{x^2} + \frac{c_{P^a}^{(4)}}{x^4}.
\end{aligned} \tag{3.27}
$$

with

$$
\begin{aligned}
c_S^{(2)} &= \frac{B_0^2 F_\pi^2 \hat{F}_\pi^2 N_f^2}{\pi^2}, \quad c_S^{(4)} = \frac{B_0^2((N_f^2-1) + N_f^2 \hat{F}_\pi^4)}{32\pi^4}, \\
c_{P^a}^{(2)} &= \frac{B_0^2 F_\pi^2}{4\pi^2}, \quad c_{P^a}^{(4)} = \frac{B_0^2 \hat{F}_\pi^2}{4\pi^4}.
\end{aligned}
$$

Here and thereafter the flavor index is understood to be held fixed. Matching the two expressions, as in (3.2) but disregarding the $1/x^2$-contribution as argued above, one deduces

$$
\Delta_S = \Delta_{P^a} = 3 - \gamma_* = 2 \quad \Leftrightarrow \quad \gamma_* = 1, \tag{3.28}
$$

the same result as in the previous section. This should not be taken for granted but as a sign of the consistency of the approach.

6. Multi-Goldstone case II: The trace of the energy momentum tensor $T^\rho{}_\rho$

The case of the TEMT is interesting as it brings several new elements. It is tempting to think that the dilaton would couple to $T^\rho{}_\rho = \frac{\beta}{2g}G^2$ but it does in a way (2.11)

$$
\langle D|T^\rho{}_\rho|0\rangle = F_D m_D^2, \tag{3.29}
$$

which vanishes when the dilaton is massless and if it is massive it would decouple from the IR physics. The next candidates in question are two pions and two dilatons. It is instructive to first consider the gluon correlator for which $\Delta_{G^2} = 4 + \beta'_*$ (2.7) is determined by the slope of the β function at the fixed point

$$
\rho_{G^2}(s) \propto s^{\Delta_{G^2} - \frac{d}{2}} \quad \Leftrightarrow \quad \langle G^2(x)G^2(0)\rangle_{\mathrm{CDQCD}} \propto \frac{1}{(x^2)^{4+\beta'_*}}. \tag{3.30}
$$

Adapting it to $T^\rho{}_\rho$ one multiplies two powers of the β function which as argued previously, leaves the x-scaling unaltered

$$
\langle T^\rho{}_\rho(x) T^\rho{}_\rho(0)\rangle_{\mathrm{CDQCD}} \propto \frac{\beta^2}{(x^2)^{4+\beta'_*}}. \tag{3.31}
$$

In the EFT it makes a difference whether one uses χPT (2.23) or dχPT in (2.22). From the LO EFTs one obtains [9], neglecting terms of higher powers in π,

$$
T^\rho{}_\rho|^{\mathrm{LO}}_{\chi\mathrm{PT}} = -\frac{1}{2}\partial^2\pi^a\pi^a, \qquad T^\rho{}_\rho|^{\mathrm{LO}}_{\mathrm{d}\chi\mathrm{PT}} = 0. \tag{3.32}
$$

Thus at LO χPT is only scale invariant but not conformal contrary to dχPT. One gets

$$
\langle T^\rho{}_\rho(x) T^\rho{}_\rho(0)\rangle^{\mathrm{LO}}_{\chi\mathrm{PT}} \propto \frac{1}{x^8}, \qquad \langle T^\rho{}_\rho(x) T^\rho{}_\rho(0)\rangle^{\mathrm{LO}}_{\mathrm{d}\chi\mathrm{PT}} \propto 0. \tag{3.33}
$$

In χPT the matching with (3.31) implies, in straightforward manner, that the slope of the β function vanishes at the IR fixed point: $\beta'_* = 0$. The case of dχPT is exceptional and corresponds to the case discussed under item (c) in Sec. III A in that the LO-term vanishes by conformal symmetry. In fact one can show that $\langle G^2(x)G^2(0)\rangle$ has no $1/x^8$-term as otherwise the β-function ought to vanish entirely. Thus one cannot argue $\beta'_* = 0$ by expanding in δg. We proceed in a different manner using RG arguments.

7. Renormalization group derivation of $\beta'_* = 0$

We first consider the GMOR-relation and the TEMT to deduce $\langle\pi|G^2|\pi\rangle = \mathcal{O}(m_q)$ and by equating to the RG version of the matrix element we deduce $\beta'_* = 0$.

We may decompose the TEMT as $T^\rho{}_\rho = \mathcal{O}'_1 + 2\mathcal{O}_2$ with $\mathcal{O}'_1 \equiv \mathcal{O}_1 - \mathcal{O}_2$ (2.4). Note that $\mathcal{O}'_1$ inherits the RG invariance of $\mathcal{O}_{1,2}$. Since $2\mathcal{O}_2$ saturates the GMOR-relation $\langle\pi|T^\rho{}_\rho|\pi\rangle = \langle\pi|2\mathcal{O}_2|\pi\rangle + \mathcal{O}(m_q^2)$, one gets that the $\mathcal{O}'_1$ matrix element must vanish to linear order in m_q. Expanding in δg, assuming $\gamma_* = 1$ and $\beta_* = 0$, we get

$$
\begin{aligned}
0 &= \langle\pi|\mathcal{O}'_1|\pi\rangle|_{m_q} \\
&= \delta g\left(\frac{\beta'_*}{2g_*}\langle\pi|G^2|\pi\rangle + \sum_q \gamma'_* m_q \langle\pi|\bar{q}q|\pi\rangle\right)\Big|_{m_q} + \mathcal{O}((\delta g)^2).
\end{aligned} \tag{3.34}
$$

Without making any assumptions on β'_* and γ'_* we deduce that $\langle\pi|G^2|\pi\rangle = \mathcal{O}(m_q)$ since we know that $\langle\pi|\bar{q}q|\pi\rangle = \mathcal{O}(m_q^0)$ from the GMOR-relation. From general RG arguments, similar to hyperscaling, e.g., [79,83–85] and also [10], one infers the m_q and the F dependence

$$\langle\pi|G^2|\pi\rangle = m_q^{\frac{\Delta_{G^2}+2d_\pi}{y_m}} h(F m_q^{-1/y_m}, \mu), \tag{3.35}$$

where μ is the RG scale, $y_m \equiv 1+\gamma_*$, $d_\pi = -1$ is the dimension of $|\pi\rangle$ [84] and h an *a priori* unknown function. In principle there is also the parameter $B_0 = -\langle\bar{q}q\rangle/F_\pi^2$ but it can be omitted since its scaling dimension vanishes $\Delta_{B_0} = \Delta_{\bar{q}q} - \Delta_{F_\pi^2} = 0$. Thus the extra powers of F^2 is what distinguishes the scaling in the QCD IR-phase from the conformal window phase.

Hence, our task is to find this power in F. We can do so by using that $\langle\pi|G^2|\pi\rangle$ and $\langle\pi|\bar{q}q|\pi\rangle$ are of the same order in F_π. From the soft-pion theorem one deduces $\langle\pi|\bar{q}q|\pi\rangle \propto \frac{1}{F_\pi^2}\langle\bar{q}q\rangle = -B_0$. Thus we conclude that $\langle\pi|G^2|\pi\rangle \propto F^0$ and we obtain

$$\langle\pi|G^2|\pi\rangle \propto m_q^{\frac{2+\beta'_*}{y_m}}, \tag{3.36}$$

and equating to $\langle\pi|G^2|\pi\rangle = \mathcal{O}(m_q)$ deduced above, one infers consistency (using $y_m = 2$) if and only if the slope vanishes

$$\beta'_* = 0. \tag{3.37}$$

Reassuringly, this is the same result obtained previously for χPT.

To gain further confidence consider $\langle\bar{q}q\rangle = -B_0F_\pi^2$ in terms of an RG analysis as in (3.35). Interpreting $\langle\bar{q}q\rangle \propto F^2$, thereby ignoring B_0, we may write

$$\langle\bar{q}q\rangle = m_q^{\frac{\Delta_{\bar{q}q}}{y_m}} H(F m_q^{-1/y_m}, \mu) \propto F^2 m_q^{\frac{\Delta_{\bar{q}q}-2}{y_m}} \propto \mathcal{O}(1), \tag{3.38}$$

which yields the correct result: no quark mass dependence at LO with $\Delta_{\bar{q}q} = 2$.

C. Consequences of $\beta'_* = 0$

In our view $\beta'_* = 0$ is an important result.

(a) It is supported by smooth matching to $\mathcal{N} = 1$ supersymmetry (cf. Sec. IV).

(b) It is the promised justification for dropping the anomaly term in the LO Lagrangian (2.15). For example, in Ref. [36] the LO anomaly terms become degenerate with the kinetic terms in the limit $\beta'_* \to 0$, and may therefore be dropped. As χPT is an EFT the EMT needs renormalization [86]. In that case three new counterterms, parametrized as a function of the Riemann tensor, have been found at NLO. We expect the same program to apply in dχPT such that the breaking of scale invariance emerges naturally at higher orders.[9] We would though expect that the terms from the Weyl or conformal anomaly [87], which are the analog of the Wess-Zumino-Witten term in QCD, would need to be matched at NLO.

(c) The slow logarithmic running (discussed below) may be interpreted in terms the mass gap among the hadrons. Once the ρ, a_0-meson, depending on the channel, decouple the theory asymptotes slowly into the free Goldstone EFT (accompanied by $1/(x^2F^2)$ power correction at NLO in accordance with the earlier discussion).

(d) It seems to make the existence of a QCD Seiberg dual in the nonsupersymmetric conformal window [14,88,89] more likely since the turnaround of the β'_* in the conformal window would go well with a weakly coupled dual IR fixed point.

Moreover, as emphasized in [90,91], $\beta'_* = 0$ also impacts on the nuclear axial current which has been the subject of scrutiny for many decades. In the light of $\beta'_* = 0$ it might be of interest to work out the strange quark mass corrections.

1. Logarithmic running

We may expand the β function around the fixed point coupling $\delta g \equiv g - g_*$

$$\beta = \beta'_*\delta g + \frac{1}{2}\beta''_*(\delta g)^2 + \mathcal{O}((\delta g)^3). \tag{3.39}$$

By the very assumption of an IR fixed point we have $\beta'_* \geq 0$ and the value in (3.37) saturates this constraint. Using $\beta'_* = 0$ gives an RG equation

$$\frac{d}{d\ln\mu}\delta g = \frac{1}{2}\beta''_*\delta g^2, \tag{3.40}$$

which solves to a logarithmic

$$\delta g(\mu) = \frac{4}{|\beta''_*| \ln\frac{\mu^2}{\lambda_{\rm QCD}^2}}, \tag{3.41}$$

instead of to a power-like form $\delta g \propto \mu^{\beta'_*}$. Above, the boundary condition $\delta g(0) = 0$ was imposed and the expression is maximally valid for $\mu < \lambda_{\rm QCD}$ where it runs into a Landau pole from below. The scale $\lambda_{\rm QCD}$ is the analog of $\Lambda_{\rm QCD}$ in the IR. Note that whereas β'_* is scheme independent under regular coupling redefinitions this is not the case for β''_*. For the sake of concreteness we may assume it to be nonvanishing, in which case it needs to be negative to assure negativity of the β function (3.39). In retrospect the logarithmic running is reassuring as it is difficult to see how (d)χPT could reproduce power-like scaling as per item (c) above.

[9] In dχPT the divergent parts of the counterterms have been computed very recently [33] but not the ones for the EMT.

2. The vanishing of β'_* implies the vanishing of γ'_*

In $\mathcal{N}=1$ supersymmetric gauge theories β'_* is proportional to $\gamma'_*=0$ due to the Novikov-Shifman-Vainshtein-Zakhariv (NSVZ) β function [92]. Hence, the vanishing of β'_* implies the vanishing of γ'_* and it is natural to ask whether this extends to the nonsupersymmetric case. Using Eq. (3.34) one infers that zero β'_* implies zero γ'_* by RG invariance:

$$\beta'_*=0 \quad \Leftrightarrow \quad \gamma'_*=0. \tag{3.42}$$

We note that this result might be more universal since they both govern the correction to hyperscaling, e.g., [85].

