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Causality, equal-time algebra, and light-cone commutators

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Extending previous work we discuss the sum rules that follow from causality, scaling, and equal-time algebra for the nonforward (spinless) single-particle matrix element of the commutator of two conserved vector currents, using the recently introduced refined infinite-momentum technique. The complete set of sum rules is found to include those obtained from light-cone commutators. It also contains some additional ones.

I. INTRODUCTION

From a quark model with vector-gluon interaction, Cornwall and Jackiw[1] derived light-cone commutators of vector currents and introduced them with the hope that their dynamical content was "far beyond that contained in conventional equal-time commutators."[1] As tests of these commutators, fixed-mass sum rules were obtained[2,3] that corrected the sum rules of current algebra derived by use of the conventional infinite-momentum procedure.[4] Reference 3 extends to the nonforward case the work of Ref. 2.

In the belief that these "corrections" to the sum rules of current algebra amounted to criticism of the conventional infinite-momentum procedure, rather than criticism of equal-time current algebra, a refinement[5] of this procedure was introduced and was shown[6] to give, in the forward case, from equal-time current algebra the same fixed-mass sum rules as obtained in Ref. 2 from light-cone commutators. The present paper is a natural extension of the work of Ref. 6 to the nonforward case and therefore directly compares to the work of Ref. 3 on light-cone commutators.

Making use of recent work[7] on the causal structure of the commutator of two conserved vector currents between spinless single-particle states of unequal momenta, we consider—to begin with—those sum rules that are purely consequences of causality and scaling. Our assumptions on scaling behavior are the same as those of Refs. 8 and 3. The fixed-mass sum rules are obtained from the general form of the sum rules through a generalization of a theorem in Ref. 6, on the refined infinite-momentum limit for the nonforward case. Our analysis is given in terms of the causal structure functions A_k^{ij}, which are related to the usual structure functions W_k^{ij} in Appendix A. The fixed-mass sum rules so obtained are given in Appendix B. This work is detailed in Sec. II.

In Sec. III we make use of equal-time current algebra

$$[J_0^i(x), J_0^j(0)]\delta(x_0) = if^{ijk}J_0^k(x)\delta^4(x) , \qquad (1.1)$$

for the time components to obtain the sum rules that follow from causality, scaling, and this algebra. The fixed-mass sum rules in this class are also summarized in Appendix B. In addition we find that the algebra (1.1), scaling, and causality impose severe restrictions on the structure of the equal-time commutator

$$[J_\mu^i(x), J_\nu^j(0)]\delta(x_0) . \qquad (1.2)$$

Section IV deals with the comparison of our results to those obtained by the authors of Ref. 3 from light-cone commutators. We find that we obtain all of their sum rules for the case under discussion. We also obtain additional sum rules—in particular some that involve the structure function W_5^{ij}—that cannot be obtained from the $(+, \nu)$ light-cone commutators, as noted in Ref. 3. Further, we observe that most of the sum rules of Ref. 3 follow purely from causality and scaling. In clarifying the role played by causality in the derivation of these sum rules, we have shown that most of them test scaling rather than light-cone or equal-time algebra. None of these sum rules can, in any case, distinguish between light-cone and equal-time commutators.

We finally remark that a treatment of the equal-time commutator, in which the difficulties of the conventional infinite-momentum limit are circumvented, has been given by Keppel-Jones[9] and, independently, by Ward.[10] The considerations of these authors, which pertain to forward inelastic neutrino-nucleon scattering, also show that corrections to fixed-mass sum rules are already contained in equal-time algebra. Further work by Ward[11] investigates the assumptions under which the refined infinite-momentum limit is valid. It is shown that while the refined limit satisfactorily handles the Z graphs that are neglected in the conventional limit, it misses the class-II states, as does the light-cone approach. It is, however, noted[11] that in principle the refined procedure allows the inclusion of all classes of intermediate states and the formalism is extended to explicitly demonstrate this.

II. THE CAUSALITY SUM RULES

A. General formulation

Consider the matrix element $\tilde{C}^{ij}_{\mu\nu}$ of the commutator of two conserved vector currents between spinless single-particle states of momenta p_1 and p_2 $(p_1{}^2=p_2{}^2=1)$,

$$\tilde{C}^{ij}_{\mu\nu}(x)=\langle p_2|[J^i_\mu(\tfrac{1}{2}x), J^j_\nu(-\tfrac{1}{2}x)]|p_1\rangle \ . \tag{2.1}$$

The Fourier transform $C^{ij}_{\mu\nu}$ of $\tilde{C}^{ij}_{\mu\nu}$, defined by

$$C^{ij}_{\mu\nu}(Q)=\frac{1}{2\pi}\int e^{iQ\cdot x}\,\tilde{C}^{ij}_{\mu\nu}(x)d^4x\ , \tag{2.2}$$

