Conserved vectors in scalar-tensor gravitational theories

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Two recently derived expressions for the Brans-Dicke scalar-tensor analog of Komar's conserved vector density are compared, and it is shown that they give different results for the total energy of the scalar-tensor Schwarzschild universe.

In a recent paper,[1] Pavelle derived a conserved Komar-type[2] vector for the Brans-Dicke[3] (BD) scalar-tensor (ST) theory. He applied variational techniques to the term $\sqrt{-g}\,\phi R$ in the BD Lagrangian to arrive at the expression[4]

$$V^i_{(P)} = \tfrac{4}{3}[(\phi\xi_{a;b} - 2\phi_{;b}\xi_a)Z^{a[ji]b}]_{;j} \,. \quad (1)$$

The term $-\sqrt{-g}\,\omega g^{ij}\phi_{;i}\phi_{;j}/\phi$ in the BD Lagrangian was not included in his considerations since its associated conserved vector is identically zero.

Using an approach which parallels that used by Komar in general relativity (GR), the author derived[5] a Komar-type conserved vector for a general ST theory which reduced, in the BD case, to

$$V^i_{(H)} = [(\phi\xi^j)^{;i} - (\phi\xi^i)^{;j}]_{;j} \,. \quad (2)$$

The above expressions can be shown to be related as follows:

$$V^i_{(P)} = V^i_{(H)} + 3[\xi^i\phi^{;c} - \xi^c\phi^{;i}]_{;c} \,. \quad (3)$$

In the GR limit, $\phi \to$ const, both $V^i_{(P)}$ and $V^i_{(H)}$ reduce to Komar's conserved vector.

Since both vectors satisfy $V^i_{;i} = 0$, they can be used to form integral conservation laws. For isolated gravitating systems, application of Gauss's theorem shows that the integrals

$$P(\xi) = \int V^i dS_i = \int V^0 dS_0 \quad (4)$$

are independent of the open timelike hypersurface over which they are evaluated and are thus conserved quantities. The identity of these conserved quantities is determined by the ξ^i used to form V^i. The ξ^i are taken to be descriptors of symmetry transformations for the gravitational field being considered, and a given ξ^i can be said to generate the conserved quantity corresponding to the transformation it describes.

In GR, the symmetry transformations of a given gravitational field are determined by considering the action of a general infinitesimal coordinate transformation $\bar{x}^i = x^i + \xi^i$ on the metric tensor g_{ij}. A symmetry transformation is defined as one in which the transformed metric, $\bar{g}_{ij}$, has the same functional form as the original metric,

$$\bar{\delta} g_{ij} \equiv \bar{g}_{ij}(x) - g_{ij}(x) = 0 \,. \quad (5)$$

In terms of descriptors the above become Killing's equations

$$\xi^{i;j} + \xi^{j;i} = 0 \,, \quad (6)$$

and the solutions of these equations are symmetry-transformation descriptors.

When considering ST gravitational fields, it would seem appropriate to define symmetry transformations as those transformations under which both the metric tensor and the scalar field, ϕ, are form-invariant. The symmetry conditions would then be[5]

$$\begin{aligned} &\bar{\delta} g_{ij} = 0 \,, \\ &\bar{\delta}\phi \equiv \bar{\phi}(x) - \phi(x) = 0 \,. \end{aligned} \quad (7)$$

In terms of descriptors, these conditions become

$$\begin{aligned} &\xi^{i;j} + \xi^{j;i} = 0 \,, \\ &\phi_{;i}\xi^i = 0 \,. \end{aligned} \quad (8)$$

We now consider the BD gravitational field outside of a spherically symmetric distribution of matter[6] expressed in isotropic coordinates:

$$\begin{aligned} g_{00} &= \left(\frac{1 - B/r}{1 + B/r}\right)^{2/\lambda}, \quad \lambda = [(C+1)^2 - C(1 - \tfrac{1}{2}\omega C)]^{1/2} \\ g_{11} &= -(1 + B/r)^4\left(\frac{1 - B/r}{1 + B/r}\right)^{(2/\lambda)(\lambda - C - 1)}, \\ g_{22} &= r^2 g_{11} \,, \\ g_{33} &= r^2 \sin^2\theta\, g_{11} \,, \\ \phi &= \phi_0\left(\frac{1 - B/r}{1 + B/r}\right)^{C/\lambda}, \end{aligned} \quad (9)$$

where B, C, ϕ_0, and ω are constants. It is readily found, by application of (8), that one of the symmetry descriptors for this field is

$$\xi^i_{(0)} = \epsilon\delta^i_0 \quad (\epsilon \ll 1) \,, \quad (10)$$

the timelike translation descriptor. By analogy with symmetry-conservation laws in flat space-time, we would expect that δ^i_0, when used in Eq.