IV. COMPARISON WITH $\mathcal{N}=1$ SUPERSYMMETRIC GAUGE THEORIES

It is instructive to reflect on the results obtained in the context of $\mathcal{N}=1$ supersymmetric gauge theories and its Seiberg dualities, for which many excellent reviews exist [93–96]. Our brief summary below, extends on the nonsupersymmetric conformal window in Sec. II B. For $SU(N_c)$ gauge theories with N_f fermions, which is referred to as the electric theory, the conformal window extends over the following region

$$\text{conformal window:} \qquad \frac{3}{2}N_c \le N_f \le 3N_c. \tag{4.1}$$

In this window there exists a Seiberg dual, referred to as the magnetic theory, which is also asymptotically free with gauge group $SU(N_f-N_c)$ and a Yukawa-superpotential interaction $\mathcal{W}=\lambda M^i_{\bar{j}} q_i \tilde{q}^{\bar{j}}$ which couples N_f chiral q_i and N_f antichiral $\tilde{q}^{\bar{j}}$ matter fields to a color neutral chiral meson superfield $M^i_{\bar{j}}$ (where i and $\bar{j}$ are $SU(N_f)_L$ and $SU(N_f)_R$ flavor indices respectively). The duality [97], in part based on nontrivial matching of anomalies, is the statement that both of these theories flow to the same CFT in the IR. The relation of the dilatation and nonanomalous R-current in the same superconformal multiplet allows to determine the scaling dimension of the bilinear (s)quark fields in terms of the known R-charges[10]

$$\Delta_{\tilde{Q}Q}=2-\gamma_*=3-\frac{3N_c}{N_f}, \quad \Delta_{\bar{q}q}=3-\gamma_*=4-\frac{3N_c}{N_f}, \tag{4.2}$$

valid in the conformal window. Alternatively, this formula can be deduced by requiring the NSVZ β function [98,99] to vanish in the domain (4.1).

Below the conformal window squark bilinears can take on VEVs and break the superconformal R-symmetry and the relations (4.2) cease to be valid. In the range

$$\text{IR-free magnetic:} \qquad N_c+1<N_f<\frac{3}{2}N_c, \tag{4.3}$$

the magnetic theory plays the role of the weakly coupled IR EFT of the strongly coupled electric theory. The magnetic theory is not asymptotically free anymore and the electric theory can be seen as its UV completion. We may think of this magnetic description as a meson EFT of the hybrid type, as it still contains colored degrees of freedom, and as such resembles the phenomenologically successful chiral quark model [100]. There are three further phases below: s-confinement for $N_f=N_c+1$ (confinement without chiral symmetry breaking), $N_f=N_c$ anomalies are matched by mesons and baryons only, $N_c>N_f>1$ has a runaway potential (no stable vacuum). We will not focus on these phases as they are quite possibly peculiar to supersymmetry itself. Further note, that for $N_f=N_c+1$ the dual gauge group in particular ceases to make sense but interestingly the anomalies can still be matched. We discuss $\gamma_*=1$, $\beta'_*=\gamma'_*=0$ under items (a) to (c) below from the supersymmetric-viewpoint reasoning whether they extend into the IR-free magnetic phase (4.3).

(a) The squark bilinear $\tilde{Q}Q$ is believed to match onto the free meson field $M^j_{\bar{i}}$ for which $\langle M(x)M(0)\rangle \propto 1/x^2$ for $x^2\to\infty$ must hold in the IR-free magnetic range (4.1). This implies $\Delta_M=\Delta_{\tilde{Q}Q}=2-\gamma_*=1$ and thus $\gamma_*=1$ must hold throughout the IR-free magnetic phase.[11]

(b) In the conformal window β'_* is nonzero. This can be established close to the electric Caswell-Banks-Zaks fixed point as its very idea is based on β'_* being small. As the coupling increases so does β'_*, e.g., [101]. Since β'_* is also the scaling exponent of the TEMT-correlator (3.31) [92] this implies the equality of the electric and magnetic slopes in the conformal window[12]

$$\beta'_*|_{\rm el}=\beta'_*|_{\rm mag}$$
$$\Leftrightarrow \quad \langle T^\rho{}_\rho(x)T^\rho{}_\rho(0)\rangle_{\rm el} \overset{\rm IR}{\leftrightarrow} \langle T^\rho{}_\rho(x)T^\rho{}_\rho(0)\rangle_{\rm mag}. \tag{4.4}$$

The value of $\beta'_*|_{\rm mag}$ corresponds to the minimal eigenvalue of the gradient matrix of the β function matrix in the gauge-Yukawa coupling space. Since the

[10] In the supersymmetry literature γ_Q, the matter-field anomalous dimension, is used rather than γ_m. They are related by $\gamma_m=-\gamma_Q$ through nonrenormalization theorems to all order in perturbation theory.

[11] As a side remark, it is curious that in $\mathcal{N}=1$ the end of the conformal window coincides with the unitarity bound whereas for QCD gauge theories the unitarity bound $\gamma_*\le 2$ is not reached. The interpretation that suggests itself [7] is that chiral symmetry breaking sets in once the scaling dimension allow the operator S^a ($\Delta_{S^a}=2$) to produce two free pions as predicted by (d)χPT.

[12] This result was derived earlier by matching the Konishi currents via an involved Kutasov construction in [102].

magnetic theory is weakly coupled at the lower edge of the conformal window, N_f toward $\frac{3}{2}N_c$, where it assumes its own perturbative fixed point this means that

$$\beta'_*|_{N_f=3} = \beta'_*|_{N_f=\frac{3}{2}} = 0, \qquad \beta'_*|_{\frac{3}{2}N_c<N_f<N_c} > 0, \quad (4.5)$$

β'_* is zero at both edges of the conformal window and positive in between which follows from the very assumption of an IR fixed point. This is curious as both, very weak and very strong coupling, seems to makes β'_* vanish. However, since $\beta'_* = 0$ also implies the logarithmic running implicit in the EFT cf. Sec. III B 6, this can be seen as a necessary result and matches the expectation of the IR-free magnetic phase (4.3). This is also the reason it should continue to hold in the IR-free magnetic phase which is though not argument inherent to supersymmetry.

(c) The NSVZ β function [98,99] is expressed in terms of known quantities and the anomalous dimension γ_m. Hence, differentiating, and using $\beta_* = 0$, give a relation [92]

$$\beta'_* = \frac{\alpha_*^2}{2\pi}\frac{N_f}{1-\frac{\alpha_*}{2\pi}N_c}\gamma'_*, \quad (4.6)$$

which must hold at least in the conformal window. ($\alpha_* = g_*^2/4\pi$ denotes the electric IR fixed point gauge coupling). Therefore, $\beta'_* = 0$ at $N_f = \frac{3}{2}N_c$ implies the same for $\gamma'_* = 0$ at this point. In IR-free magnetic phase, $\gamma'_* = 0$ might continue to hold by the same reasoning as given in Sec. III C 2. This argument is not specific to supersymmetry.

In summary, we have argued that for $\mathcal{N} = 1$ supersymmetry $\gamma_* = 1$ and $\beta'_* = \gamma'_* = 0$ hold at the boundary of the conformal window. Using Seiberg duality we provided an argument of why $\gamma_* = 1$ and $\beta'_* = 0$ carry over into the IR-magnetic phase. For $\gamma'_* = 0$, its link to β'_* which is not supersymmetric in nature was invoked with regards to the IR-magnetic phase. It is worthwhile to emphasize that there have been interesting attempts to understand the magnetic dual in terms of hidden local symmetry and low energy Goldstone physics [103,104]. It may well be that the massless 0^{++} flavor-singlet found in these cases is the dilaton.

V. SOFT DILATON THEOREMS

In this section we apply the double-soft theorem to the matrix element

$$2m_D^2 = \langle D|T^\rho{}_\rho|D\rangle, \quad (5.1)$$

which will provide some surprising model-independent results and the dilaton GMOR-relation (2.21). Equation (5.1) is usually considered a first principle formula but in the presence of a massless dilaton it does not hold for a standard hadronic state φ. The dilaton pole cancels the $2m_\varphi^2$-contribution from the kinetic term such that $\langle\varphi|T^\rho{}_\rho|\varphi\rangle = 0$ [10]. However, since here $m_D \neq 0$ is kept, this situation does not arise (which we have explicitly checked at LO). All the information needed from the potential is the mass term $V \supset \frac{1}{2}m_D^2 D^2$.

The idea of soft theorems is that one has a pseudo-Goldstone with a light mass, $m_{\rm pGB} \ll \Lambda_{\rm QCD}$, whose momenta can be approximated to be soft $q \to 0$ while keeping $m_{\rm pGB} \neq 0$ [71] (which is automatic in an EFT treatment).[13] The procedure assumes that the original matrix element does not change significantly in the soft limit. The gain is that the evaluation then proceeds by a simpler matrix element where the Goldstone emerges in a computable commutator. For the dilaton and pion they read

$$\langle D(q)\beta|\mathcal{O}(0)|\alpha\rangle = -\frac{1}{F_D}\langle\beta|i[Q_D,\mathcal{O}(0)]|\alpha\rangle + \lim_{q\to 0} iq\cdot R,$$
$$\langle\pi^a(q)\beta|\mathcal{O}(0)|\alpha\rangle = -\frac{i}{F_\pi}\langle\beta|[Q_5^a,\mathcal{O}(0)]|\alpha\rangle + \lim_{q\to 0} iq\cdot R^a, \quad (5.2)$$

where the one for pions can be found in the textbook [71]. The R's are the so-called remainders

$$R_\mu = -\frac{i}{F_D}\int d^dx e^{iq\cdot x}\langle\beta|TJ_\mu^D(x)\mathcal{O}(0)|\alpha\rangle,$$
$$R_\mu^a = -\frac{i}{F_\pi}\int d^dx e^{iq\cdot x}\langle\beta|TJ_{5\mu}^a(x)\mathcal{O}(0)|\alpha\rangle, \quad (5.3)$$

which vanish unless there are intermediate states degenerate with either α or β [106].[14] Let us focus on an operator present in the TEMT $\mathcal{O} \subset T^\rho{}_\rho$ which is responsible for generating the mass. We will refer to this operator as the mass operator. Applying the soft theorem (5.1) one gets

$$2m_D^2 = \langle D|\mathcal{O}(x)|D\rangle = -\frac{1}{F_D}\langle 0|i[Q_D,\mathcal{O}(x)]|D\rangle$$
$$= -\frac{1}{F_D}(\Delta_{\mathcal{O}} + x\cdot\partial)\langle 0|\mathcal{O}(x)|D\rangle. \quad (5.4)$$

Above we used the CFT definition of applying Q_D to a primary operator which is an assumption similar to (1.1). Now comes the main technical point. It is crucial to keep the derivative term since the matrix element, $\langle 0|\mathcal{O}(x)|D(q)\rangle = F_{\mathcal{O}}e^{-iqx}$, carries x-dependence. Thus

[13] That the soft theorem is encoded in the Lagrangian is of no coincidence as effective Lagrangians are considered a more transparent way to organize them [105].