may be written in the form

$$C^{ij}_{\mu\nu}=\sum_{k=1}^{5}L^{(k)}_{\mu\nu}A^{ij}_k\ , \tag{2.3}$$

where A^{ij}_k are invariant functions of

$$\nu=Q\cdot P,\quad t=\Delta^2,\quad Q^2,\quad \rho=\Delta\cdot Q\ ,$$

with

$$P=\tfrac{1}{2}(p_1+p_2),\quad \Delta=p_1-p_2\ ,$$

and the covariants $L^{(k)}_{\mu\nu}$ are given by

$$\begin{aligned}
L^{(1)}_{\mu\nu}&=(Q_\mu-\tfrac{1}{2}\Delta_\mu)(Q_\nu+\tfrac{1}{2}\Delta_\nu)-(Q^2-\tfrac{1}{4}t)g_{\mu\nu}\ ,\\
L^{(2)}_{\mu\nu}&=(Q^2-\tfrac{1}{4}t)P_\mu P_\nu+\nu^2 g_{\mu\nu}\\
&\quad-\nu[P_\mu(Q_\nu+\tfrac{1}{2}\Delta_\nu)+(Q_\mu-\tfrac{1}{2}\Delta_\mu)P_\nu]\ ,\\
L^{(3)}_{\mu\nu}&=(Q^2-\tfrac{1}{4}t)P_\mu\Delta_\nu+\nu(\rho-\tfrac{1}{2}t)g_{\mu\nu}\\
&\quad-(\rho-\tfrac{1}{2}t)P_\mu(Q_\nu+\tfrac{1}{2}\Delta_\nu)-\nu(Q_\mu-\tfrac{1}{2}\Delta_\mu)\Delta_\nu\ ,\\
L^{(4)}_{\mu\nu}&=(Q^2-\tfrac{1}{4}t)\Delta_\mu P_\nu+\nu(\rho+\tfrac{1}{2}t)g_{\mu\nu}\\
&\quad-(\rho+\tfrac{1}{2}t)(Q_\mu-\tfrac{1}{2}\Delta_\mu)P_\nu-\nu\Delta_\mu(Q_\nu+\tfrac{1}{2}\Delta_\nu)\ ,\\
L^{(5)}_{\mu\nu}&=(Q^2-\tfrac{1}{4}t)\Delta_\mu\Delta_\nu+(\rho^2-\tfrac{1}{4}t^2)g_{\mu\nu}\\
&\quad-(\rho+\tfrac{1}{2}t)(Q_\mu-\tfrac{1}{2}\Delta_\mu)\Delta_\nu-(\rho-\tfrac{1}{2}t)\Delta_\mu(Q_\nu+\tfrac{1}{2}\Delta_\nu)\ .
\end{aligned} \tag{2.4}$$

The invariant functions A^{ij}_k, $k\neq 2$, are causal and, provided the integrals converge, satisfy the following causality sum rules[7]:

$$\int A^{ij}_k dQ_0=0\ , \tag{2.5}$$

$$\int Q_0 A^{ij}_k dQ_0=b^{ij}_k(t)\ , \tag{2.6}$$

$$\int Q_0^2 A^{ij}_k\, dQ_0=c^{ij}_k(t)P_0+d^{ij}_k(t)\Delta_0\ , \tag{2.7}$$

where

$$b^{[ij]}_k(t)=0,\quad c^{(ij)}_k(t)=d^{(ij)}_k(t)=0\ . \tag{2.8}$$

These sum rules hold for $k\neq 2$. The brackets $[ij]$ and (ij) denote antisymmetric and symmetric parts, respectively. The sum rules (2.5)–(2.7) are general causality sum rules which do not depend on any specific assumption about current commutators. In particular they are not, in any sense, model-dependent.

These sum rules, however, hold if and only if the spectral functions $\psi^{ij}_k(u,s)$ in the Jost-Lehmann-Dyson representation for A^{ij}_k satisfy certain asymptotic conditions.[7] In particular it is necessary and sufficient for the validity of Eqs. (2.5) that $\lim_{s\to\infty}\psi^{ij}_k(u,s)=0$, $k\neq 2$. A model in which Eqs. (2.5) hold is the original quark model of Gell-Mann, as is verified by considering the explicit equal-time commutators of this model. It thus follows that in such a model the asymptotic conditions on $\psi^{ij}_k(u,s)$ are satisfied.

B. Scaling limits

Scaling behavior is usually assumed for the generally noncausal structure functions W^{ij}_k defined by[8,3]

$$C^{ij}_{\mu\nu}=\frac{1}{2\pi}\sum_{k=1}^{5}W^{ij}_k\left(g_{\mu\mu'}-\frac{q_{2\mu}q_{2\mu'}}{q_2{}^2}\right)A^k_{\mu'\nu'}\left(g_{\nu'\nu}-\frac{q_{1\nu'}q_{1\nu}}{q_1{}^2}\right), \tag{2.9}$$

where the $A^k_{\mu\nu}$ are given by

$$\begin{aligned}
A^1_{\mu\nu}&=-g_{\mu\nu}\ ,\\
A^2_{\mu\nu}&=P_\mu P_\nu\ ,\\
A^3_{\mu\nu}&=P_\mu\Delta_\nu-P_\nu\Delta_\mu\ ,\\
A^4_{\mu\nu}&=P_\mu\Delta_\nu+P_\nu\Delta_\mu\ ,\\
A^5_{\mu\nu}&=\Delta_\mu\Delta_\nu\ ,
\end{aligned} \tag{2.10}$$

and

$$q_1=Q-\tfrac{1}{2}\Delta,\quad q_2=Q+\tfrac{1}{2}\Delta\ . \tag{2.11}$$

The behavior of the functions W^{ij}_k in the scaling limit $\nu\to\infty$, $Q^2\to\infty$ at fixed t and ρ and fixed $\omega=-Q^2/2\nu$ is assumed to be[8,3]

$$W^{ij}_L\sim-\frac{1}{2\omega}F^{ij}_L(\omega,t,\rho)\ , \tag{2.12}$$

$$\nu W^{ij}_k\sim F^{ij}_k(\omega,t,\rho),\quad k=2,\ldots,5 \tag{2.13}$$

where

$$W^{ij}_L=W^{ij}_1+\frac{\nu^2}{Q^2-\frac{1}{4}t}W^{ij}_2\ . \tag{2.14}$$

The relation of the functions A^{ij}_k to W^{ij}_k is given in Appendix A. One obtains from these relations the following scaling behavior for A^{ij}_k:

$$\nu A^{ij}_1\sim\frac{1}{8\pi\omega^2}F^{ij}_L(\omega,t,\rho)\ , \tag{2.15}$$

$$\nu^2 A_k^{ij} \sim \frac{1}{4\pi} G_k^{ij}(\omega, t, \rho), \quad k=2,\ldots,5 \qquad (2.16)$$

where

$$G_2^{ij} = -\frac{1}{\omega} F_2^{ij},$$
$$G_3^{ij} = \frac{1}{\omega}\left(\frac{1}{2\omega} F_2^{ij} - F_3^{ij} - F_4^{ij}\right),$$
$$G_4^{ij} = -\frac{1}{\omega}\left(\frac{1}{2\omega} F_2^{ij} - F_3^{ij} + F_4^{ij}\right), \qquad (2.17)$$
$$G_5^{ij} = \frac{1}{2\omega^2}\left(\frac{1}{\omega} F_2^{ij} - 2F_3^{ij} - 2\omega F_5^{ij} - \frac{1}{2\omega} F_L^{ij}\right).$$

It is our aim, in the following, to make use of this scaling behavior in the causality sum rules (2.5)–(2.7). Towards this purpose we redefine some of the free variables in these sum rules. Introduce the variables α, ξ, η, and γ defined by

$$\alpha = P_0^{-1},$$
$$\xi = -P_0^{-2}\vec{P}\cdot\vec{Q},$$
$$\eta = \vec{Q}^2 - P_0^{-2}(\vec{P}\cdot\vec{Q})^2, \qquad (2.18)$$
$$\gamma = \alpha^{-1}\Delta_0.$$

These parameters vary such that

$$0 \leqslant \alpha \leqslant (1-\tfrac{1}{4}t)^{-1/2},$$
$$-\infty < \xi, \gamma < \infty, \qquad (2.19)$$
$$\eta \geqslant \frac{\xi^2}{(1-\frac{1}{4}t)^{-1} - \alpha^2};$$

and when $\alpha = (1-\frac{1}{4}t)^{-1/2}$, then $\xi = 0$ and $\eta \geqslant 0$.

