# Bethe-Salpeter equation for nucleon-nucleon scattering: Matrix Padé approximants. II*

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The matrix Padé approximants introduced in a previous paper are extended to include "off-shell" effects. The [1/1] Padé approximant becomes very accurate with the addition of only a few "off-shell" states.

## I. INTRODUCTION

What we develop in this paper is a method applicable to the summation of the series resulting from quantum field theory and powerful enough to sum the series resulting from iterating the Bethe-Salpeter equation in the ladder approximation at the first level of approximation. Earlier versions[1] of the method, which have already been applied to quantum field theory,[2-4] were not powerful enough to sum the ladder graphs at the first level of approximation. The required extension is so slight, however, that the field-theory calculations will not require much more computational effort to take it into account.

Bessis, who originally proposed the idea which we pursue in this paper, calls the sort of Padé approximants to which we refer here "operator Padé approximants." Since Bessis[5] has reviewed these approximants, since we have published a paper[6] about the Bethe-Salpeter equation, since we have published a paper[1] about the application of matrix Padé approximants to the Bethe-Salpeter equation, and since this paper constitutes an addendum to the last paper, introductory material need not be repeated.

We extend Eq. (8) of Ref. 1 to read

$$(\tan\delta)_{ij} = \frac{E}{2\hat{p}}\phi(p_i, p_{0i}, \alpha_i; p_j, p_{0j}, \alpha_j) . \qquad (1)$$

$\alpha_i$ has the range 1–4, and the $p_i, p_{0i}$ are picked out of the continuum of values lying between 0 and $\infty$ or $0i$ to $\infty i$, respectively. For $p \leq \hat{p}$, $p_{0i} = E - E(p_i)$, which is real rather than imaginary, is a possible choice, and one must use the $\phi$'s appropriate to this choice (see Ref. 6 for full details about the Bethe-Salpeter amplitude $\phi$). In Ref. 1 we took

$$\begin{aligned} &i=1, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=1, \\ &i=2, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=2, \\ &i=3, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=3, \\ &i=4, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=4, \end{aligned} \qquad (2)$$

that is, all momenta were assigned their physical values and only the spin and energy-spin variables allowed to take different values in such a way as to define a tangent matrix or operator. With this choice, we always had $p_0 = 0 = E - E(\hat{p})$, so that only one part of the Bethe-Salpeter amplitude was used. Now we propose to allow the momenta to take different values as well. By taking into account "off-shell" effects in this way, we hope to find that the sequence of Padé approximants converges much faster. But we find more than we hope: By enlarging the size of the matrices only a little the [1/1] Padé approximant becomes very accurate.

## II. MINIMAL ENLARGEMENT OF THE MATRICES

The choice of the off-shell states is completely arbitrary. In all probability, no matter what one chooses the sequence of $[N/N]$ Padé approximants will converge. The best choice is the one with fewest off-shell states, so that the [1/1] Padé approximant is accurate for, say, $g^2/4\pi < 20$. One expects that, because of the singular nature of the Bethe-Salpeter equation, a few states of high momenta will have to be chosen. We start with such a choice in this section, but we find the [1/1] accurate only for $g^2/4\pi < 9$. In Sec. III we try adding some intermediate states, and find accuracy for $g^2/4\pi < 20$; and in Sec. IV we try reducing the number of intermediate states while retaining accuracy. In this way we arrive at what we believe to be a near-optimal choice of states.

We choose

$$\begin{aligned} &i=1, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=1, \\ &i=2, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=2, \\ &i=3, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=3, \\ &i=4, \quad p=2382.2049, \quad p_0=i1518, \quad \alpha=1, \\ &i=5, \quad p=2382.2049, \quad p_0=i1518, \quad \alpha=2, \\ &i=6, \quad p=2382.2049, \quad p_0=i1518, \quad \alpha=3, \\ &i=7, \quad p=2382.2049, \quad p_0=i1518, \quad \alpha=4, \end{aligned} \qquad (3)$$

where, since $E_{\text{lab}} = 100$ MeV, $\hat{p} = 216.56408$, and all momenta are in MeV.