[14] We have checked that they vanish in the cases at hand for which it is important to keep $m_{D,\pi} \neq 0$. An example of where the R matters is sketched in the appendix of Ref. [7].

the question of how to make sense of this term? We may regard this matrix element as a test function in a distribution space and are therefore led to integrate over space by

$$\mathbb{1}_V = \frac{1}{V}\int_V d^d x. \tag{5.5}$$

This allows for integration by parts such that[15]

$$\mathbb{1}_V[x\cdot\partial\langle 0|\mathcal{O}(x)|D\rangle] = -d\frac{1}{V}\int_V d^d x\langle 0|\mathcal{O}(x)|D\rangle, \tag{5.6}$$

and assembling we get the *single-soft dilaton theorem*

$$m_D^2 F_D = \frac{1}{2}(d-\Delta_{\mathcal{O}})\langle 0|\mathcal{O}|D\rangle, \tag{5.7}$$

where the integral has been removed since the second dilaton is to be made soft as well. Applying the procedure once more one gets the *double-soft dilaton theorem*

$$m_D^2 F_D^2 = \frac{1}{2}(\Delta_{\mathcal{O}}-d)\Delta_{\mathcal{O}}\langle\mathcal{O}\rangle, \tag{5.8}$$

where this time the derivative term can be dropped since $\langle\mathcal{O}(x)\rangle$ has no x-dependence by translation invariance. It is worthwhile to stress that these two relations are model-independent by which we mean not particular to the gauge theory.

A. Consequences of the single- and the double-soft dilaton theorem

There are a number of things that one can learn from these two relations. Only items (c) and (d) are specific to the gauge theory; (a) and (b) are general.

(a) One has $\langle\mathcal{O}\rangle \le 0$ necessarily, such that $\langle T^\rho{}_\rho\rangle$ is lowered ($\mathcal{O}\subset T^\rho{}_\rho$) with respect to the perturbative vacuum. Hence, the mass squared (5.8) is indeed positive thanks to the d-term that originates from the derivative term. This gives us confidence that this term is present (cf. item (c) for a further comment).

(b) One can get a very similar relation to (5.7) by contracting (2.11)

$$m_D^2 F_D = \langle 0|\mathcal{O}|D\rangle, \tag{5.9}$$

where we have assumed $T^\rho{}_\rho \to \mathcal{O}$ since $\mathcal{O}$ is the operator that generates the mass. Inspecting (5.9) and (5.7) one infers that

$$\Delta_{\mathcal{O}} = d-2, \tag{5.10}$$

must hold which seems important. Hence, the soft theorem indicates that the dilaton can only get a mass from an operator of dimension $\Delta_{\mathcal{O}} = 2$ (in $d=4$). Alternative derivations of this results are given in Appendices B 1 and B 2 directly from the Lagrangian and from scaling arguments.

On another note, it is tempting to read the results (5.10) backward and interpret it as yet another demonstration that $\gamma_* = 1$ ($\Delta_{\bar{q}q} = 3-\gamma_*$) has to necessarily hold.

(c) Let us turn to the gauge theory where the operator

$$T^\rho{}_\rho \supset \mathcal{O}_{\bar{q}q} = (1+\gamma_*)\sum_q m_q\bar{q}q, \tag{5.11}$$

satisfies the criteria (5.10) for $\gamma_* = 1$ ($\Delta_{\bar{q}q} = 3-\gamma_*$). It is in fact tempting to read it the other way around. Namely, as another demonstration that $\gamma_* = 1$ must hold. Turning to pragmatic matters, one may use (5.8) to obtain

$$\begin{aligned} F_D^2 m_D^2 &= \Delta_{\bar{q}q}(\Delta_{\bar{q}q}-d)\frac{N_f}{2}(1+\gamma_*)m_q\langle\bar{q}q\rangle \\ &= -4N_f m_q\langle\bar{q}q\rangle, \end{aligned} \tag{5.12}$$

where $\gamma_* = 1$ and $d=4$ have been used in the last equality. Since the effective Lagrangian is rather standard, this relation has been obtained previously, e.g., [3,36,78]. What is new is the result $\gamma_* = 1$ ($\Delta_{\bar{q}q} = 2$), that there are no β'_*-terms and the derivation in itself from the double-soft theorem.

The dilaton GMOR-relation quoted in (2.21) can be obtained by going through the analogous process for the pion which has been done in [7] but we shall repeat it here for completeness. Starting with the analog of (5.1) one gets

$$\begin{aligned} 2m_\pi^2 &= \langle\pi^a|T^\rho{}_\rho|\pi^a\rangle = \langle\pi^a|\mathcal{O}_{\bar{q}q}|\pi^a\rangle = \frac{-(1+\gamma_*)m_q}{F_\pi}\langle 0|i[Q_5^a,\bar{q}\mathbb{1}_{N_f}q]|\pi^a\rangle \\ &= \frac{2m_q(1+\gamma_*)}{F_\pi}\langle 0|P^a|\pi^a\rangle = \frac{-2m_q(1+\gamma_*)}{F_\pi^2}\langle\bar{q}q\rangle, \end{aligned} \tag{5.13}$$

[15]The proper way to do this would be to form a wave packet within a finite region in x-space. In addition this makes it clear that the boundary terms that arise upon integration by parts do not contribute.

upon using $\langle \pi^a | P^b | 0 \rangle = -\frac{1}{F_\pi} \langle i[Q_5^a, P^b] \rangle = -\frac{1}{F_\pi} \langle \bar{q}q \rangle \delta^{ab}$ with P^b as in (3.3). Rewriting

$$m_\pi^2 F_\pi^2 = -(1+\gamma_*) m_q \langle \bar{q}q \rangle = -2 m_q \langle \bar{q}q \rangle, \tag{5.14}$$

the GMOR-relation emerges [73,75] upon using $\gamma_* = 1$. Combining (5.12) and (5.14)

$$F_D^2 m_D^2 = \Delta_{\bar{q}q}(d - \Delta_{\bar{q}q}) \frac{N_f}{2} F_\pi^2 m_\pi^2, \tag{5.15}$$

we get the dilaton GMOR-relation (2.21). There is a further insight hidden here. Namely the d-term, which originated from the derivative term in (5.4), in fact corresponds to the Zumino-type term in (2.20) (to see that one needs to expand to second order in D). This underlines the necessity of the integration by parts procedure applied once more. It is satisfactory and important that full consistency with the EFT is attained.

(d) Under item (b) we learned that the soft theorem demands $\Delta_{\mathcal{O}} = 2$. It is more often than not inferred from the soft theorem that the field strength tensor squared $T^\rho{}_\rho = \beta/(2g) G^2$ generates a dilaton mass in the chiral limit. However, this is not done from $\langle D | G^2 | D \rangle$ but from $\langle D | G^2 | 0 \rangle$ which gives the relation

$$m_D^2 F_D^2 = -\frac{2\beta}{g} \langle G^2 \rangle, \tag{5.16}$$

upon using $\Delta_{G^2} = 4$, sometimes referred to as the partially conserved dilatation current (PCDC) relation. Since $\Delta_{G^2} = 4$ does not meet the condition (5.10) this raises a question mark over the procedure. Either results, (5.7) and (5.8), would give zero since $d - \Delta_{G^2} \to 0$. One may wonder whether this indicates that the gluon condensate is to vanish. A logical possibility that suggests itself is that the dilaton could be massless in the chiral limit $\bar{m}_D \to 0$ and that all three matrix elements vanish, $\langle D | G^2 | D \rangle = \langle D | G^2 | 0 \rangle = \langle G^2 \rangle = 0$. We consider it worthwhile to emphasize this possibility without insisting on it (cf. the discussion in Sec. VI A).

In summary we have learned that the dilaton can only obtain a soft mass from an operator of scaling dimension two such as $\bar{q}q$ in the gauge theory. In addition, the soft theorems reproduce the dilaton GMOR-relation (5.15). Single soft theorems are equivalent to results found in more formal considerations, e.g., [107]. It could be interesting to extend the techniques of both approaches to each other.

VI. MASSIVE OR MASSLESS DILATON?

It is commonly believed that there are CFTs for which conformal symmetry is spontaneously broken, leading to a massless dilaton and massive states, e.g., [87]. The situation of when there is an RG flow into an IR fixed point cannot be regarded as settled. The problem is that the symmetry is only emergent and we are not aware of a systematic treatment of this case. Investigations in holographic approaches argue for a light but not a parametrically light dilaton [41,108–110]. However, since the answer to the question might be model-dependent we focus on our framework.[16] Parametrically, the leading effect is expected to come from the slope of the β function, which is however zero in our framework (3.37) and thus the parametric expectation moves to

$$m_D^2 \propto \mathcal{O}(\beta_*''). \tag{6.1}$$

This finding can be taken as an indication that the dilaton mass could at least be small. We are going to reflect on the question from three different points of view: the soft theorems, the large-N_c limit and the EFT-perspective. Whereas not conclusive, we hope that the reader finds the discussion instructive.

A. Soft-theorem perspective

The form of the TEMT (2.3) is correct to all orders in perturbation theory and believed to hold beyond. If we set $m_q = 0$ then there is only $T^\rho{}_\rho = \frac{\beta}{2g} G^2$ and thus we have $2m_D^2 = \langle D | \frac{\beta}{2g} G^2 | D \rangle$ in line with (5.1). However, the double-soft dilaton theorem indicates that the dilaton mass ought to originate from an operator $\mathcal{O} \subset T^\rho{}_\rho$ of scaling dimension $\Delta_{\mathcal{O}} = 2$ a result underlined by an alternative derivation from scaling in Appendix B 2. Hence, this role cannot be taken by G^2 since $\Delta_{T^\rho{}_\rho} = \Delta_{G^2} = 4 + \beta_*' = 4$.[17] As there is no other operator than G^2 present in the EMT (for $m_q = 0$) this would then seem to imply that the dilaton is

[16] There are examples of massless dilatons in lower dimensions, e.g., $d = 2$ [111] at finite temperature and $d = 3$ [112] (cf. also [113,114]) but they do not involve an RG flow. An example with a flow is given by a Gross-Neveu-Yukawa theory in $d = 3$ where spontaneous scale symmetry, emerges for certain initial conditions [114,115] (cf. also [116,117] for related work), accompanied with a massless scalar and massive fermions. Explicit studies with fundamental scalars indicate that they cannot take on the role of a dilaton [118–120] although they share some of these features. In [36] it was stressed that scalars, called scalons in [118], are not to be regarded as dilatons.

[17] As stated earlier this puts into question the use of the PCDC-type relation $m_D^2 F_D^2 = -2\beta/g \langle G^2 \rangle$ (5.16). To the best of our knowledge the relationship of $\langle G^2 \rangle$ in (5.16) to the gluon condensate introduced in phenomenology [121,122] has never been clarified. The latter has been determined empirically [123,124] and its existence is underlined by an elegant renormalon analysis within perturbation theory [125]. A similar quantity has been studied for pure SU(3) Yang Mills on the lattice [126–128] and is found to be nonzero by eleven standard deviation in the lattice scheme. The analysis is consistent with the renormalon picture.

massless.[18] We would not want to go as far as stating that this proves that the dilaton is massless, as for example there are assumptions involved such as the use of the soft dilaton formula (5.2). The near conformal scaling dimension of large charge operators (on a cylinder) is dependent on both γ_* and Δ and might give rise to additional information [130].

B. Large-N_c consideration

The large-N_c limit [71,131,132] is a useful tool for QCD as it leads to simplifications [71,133]. The following relations between two-point functions, with notation as in Eqs. (3.3) and (3.4),

$$\langle S(x)S(0)\rangle_c = \frac{2}{N_f}\langle S^a(x)S^a(0)\rangle(1+\mathcal{O}(1/N_c)) \propto \mathcal{O}(N_c),$$
$$\langle P(x)P(0)\rangle = \frac{2}{N_f}\langle P^a(x)P^a(0)\rangle(1+\mathcal{O}(1/N_c)) \propto \mathcal{O}(N_c), \tag{6.2}$$

must hold since they are distinguished by large-N_c suppressed quark-disconnected diagrams (i.e., connected by gluons only). The factor $2/N_f$ takes into account the normalization $\mathrm{Tr}[T^aT^b] = \delta^{ab}/2$ and the subscript c stands for the connected part (and serves to remove $\langle S(x)S(0)\rangle \supset \langle\bar{q}q\rangle^2 \propto \mathcal{O}(N_c^2)$ which is peculiar to the vacuum quantum numbers of S).