(4) with a suitable V^i, would generate the total energy or inertial mass of the system.

Using $V^i_{(H)}$ and $V^i_{(P)}$ in (4), we find that δ^i_0 generates the following conserved quantities (see the Appendix for details and the significance of the prime notation):

$$P'_{(H)}(\delta^i_0) = 2B\phi_0 c^4(1+\tfrac{1}{2}C)/\lambda\!\!\!^- = M_{I(H)}c^2\,, \qquad (11)$$

$$P'_{(P)}(\delta^i_0) = 2B\phi_0 c^4(1-C)/\lambda\!\!\!^- = M_{I(P)}c^2\,. \qquad (12)$$

Equation (11) is the total energy expression obtained by Ohanian[7] and the author[5] using other techniques.

The expressions (11) and (12) can be related to the gravitational mass of the system by noting that in the asymptotic region ($r\to\infty$) the gravitational field becomes

$$\begin{aligned} g_{00} &= 1-4B/\lambda\!\!\!^- r\,,\\ g_{11} &= -\left(1+\frac{4B}{r}\,\frac{1+C}{\lambda\!\!\!^-}\right),\\ g_{22} &= r^2 g_{11}\,,\\ g_{33} &= r^2\sin^2\theta\, g_{11}\,,\\ \phi &= \phi_0\left(1-\frac{2B}{r}\,\frac{C}{\lambda\!\!\!^-}\right). \end{aligned} \qquad (13)$$

The gravitational mass of the system is determined by comparing the above expression for g_{00} with the form taken in the Newtonian approximation

$$g_{00} = 1+2\Phi_{\text{Newton}} = 1-2G_0 M_G/rc^2\,.$$

Thus, $M_G = 2Bc^2/\lambda\!\!\!^- G_0$. Using this fact and the definition of G_0,

$$G_0 \equiv \frac{1}{\phi_0}\left(\frac{4+2\omega}{3+2\omega}\right),$$

we find

$$P'_{(H)}(\delta^i_0) = M_G c^2\left[(1+\tfrac{1}{2}C)\left(\frac{4+2\omega}{3+2\omega}\right)\right], \qquad (14)$$

$$P'_{(P)}(\delta^i_0) = M_G c^2\left[(1-C)\left(\frac{4+2\omega}{3+2\omega}\right)\right]. \qquad (15)$$

In the weak field case, $C\to -1/(2+\omega)$, and

$$P'_{(H)}(\delta^i_0)\to M_G c^2\,, \qquad (16)$$

$$P'_{(P)}(\delta^i_0)\to M_G c^2[1+3/(3+2\omega)]\,. \qquad (17)$$

If expression (11) is used for the total energy of the system, the weak principle of equivalence (WEP) is satisfied in the limit of weak gravitational fields, while if expression (12) is used the WEP is violated. Thus, $P'_{(P)}(\delta^i_0)$ cannot be considered a satisfactory total energy expression.

It should be noted that, in the general case,

$$P'_{(H)}(\delta^i_0) = M_{I(H)}c^2 \neq M_G c^2\,,$$

and the WEP does not hold, as has been pointed out elsewhere.[7,8]

APPENDIX

Since both $V^i_{(P)}$ and $V^i_{(H)}$ are derivable from two-index superpotentials,

$$\begin{aligned} V^i_{(P)} &= U^{[ji]}_{(P)}{}_{;j}\\ &= [\tfrac{4}{3}(\phi\,\xi_{a;b} - 2\phi_{;b}\,\xi_a)Z^{a[ji]b}]_{;j}\,, \end{aligned} \qquad (\text{A1})$$

$$\begin{aligned} V^i_{(H)} &= U^{[ji]}_{(H)}{}_{;j}\\ &= [(\phi\,\xi^j)^{;i} - (\phi\,\xi^i)^{;j}]_{;j}\,, \end{aligned} \qquad (\text{A2})$$

the volume integral (4) can be transformed, by Gauss's law, into a surface integral,

$$P = \int_V V^i dS_i = \int_V U^{[ji]}{}_{;j}dS_i = \tfrac{1}{2}\oint_S U^{[ji]}dS_{ij}\,. \qquad (\text{A3})$$

If V is a timelike hypersurface, then

$$P = \int_V V^i dS_i = \int_V V^0 dS_0 = \oint_S U^{[j0]}dS_{0j}\,, \qquad (\text{A4})$$

where S is a surface at spatial infinity, and we have used $dS_{ij} = -dS_{ji}$.

