# New Faddeev-type equations for the three-body problem*

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We present new Faddeev-type equations for the three-body problem. Although obtained from the rigorous Faddeev theory, they only require two-body bound-state wave functions and *half-off-shell* transition amplitudes as input. In addition, their "effective potentials" are independent of the three-body energy, and can easily be made real after an angular momentum decomposition. The equations are formulated in terms of physical transition amplitudes for three-body processes, except that in the breakup case the partial-wave amplitudes differ from the corresponding full amplitudes by a Watson final-state interaction factor.

## I. INTRODUCTION

The formal simplicity and transparency of the abstract formulation of scattering theory is well appreciated, and on this level the theory has been developed in considerable detail not only in the two body case but also for systems with three or more particles.

However, a more detailed study of scattering theory requires the introduction of a basis in the Hilbert space, in terms of which the abstract operators and state vectors are to be represented.

In two-body scattering theory, the only natural basis is the eigenstates of the free Hamiltonian $h_0$, i.e., the plane-wave basis $\{|\vec{p}\rangle\}$. When expressing the outgoing-wave scattering state vector $|\psi^+_{\vec{k}}\rangle$ in such a basis, $\psi^+_{\vec{k}}(\vec{p}) = \langle\vec{p}|\psi^+_{\vec{k}}\rangle$, one is naturally led to a representation in terms of a less singular amplitude,

$$\psi^+_{\vec{k}}(\vec{p}) = \delta(\vec{p}-\vec{k}) - \frac{t(\vec{p},\vec{k},\bar{k}^2+i0)}{\bar{p}^2-\bar{k}^2-i0}, \tag{1.1}$$

where $\bar{k}^2 = k^2/2\mu$, and where $t(\vec{p},\vec{k},\bar{k}^2+i0)$ is just the plane-wave matrix element of the transition operator $t(z)$ [its on-shell value, i.e., the residue at the scattering pole $\bar{p}^2 = \bar{k}^2$ in (1.1), yields the physical transition amplitude].

In three-body scattering theory the situation is more complicated. Also in this case, a plane-wave representation (corresponding to the eigenstates $|\vec{p}\,\vec{q}\rangle$ of $H_0$) is natural. A detailed analysis of the singularity structure of such a representation for the three-body wave function $\langle\vec{p}\,\vec{q}|\Psi^+\rangle$ has been carried out by Faddeev,[1] and leads to an expression similar to (1.1), but now in terms of a pair of amplitudes $\mathcal{K}_{\beta\alpha}$ and $\mathcal{G}_{\beta\alpha}$ (described in Appendix A). Just as in the two-body case, these amplitudes are closely related to the physical transition amplitudes. One can then, of course, consider the Faddeev equations these amplitudes satisfy, i.e., a three-body counterpart of the Lippmann-Schwinger equation for the two-body transition amplitude; these equations have recently been advocated by Osborn and Kowalski.[2]

However, in the three-body case other natural bases are also available, namely the complete sets of channel eigenstates $\{|\vec{p}_\beta\phi^\beta_\kappa\rangle; |\vec{p}_\beta\psi^\mp_{\vec{q}_\beta}\rangle\}$ of the channel Hamiltonians $H_\beta = H_0 + V_\beta$; $\beta = 1, 2, 3$.[3]

In this paper we consider the expansion of the three-body Faddeev wave-function components in such a basis. We show that this representation is actually more natural than the plane-wave representation mentioned before, and leads to a considerably simplified formulation of the three-body theory.

This approach leads to a new pair of amplitudes $\mathcal{K}_{\beta\alpha}$ and $\mathcal{S}_{\beta\alpha}$, which represent the nonsingular parts of the three-body wave function in a simpler way than the pair $\mathcal{K}_{\beta\alpha}$ and $\mathcal{G}_{\beta\alpha}$ do. The main advantage of this formulation, however, lies in the fact that the integral equations for the new set of amplitudes $\mathcal{K}$ and $\mathcal{S}$ are significantly simpler in structure: Their effective potentials are independent of the three-body energy, and they only require two-body half-on-shell transition amplitudes and bound-state wave functions as input. Additional convenient features become apparent after an angular momentum decomposition: By a simple redefinition of the partial-wave components of the amplitude $\mathcal{S}$, the effective potentials can be made real, and the breakup scattering amplitude is seen to exhibit explicitly a Watson final-state interaction factor in each channel.

The reasons for these simplifications can be physically understood as follows: Much of the complicated structure of the plane-wave projections of the three-body wave function is not due to true three-body dynamics, but is simply a reflection of the "spectator" two-body channel dynamics. By considering these plane-wave projections, the channel dynamics are mixed with the true three-body dynamics in a complicated way. If, however, we expand each Faddeev component of the full wave function into the complete set of eigenfunctions of the spectator Hamiltonian *in its own channel*, the two-body channel dynamics are automatically treated in a natural way by these spectator complete sets; as a consequence, the three-body entities one is left to consider when solving the three-body problem get appreciably simplified.

In Sec. II we obtain the half-off-shell amplitudes $\mathcal{K}$ and $\mathcal{S}$ from the projections of the Faddeev components into the channel eigenstates. We also define the corresponding fully-off-shell amplitudes, and derive the equations they satisfy in Sec. III. In Sec. IV we consider the angular momentum decomposition of these equations in the $S$-wave case, and show how the $\mathcal{S}$ amplitude can be redefined so as to produce equations with real effective potentials. The amplitudes for processes starting from three free particles are briefly considered in Sec. V. Finally, in Sec. VI we give an example of the kind of unitarity relations our new amplitudes satisfy; the general operator unitarity relation is given in Appendix B.