Effecting a change of variable from Q_0 to ν, the general causality sum rules (2.5)–(2.7) may be written in the form

$$\int A_k^{ij}(\nu, \alpha^2\gamma^2 - \vec{\Delta}^2, \alpha^2\nu^2 - 2\xi\nu - \eta, \gamma(\alpha^2\nu - \xi) - \vec{\Delta}\cdot\vec{Q})\, d\nu = 0, \qquad (2.20)$$

$$\alpha^2 \int \nu A_k^{ij}(\nu, \ldots)\, d\nu = b_k^{ij}(\alpha^2\gamma^2 - \vec{\Delta}^2), \qquad (2.21)$$

$$\alpha^4 \int \nu^2 A_k^{ij}(\nu, \ldots)\, d\nu = 2\xi b_k^{ij}(\alpha^2\gamma^2 - \vec{\Delta}^2) + c_k^{ij}(\alpha^2\gamma^2 - \vec{\Delta}^2) + \alpha^2\gamma d_k^{ij}(\alpha^2\gamma^2 - \vec{\Delta}^2) \qquad (2.22)$$

for $k \neq 2$.

Consider an integral of the form[6]

$$I = \int_{-\infty}^{\infty} A(\nu, \alpha^2\gamma^2 - \vec{\Delta}^2, \alpha^2\nu^2 - 2\xi\nu - \eta, \gamma(\alpha^2\nu - \xi) - \vec{\Delta}\cdot\vec{Q})\, d\nu. \qquad (2.23)$$

Dividing the integration region into the intervals $(-\infty, -R)$, $[-R, R]$, and (R, ∞), we assume that it is possible to interchange the limit $\alpha \to +0$ and the integration in the range $[-R, R]$. In the other intervals we introduce the change of variable $\nu \to \xi'$ where $\nu = -2\alpha^{-2}(\xi' - \xi)$. We then have

$$\lim_{\alpha^2\to 0} I = \int_{-R}^{R} A(\nu, -\vec{\Delta}^2, 2\xi\nu - \eta, -\gamma\xi - \vec{\Delta}\cdot\vec{Q})\, d\nu + \lim_{\alpha^2\to 0}\left(\int_{-\infty}^{\xi - \alpha^2 R/2} + \int_{\xi + \alpha^2 R/2}^{\infty}\right)[2\alpha^{-2} A(-2\alpha^{-2}(\xi' - \xi), \alpha^2\gamma^2 - \vec{\Delta}^2, 4\alpha^{-2}\xi'(\xi' - \xi) - \eta, \gamma(\xi - 2\xi') - \vec{\Delta}\cdot\vec{Q})]\, d\xi'.$$

In the second term the integral is evaluated in the scaling region. We may therefore use scaling behavior in this integral. Since the lowest value of $|\nu|$ in this term is $|\nu| = R$, we must choose $R \geqslant R_0$ where R_0 is the value of ν at which scaling behavior occurs. Thus, letting $\alpha^2 \to 0$ and then proceeding to $R \to \infty$, we obtain

$$\lim_{\alpha^2\to 0} I = \int_{-\infty}^{\infty} A(\nu, -\vec{\Delta}^2, -2\xi\nu - \eta, -\gamma\xi - \vec{\Delta}\cdot\vec{Q})\, d\nu - \mathrm{P}\int \frac{F(\xi', -\vec{\Delta}^2, \gamma(\xi - 2\xi') - \vec{\Delta}\cdot\vec{Q})}{\xi' - \xi}\, d\xi', \qquad (2.24)$$

where $\nu A \sim F$ in the scaling limit.

It has been shown[8] that causality requires the scaling functions in (2.15) and (2.16) to be independent of ρ. It is therefore safe to take the limit $\gamma \to 0$ in Eq. (2.24) obtaining

$$\lim_{\gamma\to 0}\lim_{\alpha^2\to 0} I = \int_{-\infty}^{\infty} A(\nu, -\vec{\Delta}^2, -2\xi\nu - \eta, -\vec{\Delta}\cdot\vec{Q})\, d\nu - \mathrm{P}\int \frac{F(\xi', -\vec{\Delta}^2, -\vec{\Delta}\cdot\vec{Q})}{\xi' - \xi}\, d\xi'. \qquad (2.25)$$

We now apply this theorem to the sum rules (2.20)–(2.22) taking scaling behavior into account. For $k = 3, 4, 5$ the sum rule (2.20) yields

$$\int_{-\infty}^{\infty} A_k^{ij}(\nu, -\vec{\Delta}^2, -2\xi\nu - \eta, -\vec{\Delta}\cdot\vec{Q})\, d\nu = 0 \qquad k = 3, 4, 5 \quad (2.26)$$

since $\nu A_k^{ij} \sim 0$ in the scaling limit. Condition (2.19) implies that in all integrals of the form in (2.26) $\eta \geqslant \xi^2(1 + \frac{1}{4}\vec{\Delta}^2)$.