The tangent of the phase shift,

$$\tan\delta = (\tan\delta)_{11} ,$$

calculated from the [1/1] Padé approximant, is plotted in Fig. 1, and the result may be compared with the exact result as read from the curve marked 8 in Fig. 1 of Ref. 1. The agreement is good for $|g^2/4\pi| < 9$ and very much better than the [1/1] results reported in Ref. 1. The first and second Born approximations from which the [1/1] approximant is calculated are tabulated in Table I.

## III. ENLARGEMENT OF THE MATRICES

We choose

$$\begin{aligned} &i=1-3, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=1-3, \\ &i=4-7, \quad p=433.128\,16, \quad p_0=i36.911\,285, \quad \alpha=1-4, \\ &i=8-11, \quad p=433.128\,16, \quad p_0=i181.878\,96, \quad \alpha=1-4, \\ &i=12-15, \quad p=2382.2049, \quad p_0=i12.545\,455, \quad \alpha=1-4, \\ &i=16-19, \quad p=2382.2049, \quad p_0=i138, \quad \alpha=1-4, \\ &i=20-23, \quad p=2382.2049, \quad p_0=i1518, \quad \alpha=1-4. \end{aligned} \tag{4}$$

The [1/1] Padé approximant is exact, as might be expected with such a large set of states. Since

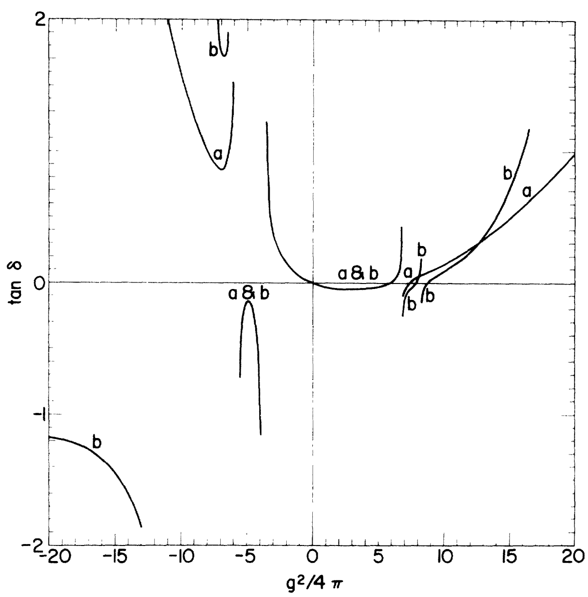


FIG. 1. Tan$\delta$ ($^1S_0$) at $E_{\text{lab}}=100$ MeV calculated from the [1/1] Padé approximant. Curve $a$ results from the choice of states shown in Eq. (3) and the first and second Born elements given in Table I. Curve $b$ results from the choice of states shown in Eq. (5). Either result should be compared with the curve marked 8 in Fig. 1 of Ref. 1, or the curve marked 19 in Fig. 2 of Ref. 1, either of these being the exact result.

the field-theory calculations require so much time, we seek an intermediate number of states resulting in good accuracy.

## IV. INTERMEDIATE ENLARGEMENTS

We choose

$$\begin{aligned} &i=1-3, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=1-3, \\ &i=4-7, \quad p=433.128\,16, \quad p_0=i1518, \quad \alpha=1-4, \\ &i=8-11, \quad p=2382.2049, \quad p_0=i1518, \quad \alpha=1-4. \end{aligned}$$

This choice gives a result almost identical to the result given by the choice Eq. (3), so that nothing has been gained by putting in the four additional states.

Therefore, we choose

$$\begin{aligned} &i=1-3, \quad p=\hat{p}, \quad p_0=0, \quad \alpha=1-3, \\ &i=4-7, \quad p=433.128\,16, \quad p_0=0, \quad \alpha=1-4, \\ &i=8-11, \quad p=2382.2049, \quad p_0=i1518, \quad \alpha=1-4. \end{aligned} \tag{5}$$

TABLE I. First and second Born approximations to $(\tan\delta)_{ij}$ for the minimal enlargement of the set of basis states. $E_{\text{lab}}=100$ MeV. Compare to a similar table in Ref. 1. These matrices are enlargements of those in Ref. 1.