## II. THE AMPLITUDES $\mathcal{K}_{\beta\alpha}$ AND $\mathcal{S}_{\beta\alpha}$

In this section we restrict ourselves to scattering processes starting from an initial state of one free particle and a two-body bound state. For this case we consider the Faddeev equation for the $\beta$ component of the three-body wave function:

$$|\Psi^+_{\beta(\alpha)}\rangle=\delta_{\beta\alpha}|\vec{p}_\alpha^{(0)}\phi^\alpha_\kappa\rangle - G_0(E+i0)\,t_\beta(E+i0)\sum_{\gamma\neq\beta}|\Psi^+_{\gamma(\alpha)}\rangle, \tag{2.1}$$

where $|\vec{p}_\alpha^{(0)}\phi^\alpha_\kappa\rangle$ describes the initial state, i.e., a bound state in channel $\alpha$ and a third free particle, and $E$ is the total energy in the initial state, $E=p_\alpha^{(0)2}/2\mu_\alpha-\kappa_\alpha^2$. Here and below we consider only one bound state per channel. Defining the complete set of channel eigenstates in channel $\beta$ by $\{|\vec{p}_\beta\phi^\beta_\kappa\rangle, |\vec{p}_\beta\psi^-_{\vec{q}_\beta}\rangle\}$, where $|\psi^-_{\vec{q}_\beta}\rangle$ is the incoming two-body scattering state with momentum $\vec{q}_\beta$, we obtain for the projection of (2.1) onto these states (recall that $G_0 t_\beta = G_\beta V_\beta$)

$$\langle\vec{p}_\beta\phi^\beta_\kappa|\Psi^+_{\beta(\alpha)}\rangle=\delta_{\beta\alpha}\delta^3(\vec{p}_\alpha-\vec{p}_\alpha^{(0)})-\frac{\mathcal{K}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(0)};E+i0)}{\bar{p}_\beta^2-\kappa_\beta^2-E-i0},$$

$$\langle\vec{p}_\beta\psi^-_{\vec{q}_\beta}|\Psi^+_{\beta(\alpha)}\rangle=-\frac{\mathcal{S}_{\beta\alpha}(\vec{p}_\beta,\vec{q}_\beta;\vec{p}_\alpha^{(0)};E+i0)}{\bar{p}_\beta^2+\bar{q}_\beta^2-E-i0}, \tag{2.2}$$

where

$$\mathcal{K}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(0)};E+i0)=\left\langle\vec{p}_\beta\phi^\beta_\kappa\middle|V_\beta\middle|\sum_{\gamma\neq\beta}\Psi^+_{\gamma(\alpha)}\right\rangle,$$

$$\mathcal{S}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(0)};E+i0)=\left\langle\vec{p}_\beta\psi^-_{\vec{q}_\beta}\middle|V_\beta\middle|\sum_{\gamma\neq\beta}\Psi^+_{\gamma(\alpha)}\right\rangle. \tag{2.3}$$

In (2.3),

$$\bar{p}_\beta^2=p_\beta^2/2n_\beta,$$

and

$$\bar{q}_\beta^2=q_\beta^2/2\mu_\beta$$

with

$$n_\beta=\frac{m_\beta(m_\gamma+m_\alpha)}{m_\alpha+m_\beta+m_\gamma}$$

and

$$\mu_\beta=m_\alpha m_\gamma/(m_\alpha+m_\gamma).$$

Using (2.2), the expansion of the plane-wave projections of the Faddeev components of the three-body wave function is obtained as

$$\langle\vec{p}_\beta\vec{q}_\beta|\Psi^+_{\beta(\alpha)}\rangle=\delta_{\beta\alpha}\delta^3(\vec{p}_\alpha-\vec{p}_\alpha^{(0)})\phi^\alpha_\kappa(\vec{q}_\alpha)-\frac{\phi^\beta_\kappa(\vec{q}_\beta)}{\bar{p}_\beta^2-\kappa_\beta^2-E-i0}\mathcal{K}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(0)};E+i0)$$
$$-\int d^3q'_\beta\;\psi^-_{\vec{q}'_\beta}(\vec{q}_\beta)\frac{1}{\bar{p}_\beta^2+\bar{q}_\beta'^2-E-i0}\mathcal{S}_{\beta\alpha}(\vec{p}_\beta\vec{q}'_\beta;\vec{p}_\alpha^{(0)};E+i0). \tag{2.4}$$

Equation (2.4) constitutes a three-body generalization of Eq. (1.1) since the amplitudes $\mathcal{K}$ and $\mathcal{S}$ of (2.3) are shown in Appendix A to be free from elastic, rearrangement, and breakup poles; i.e., they are the amplitudes in terms of which we will now formulate the three-body theory.

We first note how these amplitudes are related to the physical transition amplitudes: Recalling that the residues of the wave function at the elastic or rearrangement and breakup poles are essentially the corresponding transition amplitudes, we directly see from (2.4) that the elastic or rearrangement amplitude is simply given by the on-shell value of $\mathcal{K}_{\beta\alpha}$. In addition, it can be shown that the residue at the breakup pole $\bar{p}_\beta^2+\bar{q}_\beta^2=E$ in (2.4) is the on-shell value of $\mathcal{S}_{\beta\alpha}$, so that the breakup amplitude is given by $\sum_\beta\mathcal{S}_{\beta\alpha}$.

Having established the on-shell connection between the amplitudes $\mathcal{K}$ and $\mathcal{S}$ and the physical transition amplitudes, we look into the relationship between our amplitudes and the matrix elements of the more familiar three-body transition operators. For this purpose we recall that in the wave function formalism, the three-body operators

$K_{\beta\alpha}$ generate the Faddeev components out of the initial-state wave function,[4] i.e.,

$$|\Psi^+_{\beta(\alpha)}\rangle=[\delta_{\beta\alpha}-G_0(E+i0)K_{\beta\alpha}(E+i0)]|\vec{p}^{(0)}_\alpha\phi^\alpha_\kappa\rangle\,, \tag{2.5}$$

where $E=\tilde{p}^{(0)2}_\alpha-\kappa_\alpha{}^2$. If we take projections of (2.5) onto channel eigenstates and use the relation $G_0K_{\beta\alpha}=-G_\beta V_\beta G_0 U_{\beta\alpha}$, where $U_{\beta\alpha}$ is the Alt-Grassberger-Sandhas (AGS) transition operator,[5] we find upon comparison with (2.4) that

$$\mathcal{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}^{(0)}_\alpha;E+i0)=-\langle\vec{p}_\beta\,\phi^\beta_\kappa|V_\beta G_0(E+i0)U_{\beta\alpha}(E+i0)|\vec{p}^{(0)}_\alpha\phi^\alpha_\kappa\rangle\,, \tag{2.6}$$

$$\mathcal{S}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(0)}_\alpha;E+i0)=-\langle\vec{p}_\beta\,\psi^-_{\vec{q}_\beta}|V_\beta G_0(E+i0)U_{\beta\alpha}(E+i0)|\vec{p}^{(0)}_\alpha\phi^\alpha_\kappa\rangle\,.$$

In the on-shell limit $\langle\vec{p}_\beta\phi^\beta_\kappa|V_\beta G_0(E+i0)=-\langle\vec{p}_\beta\phi^\beta_\kappa|$, so we see that the expression for $\mathcal{H}_{\beta\alpha}$ in (2.6) reduces to the familiar expression for the elastic and rearrangement transition amplitude in terms of $U_{\beta\alpha}$.