From the scaling behavior (2.16) and the result (2.25) one gets ($k=3, 4, 5$)

$$\lim_{\gamma\to 0}\lim_{\alpha^2\to 0}\int \nu A_k^{ij}(\nu,\ldots)d\nu=\int_{-\infty}^{\infty}\nu A_k^{ij}(\nu,-\vec{\Delta}^2,-2\xi\nu-\eta,-\vec{\Delta}\cdot\vec{Q})d\nu-\frac{1}{4\pi}\,\mathrm{P}\int\frac{G_k^{ij}(\xi',-\vec{\Delta}^2,-\vec{\Delta}\cdot\vec{Q})}{\xi'-\xi}\,d\xi'$$

$$=\lim_{\gamma\to 0}\lim_{\alpha^2\to 0}\alpha^{-2}b_k^{ij}(\alpha^2\gamma^2-\vec{\Delta}^2)\,,$$

where the last equality follows from (2.21). The assumption that this limit exists implies that

$$b_k^{ij}(-\vec{\Delta}^2)=0,\quad k=3,4,5\,. \tag{2.27}$$

Thus (2.21) becomes, for $\alpha^2\gamma^2<\vec{\Delta}^2$,

$$\int \nu A_k^{ij}(\nu,\ldots)d\nu=0\,. \tag{2.28}$$

Taking the limit $\gamma\to 0$, $\alpha^2\to 0$ in this equation and using (2.25) again we finally obtain

$$\int_{-\infty}^{\infty}\nu A_k^{ij}(\nu,-\vec{\Delta}^2,-2\xi\nu-\eta,-\vec{\Delta}\cdot\vec{Q})d\nu=\frac{1}{4\pi}\,\mathrm{P}\int\frac{G_k^{ij}(\xi',-\vec{\Delta}^2,-\vec{\Delta}\cdot\vec{Q})}{\xi'-\xi}\,d\xi' \tag{2.29}$$

for $k=3, 4, 5$.

For $k=1$, the scaling behavior (2.15) and Eq. (2.20) lead, in the limit, to the sum rule

$$\int_{-\infty}^{\infty}A_1^{ij}(\nu,-\vec{\Delta}^2,-2\xi\nu-\eta,-\vec{\Delta}\cdot\vec{Q})d\nu=\frac{1}{8\pi}\,\mathrm{P}\int\frac{F_L^{ij}(\xi',-\vec{\Delta}^2,-\vec{\Delta}\cdot\vec{Q})}{\xi'^2(\xi'-\xi)}\,d\xi'\,. \tag{2.30}$$

If now, one assumes that the integral

$$\int \nu A_1^{ij}(\nu,-\vec{\Delta}^2,-2\xi\nu-\eta,-\vec{\Delta}\cdot\vec{Q})d\nu$$

exists, then Eq. (2.21) gives, in the limit, the result

$$b_1^{ij}(-\vec{\Delta}^2)=\frac{1}{4\pi}\,\mathrm{P}\int\frac{1}{\xi'^2}F_L^{ij}(\xi',-\vec{\Delta}^2,-\vec{\Delta}\cdot\vec{Q})d\xi'\,. \tag{2.31}$$

Turning now to the consideration of Eq. (2.22) for $k=3, 4, 5$, we observe that as $\alpha^2\to 0$

$$\int \nu^2A_k^{ij}(\nu,\ldots)d\nu\sim c_k^{ij}(-\vec{\Delta}^2)\alpha^{-4}+\alpha^{-2}\gamma d_k^{ij}(-\vec{\Delta}^2)\,. \tag{2.32}$$

Under the assumption that the integrals

$$\int \nu^2A_k^{ij}(\nu,-\vec{\Delta}^2,-2\xi\nu-\eta,-\gamma\xi-\vec{\Delta}\cdot\vec{Q})d\nu$$

exist ($k=3, 4, 5$), Eq. (2.32) gives

$$\int_{-\infty}^{\infty}\nu^2A_k^{ij}(\nu,-\vec{\Delta}^2,-2\xi\nu-\eta,-\gamma\xi-\vec{\Delta}\cdot\vec{Q})d\nu+\frac{1}{2\pi}\alpha^{-2}\mathrm{P}\int G_k^{ij}(\xi',-\vec{\Delta}^2,\gamma(\xi-2\xi')-\vec{\Delta}\cdot\vec{Q})d\xi'\sim c_k^{ij}(-\vec{\Delta}^2)\alpha^{-4}+\alpha^{-2}\gamma d_k^{ij}(-\vec{\Delta}^2)\,.$$

Thus

$$c_k^{ij}(-\vec{\Delta}^2)=0\,, \tag{2.33}$$

and

$$\int G_k^{ij}(\xi',-\vec{\Delta}^2,\gamma(\xi-2\xi')-\vec{\Delta}\cdot\vec{Q})d\xi'=2\pi\gamma d_k^{ij}(-\vec{\Delta}^2)\,; \tag{2.34}$$

so that

$$\int G_k^{ij}(\xi',-\vec{\Delta}^2,-\vec{\Delta}\cdot\vec{Q})d\xi'=0\,. \tag{2.35}$$

The consideration of Eq. (2.22) for $k=1$, in the limit, would require the existence of the integral

$$\int \nu^2A_1^{ij}(\nu,-\vec{\Delta}^2,-2\xi\nu-\eta,-\gamma\xi-\vec{\Delta}\cdot\vec{Q})d\nu\,.$$

Under this assumption one obtains Eq. (2.31) for b_1^{ij} as well as the results

$$c_1^{ij}(-\vec{\Delta}^2)=-\frac{1}{2\pi}\,\mathrm{P}\int F_L^{ij}(\xi',-\vec{\Delta}^2,-\vec{\Delta}\cdot\vec{Q})\frac{d\xi'}{\xi'}\,, \tag{2.33$'$}$$

and

$$d_1^{ij}(-\vec{\Delta}^2)=0\,.$$

C. Fixed-mass sum rules

The main sum rules obtained in the previous section are

$$\int A_k^{ij}(\nu,-\vec{\Delta}^2,-2\xi\nu-\eta,-\vec{\Delta}\cdot\vec{Q})d\nu=0,\quad k=3,4,5 \tag{2.26}$$

$$\int \nu A_k^{ij}(\nu,-\vec{\Delta}^2,-2\xi\nu-\eta,-\vec{\Delta}\cdot\vec{Q})d\nu=\frac{1}{4\pi}\,\mathrm{P}\int\frac{G_k^{ij}(\xi',-\vec{\Delta}^2,-\vec{\Delta}\cdot\vec{Q})}{\xi'-\xi}d\xi'\,,\quad k=3,4,5 \tag{2.29}$$

$$\int A_1^{ij}(\nu, -\vec{\Delta}^2, -2\xi\nu - \eta, -\vec{\Delta}\cdot\vec{Q})d\nu = \frac{1}{8\pi}\,\mathrm{P}\int \frac{F_L^{ij}(\xi', -\vec{\Delta}^2, -\vec{\Delta}\cdot\vec{Q})}{\xi'^2(\xi'-\xi)}d\xi' , \quad (2.30)$$

in addition to the sum rules (2.27), (2.31), (2.33), (2.33′), and (2.35).