| | | |
|---|---|---|
| $T^{(1)}_{11}=-0.042\,63$, | $T^{(1)}_{12}=-2.607$, | $T^{(1)}_{13}=-0.2611$, |
| $T^{(1)}_{14}=-0.090\,87$, | $T^{(1)}_{15}=-0.1958$, | $T^{(1)}_{16}=-0.1830$, |
| $T^{(1)}_{17}=0.0$, | $T^{(1)}_{22}=-0.042\,63$, | $T^{(1)}_{23}=0.2611$, |
| $T^{(1)}_{24}=-0.1958$, | $T^{(1)}_{25}=-0.090\,87$, | $T^{(1)}_{26}=0.1830$, |
| $T^{(1)}_{27}=0.0$, | $T^{(1)}_{33}=1.050$, | $T^{(1)}_{34}=-0.008\,815$, |
| $T^{(1)}_{35}=0.008\,815$, | $T^{(1)}_{36}=0.064\,39$, | $T^{(1)}_{37}=0.0$, |
| $T^{(1)}_{44}=-0.7164$, | $T^{(1)}_{45}=-3.909$, | $T^{(1)}_{46}=-0.3989$, |
| $T^{(1)}_{47}=0.0$, | $T^{(1)}_{55}=-0.7164$, | $T^{(1)}_{56}=0.3989$, |
| $T^{(1)}_{57}=0.0$, | $T^{(1)}_{66}=4.404$, | $T^{(1)}_{67}=0$, |
| $T^{(1)}_{77}=2.693$. | | |
| $T^{(2)}_{11}=0.010\,38$, | $T^{(2)}_{12}=0.078\,66$, | $T^{(2)}_{13}=0.008\,403$, |
| $T^{(2)}_{14}=0.046\,39$, | $T^{(2)}_{15}=0.073\,92$, | $T^{(2)}_{16}=0.046\,80$, |
| $T^{(2)}_{17}=0.020\,56$, | $T^{(2)}_{22}=0.6027$, | $T^{(2)}_{23}=-0.094\,34$, |
| $T^{(2)}_{24}=0.1839$, | $T^{(2)}_{25}=0.4024$, | $T^{(2)}_{26}=0.2353$, |
| $T^{(2)}_{27}=-0.020\,56$, | $T^{(2)}_{33}=-0.033\,57$, | $T^{(2)}_{34}=0.007\,504$, |
| $T^{(2)}_{35}=0.015\,36$, | $T^{(2)}_{36}=-0.002\,137$, | $T^{(2)}_{37}=-0.007\,359$, |
| $T^{(2)}_{44}=0.5048$, | $T^{(2)}_{45}=0.4336$, | $T^{(2)}_{46}=0.080\,83$, |
| $T^{(2)}_{47}=0.042\,85$, | $T^{(2)}_{55}=1.461$, | $T^{(2)}_{56}=0.1959$, |
| $T^{(2)}_{57}=-0.042\,85$, | $T^{(2)}_{66}=-0.3427$, | $T^{(2)}_{67}=-0.4645$, |
| $T^{(2)}_{77}=0.1213$. | | |

Now the improvement is substantial, and the result, also plotted in Fig. 1, is very similar to the exact result for $|g^2/4\pi| < 20$, and we conclude that such a choice gives the best results with a minimum number of additional states.

## V. SOME DETAILS OF THE CALCULATION

Most details are exactly as in Ref. 5 and 6. The details to be coped with arise from the Wick rotation, the symmetry in $p_0$ which exists for two identical particles, and the desire to have real symmetric matrices. The Bethe-Salpeter is extended to read

$$\phi = \pm K + \frac{1}{\pi^2} K S \phi , \tag{6}$$

where the minus sign in $\pm K$ is used for $K(p, p_0, a, q, iq_4, 4)$ and $K(p, p_0, 4, q, iq_4, 4)$, where $p_0$ is either real $[E - E(p)]$ or imaginary $(ip_4)$. A real symmetric tangent matrix results with these real, symmetric $K$'s (again, $q_0$ may be real $[E - E(q)]$ or imaginary $(iq_4)$):