In addition, the half-on-shell singularity-free amplitude $\mathcal{K}_{\beta\alpha}$ that in Faddeev's treatment[1,2] yields the breakup-amplitude component can be written in operator form as

$$\mathcal{K}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(0)}_\alpha;E+i0)=\langle\vec{p}_\beta\vec{q}_\beta|t_\beta(E+i0)G_0(E+i0)U_{\beta\alpha}(E+i0)|\vec{p}^{(0)}_\alpha\phi^\alpha_\kappa\rangle\,. \tag{2.7}$$

The breakup-amplitude component is obtained by taking the function $\mathcal{K}_{\beta\alpha}$ fully on-shell, i.e., for $\tilde{p}_\beta{}^2+\tilde{q}_\beta{}^2=\tilde{p}^{(0)2}_\alpha-\kappa_\alpha{}^2=E$. Since in that case $\langle\vec{p}_\beta\vec{q}_\beta|t_\beta(E+i0)=\langle\vec{p}_\beta\psi^-_{\vec{q}_\beta}|V_\beta$, we again obtain here that, on-shell, $\mathcal{S}_{\beta\alpha}$ yields the $\beta$ component of the breakup amplitude.

The factors $V_\beta G_0$ on the left in the amplitudes (2.6) are present to ensure that the half-off-shell amplitudes $\mathcal{H}_{\beta\alpha}$ and $\mathcal{S}_{\beta\alpha}$ do not contain singularities (poles) in the off-shell variable $\vec{q}_\beta$.

The off-shell extensions of the amplitudes (2.6) are defined as

$$\mathcal{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}^{(0)}_\alpha;z)=\langle\vec{p}_\beta\phi^\beta_\kappa|V_\beta G_0(z)U_{\beta\alpha}(z)G_0(z)V_\alpha|\vec{p}^{(0)}_\alpha\phi^\alpha_\kappa\rangle\,, \tag{2.8}$$

$$\mathcal{S}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(0)}_\alpha;z)=\langle\vec{p}_\beta\psi^-_{\vec{q}_\beta}|V_\beta G_0(z)U_{\beta\alpha}(z)G_0(z)V_\alpha|\vec{p}^{(0)}_\alpha\phi^\alpha_\kappa\rangle\,.$$

In Appendix A it is shown that the amplitudes (2.8) are free from elastic, rearrangement, and breakup poles. In fact, $\mathcal{K}_{\beta\alpha}$ in (2.8) coincides with Faddeev's fully-off-shell amplitude $\mathcal{H}_{\beta\alpha}$. On the other hand, the amplitude $\mathcal{S}_{\beta\alpha}$ in (2.8) and the Faddeev fully-off-shell amplitude $\mathcal{K}_{\beta\alpha}$ are different; it is this different choice of off-shell extensions that enables us to write remarkably simple Faddeev equations for $\mathcal{H}$ and $\mathcal{S}$, as we show in the next section.

## III. EQUATIONS FOR $\mathcal{H}_{\beta\alpha}$ AND $\mathcal{S}_{\beta\alpha}$

Inserting the expansion (2.4) into Faddeev's equations (2.1), a system of coupled integral equations for the half-on-shell amplitudes $\mathcal{H}$ and $\mathcal{S}$ can be immediately obtained. However, as it will be more convenient for the discussion of their properties, we present here the corresponding equations for the fully-off-shell amplitudes.

Such equations can be obtained from the Faddeev equations for the operators $U_{\beta\alpha}$,[5]

$$U_{\beta\alpha}(z)=-\bar{\delta}_{\beta\alpha}G_0{}^{-1}(z)-\sum_\gamma\bar{\delta}_{\beta\gamma}t_\gamma(z)G_0(z)U_{\gamma\alpha}(z)\,, \tag{3.1}$$

where $\bar{\delta}_{\beta\gamma}=1-\delta_{\beta\gamma}$. Multiplying (3.1) with the appropriate operators and taking the matrix elements indicated by the definitions (2.8), (again, recall that $G_0t_\beta=G_\beta V_\beta$ and that the channel eigenstates form a complete set) we get

$$\begin{aligned}\mathcal{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}^{(0)}_\alpha;z)=\mathcal{H}^{(0)}_{\beta\alpha}(\vec{p}_\beta;\vec{p}^{(0)}_\alpha;z)&-\sum_{\gamma\neq\beta}\int d^3p'_\gamma\,\mathfrak{V}^{\mathcal{H}\mathcal{H}}_{\beta\gamma}(\vec{p}_\beta;\vec{p}'_\gamma)\frac{1}{\tilde{p}'^2_\gamma-\kappa_\gamma{}^2-z}\mathcal{H}_{\gamma\alpha}(\vec{p}'_\gamma;\vec{p}^{(0)}_\alpha;z)\\&-\sum_{\gamma\neq\beta}\int\int d^3p'_\gamma d^3q'_\gamma\,\mathfrak{V}^{\mathcal{H}\mathcal{S}}_{\beta\gamma}(\vec{p}_\beta;\vec{p}'_\gamma\vec{q}'_\gamma)\frac{1}{\tilde{p}'^2_\gamma+\tilde{q}'^2_\gamma-z}\mathcal{S}_{\gamma\alpha}(\vec{p}'_\gamma\vec{q}'_\gamma;\vec{p}^{(0)}_\alpha;z)\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathcal{S}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(0)}_\alpha;z)=\mathcal{S}^{(0)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(0)}_\alpha;z)&-\sum_{\gamma\neq\beta}\int d^3p'_\gamma\,\mathfrak{V}^{\mathcal{S}\mathcal{H}}_{\beta\gamma}(\vec{p}_\beta\vec{q}_\beta;\vec{p}'_\gamma)\frac{1}{\tilde{p}'^2_\gamma-\kappa_\gamma{}^2-z}\mathcal{H}_{\gamma\alpha}(\vec{p}'_\gamma;\vec{p}^{(0)}_\alpha;z)\\&-\sum_{\gamma\neq\beta}\int\int d^3p'_\gamma d^3q'_\gamma\,\mathfrak{V}^{\mathcal{S}\mathcal{S}}_{\beta\gamma}(\vec{p}_\beta\vec{q}_\beta;\vec{p}'_\gamma\vec{q}'_\gamma)\frac{1}{\tilde{p}'^2_\gamma+\tilde{q}'^2_\gamma-z}\mathcal{S}_{\gamma\alpha}(\vec{p}'_\gamma\vec{q}'_\gamma;\vec{p}^{(0)}_\alpha;z)\,,\end{aligned}$$