To obtain the fixed-mass sum rules one proceeds to the limit $\xi \to 0$. If one simply sets $\xi = 0$, one gets the following set of fixed mass sum rules:

$$\int A_1^{ij}(\nu, t, Q^2, \rho)d\nu = \frac{1}{8\pi}\,\mathrm{P}\int \frac{1}{\omega^3}F_L^{ij}(\omega, t, \rho)d\omega , \quad (2.36)$$

$$\int A_k^{ij}(\nu, t, Q^2, \rho)d\nu = 0, \quad k = 3, 4, 5 \quad (2.37)$$

$$\int \nu A_k^{ij}(\nu, t, Q^2, \rho)d\nu = \frac{1}{4\pi}\,\mathrm{P}\int \frac{1}{\omega}G_k^{ij}(\omega, t, \rho)d\omega , \quad k = 3, 4, 5 \quad (2.38)$$

where $t, Q^2 < 0$.

In terms of the structure functions W_k^{ij} and the scaling functions F_k^{ij} [see Appendix A and Eqs. (2.17)] these sum rules may be written in the following form:

$$\int W_4^{ij}(\nu, t, Q^2, \rho)d\nu = 0 , \quad (2.39)$$

$$\int \left[\frac{\nu W_2^{ij}(\nu, t, Q^2, \rho)}{Q^2 - \frac{1}{4}t} + W_3^{ij}(\nu, t, Q^2, \rho)\right]d\nu = 0 , \quad (2.40)$$

$$\int \nu\left[\frac{\nu W_2^{ij}(\nu, t, Q^2, \rho)}{Q^2 - \frac{1}{4}t} + W_3^{ij}(\nu, t, Q^2, \rho)\right]d\nu = \tfrac{1}{2}(Q^2 + \tfrac{1}{4}t)\mathrm{P}\int F_{23}^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} + \tfrac{1}{2}\rho\,\mathrm{P}\int F_4^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} , \quad (2.41)$$

$$-\int \nu W_4^{ij}(\nu, t, Q^2, \rho)d\nu = \tfrac{1}{2}(Q^2 + \tfrac{1}{4}t)\mathrm{P}\int F_4^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} + \tfrac{1}{2}\rho\,\mathrm{P}\int F_{23}^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} , \quad (2.42)$$

$$\int \left[W_L^{ij}(\nu, t, Q^2, \rho) + (Q^2 - \tfrac{1}{4}t)W_5^{ij}(\nu, t, Q^2, \rho)\right]d\nu = (Q^2 + \tfrac{1}{4}t)\mathrm{P}\int F_{23}^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} + \rho\,\mathrm{P}\int F_4^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} . \quad (2.43)$$

$$\int \left[(Q^2 + \tfrac{3}{4}t)W_L^{ij}(\nu, t, Q^2, \rho) + (\rho^2 - \tfrac{1}{4}t^2)W_5^{ij}(\nu, t, Q^2, \rho)\right]d\nu$$
$$= \tfrac{1}{4}[(Q^2 + \tfrac{1}{4}t)^2 - \rho^2]\,\mathrm{P}\int F_L^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^3} + \tfrac{1}{2}t(Q^2 + \tfrac{1}{4}t)\mathrm{P}\int F_{23}^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} + \tfrac{1}{2}\rho t\,\mathrm{P}\int F_4^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} , \quad (2.44)$$

$$\int \nu\left[W_L^{ij}(\nu, t, Q^2, \rho) - \frac{2\nu^2 W_2^{ij}(\nu, t, Q^2, \rho)}{Q^2 - \frac{1}{4}t} - 2\nu W_3^{ij}(\nu, t, Q^2, \rho) + (Q^2 - \tfrac{1}{4}t)W_5^{ij}(\nu, t, Q^2, \rho)\right]d\nu$$
$$= \tfrac{1}{4}[(Q^2 + \tfrac{1}{4}t)^2 - \rho^2]\,\mathrm{P}\int \frac{1}{\omega^3}\left[F_{23}^{ij} - 2\omega F_5^{ij} - \frac{1}{2\omega}F_L^{ij}\right]d\omega . \quad (2.45)$$

where

$$F_{23}^{ij}(\omega, t, \rho) = \frac{1}{2\omega}F_2^{ij} - F_3^{ij} . \quad (2.46)$$

In terms of F_k^{ij} the sum rules (2.35) read

$$\int F_{23}^{ij}(\omega, t, \rho)\frac{d\omega}{\omega} = 0 , \quad (2.47)$$

$$\int F_4^{ij}(\omega, t, \rho)\frac{d\omega}{\omega} = 0 , \quad (2.48)$$

$$\int \frac{1}{\omega^2}\left[F_{23}^{ij}(\omega, t, \rho) - \omega F_5^{ij}(\omega, t, \rho) - \frac{1}{4\omega}F_L^{ij}(\omega, t, \rho)\right]d\omega = 0 . \quad (2.49)$$

Under our assumptions on scaling behavior the above constitute a complete set of sum rules that are obtainable from causality for the matrix element under consideration. In the next section we consider extra sum rules that follow from the additional hypothesis of equal-time current algebra.

A full discussion of all these sum rules and their relation to those previously obtained from light-cone commutators is then given in Sec. IV.

III. EQUAL-TIME ALGEBRA

A. General

When the equal-time commutator

$$\langle p_2|[J_0^i(\tfrac{1}{2}x), J_0^j(-\tfrac{1}{2}x)]|p_1\rangle\delta(x_0) \quad (3.1)$$

is nonzero, the invariant amplitude A_2^{ij} cannot be causal.[7] Assuming that the Fourier transform E_{00}^{ij} of (3.1),

$$E_{00}^{ij} = \int C_{00}^{ij}(Q)dQ_0 , \quad (3.2)$$

is given by the equal-time algebra

$$E_{00}^{ij}=if^{ijk}F_k(t)P_0\ , \qquad (3.3)$$

one can identify[7] the noncausal part, $A_2^{ij,\mathrm{nc}}$, of A_2^{ij} as

$$A_2^{ij,\mathrm{nc}}=\frac{if^{ijk}F_k(t)}{Q^2-\frac{1}{4}t}[\epsilon(P_0+Q_0)\delta(Q^2-\tfrac{1}{4}t+2\nu)+\epsilon(P_0-Q_0)\delta(Q^2-\tfrac{1}{4}t-2\nu)]\ . \qquad (3.4)$$

Since this noncausal part is completely antisymmetric in the internal indices, the symmetric component, $A_2^{(ij)}$, of A_2^{ij} is causal and satisfies the following causality sum rules

$$\int A_2^{(ij)}dQ_0=0\ , \qquad (3.5)$$

$$\int Q_0A_2^{(ij)}dQ_0=b_2^{(ij)}(t)\ , \qquad (3.6)$$

$$\int Q_0{}^2A_2^{(ij)}dQ_0=0\ , \qquad (3.7)$$

corresponding to the general causality sum rules (2.5)–(2.8).

