Derivation of the string equation of motion in general relativity*

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It is shown that the equation of motion for a Nambu string in curved space can be derived from the gravitational field equation following the method of Einstein, Infeld, and Hoffmann.

INTRODUCTION

A great conceptual simplification was achieved in general relativity when Einstein, Infeld, and Hoffmann succeeded in deriving the geodesic equation of motion for a world-line singularity of the field from Einstein's gravitational equations for the metric.[1] To this end the energy-momentum tensor $T^{\mu\nu}$ in the right-hand side of the field equations

$$G^{\mu\nu}=\kappa T^{\mu\nu} \tag{1}$$

is taken as

$$\sqrt{-g}T^{\mu\nu}=\int\epsilon\frac{dz^\mu}{ds}\frac{dz^\nu}{ds}\delta^{(4)}(x-z(s))ds, \tag{2}$$

where

$$\begin{aligned}\rho&=\epsilon\delta^{(4)}(x-z(s))\\&=\epsilon\delta(x^0-z^0(s))\delta(x^1-z^1(s))\\&\quad\times\delta(x^2-z^2(s))\delta(x^3-z^3(s))\end{aligned}$$

is the mass density which is infinite on the world line $z^\alpha=z^\alpha(s)$ of the singularity and zero elsewhere. Because of the scalar density character of the δ function the left-hand side of Eq. (2) is a tensor density. From Eq. (1) follows the conservation law

$$\nabla_\mu T^{\mu\nu}=0, \tag{3}$$

or

$$\partial_\mu(\sqrt{-g}T^{\mu\nu})+\sqrt{-g}\Gamma^\nu_{\mu\alpha}T^{\mu\alpha}=0\,. \tag{4}$$

Integration of (4) over a tubular space-time region gives, in the test-particle approximation,

$$\int ds\left(\frac{d^2z^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dz^\alpha}{ds}\frac{dz^\beta}{ds}\right)=0\,, \tag{5}$$

which is nothing but the geodesic equation,

$$\delta\int ds=\delta\int\left(\frac{dz^\mu}{ds}\frac{dz^\nu}{ds}g_{\mu\nu}\right)^{1/2}ds=0\,, \tag{6}$$

where

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu \tag{7}$$

is the line element.

When a string moves it traces a two-dimensional world sheet instead of a world line generated by the motion of a point mass. The equation of motion for the string was postulated to be

$$\delta\int\!\!\int(d\sigma_{\alpha\beta}d\sigma^{\alpha\beta})^{1/2}=\delta\int\!\!\int d\tau\, d\xi(\sigma_{\alpha\beta}\sigma^{\alpha\beta})^{1/2}=0 \tag{8}$$

by Nambu[2] as a generalization of Eq. (6). The equation defining the world sheet is taken to be

$$z^\mu=z^\mu(\tau,\,\xi)\,, \tag{9}$$

and

$$\sigma^{\alpha\beta}=u^\alpha v^\beta-u^\beta v^\alpha\,, \tag{10}$$

where

$$u^\alpha=\frac{\partial z^\alpha}{\partial\tau},\quad v^\alpha=\frac{\partial z^\alpha}{\partial\xi}\,. \tag{11}$$

The string equation of motion was further studied by many authors[3-5] in connection with duality. The purpose of this note is to show that the world-sheet equation (8) can be derived in general relativity following Einstein's method for the geodesic world line.

MOTION OF A STRING IN A GRAVITATIONAL FIELD

To find the general-relativistic equation of motion of a string we start with the conservation law in its form of Eq. (4). We now take the energy-momentum tensor of the string, which is only nonzero on the world sheet of the string. The metric $g_{\mu\nu}$ represents the external gravitational field in which the test string is moving. Following Kalb and Ramond[5] we define the energy-momentum tensor of the string as

$$\sqrt{-g}T^{\mu\nu}=\int\epsilon\frac{\sigma^{\mu\alpha}\sigma^\nu_{\ \alpha}}{(\frac{1}{2}\sigma^{\beta\gamma}\sigma_{\gamma\beta})^{1/2}}\delta^{(4)}(x-z(\tau,\,\xi))\,d\tau\, d\xi\,, \tag{12}$$

where

$$\rho=\epsilon\delta(x-z(\tau,\,\xi))$$

is the mass density of the string, which is only nonzero on the world sheet of the string. $\sigma^{\mu\alpha}$ is defined by Eq. (10), with τ and ξ chosen in such a way[2-4] that

$$u^\mu u_\mu + v^\mu v_\mu = 0\,, \quad v^\mu u_\mu = 0\,. \tag{13}$$

In accordance with the test-string approximation, we have made two assumptions. Firstly, we assumed that the string is very far away from the matter which creates the gratitational field $g_{\mu\nu}$. Secondly, we have not included the gravitational field of the string itself into $g_{\mu\nu}$.