where the "effective potentials" $\mathcal{V}$ are given by

$$\mathcal{V}^{\mathcal{H}\mathcal{H}}_{\beta\gamma}(\vec{p}_\beta;\vec{p}\,'_\gamma) = -\frac{\Phi^\beta_\kappa(\vec{q}^{\,(1)}_\beta)\Phi^\gamma_\kappa(\vec{q}^{\,(2)}_\gamma)}{(\vec{q}^{\,(2)}_\gamma)^2+\kappa_\gamma^2}\,,$$

$$\mathcal{V}^{\mathcal{H}\mathcal{E}}_{\beta\gamma}(\vec{p}_\beta;\vec{p}\,'_\gamma\vec{q}\,'_\gamma) = \Phi^\beta_\kappa(\vec{q}^{\,(1)}_\beta)\,\psi^-_{\vec{q}\,'_\gamma}(\vec{q}^{\,(2)}_\gamma)\,,$$

$$\mathcal{V}^{\mathcal{E}\mathcal{H}}_{\beta\gamma}(\vec{p}_\beta\vec{q}_\beta;\vec{p}\,'_\gamma) = -t_\beta(\vec{q}_\beta;\vec{q}^{\,(1)}_\beta;\bar{q}_\beta^{\,2}+i0)\,\frac{\Phi^\gamma_\kappa(\vec{q}^{\,(2)}_\gamma)}{(\vec{q}^{\,(2)}_\gamma)^2+\kappa_\gamma^2}\,, \tag{3.3}$$

$$\mathcal{V}^{\mathcal{E}\mathcal{E}}_{\beta\gamma}(\vec{p}_\beta\vec{q}_\beta;\vec{p}\,'_\gamma\vec{q}\,'_\gamma) = t_\beta(\vec{q}_\beta;\vec{q}^{\,(1)}_\beta;\bar{q}_\beta^{\,2}+i0)\,\psi^-_{\vec{q}\,'_\gamma}(\vec{q}^{\,(2)}_\gamma)\,.$$

The driving term $\mathcal{H}^{(0)}_{\beta\alpha}$ $(\mathcal{E}^{(0)}_{\beta\alpha})$ vanishes if $\alpha=\beta$, and is otherwise obtained from the expression for $\mathcal{V}^{\mathcal{H}\mathcal{H}}_{\beta\gamma}$ $(\mathcal{V}^{\mathcal{E}\mathcal{H}}_{\beta\gamma})$ in (3.3), taking $\gamma=\alpha$ and replacing $\kappa_\gamma^2$ by $\bar{p}_\alpha^{(0)2}-z$. In (3.3),

$$\vec{q}^{\,(1)}_\beta = [m_\gamma/(m_\alpha+m_\gamma)]\,\vec{p}_\beta+\vec{p}\,'_\gamma\,,$$

$$\vec{q}^{\,(2)}_\gamma = -\vec{p}_\beta-[m_\beta/(m_\alpha+m_\beta)]\,\vec{p}\,'_\gamma\,,$$

and $\Phi^\beta_\kappa$ is the two-body bound-state vertex function, defined as

$$\Phi^\beta_\kappa(\vec{q}_\beta) = -(\bar{q}_\beta^{\,2}+\kappa_\beta^2)\,\phi^\beta_\kappa(\vec{q}_\beta)\,.$$

As was mentioned before, we can see in (3.2) and (3.3) how the formulation of the three-body theory gets simplified when it is expressed in terms of the new pair of amplitudes $\mathcal{H}$ and $\mathcal{E}$. In fact, Eqs. (3.2) have the following features:

(i) The effective potentials are all independent of the energy parameter $z$. This fact simplifies the structure of the equations and has obvious computational advantages.

(ii) The input consists solely of two-body bound-state wave functions and half-off-shell transition amplitudes. The completely off-shell amplitudes—in particular for arbitrarily large negative energies—occurring in the usual treatments of the Faddeev equations are therefore completely eliminated.

Additional convenient features become evident after an angular momentum decomposition of Eqs. (3.2) is carried out. This is discussed in detail in the next section.

## IV. ANGULAR MOMENTUM DECOMPOSITION

In this section we consider the angular momentum decomposition of Eqs. (3.2). Since the properties we want to discuss are present in all terms of such a decomposition, we only consider the simplest situation, i.e., the $S$-wave case: We assume that the total angular momentum $J$ is zero, and that only $S$-wave two-body interactions are present.

It will be remembered from Eqs. (2.8) that the breakup-amplitude component $\mathcal{E}_{\beta\alpha}$ is obtained by projecting to the left onto scattering channel eigenstates $\langle\psi^-_{\vec{\beta}}|$. As is well known from two-body scattering theory,[6] the coordinate-space representation of this solution can be expressed in the $S$-wave case as

$$\psi^-_{q_\beta}(r) = \frac{q_\beta\phi_{q_\beta}(r)}{\mathcal{L}_-(q_\beta)}\,, \tag{4.1}$$

where $\phi_{q_\beta}(r)$ is the $S$-wave regular solution to the partial-wave Schrödinger equation [satisfying boundary conditions at the origin $\phi_{q_\beta}(0)=0$, $\phi_{q_\beta}'(0)=1$], and $\mathcal{L}_-(q_\beta)$ is the two-body Jost function. A similar relation holds, of course, in every partial wave.