Using the explicit expression (3.4) for $A_2^{ij,\mathrm{nc}}$ and the above causality sum rules one then obtains the following sum rules for the antisymmetric component $A_2^{[ij]}$ (see Ref. 7):

$$\int A_2^{[ij]}dQ_0=\frac{if^{ijk}F_k(t)P_0}{(\vec{P}\cdot\vec{Q})^2-(\frac{1}{4}t+\vec{Q}^2)P_0{}^2}\ , \qquad (3.8)$$

$$\int Q_0A_2^{[ij]}dQ_0=\frac{if^{ijk}F_k(t)\vec{P}\cdot\vec{Q}}{(\vec{P}\cdot\vec{Q})^2-(\frac{1}{4}t+\vec{Q}^2)P_0{}^2}\ , \qquad (3.9)$$

and

$$\int Q_0{}^2A_2^{[ij]}dQ_0=\frac{i(\vec{Q}^2+\frac{1}{4}t)f^{ijk}F_k(t)}{(\vec{P}\cdot\vec{Q})^2-(\frac{1}{4}t+\vec{Q}^2)P_0{}^2}P_0+c_2^{[ij]}(t)P_0+d_2^{[ij]}(t)\Delta_0\ . \qquad (3.10)$$

One notes that all the sum rules (3.5)–(3.10) are consequences of causality *and* current algebra. In particular the causality sum rules (3.5)–(3.7) follow only on imposing the equal-time algebra (3.3). The presence of a symmetric Schwinger contribution on the right-hand side of (3.3), for example, would introduce a symmetric noncausal part and consequently invalidate the sum rules (3.5)–(3.7), albeit in a definite manner.

B. Scaling

Transforming to the variables of Sec. (II B) we write the above sum rules in the following form:

$$\int A_2^{(ij)}(\nu,\ \alpha^2\gamma^2-\vec{\Delta}^2,\ \alpha^2\nu^2-2\xi\nu-\eta,\ \gamma(\alpha^2\nu-\xi)-\vec{\Delta}\cdot\vec{Q})d\nu=0\ , \qquad (3.11)$$

$$\alpha^2\int\nu A_2^{(ij)}(\nu,\ldots)d\nu=b_2^{(ij)}(\alpha^2\gamma^2-\vec{\Delta}^2)\ , \qquad (3.12)$$

$$\alpha^4\int\nu^2A_2^{(ij)}(\nu,\ldots)d\nu=2\xi b_2^{(ij)}(\alpha^2\gamma^2-\vec{\Delta}^2)\ , \qquad (3.13)$$

$$\int A_2^{[ij]}(\nu,\ldots)d\nu=\frac{-4if^{ijk}F_k(\alpha^2\gamma^2-\vec{\Delta}^2)}{4\eta-\vec{\Delta}^2+\alpha^2\gamma^2}\ , \qquad (3.14)$$

$$\int\nu A_2^{[ij]}(\nu,\ldots)d\nu=0\ , \qquad (3.15)$$

$$\alpha^4\int\nu^2A_2^{[ij]}(\nu,\ldots)d\nu=-i\alpha^2f^{ijk}F_k(\alpha^2\gamma^2-\vec{\Delta}^2)+c_2^{[ij]}(\alpha^2\gamma^2-\vec{\Delta}^2)+\alpha^2\gamma d_2^{[ij]}(\alpha^2\gamma^2-\vec{\Delta}^2)\ . \qquad (3.16)$$

To these sum rules we apply the method of Sec. (II B) noting the scaling behavior (2.16) for A_2^{ij}. We obtain

$$\int A_2^{(ij)}(\nu,\ -\vec{\Delta}^2,\ -2\xi\nu-\eta,\ -\vec{\Delta}\cdot\vec{Q})d\nu=0\ , \qquad (3.17)$$

from Eq. (3.11);

$$\int G_2^{(ij)}(\xi',\ -\vec{\Delta}^2,\ -\vec{\Delta}\cdot\vec{Q})d\xi'=0\ , \qquad (3.18)$$

from Eq. (3.13);

$$b_2^{(ij)}(-\vec{\Delta}^2)=0\ , \qquad (3.19a)$$

and

$$\int\nu A_2^{ij}(\nu,\ -\vec{\Delta}^2,\ -2\xi\nu-\eta,\ -\vec{\Delta}\cdot\vec{Q})d\nu=\frac{1}{4\pi}\,\mathrm{P}\int\frac{G_2^{ij}(\xi',\ -\vec{\Delta}^2,\ -\vec{\Delta}\cdot\vec{Q})}{\xi'-\xi}d\xi'\ , \qquad (3.19b)$$

from (3.12) and (3.15);

$$\int A_2^{[ij]}(\nu,\ -\vec{\Delta}^2,\ -2\xi\nu-\eta,\ -\vec{\Delta}\cdot\vec{Q})d\nu=\frac{if^{ijk}F_k(-\vec{\Delta}^2)}{\frac{1}{4}\vec{\Delta}^2-\eta}\ , \qquad (3.20)$$

from Eq. (3.14);

$$c_2^{[ij]}(-\vec{\Delta}^2)=0\ , \qquad (3.21)$$

and

$$\frac{1}{2\pi}\int G_2^{[ij]}(\xi',\ -\vec{\Delta}^2,\ -\vec{\Delta}\cdot\vec{Q})d\xi'=-if^{ijk}F_k(-\vec{\Delta}^2)\ , \qquad (3.22)$$

from Eq. (3.16).