The leading graphs in (6.2) are the connected planar ones of $\mathcal{O}(N_c)$ as nonplanar ones are $\mathcal{O}(g^4) \propto \mathcal{O}(1/N_c^2)$ suppressed since g^2N_c is held fixed. We point out that quark-disconnected graphs arising from S and P are of $\mathcal{O}(N_c^0)$ and falls in between the N_c-counting above.

Famously, one can deduce in this way that the η' mass must go to zero in the large-N_c limit. In the deep-IR the P^a-correlator is dominated by the pion

$$\langle P^a(x)P^a(0)\rangle \propto \frac{|\langle 0|P^a|\pi^a\rangle|^2}{x^2} \propto \mathcal{O}(N_c), \tag{6.3}$$

and the scaling is deduced from the soft pion theorem $|\langle 0|P^a|\pi^a\rangle|^2 \propto \langle\bar{q}q\rangle/F_\pi \propto \mathcal{O}(N_c^{1/2})$ with $\langle\bar{q}q\rangle \propto \mathcal{O}(N_c)$ and $F_\pi^2 \propto \mathcal{O}(N_c)$ [73,86]. Since it is part of the leading contribution its behavior must be mirrored by the P-correlator and one concludes that $m_{\eta'} \to m_\pi = 0$ and $F_\pi \to F_{\eta'}$ for $N_c \to \infty$.

Does the same thing happen in the scalar channel? If the answer is yes, then a massless dilaton would imply that the corresponding flavored states would approach zero in the large-N_c limit. This is however in contradiction with a lattice $SU(N_c)$-study where the a_0-meson, the 0^{++}-cousin of the pion, shows no sign of becoming massless for increasing N_c (cf. [134] Sec. III. 5 and also Fig. 14 in [133]). The only caveat is that these studies are performed in the quenched approximation but one cannot expect these qualitative features to be overturned by unquenching. We therefore conclude that either the dilaton cannot be massless or that the dilaton must be subleading in the connected S-correlator. We will argue for the latter.

In clarifying at what order the dilaton appears in the large-N_c counting we consider the analog of (6.3) in the deep-IR. Using the soft theorem (B10),

$$\langle S(x)S(0)\rangle \propto \frac{|\langle 0|S|D\rangle|^2}{x^2} \propto \frac{\langle\bar{q}q\rangle^2}{F_D^2}\frac{1}{x^2}. \tag{6.4}$$

we learn that the scaling is hidden in F_D and as the latter is defined from the coupling to the EMT (2.11), we are lead to consider the EMT-correlator

$$\langle T_{\mu\nu}(x)T_{\rho\lambda}(0)\rangle = t^{(0)}_{\mu\nu\rho\lambda}\Gamma^{(0)}_{TT}(x^2) + t^{(2)}_{\mu\nu\rho\lambda}\Gamma^{(2)}_{TT}(x^2). \tag{6.5}$$

The tensor structures $t^{(0,2)}_{\mu\nu\rho\lambda}$ correspond to spin 0 and 2 respectively and are dependent on x_α and $\eta_{\alpha\beta}$. Since the dilaton is of spin 0 we have

$$\Gamma^{(0)}_{TT}(x^2) \propto \frac{F_D^2}{x^2}, \tag{6.6}$$

and the large-N_c behavior of F_D follows from corresponding behavior of the correlator. To infer this we must have a look at the EMT of a non-Abelian gauge theory which assumes the form

$$T_{\mu\nu} = \left(\frac{1}{4}g_{\mu\nu}G^2 - G_{\mu\lambda}G^\lambda_\nu\right) + \frac{i}{4}\bar{q}(\gamma_{\{\mu}\vec{D}_{\nu\}} - \gamma_{\{\mu}\overleftarrow{D}_{\nu\}})q + \ldots, \tag{6.7}$$

where $\vec{D}_\nu = (\vec{\partial} + igA)_\nu$, $\overleftarrow{D}_\nu = (\overleftarrow{\partial} - igA)_\nu$ and the dots stand for terms which vanish on physical states. The main point is that the gluonic part is in the adjoint and the quark part in the fundamental representation of the $SU(N_c)$ gauge group. Hence, one expects

$$\Gamma_{TT} = AN_c^2 + BN_c + \mathcal{O}(N_c^0), \tag{6.8}$$

and matching with (6.6) one infers that generically

$$F_D^2 = aN_c^2 + bN_c + \mathcal{O}(N_c^0), \tag{6.9}$$

is expected, implying $F_D \propto \mathcal{O}(N_c)$.[19] With (6.4) it follows that the dilaton contribution is subleading in the large-N_c limit

[18] It is in principle conceivable that another operator becomes relevant and it its scaling dimension was $\Delta = 2$ then this could give rise to a mass. We consider this possibility rather unlikely as lattice studies of four fermi operators for example do not indicate large anomalous dimensions [129].

[19] This scaling is identified with glueballs as opposed to $F \propto \sqrt{N_c}$ which are referred to as $\bar{q}q$-states. In the literature one can find $F_D \propto N_c$ [30] and $F_D \propto \sqrt{N_c}$ [49]. We agree with the former reference who use the same argument without making it explicit.

$$\langle S(x)S(0)\rangle \propto \frac{\mathcal{O}(N_c^0)}{x^2}. \tag{6.10}$$

We therefore conclude that a finite value in m_{a_0} at large N_c does not exclude a massless dilaton.

C. Is dilaton-χPT consistent with a massless dilaton?

Another way to assess whether a dilaton could be massless is to seek for contradictions with the EFT. Are quantum fluctuations going to induce a mass term? In the chiral limit of χPT it is simply impossible to write down a potential for the pion respecting the symmetries within the coset construction U (2.8) (due to the shift symmetry of the Goldstone). Hence the zero pion mass is built into χPT naturally since chiral symmetry is present at all scales. Returning to the dilaton we may observe that since there is no scale in the LO Lagrangian, in the chiral limit, no dilaton mass can be generated either. This conclusion is however too quick since scales enter through hadron masses. The dilaton couples to the nucleon mass term, e.g., [10],

$$\delta\mathcal{L}_{m_N} = -\hat{\chi} m_N \bar{N} N. \tag{6.11}$$

Generically such a term will induce a dilaton mass term (e.g., a single nucleon loop). There are several loopholes in this argument. First of all nucleon chiral perturbation theory [73,135,136] is designed to compute nucleon properties due to a light pion cloud and not the other way around. Whereas the nucleon is the only other stable hadron made out of light quarks there are of course many other resonances such as the $\rho, \omega, \ldots$ with light quark content. That opens the door to potential cancellations. This does in fact happen since bosons and fermions contribute with opposite sign as exploited in supersymmetry and the Veltman condition for the Higgs mass [137]. Moreover, in the case where a CFT is spontaneously broken, e.g., [87], the same problems would be apparent, but only apparent, as the dilaton is believed to be a true massless Goldstone in that case. Hence, one has to conclude that the EFT is not capable of making a definite statement about the dilaton mass due to (potential) hidden cancellations.

VII. THE DILATON CANDIDATE IN QCD: $\sigma \equiv f_0(500)$

Let us turn to the dilaton candidate in QCD, the σ-meson or $f_0(500)$ by official PDG-terminology [138]. We replace $D \to \sigma$ honouring the name used by many particle physicists. The σ-meson has captured the interest and imagination of particle physicists for long as testified by its history and properties in a dedicated physics reports [139]: it is very broad, it does not fit well into a nonet structure, it defies Regge trajectories as well as qualitative aspects of the large-N_c limit [139].[20] The goal of this section is to apply dχPT at LO and try to see whether one can understand its mass and width semiquantitatively.

The meaning of the mass and the width are given by its pole $\sqrt{s_\sigma} = m_\sigma - \frac{i}{2}\Gamma_\sigma$, on the second Riemann sheet, in $\pi\pi$-scattering. Its current PDG value [138] is

$$\sqrt{s_\sigma} = (400 - 550) - i(200 - 350)\ \text{MeV}. \tag{7.1}$$

This range is noticeably larger than the specific determination from Roy equations $\sqrt{s_\sigma} = (441^{+16}_{-8} - i272^{+9}_{-12.5})$ MeV [140], which is considered to have settled the issue of its existence. Earlier determinations were compatible within larger uncertainties, e.g., [141]. It is often noted with regard to the σ being a pseudo-Goldstone that its mass is notably heavier than that of the pion. However, this is not the right comparison since the σ, just like the η, is an $SU(3)_F$-singlet for degenerate quark masses and thus retains sensitivity to the strange quark mass. This can be seen in the dilaton GMOR-relation (2.21).

As singlet-octet mixing will be relevant for the width, it is instructive to discuss the nonet structure and comparing it with the ρ-meson family (cf. Tab. I). The qualitative differences are apparent: (i) the $u\bar{d}$-mesons ($I = 1$) are lighter than the $u\bar{s}$-mesons ($I = 1/2$) for the vectors but heavier for the scalars, (ii) the ratio of the $I = 1$ octet to $I = 0$ singlet is roughly one for the vectors but a factor of two for the scalars. These aspects challenge the quark model picture and can be seen as one the motivations for introducing the phenomenologically successful tetraquark model [142]. The widths also follow interesting patterns. The decays $a_0(968), f_0(980) \to \pi\pi$ are suppressed by G-parity and being mostly an $\bar{s}s$-state respectively. This is analogous to the $\omega(780)$ and the $\phi(1200)$-meson in the vector channel. The κ is the $K^*(895)$-analog and indeed rather broad $\Gamma_\kappa \approx 600(80)$ MeV, an aspect which we will understand better when considering the mixing in the next section.

We now turn to the decay constant F_σ. For the unstable σ-meson, the matrix element (2.11) is not well defined. However, the residue at the complex pole, which is generally complex, is well defined and accessible via proper analytic continuation. It has been extracted in QCD through $\langle\pi|\bar{q}q|\pi\rangle$ and $\langle\pi|T^\rho{}_\rho|\pi\rangle$ form factor input up to $\mathcal{O}(q^2)$ [143]. It is not straightforward to interpret this result in the context of this paper as it requires us to understand the meaning of an unstable pseudo-Goldstone and how this affects its representation. The same remark applies to the $g_{\sigma NN}$-residue extracted from $\pi\pi \to NN$ scattering [144]. The clarification thereof seems important and we hope to return to this question in a future publication. We may get an indirect estimate from relation

[20] By nonet one means the union of the $SU(3)_F$-octet and -singlet which mix when $SU(3)_F$ is broken by nondegenerate quark masses.

TABLE I. Nonet of $SU(3)_F$ which illustrates the special character of the $J^{PC} = 0^{++}$ mesons versus the more familiar and understood $J^{PC} = 1^{--}$ vector mesons, given in PDG-notation [138]. The double bar separates the singlet from the octet states in the $SU(3)_F$ limit. It seems relevant to mention that higher f_0-resonances all have considerably smaller widths than the σ-meson: $\Gamma_{f_0(1370)} = 350(150)$ MeV, $\Gamma_{f_0(1500)} = 108(33)$ MeV, $\Gamma_{f_0(1710)} = 150(12)$ MeV and $\Gamma_{f_0(2020)} = 180(60)$ MeV [138].