In this way, we see from (4.1) that a Jost function factor $1/\mathcal{L}_{\beta+}$ can naturally be extracted from each partial-wave component of $\mathcal{E}_{\beta\alpha}$. Redefining these amplitudes accordingly,

$$\mathcal{E}_{\beta\alpha} = \frac{1}{\mathcal{L}_{\beta+}}\,\hat{\mathcal{E}}_{\beta\alpha}\,, \tag{4.2}$$

the new amplitudes $\hat{\mathcal{E}}_{\beta\alpha}$ are obtained in each partial tial wave by projecting onto the regular solutions rather than onto the scattering solutions. The resulting equations for the amplitudes $\mathcal{H}_{\beta\alpha}$ and $\hat{\mathcal{E}}_{\beta\alpha}$ in the $S$-wave case are

$$\begin{aligned}\mathcal{H}_{\beta\alpha}(p_\beta;p_\alpha^{(0)};z) = {}&\mathcal{H}^{(0)}_{\beta\alpha}(p_\beta;p_\alpha^{(0)};z) - \sum_{\gamma\neq\beta}\int_0^\infty p_\gamma'^2\,dp_\gamma'\,\mathcal{V}^{\mathcal{H}\mathcal{H}}_{\beta\gamma}(p_\beta;p_\gamma')\,\frac{1}{\bar{p}_\gamma'^2-\kappa_\gamma^2-z}\,\mathcal{H}_{\gamma\alpha}(p_\gamma';p_\alpha^{(0)};z)\\ &-\sum_{\gamma\neq\beta}\int_0^\infty\int_0^\infty p_\gamma'^2\,dp_\gamma'\,q_\gamma'^2\,dq_\gamma'\,\hat{\mathcal{V}}^{\mathcal{H}\mathcal{E}}_{\beta\gamma}(p_\beta;p_\gamma',q_\gamma')\,\frac{1/|\mathcal{L}_+(q_\gamma')|^2}{\bar{p}_\gamma'^2+\bar{q}_\gamma'^2-z}\,\hat{\mathcal{E}}_{\gamma\alpha}(p_\gamma',q_\gamma';p_\alpha^{(0)};z)\,,\\ \hat{\mathcal{E}}_{\beta\alpha}(p_\beta,q_\beta;p_\alpha^{(0)};z) = {}&\hat{\mathcal{E}}^{(0)}_{\beta\alpha}(p_\beta,q_\beta;p_\alpha^{(0)};z) - \sum_{\gamma\neq\beta}\int_0^\infty p_\gamma'^2\,dp_\gamma'\,\hat{\mathcal{V}}^{\mathcal{E}\mathcal{H}}_{\beta\gamma}(p_\beta,q_\beta;p_\gamma')\,\frac{1}{\bar{p}_\gamma'^2-\kappa_\gamma^2-z}\,\mathcal{H}_{\gamma\alpha}(p_\gamma';p_\alpha^{(0)};z)\\ &-\sum_{\gamma\neq\beta}\int_0^\infty\int_0^\infty p_\gamma'^2\,dp_\gamma'\,q_\gamma'^2\,dq_\gamma'\,\hat{\mathcal{V}}^{\mathcal{E}\mathcal{E}}_{\beta\gamma}(p_\beta,q_\beta;p_\gamma',q_\gamma')\,\frac{1/|\mathcal{L}_+(q_\gamma')|^2}{\bar{p}_\gamma'^2+\bar{q}_\gamma'^2-z}\,\hat{\mathcal{E}}_{\gamma\alpha}(p_\gamma'q_\gamma';p_\alpha^{(0)};z)\,.\end{aligned} \tag{4.3}$$

The partial-wave components of the effective potentials of the original equations are redefined accordingly, and the resulting potentials in (4.3) are

$$\mathcal{V}^{\mathcal{K}\mathcal{K}}_{\beta\gamma}(p_\beta;p'_\gamma)=-\tfrac{1}{2}\int_{-1}^{1}d(\cos\vartheta_{\beta\gamma})\,\frac{\Phi^\beta_\kappa(q^{(1)}_\beta)\Phi^\gamma_\kappa(q^{(2)}_\beta)}{\bar q^{(2)2}_\gamma+\kappa_\gamma^2}\,,$$

$$\hat{\mathcal{V}}^{\mathcal{K}\mathcal{E}}_{\beta\gamma}(p_\beta;p'_\gamma,q'_\gamma)=\tfrac{1}{2}\left[\int_{-1}^{1}d(\cos\vartheta_{\beta\gamma})\,\Phi^\beta_\kappa(q^{(1)}_\beta)\,\psi^-_{q'_\gamma}(q^{(2)}_\gamma)\right]\mathcal{L}_-(q'_\gamma)\,,$$

$$\hat{\mathcal{V}}^{\mathcal{E}\mathcal{K}}_{\beta\gamma}(p_\beta,q_\beta;p'_\gamma)=-\tfrac{1}{2}\mathcal{L}_+(q_\beta)\left[\int_{-1}^{1}d(\cos\vartheta_{\beta\gamma})\,t_\beta(q_\beta,q^{(1)}_\beta;\bar q_\beta{}^2+i0)\,\frac{\Phi^\gamma_\kappa(q^{(2)}_\gamma)}{\bar q^{(2)2}_\gamma+\kappa_\gamma{}^2}\right]\,, \tag{4.4}$$

$$\hat{\mathcal{V}}^{\mathcal{E}\mathcal{E}}_{\beta\gamma}(p_\beta,q_\beta;p'_\gamma q'_\gamma)=\tfrac{1}{2}\mathcal{L}_+(q_\beta)\left[\int_{-1}^{1}d(\cos\vartheta_{\beta\gamma})\,t_\beta(q_\beta,q^{(1)}_\beta;\bar q_\beta{}^2+i0)\,\psi^-_{q'_\gamma}(q^{(2)}_\gamma)\right]\mathcal{L}_-(q'_\gamma)\,,$$

where

$$q^{(1)}_\beta=\left[\left(\frac{\mu_\beta}{m_\alpha}p_\beta\right)^2+p'^2_\gamma+2\frac{\mu_\beta}{m_\alpha}p_\beta p'_\gamma\cos\vartheta_{\beta\gamma}\right]^{1/2}$$
$$q^{(2)}_\gamma=\left[p_\beta{}^2+\left(\frac{\mu_\gamma}{m_\alpha}p'_\gamma\right)^2+2\frac{\mu_\gamma}{m_\alpha}p_\beta p'_\gamma\cos\vartheta_{\beta\gamma}\right]^{1/2} \tag{4.5}$$

This redefinition of the $\mathcal{E}$ amplitudes has the following advantages: First, the phase of the Jost function is precisely the two-body phase shift, i.e.,

$$\mathcal{L}_\pm(q_\beta)=|\mathcal{L}_\pm(q_\beta)|\,e^{\mp i\delta(q_\beta)}\,. \tag{4.6}$$

Since the same phase is carried by the two-body half-on-shell $t$ matrix and two-body scattered wave function, we see that all these phases cancel out in the expression for the potentials. That is to say, the potentials (4.4) in the equations for $\hat{\mathcal{E}}$ and $\mathcal{K}$ are not only $z$-independent, but are also *real*. In addition to the computational simplifications entailed by such a situation, problems related to unitarity (such as the construction of unitary approximation schemes) become easier to handle.