C. Fixed-mass sum rules

From the sum rules of the previous subsection one obtains the following fixed-mass sum rules:

$$\int A_2^{(ij)}(\nu, t, Q^2, \rho)d\nu = 0 \;, \qquad (3.23)$$

$$\int \nu A_2^{ij}(\nu, t, Q^2, \rho)d\nu = \frac{1}{4\pi}\, \mathrm{P} \int G_2^{ij}(\omega, t, \rho)\frac{d\omega}{\omega} \;, \qquad (3.24)$$

$$\int A_2^{[ij]}(\nu, t, Q^2, \rho)d\nu = \frac{if^{ijk}F_k(t)}{Q^2 - \frac{1}{4}t} \;, \qquad (3.25)$$

where $t, Q^2 < 0$. In addition one has the results (3.18), (3.19a), (3.21), and (3.22).

We thus stress that of all the fixed-mass sum rules that one may derive for the structure functions A_k^{ij} using scaling behavior, it is only the sum rules of the present section that require *current algebra* as well as causality. In particular of the ten fixed-mass sum rules (2.36)–(2.38) and (3.23)–(3.25) directly involving integrals over the structure functions A_k^{ij}, only three [namely (3.23)–(3.25)] require the additional assumption of current algebra.

Finally we rewrite the sum rules (3.22)–(3.25) in terms of the structure function W_2^{ij} and the scaling function F_2^{ij}:

$$\frac{1}{2\pi} \int F_2^{[ij]}(\omega, t, \rho)\frac{d\omega}{\omega} = if^{ijk}F_k(t) \;, \qquad (3.26)$$

$$\int W_2^{(ij)}(\nu, t, Q^2, \rho)d\nu = 0 \;, \qquad (3.27)$$

$$\frac{1}{\frac{1}{4}t - Q^2}\int \nu W_2^{ij}(\nu, t, Q^2, \rho)d\nu = \tfrac{1}{2}\,\mathrm{P}\int F_2^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2}, \qquad (3.28)$$

$$\frac{1}{2\pi}\int W_2^{[ij]}(\nu, t, Q^2, \rho)d\nu = if^{ijk}F_k(t) \;. \qquad (3.29)$$

We next include a short subsection on the restrictions placed by causality, scaling, and the form of the equal-time commutator E_{00}^{ij} on the structure of equal-time algebra of other current components.

D. Equal-time commutators $E_{\mu\nu}^{ij}$

So far we have restricted our considerations to the time-time algebra E_{00}^{ij}, for which the structure (3.3) is assumed. This structure, together with causality and scaling, imposes considerable contraints on the form of the equal-time algebra E_{0r}^{ij} involving a time and a space component. For, the results (2.27), (2.31), and (3.19a)—consequences of causality, scaling, and the algebra E_{00}^{ij} of (3.3)—reduce E_{0r}^{ij} to the form[7]

$$E_{0r}^{ij} = \int C_{0r}^{ij}(Q)dQ_0 = if^{ijk}F_k(t)P_r + \frac{1}{4\pi}\,\mathrm{P}\int F_L^{(ij)}(\omega, t, \rho)\frac{d\omega}{\omega^2}\,[Q_r + \tfrac{1}{2}\Delta_r] \;, \qquad (3.30)$$

clearly exhibiting the Schwinger-term sum rule.[12] If one now requires that the equal-time commutator $E_{0\nu}^{ij}$ be of the form

$$E_{0\nu}^{ij} = if^{ijk}F_k(t)P_\nu \;, \qquad (3.31)$$

then one obtains

$$\mathrm{P}\int F_L^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} = 0 \;. \qquad (3.32)$$

Assuming that this requires $F_L^{ij} \equiv 0$, it follows that $\nu^2 A_1^{ij}$ may then scale, $\nu^2 A_1^{ij} \sim G_1^{ij}$, so that our analysis on A_k^{ij} ($k = 3, 4, 5$) is also applicable to A_1^{ij}. This leads to the sum rules

$$\int G_1^{ij}(\omega, t, \rho)d\omega = 0 \;, \quad c_1^{ij}(t) = d_1^{ij}(t) = 0 \;. \qquad (3.33)$$

If in addition one takes the scaling functions to be independent of ρ, so that $d_k^{ij} = 0$ from (2.34) and (3.16), one can determine the structure of the space-space equal-time commutator E_{rs}^{ij} completely.[7] One finds

$$E_{rs}^{ij} = if^{ijk}F_k(t)P_0\delta_{rs} \;, \qquad (3.34)$$

which flatly rejects the space-space equal-time commutators of field algebra.[13] However, for $r = s$, this expression coincides with the result obtained from the quark model.[14] It thus appears that, when combined with scaling, the hypothesis of field algebra is inconsistent with experiment, since the right-hand side of (3.34) cannot identically vanish. Such conflict of field algebra and scaling with experiment has previously been suspected in other contexts.[15]

IV. COMPARISON WITH LIGHT-CONE ANALYSIS

As tests of light-cone commutators, Dicus and Teplitz[3] have extended previous work on the forward case[2] to derive a complete set of fixed-mass sum rules for the nonforward matrix element $C_{\mu\nu}^{ij}$ using the same assumptions on scaling behavior as in this paper. This analysis is based on the $(+, \nu)$ light-cone commutators of a vector-gluon fermion-quark model.[1] The sum rules they obtain may be rearranged to read as follows:

$$\frac{1}{\pi}\int_0^\infty \overline{W}_2^{[ij]}d\nu = if^{ijk}F_k(t) \;, \qquad (4.1)$$

$$\frac{1}{\frac{1}{4}t - Q^2}\int_0^\infty \nu\overline{W}_2^{(ij)}d\nu = \tfrac{1}{2}\,\mathrm{P}\int_0^\infty \overline{F}_2^{(ij)}\frac{d\omega}{\omega^2} \;, \qquad (4.2)$$

$$\frac{1}{\frac{1}{4}t - Q^2}\int_0^\infty \nu\overline{W}_2^{(ij)}d\nu - \int_0^\infty \overline{W}_3^{(ij)}d\nu = 0 \;, \qquad (4.3)$$