$J^{PC}\backslash I$	0	0	1/2	1
0^{++}	$\sigma \equiv f_0(500)$	$f_0(980)$	$\kappa \equiv K_0^*(700)$	$a_0(980)$
$\Gamma_{0^{++}}$	550(150) MeV	55(15) MeV	600(80) MeV	90(50) MeV
1^{--}	$\omega(780)$	$\phi(1020)$	$K^*(895)$	$\rho(770)$
$\Gamma_{1^{--}}$	6.86(13) MeV	0.016 MeV	52(12) MeV	145(3) MeV

$g_{\sigma NN} = \frac{m_N}{F_\sigma}$, e.g., [10] (akin to the Goldberger-Treiman relation $g_{\pi NN} = \frac{m_N}{F_\pi}$). In nuclear physics there are approaches using $g_{\sigma NN}$ in their LO Lagrangian such as the one-boson exchange model describing nucleon-nucleon scattering [145]. The σ-meson, among π, ρ and ω, describe the scattering phase shifts reasonably well. From the range in Tab. III [145] one infers $g_{\sigma NN} = 10(2)$ (cf. [146] for compatible results and also the fact that the real part in [144] is in agreement as well) as a reasonable estimate. Remarkably, using $m_N = 0.93$ GeV,

$$F_\sigma = \frac{m_N}{g_{\sigma NN}} \approx 93(19)\ \text{MeV}, \qquad (7.2)$$

a value close to $F_\pi = 93$ MeV emerges. However, one ought to be cautious as this is a very difficult subject where systematics are difficult to estimate.

A. The width of the σ-meson

The width of the σ-meson has been one of the early qualitative successes of the dilaton approach. It is well approximated by $\Gamma_\sigma \approx \Gamma_{\sigma\to\pi\pi}$ since the photon channel $\Gamma_{\sigma\to\gamma\gamma} = 1.7(4)$ keV [147] is highly suppressed as one would expect. Its interest lays more in the possibility to learn about the σ-meson substructure [139]. The amplitude into two pions is described the effective coupling, $\mathcal{L}_{\rm eff} = \frac{1}{2} g_{\sigma\pi\pi}\sigma\pi^a\pi^a$, which we can read off from the Lagrangian (2.22)[21]

$$g_{\sigma\pi\pi} = \frac{1}{F_\sigma}(m_\sigma^2 + (1-\gamma_*)m_\pi^2 + \mathcal{O}(\beta_*'', m_q^2)) \to \frac{m_\sigma^2}{F_\sigma}(1 + \mathcal{O}(\beta_*'', m_q^2)), \qquad (7.3)$$

and resembles earlier expressions [3,50,76]. Differences are that $\gamma_* = 1$ is an open parameter and that β_*'-corrections with unknown coefficients are parametrized, e.g., [50]. The rate

$$\Gamma_{\sigma\to\pi\pi}|_{SU(2)} = \frac{3|g_{\sigma\pi\pi}|^2}{32\pi m_\sigma}\sqrt{1-4\hat{m}_\pi^2}, \qquad (7.4)$$

follows from the $1\to 2$ decay $d\Gamma = \sum_{\pi\pi}\frac{|\mathcal{A}_{\sigma\to\pi\pi}|^2}{32\pi^2}\frac{|\vec{p}|}{m_\sigma^2}d\Omega$ [138] where $\frac{|\vec{p}|}{m_\sigma} = \frac{1}{2}\sqrt{1-4\hat{m}_\pi^2}$ is the velocity in the frame of the σ-meson, $\int d\Omega \to 2\pi$ as the pions are identical particles and the factor of 3 results from the three pion channels. The case with and without mixing are denoted by $SU(3)_F$ and $SU(2)_F$ respectively.

No mixing: using $m_\sigma = 440$ MeV and $F_\sigma = 93$ MeV one gets

$$\Gamma_{\sigma\to\pi\pi}|_{SU(2)} = 227\ \text{MeV} \qquad (7.5)$$

where uncertainties are not given since the mixing is neglected. The rate is a factor of ≈2.5 lower compared to (7.1), which corresponds to a factor ≈1.5 in the amplitude and not a bad results in view of the crudeness of the approach. The question is whether the singlet-octet refinement will improve or worsen it.

Singlet-octet mixing: the effect of singlet-octet mixing is driven by the breaking of $SU(3)_F$, i.e., $m_s \gg m_{u,d}$. It is known that this effect is not negligible from the $\eta - \eta'$ system. In the same way the $\sigma - f_0(980)$ system may be parametrized by a single angle θ

$$|\sigma\rangle = \cos\theta|S_1\rangle + \sin\theta|S_8\rangle,$$
$$|f_0(980)\rangle = -\sin\theta|S_1\rangle + \cos\theta|S_8\rangle, \qquad (7.6)$$

in terms of the $SU(3)_F$ eigenstates. The isospin breaking mixing with the a_0, the analog of π^0 in the $\eta - \eta'$ system, is neglected. We follow the approach by Oller [148] based on a Wigner-Eckart decomposition with two reduced matrix element g_1 and g_8 for which one has

$$\mathcal{A}_{\sigma\to(\pi\pi)_0} = -\frac{\sqrt{3}}{4}\cos\theta g_1 - \sqrt{\frac{3}{10}}\sin\theta g_8, \qquad (7.7)$$

and similarly for $f_0(980)$ including all open channels. The zero subscript stands for $I = 0$. The three unknowns $g_{1,8}$ and θ when fitted to experiment are (Eq. (2.30) [148])[22,23]

[21] In an EFT framework (for off-shell σ, e.g., [9]) $m_\sigma^2 \to q^2$ where q^2 is the momentum entering the σ-field.

[22] There is a second determination $\theta \approx 21°$ in an $U(3)\times U(3)$ σ-model [149]. Whereas no error is given in this determination, presumably due to model-dependence, the agreement with (7.8) is encouraging.

[23] We have used the singlet-octet to $\bar{s}s - \mathrm{n}\bar{s}s$ basis conversion $\theta = \phi + 35.264°$ for $\phi \approx -14°$. I am grateful to Oller in assisting in the conversion which incidentally is not the same as in the standard $\eta' - \eta$ mixing.

$$g_1 = 3.9(8)\ \text{GeV}, \qquad g_8 = 8.2(8)\ \text{GeV}, \qquad \theta = 19(5)^\circ. \tag{7.8}$$

We note the remarkable enhancement of the octet component $g_8/g_1 \approx 2$. Its effect may be estimated by the ratio of mixing versus no mixing

$$r_{18} = \left| \frac{\mathcal{A}^{\theta}_{\sigma\to(\pi\pi)_0}}{\mathcal{A}^{\theta=0}_{\sigma\to(\pi\pi)_0}} \right| = 1.81^{-0.18}_{+0.20}. \tag{7.9}$$

Using this value we get

$$\Gamma_{\sigma\to\pi\pi}|_{SU(3)} = r_{18}^2 \Gamma_{\sigma\pi\pi}|_{SU(2)} = 744^{-108}_{+146} \pm 40\%\ \text{MeV}, \tag{7.10}$$

where uncertainties were obtained by adding the ones due to g_1 and θ in quadrature plus 40% for F_σ. Whereas the estimate is crude that is errors are large, it is fair to state that its central value is considerably improved due to the mixing and compares favorable with $\Gamma_{\sigma\to\pi\pi} \approx 544^{+18}_{-25}$ MeV from the Roy equation [140]. The enhanced octet versus singlet matrix element explains the large κ rate (Table I). It is enhanced by $(g_8/g_1)^2$ and its natural value when compared with the σ-meson is therefore 140 MeV rather than 600 MeV. It would be interesting to understand the octet enhancement qualitatively.

B. The mass of the σ-meson

In QCD the GMOR-type mass relation, which are the χPT LO expressions [71,73],

$$m_\pi^2 = (m_u + m_d)B_0, \qquad m_{K^+}^2 = (m_u + m_s)B_0,$$
$$m_{K^0}^2 = (m_d + m_s)B_0, \tag{7.11}$$

work rather well for pions and kaons (recall $B_0 = -\langle \bar{q}q \rangle / F_\pi^2$). This raises hopes that the dilaton GMOR-relation (5.12) will give a good for value of the σ-mass. We shall see and understand that this is not necessarily the case. Let us first adapt (5.12) to non-degenerate quark flavors, using (7.11), and deduce

$$F_\sigma^2 m_\sigma^2 = (1+\gamma_*)(3-\gamma_*) F_\pi^2 \sum_{q=u,d,s} m_q B_0 \Big|_{\gamma_*=1}$$
$$= 2F_\pi^2 (m_{K^0}^2 + m_{K^+}^2 + m_\pi^2). \tag{7.12}$$

With $m_\pi = 140$ MeV, $m_K = 495$ MeV, $F_\pi = 93$ MeV and $F_\sigma = 93$ MeV as input on gets,

$$m_\sigma|_{\text{LO}} \approx 1\ \text{GeV} \left(\frac{93\ \text{MeV}}{F_\sigma} \right), \tag{7.13}$$

a value which is about a factor of two larger than the in the real world. There is some irony here as often the σ-meson is regarded as being too heavy to be considered a pseudo-Goldstone. Formula (7.12) has been obtained in [150] but the difference is that γ_* and β'_*-corrections were undetermined and thus the large value has not come to attention.

One might wonder whether dχPT would be convergent with such a large LO value since it is well-known that $SU(3)-\chi$PT is not as efficient as $SU(2)-\chi$PT because of the proximity of m_K to $m_\rho \approx 770$ MeV. The separation of the Goldstone sector is not strong in actual numbers. To get an idea we might want to use the NLO formula for $SU(2)-\chi$PT [73,151]

$$m_\pi^2|_{\text{NLO}} = m_\pi^2 \left(1 - \frac{1}{32\pi^2} \frac{m_\pi^2}{F_\pi^2} \bar{\ell}_3 \right), \tag{7.14}$$

with $\bar{\ell}_3 \equiv \ln \Lambda_3^2 / m_\pi^2 = 3.53(26)$ [152] is sizeable due to a chiral logarithm. One infers NLO-corrections factors of (0.025,0.21,0.25,1.4), using LO masses (m_π, m_K, m_σ, 1 GeV).[24] The large correction factor 1.4 for $m_\sigma|_{\text{LO}}$ is telling us that convergence cannot be expected. The situation is unsatisfactory but deserves some more contemplation.

The large value obtained is driven by m_K which is comparatively large due the $m_s \gg m_{u,d}$. The GMOR-type formulas are expansions m_q (or Goldstone masses) and not in $1/m_q$. That is, there is no built-in decoupling limit but rather one decouples by hand in excluding a quark from the sum in (7.12). Can we assess this in another way? Yes, through the mixing if we are willing to commit to a quark mass picture. In Appendix C it was argued that the mixing angle supports the $\bar{q}q$-state interpretation. Equation (C2) indicates suppression of the $\bar{s}s$ versus the $\bar{u}u$,$\bar{d}d$-contribution. The strange quark decoupling angle $\theta_{\text{dec}} \approx 35.7^\circ$ is close but not too close to $\theta = 19^\circ$.[25] Indeed if we were to completely decouple the strange quark we would expect to replace $2m_K^2 + m_\pi^2 \to 2m_\pi^2$ in (7.12) which does yield $m_\sigma \approx 2m_\pi = 280$ MeV an underestimate.

In summary the situation remains inconclusive but we pointed out why we cannot expect formula (7.12) to give us a good number. First from the convergence in the EFT and second the formula overestimates the role of the strange quark since it has no decoupling built in.

VIII. OUTLOOK

There are a few directions in which the work begun in this paper can be extended, which is to investigate the m_q

[24] This procedure only gives realistic numbers for the pions since the kaon requires $SU(3)-\chi$PT [153] and for the σ dχPT is required which has not been fully developed at NLO yet. We believe however that the numbers give a reasonable estimate of the size of the NLO corrections.

[25] There is indirect evidence for strangeness in the σ-meson. The $\langle \bar{s}s(x)\bar{u}u(0)\rangle$-correlator is nonzero suggesting a light state which could correspond to the σ-meson [154].

dependence of the σ-mass, extending the LO Lagrangian (2.22) to include $\bar{m}_D$-mass and investigate whether the Higgs boson could be a dilaton. Below we give a brief outlook on some of these matters.

A. Quark mass dependence of the σ-meson(s) on the lattice and beyond

Investigating the m_q-dependence of the σ-meson in QCD and other gauge theories with chiral symmetry breaking is interesting and still largely open. There are several avenues and it is to be hoped for efforts to continue and for new ones to start.