Obviously, to obtain real potentials it is only necessary to factor out the phase of the Jost function from the original $\mathcal{E}$ amplitude. However, we believe it is useful to factor out also the modulus of the Jost function, as we have done above. The reason is that the regular solution $\phi_{q_\beta}(r)$ of (4.1) is analytic everywhere in the complex $q_\beta$ plane, i.e., it has no bound-state or resonance poles, nor any branch points. Instead, this structure of the two-body scattering wave function is carried by the Jost function denominator. Thus, the amplitudes $\hat{\mathcal{E}}_{\beta\alpha}$ are more smoothly varying functions of $q_\beta$ than the corresponding $\mathcal{E}_{\beta\alpha}$ amplitudes.

The same two-body structure is also absent from the potentials in (4.4), since they carry factors $\mathcal{L}_+t$ and $\psi^-\mathcal{L}_-$. In this manner, the two-body bound-state and resonance singularities are predominantly carried by the factor $1/|\mathcal{L}|^2$ in Eq. (4.3).

We conclude by writing the expression for the breakup amplitude in terms of the new amplitudes $\hat{\mathcal{E}}_{\beta\alpha}$ in the $S$-wave case:

$$\mathcal{B}_{0\alpha}=\sum_\beta\frac{1}{\mathcal{L}_{\beta+}}\hat{\mathcal{E}}_{\beta\alpha}\,. \tag{4.7}$$

We see in (4.7) that $\hat{\mathcal{E}}_{\beta\alpha}$ differs from the corresponding breakup-amplitude component by a Watson final-state interaction factor.

## V. THE 3-3 AND 3-2 AMPLITUDES

For the sake of completeness, we consider in this section the amplitudes for processes starting from three free particles. For this purpose we recall expression (2.8) for the amplitudes corresponding to processes starting from a bound state and a third free particle, i.e.,

$$\begin{aligned}2\to2:\ &\mathcal{K}_{\beta\alpha}=\langle\vec p_\beta\phi^\beta_\kappa|V_\beta G_0U_{\beta\alpha}G_0V_\alpha|\vec p^{(0)}_\alpha\phi^\alpha_\kappa\rangle\,,\\ 2\to3:\ &\mathcal{E}_{\beta\alpha}=\langle\vec p_\beta\psi^-_{\vec q_\beta}|V_\beta G_0U_{\beta\alpha}G_0V_\alpha|\vec p^{(0)}_\alpha\phi^\alpha_\kappa\rangle\,.\end{aligned} \tag{5.1}$$

The remaining amplitudes are now defined as

$$\begin{aligned}3\to2:\ &\tilde{\mathcal{E}}_{\beta\alpha}=\langle\vec p_\beta\phi^\beta_\kappa|V_\beta G_0U_{\beta\alpha}G_0V_\alpha|\vec p^{(0)}_\alpha\psi^+_{\vec q^{(0)}_\alpha}\rangle\,,\\ 3\to3:\ &\mathcal{T}_{\beta\alpha}=\langle\vec p_\beta\psi^-_{\vec q_\beta}|V_\beta G_0U_{\beta\alpha}G_0V_\alpha|\vec p^{(0)}_\alpha\psi^+_{\vec q^{(0)}_\alpha}\rangle\,.\end{aligned} \tag{5.2}$$

That the 3-3 amplitude of (5.2) directly yields the connected part of the 3-3 transition amplitude can be seen as follows: In Faddeev's treatment, this 3-3 amplitude is obtained by taking the fully-on-shell plane-wave matrix elements of the operator $M_{\beta\alpha}$, i.e.,

$$T=\sum_{\beta\alpha}\langle\vec p_\beta\vec q_\beta|M_{\beta\alpha}(E+i0)|\vec p^{(0)}_\alpha\vec q^{(0)}_\alpha\rangle\,, \tag{5.3}$$

where $\bar p_\beta{}^2+\bar q_\beta{}^2=\bar p^{(0)2}_\alpha+\bar q^{(0)2}_\alpha$.

Since $M_{\beta\alpha}=\delta_{\beta\alpha}t_\beta+W_{\beta\alpha}$, where $W_{\beta\alpha}$ is the connected three-body Faddeev operator, related to $U_{\beta\alpha}$ through $W_{\beta\alpha}=t_\beta G_0 U_{\beta\alpha} G_0 t_\alpha$, we see that

$$T=\sum_\beta \langle \vec{p}_\beta \vec{q}_\beta | t_\beta | \vec{p}_\alpha^{(0)} \vec{q}_\alpha^{(0)}\rangle + \sum_{\beta\alpha} \langle \vec{p}_\beta \vec{q}_\beta | t_\beta G_0 U_{\beta\alpha} G_0 t_\alpha | \vec{p}_\alpha^{(0)} \vec{q}_\alpha^{(0)}\rangle . \tag{5.4}$$

However, since (5.4) is fully-on-shell, we can write the second term as

$$\sum_{\beta\alpha} \langle \vec{p}_\beta \psi^-_{\vec{q}_\beta} | V_\beta G_0 U_{\beta\alpha} G_0 V_\alpha | \vec{p}_\alpha^{(0)} \psi^+_{\vec{q}_\alpha^{(0)}}\rangle \tag{5.5}$$

so that the 3-3 amplitude is simply given by

$$T=\sum_{\beta\alpha} \mathcal{T}_{\beta\alpha} + \sum_\beta t_\beta . \tag{5.6}$$

Off-shell, of course, again $\mathcal{T}_{\beta\alpha}$ and the plane-wave matrix elements of $W_{\beta\alpha}$ differ.