In order to make the derivation simpler, we consider the coordinate transformation

$$x^\mu = f^\mu(\xi^\lambda)\,, \tag{14}$$

where

$$\xi^\lambda = (\tau, \omega^1, \omega^2, \xi) = F^\lambda(x^\mu)\,. \tag{15}$$

We assume that the inverse function $F^\lambda(x^\mu)$ exists. The additional coordinates ω^1 and ω^2 are defined in such a way that they are zero on the world sheet. Then we can write

$$z^\mu(\tau, \xi) = \int f^\mu(\tau, \xi, \omega)\delta^{(2)}(\omega)d^2\omega = f^\mu(\tau, \xi, 0, 0)\,,$$

where

$$\omega = (\omega_1, \omega_2)\,.$$

The energy-momentum tensor in Eq. (12) can be written as[6]

$$\begin{aligned}\sqrt{-g}T^{\mu\nu} &= \int H^{\mu\nu}(\tau, \xi)\delta^{(4)}(x^\mu - f^\mu(\tau, \xi, 0, 0))\,d\tau\, d\xi \\ &= \int H^{\mu\nu}(\tau, \xi)\delta^{(4)}(x^\mu - f^\mu(\tau, \xi, \omega))\delta^{(2)}(\omega)d^4\xi\,,\end{aligned} \tag{16}$$

where

$$\begin{aligned}H^{\mu\nu} &= \frac{\sigma^{\mu\alpha}\sigma^\nu{}_\alpha}{(-\frac{1}{2}\sigma^{\gamma\beta}\sigma_{\gamma\beta})^{1/2}} = u^\mu u^\nu - v^\mu v^\nu\,, \\ d^4\xi &= d\tau\, d\xi\, d\omega^1 d\omega^2\,.\end{aligned} \tag{17}$$

Now let us go back to Eq. (4) and write it as

$$I^\nu + J^\nu = 0\,,$$

where

$$I^\nu = \int_{V_4} \partial_\mu(\sqrt{-g}T^{\mu\nu})d^4x = \oint_S \sqrt{-g}T^{\mu\nu}dS_\mu\,, \tag{18}$$

$$\begin{aligned}J^\nu &= \int_{V_4} \sqrt{-g}\,\Gamma^\nu_{\mu\alpha}T^{\mu\alpha}d^4x \\ &= \int_{\xi_1}^{\xi_2}\int_{\tau_1}^{\tau_2} d\tau\, d\xi\, H^{\mu\alpha}\Gamma^\nu_{\mu\alpha}\,.\end{aligned} \tag{19}$$

For the last equality we used the form of $T^{\mu\nu}$ in Eq. (16). The surface S in Eq. (18), which covers V_4, can be chosen in such a way that $S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$ and $T^{\mu\nu}$ vanishes on the surfaces S_5 and S_6. Then we define the remaining surfaces as follows:

$\tau = \tau_1$ is the equation of S_1, with $dS_\mu = -u_\mu d\xi\, d\omega^1 d\omega^2$;
$\tau = \tau_2$ is the equation of S_2, with $dS_\mu = u_\mu d\xi\, d\omega^1 d\omega^2$;
$\xi = \xi_1$ is the equation of S_3, with $dS_\mu = v_\mu d\tau\, d\omega^1 d\omega^2$;
$\xi = \xi_2$ is the equation of S_4, with $dS_\mu = -v_\mu d\tau\, d\omega^1 d\omega^2$.