Before the advent of the LHC investigations in lattice gauge theory started in order to obtain reliable information for walking technicolor [65,66,155] and composite Higgs models [156]. In recent years light scalars with σ quantum numbers were reported for decreasing quark masses [16–29]. There are also studies of the QCD σ-meson on the lattice [157–160] which is challenging because of its large width. As far as we know all simulations are performed at physical strange quark mass but larger $m_{u,d}$ masses. Concretely, in [157,158] a pion mass $m_\pi = 391$ MeV leads to $m_\sigma = 745(5)$ MeV and $m_{f_0(980)} = 1166(45)$ MeV. Another interesting avenue is to measure the matrix element $\langle 0|\bar{q}q|D\rangle$, from which the σ-mass is extracted. If the dilaton chiral mass is zero then there is a simple relation with F_σ of this matrix element cf. Appendix B 3. In fact, the $SU(3)$ and $N_f = 8$ lattice results is compatible with a very light or possibly massless dilaton.

Among the analytic methods the Roy equations [140,161], which give the most precise determination of the σ-pole, would seem the most promising method. They do rely though, as many dispersive methods, on a mixture of experimental and theoretical hadronic input. Hence the success would depend on how well one could control the input as a function of the quark masses. The lattice could play an indirect but important role in providing input for the Roy equations. For example, recently the data from [157,158] has been used in Roy equations [162] and the $m_\sigma = 745(5)$ MeV consistent with [157,158] has been found. Another possible avenue is the analytic S-matrix bootstrap [163,164] which would equally depend on input. Analytic methods such as Dyson-Schwinger equation [165,166] and functional renormalization group methods based on elastic unitarity, $\pi\pi$-scattering $s < 4m_\pi^2$, have been applied (reviewed in [139]). Concrete studies include the inverse amplitude method with NLO χPT input for fixed m_s and varying pion mass [167], unitarized chiral perturbation theory [168] with varying m_s-dependence and the N/D-method with LO χPT -input [148].

The numbers from lattice [157,158,169] and analytic methods indicate that σ could decrease by $\mathcal{O}(100$ MeV$)$ for in $m_{u,d} \to 0$. Hence, the role of the strange quark mass might be important for which information is sparse. The above mentioned N/D method [148] gives most concrete information where the nonet, in mass and widths, are continuously deformed to become $SU(3)_F$-symmetric at $m_{\pi,K} = 350$ MeV (corresponding to a degenerate quark mass $\approx$23 MeV, an increase in $m_{u,d}$ and a decrease in m_s). At this point $m_\sigma \approx 300$ MeV as can be inferred from the plot in Fig. 2 in [148]. This is a significant reduction in view of the expected increase due to $m_{u,d}$.[26] This is somewhat at variation with [168] where (small) m_s-changes in mass and width were found to be very small. Hence the situation is not entirely conclusive but rather motivates to further investigate m_σ as a function of the light quark masses.

B. The Higgs as a dilaton?

It has been appreciated for a long time that in the absence of the Higgs VEV the SM is conformal up to the logarithmic running of the couplings. The dilaton therefore fits the role of the Higgs naturally as it couples to mass and is associated with a VEV. There are several realizations of this scenario but they all have in common that the SM Higgs sector is replaced by a (strongly coupled) sector which undergoes spontaneous scale and electroweak symmetry breaking at a scale F_D (not necessarily equal to the Higgs VEV $v \approx 250$ GeV). The most relevant change is that in the LO-SM Lagrangian the dilaton replaces the Higgs as follows

$$1 + \frac{h}{v} \to e^{-\frac{D}{F_D}} \to 1 + \frac{h}{F_D}, \qquad (8.1)$$

where in the second arrow the freedom of field redefinition has been made use of.[27] This particle behaves like a SM Higgs with all couplings rescaled by the ratio of the two scales: $r_v = \frac{v}{F_D}$. Post LHC we know that this ratio has to be close to one within approximately ten percent which would equally tame tensions with electroweak precision observables (e.g., [170] where $r_v = 1$ implies $\kappa_W = 1$). This raises the question: is $r_v \approx 1$ natural? Other important low energy parameters are the dilaton mass and more generally its potential. The answer to all of these questions is model-dependent and we thus restrict attention to our framework.

We consider a gauge theory with a gauge group G' which undergoes chiral symmetry breaking close to the electroweak scale. This means that the W- and Z-boson masses are generated in the same way as in technicolor [65,66,155]. The Higgs VEV and the pion decay constant of the new sector are related by $v^2 = n_d F_\pi^2$, where n_d is the number of techniquark electroweak doublets. Since the Higgs width

[26] Note that for small deformation it is the width rather than the mass that decreases. This also explains why the σ-mass shows little variation for small m_s in the in Ref. [168].

[27] This is legitimate as long as we are interested in on-shell matrix elements and for small fluctuations ($h < -F_D$ would violate the positivity of the exponential).

has been measured to be narrow $\Gamma_h = 3.2^{+2.8}_{-2.2}$ MeV [138] there can only be one doublet, which takes on the role of the longitudinal degrees of freedom of the gauge bosons. Otherwise the Higgs/dilaton would disintegrate fast into the additional $\pi'\pi'$-pairs giving rise to a width $\Gamma_h = \mathcal{O}(100\ \mathrm{GeV})$. Hence, $v = F_\pi$ and therefore the difference of the dilaton versus SM-Higgs coupling are parametrized by the ratio

$$r_{N_f} = \frac{F_\pi}{F_D}, \tag{8.2}$$

of pion to dilaton decay constants. Besides the indicated N_f-dependence there is also an implicit G'-dependence which is less important but has to be kept in mind. Whereas it is likely that $r_{N_f} = \mathcal{O}(1)$ there is no known reason why it should be numerically close to one. The quantity we are interested in is r_2 and we resort to actual QCD for guidance. With F_D as in (7.2) and $F_\pi = 93$ MeV one finds a number $r_{2\text{–}3}|_{\mathrm{QCD}} = 1.0(2)$ which is compatible with one within uncertainties. Cautionary remarks apply. First it is difficult to estimate the systematics of (7.2). Second whereas we are interested in $N_f = 2$ massless quarks, in QCD we have two light quarks and a light but sizeable strange quark.[28] The most pragmatic interpretation is that this motivates trying to find a reason for r_2 being close to one.

So far we have not addressed how the coupling (8.1) arises.[29] The starting point is the Higgsless SM developed as an EFT for a heavy Higgs [173–175]. This is analogous to decoupling the σ-particle in the linear σ-model which gives a for of χPT. The would-be Goldstone bosons $U = \exp(i2T^a\pi^a/F_\pi)$ transform as $U \to V_L U V_Y$ under the SM gauge group $SU(2)_L \times U(1)_Y$ with $V_Y = e^{iyT_3}$ such that the condition $\det U = 1$ is preserved (since $\det V_L = \det V_Y = 1$). The effective Lagrangian contains terms of the form

$$\mathcal{L} \supset \frac{1}{4} v^2 \mathrm{Tr}[D^\mu U D_\mu U^\dagger] - v\bar{q}_L Y_d U \mathcal{D}_R + \ldots, \tag{8.3}$$

where $\bar{\mathcal{D}}_R \equiv (0, \bar{d}_R)$ and the covariant derivatives assure $SU(2)_L \times U(1)_Y$ gauge invariance. The dots correspond to similar fermionic and gauge kinetic terms. Equation (8.3) resembles the corresponding dχPT term in (2.16) but lacks the dilaton nota bene. It seems to us that one cannot resort to the compensator argument in coupling the dilaton since it is only the scale symmetry of G' which is broken. However, the same rationale that would imply $r_2 \approx 1$ ought to group the pions and the dilaton under one and the same symmetry. Assuming this to be true then gives

$$\mathcal{L} \supset \frac{1}{4} v^2 e^{-2D/F_D} \mathrm{Tr}[D^\mu U D_\mu U^\dagger] - v e^{-D/F_D} \bar{q}_L Y_d U \mathcal{D}_R + \ldots, \tag{8.4}$$

an effective Lagrangian equivalent to the SM at LO modulo the Higgs potential. Hence, finding an (approximate) symmetry assuring $r_2 \approx 1$ would therefore help in both ways, explaining the required closeness to the SM and justifying the Lagrangian in (8.4). Our reasoning is the same as in [36], except that we give provide a reason for dilaton interactions.

Let us turn to the dilaton potential which consists of the pure G'-part and the one induced by the mixed Lagrangian (8.4). For the former the situation is similar to CDQCD, we do not know anything for certain other than how to incorporate q' mass terms. Now, since $m_{q'} \neq 0$ breaks $SU(2)_L$, these terms are absent in the most straightforward setting. If the dilaton were to acquire a chiral mass $\bar{m}_D \neq 0$ then V_Δ (2.19) is a simple potential describing it. However, since the dilaton soft theorems (in Sec. V) indicate that $\Delta = 2$ (which corresponds to the SM Higgs potential), it remains unclear which operator would take on this role in the G' gauge theory. Let us turn to the potential induced by the mixed Lagrangian (8.3). As is the case in composite Higgs models [156] the coupling to the W, Z-boson and the top quark would induce sizeable corrections to the Higgs mass which are quadratically sensitive to the cut-off $\Lambda' = 4\pi F_D$ of the G'-confinement scale. It is difficult to say anything concrete other than parametrically these contribution are of order $\mathcal{O}(v^2)$ subject to further sizeable NLO corrections. In view of the generally large and negative contribution of the top mass a sizeable chiral mass $\bar{m}_D$ could potentially be required. A more thorough assessment might necessitate to find a UV completion at some scale M^2, explaining the origin of the coupling (8.4) through terms of the type $\mathcal{L}_{\mathrm{eff}} \supset \frac{\mathcal{O}(1)}{M^2} \bar{q}' q' \bar{t} t$.

There are further phenomenological aspects that needs attention. In the standard dilaton scenarios the β function contributions to the SM radiative processes $gg \to h$ and $h \to \gamma\gamma$ give rise to severe constraints, e.g., [172]. In our scenario the q'-fermions are not charged under $SU(3)_c$, implying that $gg \to h$ is truly loop-suppressed (with respect to the SM) and therefore we do not expect tensions with the LHC. The process $h \to \gamma\gamma$ is more subtle as the q'-fermions are generally electrically charged (since q'_L is charged under $SU(2)_L$). Its calculation is a formidable task as nonperturbative. Early assessments within QCD go back to the discovery of the trace anomaly [52–54]. This LO evaluation gives the correct order of magnitude, e.g., [49],

[28] The N_f-dependence comes from the σ-meson being a singlet $SU(N_f)$ as manifested in (2.21). Hence on expects $r_{N_f} \propto 1/\sqrt{N_f}$. In a lattice fit [171] $r_8 \approx 0.33$ found for which the scaling gives $r_2 = 2r_8 = 0.66$ a value slightly lower than 1. There could be many reasons, one of which is that the (2.21) is only a LO relation and the analysis in Sec. VI B indicates that the counting is not that straightforward.

[29] In the most commonly used dilaton scenarios this follows from the conformality of the total SM and extended sector. This is not the route we have in mind as this scenario has sizeable contributions in $gg \to h$ for example, in tension with the findings at the LHC [172].

but its precise value is extracted indirectly from scattering data (e.g., [147,176]). Since there is no precise method of direct computation and the rate might well be m_σ/m_N-dependent, it seems that one cannot easily borrow an estimate from QCD. The QCD-value $\Gamma_{\sigma\to\gamma\gamma} = 1.7(4)$ keV indicates however that it could be sizeable and therefore deserves attention. There is another challenge and that is the nondiscovery of hadronic G'-resonances. Whereas the dilaton itself might be light, the basic scale is set by the Higgs VEV $v = F_\pi \approx 250$ GeV. A 2 TeV-benchmark gives a ratio $2\ \mathrm{TeV}/F_\pi|_{G'} = 8$ which is comparable to the one in QCD: $m_\rho/F_\pi|_{\mathrm{QCD}} \approx 8.3$. Without a phenomenological analysis and a more precise assessment of the ratio it seems difficult to make a statement other than that these resonances may be within the future LHC-reach.