$$\begin{aligned}
&K(p, ip_4, a, q, q_0, b) = \tfrac{1}{2}[G(p, ip_4, a, q, q_0, b) + G(p, ip_4, a, q, -q_0, b)],\\
&K(p, E-E(p), a, q, iq_4, b) = \mathrm{Re}G(p, E-E(p), a, q, iq_4, b),\\
&K(p, p_0, a, q, iq_4, 4) = \tfrac{1}{2}\mathrm{Re}[-iG(p, p_0, a, q, iq_4, 4) + iG(p, p_0, a, q, -iq_4, 4)],\\
&K(q, iq_4, 4, p, p_0, a) = \tfrac{1}{2}[iG(q, iq_4, 4, p, p_0, a) + iG(q, iq_4, 4, p, -p_0, a)],\\
&K(p, E-E(p), a, q, E-E(q), b) = \tfrac{1}{2}[G(p, E-E(p), a, q, E-E(q), b) + G(p, E-E(p), a, q-[E-E(q)], b)],\\
&K(p, E-E(p), 4, q, q_0, a) = \tfrac{1}{2}[G(p, E-E(p), 4, q, q_0, a) + G(p, E-E(p), 4, q, -q_0, a)],\\
&K(p, ip_4, 4, q, iq_4, 4) = \tfrac{1}{2}[G(p, ip_4, 4, q, iq_4, 4) - G(p, ip_4, 4, q, -iq_4, 4)],\\
&K(p, E-E(p), 4, q, iq_4, 4) = \tfrac{1}{2}\mathrm{Im}[G(p, E-E(p), 4, q, iq_4, 4) - G(p, E-E(p), 4, q, -iq_4, 4)],\\
&K(p, E-E(p), 4, q, E-E(q), 4) = \tfrac{1}{2}[G(p, E-E(p), 4, q, E-E(q), 4) - G(p, E-E(p), 4, q, -[E-E(q)], 4)],\\
&K(p, p_0, a, q, E-E(q), 4) = \tfrac{1}{2}[G(p, p_0, a, q, E-E(q), 4) - G(p, p_0, a, q, -[E-E(q)], 4)],\\
&K(p, ip_4, 4, q, E-E(q), 4) = \tfrac{1}{2}i[G(p, ip_4, 4, q, E-E(q), 4) - G(p, ip_4, 4, q, -[E-E(q)], 4)].
\end{aligned} \tag{7}$$

Also,

$$S(q, iq_4, 34) = -S(q, iq_4, 43) = \frac{q_4}{2E}\left\{-\frac{1}{[E-E(q)]^2 + q_4^2} + \frac{1}{[E+E(q)]^2 + q_4^2}\right\}. \tag{8}$$

## VI. CONCLUSIONS

What we have given are sets of states [in Eqs. (3), (4), and (5)] chosen so that the [1/1] matrix Padé approximant solves the Bethe-Salpeter equation accurately. The field-theory calculations (Refs. 2, 3, and 4) may be extended without any modification to take account of these states. We have also given the rules for calculating a real, symmetric tangent matrix from a Feynman graph in Eqs. (6) and (7). Thus we hope that we have made the idea of Bessis which he reviews in Ref. 5 more concrete and more readily applicable to the field theory of nucleon-nucleon scattering.

It is important to note that the two-rung ladder graph from which the [1/1] Padé approximant is calculated is gotten[1,6] by iterating the Bethe-Salpeter equation using certain finite meshes. When we refer to an "exact" solution of the Bethe-Salpeter equation, we refer to these same finite meshes. The use of finite meshes reduces the high-momentum singularity of the Bethe-Salpeter equation, but it is hoped that actual field theory is not so singular, not even so singular as the Bethe-Salpeter equation with the finite but very fine meshes which we use.

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[1] J. L. Gammel and M. T. Menzel, Phys. Rev. D 7, 565 (1973).

[2] D. Bessis, G. Turchetti, and W. R. Wortman, Phys. Lett. 39B, 601 (1972).

[3] J. Fleischer, J. L. Gammel, and M. T. Menzel, Phys. Rev. D 8, 1545 (1973). This paper criticizes an approximation in Ref. 2.

[4] A. Gersten and Z. Solow, Phys. Rev. D 10, 1031 (1974). This paper revises the calculations of Ref. 2. The authors of Ref. 2 plan to extend their calculations. See

D. Bessis, P. Mery, and G. Turchetti, *ibid.* 10, 1992 (1974). A detail about the way the nucleon self-energy graph is put into the calculation is handled differently in this reference than in Ref. 2 and 3.

[5]D. Bessis, *Padé Approximants*, edited by P. R. Graves-Morris (The Institute of Physics, London, 1973), p. 19.

[6]J. L. Gammel, M. T. Menzel, and W. R. Wortman, Phys. Rev. D 3, 2175 (1971).