The energy-momentum tensor in Eq. (16) can also be written as

$$\sqrt{-g}T^{\mu\nu} = \frac{1}{J}H^{\mu\nu}\delta^{(2)}(\omega)\,. \tag{20}$$

This can be obtained easily by using the identity

$$d^4x = J\, d^4\xi\,,$$

where J is the determinant of the transformation matrix $\partial x^\mu/\partial\xi^\nu = A^\mu{}_\nu$. Then after some simple manipulations we find

$$\begin{aligned}I^\nu &= \int_{\tau_1}^{\tau_2} d\tau \frac{\partial}{\partial\tau}\int_{\xi_1}^{\xi_2} \frac{\phi^2}{J}u^\nu d\xi \\ &\quad - \int_{\xi_1}^{\xi_2} d\xi \frac{\partial}{\partial\xi}\int_{\tau_1}^{\tau_2} \frac{\phi^2}{J}v^\nu d\tau\,,\end{aligned} \tag{21}$$

where

$$\phi^2 = u^\nu u_\nu = -v^\nu v_\nu\,. \tag{22}$$

Combining Eqs. (19) and (21) we get

$$\frac{\partial}{\partial\tau}(\phi^2 J^{-1}u^\nu) - \frac{\partial}{\partial\xi}(\phi^2 J^{-1}v^\nu) + H^{\mu\alpha}\Gamma^\nu_{\mu\alpha} = 0\,.$$

Now we complete the determination of the functions $f^\mu(\xi^\lambda)$ in Eq. (14) by choosing $J = \phi^2$ to obtain the equation of motion of a string in curved space-time

$$\frac{\partial^2 z^\mu}{\partial\tau^2} - \frac{\partial^2 z^\mu}{\partial\xi^2} + \Gamma^\mu_{\alpha\beta}H^{\alpha\beta} = 0\,. \tag{23}$$

The form (23) in a general coordinate system is due to Ramond.[7]

Another way of obtaining the same result is through the use of the variational principle. To this end we take Eq. (8) and vary with respect to τ and ξ, that is, $\delta = \delta_\tau + \delta_\xi$. It remains to show that this action is the one for the string singularities in curved space-time. In general relativity, to obtain the field equations (1) we use the action

$$I = \int \sqrt{-g}R\, d^4x, \quad R = +g_{\mu\nu}R^{\mu\nu} = -\kappa T^\mu{}_\mu$$

and take the variation with respect to the metric. This same action should also be used to obtain the equation of motion of the string singularities. Using the field equations, we can also write the action I as

$$I=-\kappa\int\sqrt{-g}\,T^{\mu}{}_{\mu}\,d^4x\,.$$

Equation (16) gives

$$I=-\kappa\int_{\text{world sheet}}H^{\mu}{}_{\mu}(\tau,\xi)d\tau\,d\xi\,. \tag{24}$$

From the invariance of I under general coordinate transformations in τ and ξ the integrand in Eq. (24) can only be the differential area element on the world sheet. In general, for an n-dimensional body which traces an $(n+1)$-dimensional world volume in space-time, Eq. (24) becomes

$$I=-\kappa\int H^{\mu}{}_{\mu}(y^1,\ldots,y^{n+1})\prod_{i=1}^{n+1}dy^i\,, \tag{25}$$

where $H^{\mu\nu}(y^1,\ldots,y^{n+1})$ is a symmetric tensor which is defined on the $(n+1)$-dimensional volume, and

$$H^{\mu}{}_{\mu}\prod_{i=1}^{n+1}dy^i$$

represents the $(n+1)$-dimensional invariant volume element. For the values of $n=0, 1, 2, 3,$ we obtain the actions for a world line, a world sheet, a world 3-volume, and the world 4-volume, respectively.

Finally, we would like to add a remark on the decomposition of the energy-momentum tensor $T^{\mu\nu}$ which includes contributions from electrically neutral matter and neutral fields other than the gravitational field. In general, $T^{\mu\nu}$, which has ten independent components, can be written as the sum of the energy-momentum tensor of a perfect fluid (which has five independent components) and the energy-momentum (Maxwell) tensor associated with an antisymmetrical tensor field, which depends on five independent components[8] since the antisymmetrical tensor is only defined up to a duality rotation. Hence the energy-momentum tensor of neutral matter, confined in a space-time region, can be written as

$$T^{\mu\nu}=T_p^{\mu\nu}+T_s^{\mu\nu}+Bg^{\mu\nu}\,,$$

where $T_p^{\mu\nu}$ is the energy-momentum tensor due to point mass particles of the type given in Eq. (2), $T_s^{\mu\nu}$ is the energy-momentum tensor of a test string, given by Eq. (12), and B is like a "bag" term.[9] In the most general expression for charged matter we would also need an electromagnetic contribution in the form of a traceless Maxwell tensor of electric type in contrast to the string tensor which is of magnetic type. The two types of Maxwell tensors add up to the most general energy-momentum tensor associated with an antisymmetrical tensor field.

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