In summary the by far most important task is to investigate whether there is a reason for $r_2 \approx 1$ in (8.2). If this were true then our rough assessment indicates that the model might pass current constraints. We hope to return to these topics in future work.

IX. CONCLUSIONS AND SUMMARY

In this paper we further explored the possibility that gauge theories in the chirally broken phase admit an IR fixed point interpretation (cf. Fig. 1). In Sec. IV it was argued that this idea makes sense in $\mathcal{N} = 1$ supersymmetric based on the free meson and the squark-bilinear duality in the infrared-free magnetic and electric phase.

Scaling dimensions were deduced by matching the gauge theory fixed-point behavior with the effective (dilaton)-χPT (Sec. III). The mass anomalous dimension at the fixed point is found to be $\gamma_* = 1$, providing a more in-depth analysis of our earlier findings [7]. The vanishing of the β- and γ_m-slopes at the fixed point: $\beta'_* = \gamma'_* = 0$, cf. (3.37) and (3.42), are new results. In particular, $\beta'_* = 0$ has many attractive features: (i) it suits a mass-gap interpretation, (ii) it is consistent with $\mathcal{N} = 1$ supersymmetry (Sec. IV)[30] and, (iii) it implies logarithmic running of the fixed-point coupling $\delta g \equiv g - g^*$ (3.41). The latter makes it plausible that the trace anomaly is reproduced by the generalization of the energy-momentum tensor for chiral theories (to include a dilaton).

In the second part of the paper we focus on the dilaton D, the (pseudo) Goldstone boson due to spontaneous scale symmetry breaking. From soft dilaton theorems in Sec. V it was deduced that a dilaton mass generating operator $\mathcal{O} \subset T^\rho{}_\rho$ is necessarily of scaling dimension $\Delta_\mathcal{O} = 2$. An important role is played by the partial derivative $x\cdot\partial$-term in the dilatation commutator (5.4): (i) it renders the mass positive and (ii) it is the counterpart of the Zumino-term (2.20) in dχPT. These results are model-independent. For $m_q = 0$ there is no operator of scaling dimension two and this sets a question mark on the PCDC-type relations (5.16) based in $\Delta_{G^2} = 4$. For $m_q \neq 0$ the $\mathcal{O} = m_q\bar{q}q$ takes on the role of $\Delta_\mathcal{O} = 3 - \gamma_* = 2$ and might be seen as another reasoning for $\gamma_* = 1$. In Sec. VI we contemplate whether the dilaton could be massless or not. We found that combining lattice data and $F_D \propto N_c$ does in principle still allow for a massless dilaton. Reasoning in terms of spontaneously broken CFT suggests that the EFT entails subtle cancellations.

In the third part applications of the dilaton are explored. In Sec. VII we consider whether the σ-meson in QCD could be a dilaton. Singlet-octet mixing is found to be important as it considerably improves the prediction of the width (7.10). The leading order σ-mass, which follows from the dilaton GMOR-relation (7.12), is rather large and indicates convergence issues in the EFT (due to the large strange quark mass). The dilaton as the low energy part of a new gauge sector, can take on the role of the Higgs boson provided that the ratio of pion to dilaton decay constants $r_{N_f} \equiv F_\pi/F_D$ is unity for two flavors (cf. Sec. VIII). Whereas in QCD indications are that $r_2 \approx 1$ there is no underlying principle known why this ought to happen. An assessment beyond leading order would give rise to a potential and corrections to radiative processes such as $gg \to h$ and $h \to \gamma\gamma$ which might pass current LHC-constraints.

There are many directions to explore but among the unresolved questions the most outstanding ones are: (i) what is the chiral dilaton (or σ-meson) mass? Is it zero or at least considerably smaller than the nucleon mass and how does it depend on the distance to the lower boundary of the conformal window? (ii) is there a principle that would imply $r_2 \approx 1$? (iii) a more systematic investigation gauge theories away from the near-conformal regime (cf. Appendix B 3 for motivation and one route) These questions might well be very difficult to answer by pure reasoning. Hopefully lattice Monte Carlo simulations and methods using analyticity and unitarity can be of some guidance.

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[30]Furthermore, we argue that $\gamma_* = 1$ in the chirally broken phase provides a correct description of $\mathcal{N} = 1$ Seiberg-duality aspects.

APPENDIX A: CONVENTIONS

The Minkowski metric $\eta_{\mu\nu}$ reads $\mathrm{diag}(1,-1,-1,-1)$, $g=\det(g_{\mu\nu})$ is the determinant. Weyl transformations are defined by

$$g_{\mu\nu}\to e^{-2\alpha}g_{\mu\nu}, \qquad \hat{D}\to\hat{D}-\alpha, \tag{A1}$$

where the normalized dilaton field is $\hat{D}\equiv D/F_D$. Kinetic terms are shortened as $(\partial\varphi)^2\equiv\partial_\mu\varphi\partial^\mu\varphi$. The β function is defined in (2.2) and is given by

$$\frac{\beta}{g}=-\left(\beta_0\frac{g^2}{(4\pi)^2}+\mathcal{O}(g^4)\right),\quad \beta_0=\left(\frac{11}{3}C_A-\frac{4}{3}N_F T_F\right), \tag{A2}$$

where for QCD with $N_f=3$ and $G=SU(3)$ we have $\beta_0=9$ since $T_F=\frac{1}{2}$ and $C_A=N_c$. Further to that we derive γ_{G^2} in (2.5). It departs from the observation that $T^\rho{}_\rho=\frac{\beta}{2g}G^2$ (for $m_q=0$) is an RG invariant

$$0=\frac{d}{d\ln\mu}T^\rho{}_\rho=\frac{d}{d\ln\mu}\left(\frac{\beta}{2g}G^2\right)=\left(\frac{\beta}{2g}\right)'\beta G^2+\frac{\beta^2}{2g}(G^2)', \tag{A3}$$

where the prime denotes differentiation with respect to g and from (2.2) one then gets

$$\gamma_{G^2}=2g\left(\frac{\beta}{2g}\right)'=\beta'-\frac{\beta}{g}, \tag{A4}$$

in accordance with (2.5).

APPENDIX B: MORE ON SOFT THEOREMS

In this Appendix we take a look at the results in Sec. V from a slightly different angle. First in deriving formula (5.8) directly from dχPT and second by considering the soft theorem for a single dilaton. The common theme is the matching of

$$\mathcal{O}_{\bar{q}q}\equiv(1+\gamma_*)\sum_q m_q\bar{q}q, \tag{B1}$$

which we can write as $\mathcal{O}_{\bar{q}q}=(1+\gamma_*)\partial_{\ln m_q}\mathcal{L}_{\mathrm{QCD}}$ in QCD to dχPT (2.22)

$$\begin{aligned}\mathcal{O}_{\bar{q}q}\to(1+\gamma_*)\partial_{\ln m_q}\mathcal{L}_{\mathrm{LO}}^{\mathrm{d}\chi\mathrm{PT}}&=(1+\gamma_*)\sum_q m_q\langle\bar{q}q\rangle\hat{\chi}^{\Delta_{\bar{q}q}}\\&=-\frac{1}{2}m_D^2F_D^2\hat{\chi}^{\Delta_{\bar{q}q}},\end{aligned} \tag{B2}$$

where the pions are neglected as they play no role in this Appendix. In the last equality (5.14) was used and we note that since $\langle\hat{\chi}^{\Delta_{\bar{q}q}}\rangle=1$ the VEV $\langle\mathcal{O}_{\bar{q}q}\rangle=-\frac{1}{2}m_D^2F_D^2$ is correctly reproduced. A peculiar aspect is that this operator contains tadpoles

$$\mathcal{O}_{\bar{q}q}=-\frac{1}{2}m_D^2F_D^2\left(1-\Delta_{\bar{q}q}\hat{D}+\frac{1}{2}\Delta_{\bar{q}q}^2\hat{D}^2+\mathcal{O}(\hat{D}^3)\right). \tag{B3}$$

We note that whereas in a Lagrangian tadpoles are not acceptable as they signal a false vacuum, there is nothing that prevent tadpoles in operator matching. There is another way to look at this since we state $T^\rho{}_\rho=\mathcal{O}_{\bar{q}q}$ the same must result from dχPT (2.22). Using $T^\rho{}_\rho=4V-\partial_{\ln\chi}V$ [9] we get $T^\rho{}_\rho=-\frac{1}{\Delta_{\bar{q}q}}m_D^2F_D^2\hat{\chi}^{\Delta_{\bar{q}q}}$ which indeed matches provided $\Delta_{\bar{q}q}=2$ (or $\gamma_*=1$). In this matching we have used $d=4$ for simplicity.

1. The double-soft theorem from dilaton-χPT

We consider it important to make contact with the result (5.8) directly from the Lagrangian. Whereas in a general CFT setup (5.8) holds for any $\Delta_{\mathcal{O}}$ this is not the case here as it was already concluded that $\Delta_{\mathcal{O}}=d-2$. We aim to reproduce this result.

Using (B3) and dχPT (2.22) for the tadpoles we find

$$\begin{aligned}\langle D|\mathcal{O}_{\bar{q}q}|D\rangle&=\left(\langle D|\frac{1}{2}\hat{D}^2(\Delta_{\bar{q}q}^2-\Delta_{\bar{q}q}(d+\Delta_{\bar{q}q}))-\frac{1}{2}\frac{(\partial\hat{D})^2}{m_D^2}\Delta_{\bar{q}q}(d-2)|D\rangle\right)\langle\mathcal{O}_{\bar{q}q}\rangle\\&=\frac{1}{F_D^2}(\Delta_{\bar{q}q}^2-\Delta_{\bar{q}q}(d+\Delta_{\bar{q}q})+\Delta_{\bar{q}q}(d-2))\langle\mathcal{O}_{\bar{q}q}\rangle=-\frac{1}{F_D^2}2\Delta_{\bar{q}q}\langle\mathcal{O}_{\bar{q}q}\rangle,\end{aligned} \tag{B4}$$

for $\mathcal{O}_{\bar{q}q}$ defined in (B1) and subtleties due to tadpoles are comment further below. Using $\Delta_{\bar{q}q}=d-2$ one finds indeed consistency with Eqs. (5.1) and (5.8) and our goal is achieved. Namely, we have shown that the soft theorem manipulations follow from the EFT provided that $\Delta_{\bar{q}q}=d-2$.

We consider it worthwhile to comment on the origin of the terms in (B4). The $\Delta_{\bar{q}q}^2$-term is straightforward as it corresponds to the D^2-term in (B3). The remaining two originate from tadpole diagrams due to the linear $\hat{D}$-term in (B3). For those the dilaton propagator assumes the form $\Delta_F(q^2=0)\to -i/m_D^2$ and the mass term gets canceled against a corresponding term in the Lagrangian. Specifically, the $\widehat{D^2}\Delta(d+\Delta_{\bar{q}q})$- and $(\partial\hat{D})^2\Delta_{\bar{q}q}(d-2)$-terms are due to the two interaction terms

$$\mathcal{L}_{\rm LO}^{\rm d\chi PT} \supset (d+\Delta_{\bar{q}q}) m_D^2 F_D^2 \frac{\hat{D}^3}{3!} - (d-2) F_D^2 \hat{D} \frac{(\partial \hat{D})^2}{2}. \quad \text{(B5)}$$

2. The mass operator in the deep infrared

We aim to derive the result (5.10) using the intuitive idea that the free field mass operator $\mathcal{L} \supset \frac{1}{2} m^2 \phi^2$ is of scaling dimension two. This is done by matching correlators in the full and in the effective theory in the deep-IR. We assume that the mass is generated by an operator $\lambda \mathcal{O} \subset T^\rho{}_\rho$ where λ serves as a bookkeeping parameter. (If $\lambda = m_q$ then $\mathcal{O} = \bar{q}q$ for example).