The Faddeev equations for $\tilde{\mathcal{E}}_{\beta\alpha}$ and $\mathcal{T}_{\beta\alpha}$ can be obtained from (3.2) by replacing $\mathcal{K}_{\beta\alpha}$ by $\tilde{\mathcal{E}}_{\beta\alpha}$ and $\mathcal{E}_{\beta\alpha}$ by $\mathcal{T}_{\beta\alpha}$. In addition, the driving terms must also be replaced: For example, the driving term in the $\mathcal{T}_{\beta\alpha}$ equations is given by

$$\mathcal{T}^{(0)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(0)}\vec{q}_\alpha^{(0)};z)=-\bar{\delta}_{\beta\alpha} t_\beta(\vec{q}_\beta,\vec{q}_\beta^{(1)};\bar{q}_\beta^{\,2}+i0)\frac{1}{\bar{p}_\beta^{\,2}+\bar{q}_\beta^{(1)2}-z} t_\alpha(\vec{q}_\alpha^{(2)},\vec{q}_\alpha^{(0)};\bar{q}_\alpha^{(0)2}+i0) \tag{5.7}$$

(note that $\bar{p}_\beta^{\,2}+\bar{q}_\beta^{(1)2}=\bar{p}_\alpha^{(0)2}+\bar{q}_\alpha^{(2)2}$).

Similarly, it can be shown that the amplitude $\tilde{\mathcal{E}}_{\beta\alpha}$ of (5.2), when taken on-shell, is a component of the 3-2 transition amplitude.

Returning to Eqs. (5.1) and (5.2), we observe that the amplitudes for all possible three-body processes are obtained by taking matrix elements of the operator $V_\beta G_0 U_{\beta\alpha} G_0 V_\alpha$ between channel eigenstates appropriate to the initial and final states. Since we have at our disposal both incoming and outgoing scattering states, it should be noted that it is also possible to define amplitudes with a choice of $\psi^-$ and $\psi^+$ states that is different from the choice used in (5.1) and (5.2). However, such amplitudes are not as simply related to the physical transition amplitudes. The physical reason for this is that the three-body $S$ matrix involves inner products of incoming and outgoing three-body scattering states in the same order as they are expanded in (5.1) and (5.2).

## VI. UNITARITY RELATIONS

As we have seen in the previous section, the amplitudes that describe all physical three-body processes can be obtained by taking appropriate matrix elements of the operator $V_\beta G_0 U_{\beta\alpha} G_0 V_\alpha$. It is thus possible to obtain general unitarity relations in operator form; they are given in Appendix B. From these operator relations one can, of course, obtain fully-off-shell unitarity relations for all the amplitudes $\mathcal{K}$, $\mathcal{E}$, $\tilde{\mathcal{E}}$, and $\mathcal{T}$.

As an example, we give in this section the form the unitarity relations for $\mathcal{E}_{\beta\alpha}$ take when going fully on-shell:

$$\begin{aligned}
&\mathcal{E}_{\beta\alpha}(\vec{p}_\beta,\vec{q}_\beta;\vec{p}_\alpha^{(0)};E+i0)-\mathcal{E}_{\beta\alpha}(\vec{p}_\beta,\vec{q}_\beta;\vec{p}_\alpha^{(0)};E-i0)\\
&\quad=-2\pi i\Big\{\sum_\gamma \int d^3p'_\gamma\, \mathcal{E}_{\beta\gamma}(\vec{p}_\beta,\vec{q}_\beta;\vec{p}'_\gamma;E+i0)\delta(\bar{p}_\gamma^{\,2}-\kappa_\gamma^{\,2}-E)\mathcal{K}^*_{\alpha\gamma}(\vec{p}_\alpha^{(0)};\vec{p}'_\gamma;E+i0)]\\
&\qquad+\int\int d^3p'd^3q'\sum_\gamma [\mathcal{T}_{\beta\gamma}(\vec{p}_\beta,\vec{q}_\beta;\vec{p}'_\gamma\vec{q}'_\gamma;E+i0)+\delta_{\beta\gamma}\delta^3(\vec{p}_\gamma-\vec{p}'_\gamma)t_\gamma(\vec{q}_\gamma,\vec{q}'_\gamma;\bar{q}_\gamma^{\,2}+i0)]\\
&\qquad\times\delta(\bar{p}'^2+\bar{q}'^2-E)\sum_{\gamma'}\tilde{\mathcal{E}}^*_{\alpha\gamma'}(\vec{p}_\alpha^{(0)};\vec{p}'_{\gamma'}\vec{q}'_{\gamma'};E+i0)\Big\}.
\end{aligned} \tag{6.1}$$

As the on-shell amplitudes $\mathcal{E}_{\beta\alpha}$ directly yield the breakup scattering amplitudes, Eq. (6.1), when summed over $\beta$, has the form one would physically expect for the breakup case.[7]

## VII. CONCLUSIONS

We have seen in the preceding sections how the use of the complete sets of eigenstates of the channel Hamiltonians significantly simplifies the formulation of three-body scattering theory. By using this representation we have obtained a new set of amplitudes for all three-body processes that coincide on-shell with the physical transition amplitudes. We have further shown how these amplitudes satisfy integral equations that are simpler than the usual Faddeev equations:

(i) The effective potentials are all independent of the three-body energy;

(ii) the input consists solely of two-body bound-state wave functions and half-off-shell transition amplitudes; and

(iii) our choice of partial-wave components of the three-body amplitudes satisfies equations with *real* effective potentials. In addition, the breakup amplitudes explicitly exhibit a Watson final-state interaction factor.

Finally, we expect that by the nature of the input to these equations, they will be particularly useful in understanding the dependence of three-body observables on the off-shell two-body input. In addition, the simplified structure of our equations suggests that the problem of constructing approximation schemes should now be reconsidered.

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## APPENDIX A

Here we give an outline of the proof of the fact that the new amplitudes $\mathcal{E}_{\beta\alpha}$ are free from primary singularities.[8] Similar proofs can be obtained for the remaining new amplitudes $\tilde{\mathcal{E}}_{\beta\alpha}$ and $\mathcal{T}_{\beta\alpha}$.