As pointed out previously,[5] expression (A4) does not have dimensions appropriate to total energy, and the proper expression should be

$$P' = \frac{c^4}{8\pi}P\,. \qquad (\text{A5})$$

If we take S to be a sphere of radius $R\to\infty$,

$$\begin{aligned} dS_{01} &= R^2\sin\theta\, d\theta d\phi\\ &= R^2 d\Lambda\,,\\ dS_{02} &= dS_{03} = 0\,, \end{aligned}$$

and

$$\begin{aligned} P'_{(H)}(\xi) &= \left(\frac{c^4}{8\pi}\right)\oint_S U^{[10]}_{(H)}R^2 d\Lambda\\ &= \left(\frac{c^4}{8\pi}\right)\oint_S [(\phi\,\xi^1)^{;0} - (\phi\,\xi^0)^{;1}]R^2 d\Lambda\\ &= \left(\frac{c^4}{8\pi}\right)\oint_S [g^{0k}(\phi\,\xi^1)_{,k} + g^{0k}\Gamma^1_{kl}\phi\,\xi^l\\ &\qquad - g^{1k}(\phi\,\xi^0)_{,k} - \Gamma^0_{kl}\phi\,\xi^l g^{1k}]R^2 d\Lambda\,. \end{aligned} \qquad (\text{A6})$$

Letting $\xi^i = \delta^i_0$ and using (13) for the fields in the asymptotic region, we find

$$\begin{aligned} P'_{(H)}(\delta^i_0) &= \left(\frac{c^4}{8\pi}\right)\oint_S [\phi g^{00}\Gamma^1_{00} - g^{11}\phi_{,1} - g^{11}\Gamma^0_{10}\phi]R^2 d\Lambda\\ &= \left(\frac{c^4}{8\pi}\right)\oint_S [-g^{11}\phi_{,1} - \phi g^{00}g^{11}g_{00,1}]R^2 d\Lambda\\ &= 2B\phi_0\, c^4(1+\tfrac{1}{2}C)/\lambda\!\!\!^-\,. \end{aligned} \qquad (\text{A7})$$

A similar analysis yields

$$\begin{aligned}
P'_{(P)}(\delta_0^i) &= \left(\frac{c^4}{8\pi}\right)\oint_S U_{(P)}^{[10]}R^2 d\Lambda \\
&= \left(\frac{c^4}{8\pi}\right)\oint_S \tfrac{4}{3}(\phi\,\xi_{a;b} - 2\phi_{;b}\,\xi_a)Z^{a[10]b}R^2 d\Lambda \\
&= \left(\frac{c^4}{8\pi}\right)\oint_S (\phi\,\xi_{a;b} - 2\phi_{;b}\xi_a)(g^{a1}g^{0b} - g^{a0}g^{1b})R^2 d\Lambda \\
&= \left(\frac{c^4}{8\pi}\right)\oint_S [\phi\,\xi^1_{;b}g^{0b} - 2\phi_{;b}\xi^1 g^{0b} - \phi\,\xi^0_{;b}g^{1b} + 2\phi_{;b}\xi^0 g^{1b}]R^2 d\Lambda \qquad \text{(A8)}\\
&= \left(\frac{c^4}{8\pi}\right)\oint_S [\phi g^{00}\Gamma^1_{00} - \phi g^{11}\Gamma^0_{01} + 2g^{11}\phi_{,1}]R^2 d\Lambda \\
&= \left(\frac{c^4}{8\pi}\right)\oint_S [2g^{11}\phi_{,1} - \phi g^{00}g^{11}g_{00,1}]R^2 d\Lambda \\
&= \frac{2B\phi_0 c^4}{\hbar}(1-C)\,.
\end{aligned}$$

[1]R. Pavelle, Phys. Rev. D 8, 2369 (1973).

[2]A. Komar, Phys. Rev. 113, 934 (1959).

[3]C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

[4]This expression differs by a factor of 2 from that of Ref. 1. The semicolon denotes covariant differentiation, the comma denotes ordinary differentiation, and

$$Z^{a[ji]b} = \tfrac{1}{2}(Z^{ajib} - Z^{aijb}),$$

where

$$Z^{ajib} = g^{aj}g^{ib} - \tfrac{1}{2}g^{ai}g^{jb} - \tfrac{1}{2}g^{ab}g^{ji}\,.$$

[5]H. B. Hart, Phys. Rev. D 5, 1256 (1972).

[6]See Ref. 3, p. 931.

[7]H. C. Ohanian, Ann. Phys. (N.Y.) 67, 648 (1971).

[8]K. Nordvedt, Jr., Phys. Rev. 180, 1293 (1969).