(i) *EFT:* the TEMT at LO assumes the form

$$T^\rho{}_\rho|^{\rm LO}_{d\chi{\rm PT}} = m_\pi^2 \pi^2 + F_D m_D^2 D - \frac{1}{2} \Delta_{\mathcal{O}} m_D^2 D^2 \ldots, \quad \text{(B6)}$$

where the dots stand for terms which give suppressed contributions in the deep-IR. Assembling all the information one gets

$$\langle T^\rho{}_\rho(x) T^\rho{}_\rho(0) \rangle^{\rm LO}_{\rm d\chi PT} \propto \frac{F_D^2}{(x^2)^{d/2-1}} e^{-m_D x} + \frac{1}{(x^2)^{d-2}} (c'_{2\pi} e^{-2m_\pi x} + c'_{2D} e^{-2m_D x}), \quad \text{(B7)}$$

where the exponential behavior follows from the asymptotic limit of the Euclidean scalar propagator $\Delta_E(x,m) \propto e^{-mx}(x^2)^{1-d/2}$.

(ii) *RG analysis:* with arguments similar to the ones in Sec. III B 7, the TEMT correlator assumes the form

$$\langle T^\rho{}_\rho(x) T^\rho{}_\rho(0) \rangle_{\rm CDQCD} = \frac{\mu_0^8}{(\hat{x}^2)^d} g(\hat{F} \hat{\lambda}^{-1/y_\lambda} \hat{F} \hat{m}_q^{-1/y_m}, \hat{F} \hat{x}^{d/2-1}, \mu), \quad \text{(B8)}$$

with $y_\lambda = d - \Delta_{\mathcal{O}}$, $y_m = d - 3 + \gamma_*$, g a dimensionless function and μ_0 some arbitrary reference scale used such that all hatted quantities are dimensionless. Above we used that F_D is of mass dimension $d/2 - 1$ and $\Delta_{T^\rho{}_\rho} = d$ (3.6) since $\beta'_* = 0$. In order to regain predictiveness we must know the λ- and the F-behavior. First, since the TEMT is proportional to $m_{\pi,D}^2 = \mathcal{O}(\lambda)$ it follows that $\langle T^\rho{}_\rho(x) T^\rho{}_\rho(0) \rangle = \mathcal{O}(\lambda^2)$ and thus the scaling exponent effectively changes from $d \to d - y_\lambda = \Delta_{\mathcal{O}}$ as one would expect. We may then drop the first argument in g and focus on the F-dependence. For the F-dependence we need to think in terms of the EFT. For the two Goldstone case there is no F-dependence (B6) and thus no further change in the scaling. The single dilaton intermediate state is $\mathcal{O}(F_D^2)$ which can be formed as the ratio of the first and the second entry in g to yield $x^2 F^2$ and thus $d \to d - y_\lambda - (d/2 - 1) = \Delta_{\mathcal{O}} - (d/2 - 1)$ in that case. Finally, one gets

$$\langle T^\rho{}_\rho(x) T^\rho{}_\rho(0) \rangle_{\rm CDQCD} \propto \frac{c_D e^{-m_D x}}{(x^2)^{\Delta_{\mathcal{O}} - (d/2-1)}} + \frac{1}{(x^2)^{\Delta_{\mathcal{O}}}} (c_{2\pi} e^{-2m_D x} + c_{2D} e^{-2m_D x}), \quad \text{(B9)}$$

where $c_D = \mathcal{O}(F_D^2)$ and $c_{2\pi,D}$ are $F_{\pi,D}$-independent as of above. The exponential behavior follows from dominance of the lowest state in the specific channel and is well known from the study of Euclidean correlation functions in lattice applications.

Equating (B7) with (B9) one gets the result $\Delta_{\mathcal{O}} = d - 2$ in accordance with (5.10), serving as another consistency check.

3. The single soft theorem, $\langle 0 | \bar{q}q | D \rangle$ and the dilaton decay constant

We may apply the single dilaton soft theorem (5.2) to $\mathcal{O}_{\bar{q}q}$ (B1) which yields

$$\langle 0 | \mathcal{O}_{\bar{q}q} | D \rangle = -\frac{1}{F_D} \langle [Q_D, \mathcal{O}_{\bar{q}q}] \rangle = -\frac{\Delta_{\bar{q}q}}{F_D} \langle \mathcal{O}_{\bar{q}q} \rangle, \quad \text{(B10)}$$

as the remainder vanishes and we notice that the same result would follow from expanding (B2) to linear order in the dilaton field. We may now make contact with [15,171] where the following matrix element was defined

$$\langle 0 | \sum_q m_q \bar{q}q | D \rangle = m_D^2 F_S, \quad \text{(B11)}$$

such that F_S is RG invariant. Combining this equation together with (5.14) and (B10) one gets

$$F_S = N_f \frac{\Delta_{\bar{q}q}}{(1+\gamma_*)} \left(\frac{m_\pi^2 F_\pi^2}{m_D^2 F_D^2} \right) F_D. \quad \text{(B12)}$$

It has previously been obtained in Eq. 5 of [177] from a soft-theorem and from the EFT cf. Eq. (9) in [171] ($y = \Delta_{\bar{q}q}$ in their notation).[31] We may use this formula to assess the zero chiral mass hypothesis ($\bar{m}_D = 0$) as then the dilaton mass ought to be well approximated by the

[31] A distinctive feature is that in [171] a generic potential (2.19) is assumed on top of $m_q \neq 0$ with $\Delta_{\bar{q}q} = y$ and this leads to $f_{\pi,D}$ and $F_{\pi,D}$ where the lower case quantities are the one in the chiral limit. Using that $F_\pi / F_D = f_\pi / f_D$ [171] we can escape these difficulties in principle and if we use the values in [15] we of get an agreement of the right- and left-hand within 7% which is well within the errors. This is of no surprise as according to our understanding F_S went into the fit in [171]. This is just a confirmation of their numbers.

dilaton GMOR-relation (2.21). Using the latter in (B12) we get a remarkably simple relation between F_D and F_S (with $\gamma_* = 1$)[32]

$$F_D|_{\bar{m}_D=0} = 2F_S, \quad \text{(B13)}$$

that is the coupling of the dlaton to the EMT- and the $\bar{q}q$-operator respectively. We may then define the quantity

$$R_{S,N_f} = \sqrt{\frac{N_f}{2}}\frac{m_\pi F_\pi}{m_D F_S}, \qquad R_{S,N_f}|^{\rm LO}_{\bar{m}_D=0} = 1, \quad \text{(B14)}$$

which is unity in the $\bar{m}_D \to 0$ limit at LO. Using the LSD-data [15] one finds

$$R_{S,8}|_{\rm LSD} \approx 1.18 \pm 5\%, \quad \text{(B15)}$$

where we have added the lattice uncertainties in quadrature. Note that since $m_\pi/F_\pi \approx 4$ is a similar to the ratio to the kaon mass to the pion decay constant in QCD NLO-corrections could easily amount to 30% (e.g., $m_D^2/m_\rho^2 \approx 0.4$). Hence, the 20%-proximity to unity in (B15) is thus remarkable. Notice that for QCD with $m_\pi/F_\pi \approx 4$ the dilaton (or σ-meson) is a stable bound state [169], suggesting that the same is the case in these simulations. Moreover, $\bar{m}_D \neq 0$ would lead to

$$R_{S,N_f} = \frac{1}{\sqrt{1+x^2}} \leq 1, \qquad x \equiv \frac{\bar{m}_D}{(m_D)_{\rm GMOR}}, \quad \text{(B16)}$$

where $(m_D)_{\rm GMOR}$ is the mass from the dilaton GMOR-relation (2.21). We conclude that for $N_f = 8$ the data do not exclude a massless dilaton in the chiral limit.[33] It would be interesting to extend this analysis to the other simulations such as the $SU(3)$-case with $N_f = 4$ [24,25] or the $SU(3)$ sextet-representation [178–180]. One might be hopeful that this will happen in the foreseeable future.

APPENDIX C: ON SINGLET-OCTET MIXING

The aim of this Appendix is to give some more interpretation with regards to the quark content of the σ-meson. Unlike for flavored mesons this is not a well-defined question. There are many ways to think about it as summarized in the excellent review [139].

An instructive starting point is the following ratio of rates [138]

$$r_{f_0\to KK/\pi\pi} = \frac{\Gamma(f_0(980)\to K^+K^-)}{\Gamma(f_0(980)\to \pi^+\pi^-)} = 0.69(32). \quad \text{(C1)}$$

If the $f_0(980)$ roughly equal parts of u, d, s-quarks then phase space would dictate $r_{f_0\to KK/\pi\pi} \ll 1$. The surprisingly large rate to kaons could be explained by the f_0 having a large mount of $\bar{s}s$-quarks versus $\bar{q}q$-quarks ($q = u, d$). It is indeed the commonly accepted view that $f_0(980)$ has a high strange quark content.

It is helpful to consider the octet and singlet in a $\bar{q}q$- and a tetraquark-basis

$$|S_1^{I=0}\rangle_{\bar{q}q} = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d + \bar{s}s), \qquad |S_1^{I=0}\rangle_{\bar{q}q\bar{q}q} = \frac{1}{\sqrt{6}}(\bar{s}s\bar{u}u + \bar{s}s\bar{d}d + \bar{u}u\bar{d}d),$$
$$|S_8^{I=0}\rangle_{\bar{q}q} = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s), \qquad |S_8^{I=0}\rangle_{\bar{q}q\bar{q}q} = \frac{1}{\sqrt{6}}(\bar{s}s\bar{u}u + \bar{s}s\bar{d}d - 2\bar{u}u\bar{d}d). \quad \text{(C2)}$$

The difference in role of the strange quark in the two bases is apparent. With the central value from the $SU(3)_F$-analysis (7.8), one finds the following quark compositions

$$|\sigma\rangle_{\bar{q}q} \approx 0.67(\bar{u}u + \bar{d}d) + 0.28\bar{s}s, \qquad |\sigma\rangle_{\bar{q}q\bar{q}q} \approx 0.67(\bar{s}s\bar{u}u + \bar{s}s\bar{d}d) + 0.28\bar{u}u\bar{d}d,$$
$$|f_0\rangle_{\bar{q}q} \approx 0.20(\bar{u}u + \bar{d}d) - 0.95\bar{s}s, \qquad |f_0\rangle_{\bar{q}q\bar{q}q} \approx 0.20(\bar{s}s\bar{u}u + \bar{s}s\bar{d}d) - 0.95\bar{u}u\bar{d}d. \quad \text{(C3)}$$

We see that the strange quark content in the $f_0(980)$ is enhanced in the $\bar{q}q$-states and suppressed for the tetraquarks. Hence the $\bar{q}q$-state interpretation harmonizes with the commonly accepted picture that the $f_0(980)$ has a large strange quark content.[34] Other analyses finding support for the $\bar{q}q$-interpretation, which is not the common view, are for example [143,181,182].

[32] The relation (B13) is consistent with the large-N_c considerations in Sec. VI B. If $F_D = \mathcal{O}(N_c)$ then the same holds for $F_S = \mathcal{O}(N_c)$ since the dilaton does not couple to the closed $\bar{q}q$-correlator at leading order in N_c.

[33] If we take $F_\pi^2/F_D^2 = 0.1089(41)$ [171] and $F_\pi = 0.021677(40)$ [15] and get $F_S \approx 0.033$ from (B13) which is 30% off from $F_S = 0.0254(17)$ [15]. In our view the analysis above is preferable since it is independent of the potential.

[34] In fact the angle where this is picture is extremized is $\theta \approx 35.7°$, also known as ideal mixing. The angle, in the original proposal of Jaffe, is $\theta \approx -54.7°$, where $|\sigma\rangle_{\bar{q}q\bar{q}q} = \bar{u}u\bar{d}d$.

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