We start by noting that the amplitudes $\mathcal{H}$ and $\mathcal{G}$ in terms of which Faddeev carries out the singularity analysis of the three-body wave function are defined by the first of Eq. (2.8) and

$$\mathcal{G}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(0)};z) = -\langle\vec{p}_\beta\vec{q}_\beta|\hat{t}_\beta(z)G_0(z)U_{\beta\alpha}(z)G_0(z)V_\alpha|\vec{p}_\alpha^{(0)}\varphi_\kappa^\alpha\rangle, \tag{A1}$$

where $\hat{t}_\beta$ is obtained by splitting the two-body transition operator $t_\beta$ into a term $t_\beta^p$ containing the bound-state pole and a remainder $\hat{t}_\beta$. The representation of the three-body wave-function component in terms of $\mathcal{H}$ and $\mathcal{G}$ can be obtained from (2.5) and the relations $G_0K_{\beta\alpha}=-G_0t_\beta G_0U_{\beta\alpha}$, $t_\beta=t_\beta^p+\hat{t}_\beta$, with the result

$$\langle\vec{p}_\beta\vec{q}_\beta|\Psi_\beta^{(\alpha)}\rangle=\delta_{\beta\alpha}\delta^3(\vec{p}_\alpha-\vec{p}_\alpha^{(0)})\phi_\kappa^\alpha(\vec{q}_\alpha)-\frac{1}{\bar{p}_\beta^{\,2}+\bar{q}_\beta^{\,2}-E-i0}\,\frac{(\bar{q}_\beta^{\,2}+\kappa_\beta^2)\phi_\kappa^\beta(\vec{q}_\beta)}{\bar{p}_\beta^{\,2}-\kappa_\beta^2-E-i0}\mathcal{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(0)};E+i0)$$
$$+\frac{1}{\bar{p}_\beta^{\,2}+\bar{q}_\beta^{\,2}-E-i0}\mathcal{G}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(0)};E+i0)\,. \tag{A2}$$

By comparing (A2) with (2.4), it can be seen how the choice of the new set $\mathcal{H}$ and $\mathcal{E}$ instead of the set $\mathcal{H}$ and $\mathcal{G}$ simplifies the singularity structure of the expansion.

Now, recalling Eqs. (2.8), (3.1), and (A1), we find that

$$\mathcal{E}_{\beta\alpha}(\vec{p}_\beta,\vec{q}_\beta;\vec{p}_\alpha^{(0)};z)$$
$$=-\bar{\delta}_{\beta\alpha}t_\beta(\vec{q}_\beta,\vec{q}_\beta^{(1)};\bar{q}_\beta^{\,2}+i0)\frac{\Phi_\kappa^\alpha(\vec{q}_\alpha^{(2)})}{\bar{p}_\alpha^{(0)2}+\bar{q}_\alpha^{(2)2}-z}$$
$$+\sum_{\gamma\neq\beta}\int\int d^3p'_\beta d^3q'_\beta t_\beta(\vec{q}_\beta,\vec{q}'_\beta;\bar{q}_\beta^{\,2}+i0)\frac{\delta^3(\vec{p}_\beta-\vec{p}'_\beta)}{\bar{p}'^2_\beta+\bar{q}'^2_\beta-z}\left[\mathcal{G}_{\gamma\alpha}(\vec{p}'_\gamma,\vec{q}'_\gamma;\vec{p}_\alpha^{(0)};z)+\frac{\Phi_\kappa^\gamma(\vec{q}'_\gamma)}{\bar{p}'^2_\gamma-\kappa_\gamma^2-z}\mathcal{H}_{\gamma\alpha}(\vec{p}'_\gamma;\vec{p}_\alpha^{(0)};z)\right]. \tag{A3}$$

Consider the first term in (A3). Since neither the vertex function $\Phi_\kappa^\alpha$ nor the half-off-shell $t_\beta$ have any real singularities as functions of the momenta, only secondary singularities occur in this term. Turning to the second term in (A3), we note that it is identical to the expression (6.26) of Ref. 1, with the half-off-shell $t_\beta$ instead of the off-shell $\hat{t}_\beta$, and the functions $\mathcal{F}^{(3)}$, $\mathcal{F}^{(2)}$, and $\mathcal{G}^{(2)}$ replaced by the functions $\mathcal{E}$, $\mathcal{G}$, and $\mathcal{H}$, respectively. These changes do not affect the character of the estimates used in the subsequent discussion of (6.26); therefore, arguments similar to those of Faddeev enable us to conclude that the singularity structure of $\mathcal{E}$ is similar to that of the Faddeev amplitudes $\mathcal{H}$ and $\mathcal{G}$. In particular, $\mathcal{E}$ is free from primary singularities.

## APPENDIX B

Here we give the general unitarity relations for our amplitudes in operator form. In order to do so we define an operator

$$T_{\beta\alpha}(z)=V_\beta G_0(z)U_{\beta\alpha}(z)G_0(z)V_\alpha. \tag{B1}$$

It can be shown after some algebra that $T_{\beta\alpha}$ satisfies the relation[9]

$$T_{\beta\alpha}(E+i0)-T_{\beta\alpha}(E-i0)$$
$$=-2\pi i\Big\{\sum_\gamma T_{\beta\gamma}(E+i0)\Delta_\gamma^{(\kappa)}(E)T_{\gamma\alpha}(E-i0)$$
$$+\sum_\gamma T_{\beta\gamma}(E+i0)[1-G_0(E+i0)t_\gamma(E+i0)]\Delta_0(E)\sum_{\gamma'}[1-t_{\gamma'}(E-i0)G_0(E-i0)]T_{\gamma'\alpha}(E-i0)$$
$$+\bar\delta_{\beta\alpha}V_\beta\Delta_0(E)V_\alpha+V_\beta\Delta_0(E)\sum_{\gamma'}\bar\delta_{\beta\gamma'}[1-t_{\gamma'}(E-i0)G_0(E-i0)]T_{\gamma'\alpha}(E-i0)$$
$$+\sum_\gamma\bar\delta_{\gamma\alpha}T_{\beta\gamma}(E+i0)[1-G_0(E+i0)t_\gamma(E+i0)]\Delta_0(E)V_\alpha\Big\}\,, \tag{B2}$$

where

$$\Delta_0(E)=\iint|\vec p\,\vec q\rangle\,d^3p\,d^3q\,\delta(\vec p^{\,2}+\vec q^{\,2}-E)\langle\vec p\,\vec q|\,,$$
$$\Delta_\gamma^{(\kappa)}(E)=\int|\vec p\,\phi_\kappa^\gamma\rangle\,d^3p\,\delta(\vec p^{\,2}-\kappa_\gamma^{\,2}-E)\langle\vec p\,\phi_\kappa^\gamma|\,. \tag{B3}$$

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[8]See Ref. 1. Primary singularities (i.e., elastic, rearrangement, or breakup poles) can be distinguished from secondary singularities by the fact that the former are present in all terms of an iterative series of the amplitudes, while the latter get progressively weaker and disappear after a few iterations.

[9]We are grateful to Professor W. Sandhas for communicating to us the approach used by Alt, Grassberger, and Sandhas for obtaining three-body unitarity relations (unpublished).