$$\int_0^\infty \overline{W}_4^{(ij)}d\nu = 0 \;, \qquad (4.4)$$

$$2\int_0^\infty \overline{W}_L^{[ij]}d\nu - \frac{[t(Q^2+\frac{1}{4}t)-2\rho^2]}{(Q^2+\frac{1}{4}t-\rho)(Q^2+\frac{1}{4}t+\rho)(Q^2-\frac{1}{4}t)}\int_0^\infty \nu^2\overline{W}_2^{[ij]}d\nu + \frac{2\rho(Q^2-\frac{1}{4}t)}{(Q^2+\frac{1}{4}t-\rho)(Q^2+\frac{1}{4}t+\rho)}\int_0^\infty \nu\overline{W}_4^{[ij]}d\nu$$
$$+\frac{2\rho^2-t(Q^2+\frac{1}{4}t)}{(Q^2+\frac{1}{4}t-\rho)(Q^2+\frac{1}{4}t+\rho)}\int_0^\infty \nu\overline{W}_3^{[ij]}d\nu = \tfrac{1}{2}t\,\mathrm{P}\int_0^\infty \overline{F}_{23}^{[ij]}\frac{d\omega}{\omega^2}\,, \quad (4.5)$$

$$\frac{2\rho}{(Q^2+\frac{1}{4}t-\rho)(Q^2+\frac{1}{4}t+\rho)}\int_0^\infty \nu^2\overline{W}_2^{[ij]}d\nu + \frac{2\rho(Q^2-\frac{1}{4}t)}{(Q^2+\frac{1}{4}t-\rho)(Q^2+\frac{1}{4}t+\rho)}\int_0^\infty \nu\overline{W}_3^{[ij]}d\nu$$
$$-\frac{t(Q^2+\frac{1}{4}t)-2\rho^2}{(Q^2+\frac{1}{4}t-\rho)(Q^2+\frac{1}{4}t+\rho)}\int_0^\infty \nu\overline{W}_4^{[ij]}d\nu = \rho\,\mathrm{P}\int_0^\infty \overline{F}_{23}^{[ij]}\frac{d\omega}{\omega^2}\,. \quad (4.6)$$

In these sum rules,

$$\overline{W}_m^{ij} = \tfrac{1}{2}[W_m^{ij}(\nu,t,Q^2,\rho)+W_m^{ij}(\nu,t,Q^2,-\rho)], \quad m=L,2,3,5 \quad (4.7)$$

$$\overline{W}_4^{ij} = \tfrac{1}{2}[W_4^{ij}(\nu,t,Q^2,\rho)-W_4^{ij}(\nu,t,Q^2,-\rho)], \quad t,Q^2<0 \quad (4.8)$$

of which the ij-symmetric and ij-antisymmetric components have definite symmetry properties under the transformation $\nu\to-\nu$ in view of the fact that the W_k^{ij} satisfy[3]

$$W_k^{(ij)}(\nu,t,Q^2,\rho) = -W_k^{(ij)}(-\nu,t,Q^2,-\rho)\,, \quad k=L,2,4,5 \quad (4.9)$$

$$W_3^{(ij)}(\nu,t,Q^2,\rho) = W_3^{(ij)}(-\nu,t,Q^2,-\rho)\,, \quad (4.10)$$

$$W_k^{[ij]}(\nu,t,Q^2,\rho) = W_k^{[ij]}(-\nu,t,Q^2,-\rho)\,, \quad k=L,2,4,5 \quad (4.11)$$

$$W_3^{[ij]}(\nu,t,Q^2,\rho) = -W_3^{[ij]}(-\nu,t,Q^2,-\rho)\,. \quad (4.12)$$

In addition to these sum rules the authors of Ref. 3 obtain a corresponding set, identical in form to (4.1)–(4.6) with $\overline{W}_k^{ij}$ replaced by $\tilde{W}_k^{ij}$, where

$$\tilde{W}_m^{ij} = \tfrac{1}{2}[W_m^{ij}(\nu,t,Q^2,\rho)-W_m^{ij}(\nu,t,Q^2,-\rho)]\,, \quad m=L,2,3,5 \quad (4.13)$$

$$\tilde{W}_4^{ij} = \tfrac{1}{2}[W_4^{ij}(\nu,t,Q^2,\rho)+W_4^{ij}(\nu,t,Q^2,-\rho)]\,, \quad (4.14)$$

and with the structure functions having the opposite symmetry in i and j and the right-hand sides of the sum rules set to zero.

We now assert that the above sum rules are already among the ones we have obtained on the basis of causality, scaling, and current algebra. The sum rule (B11) (see Appendix B), e.g.,

$$\frac{1}{2\pi}\int_{-\infty}^\infty W_2^{ij}(\nu,t,Q^2,\rho)d\nu = if^{ijk}F_k(t)\,,$$

gives on writing

$$W_2^{ij} = \overline{W}_2^{ij} + \tilde{W}_2^{ij}$$

and noting that $\overline{W}_2^{(ij)}$ and $\tilde{W}_2^{[ij]}$ are antisymmetric in ν whereas $\tilde{W}_2^{(ij)}$ and $\overline{W}_2^{[ij]}$ are symmetric in ν

$$\frac{1}{\pi}\int_0^\infty \overline{W}_2^{[ij]}d\nu = if^{ijk}F_k(t)$$

and

$$\int_0^\infty \tilde{W}_2^{(ij)}d\nu = 0\,;$$

i.e., (B11) is equivalent to (4.1) and its counterpart involving $\tilde{W}_2^{(ij)}$. Similarly the sum rules (4.2), (4.3), (4.4), and their counterparts are equivalent to (B12), (B2), and (B1), respectively. In showing the equivalence one uses the fact that causality requires the scaling functions to be independent of ρ.[8]

The sum rules (4.5) and (4.6), and their counterparts, may—on using Eqs. (A6)–(A9) of Appendix A—be written in the form

$$2\int_{-\infty}^\infty [(Q^2-\tfrac{1}{4}t)A_1^{ij}-(\rho^2-\tfrac{1}{4}t^2)A_5^{ij}]d\nu - \int_{-\infty}^\infty \nu[(\rho-\tfrac{1}{2}t)A_3^{ij}+(\rho+\tfrac{1}{2}t)A_4^{ij}]d\nu = \frac{t}{4\pi}\,\mathrm{P}\int_{-\infty}^\infty F_{23}^{ij}\frac{d\omega}{\omega^2} \quad (4.15)$$

and

$$\int_{-\infty}^\infty \nu[(\rho-\tfrac{1}{2}t)A_3^{ij}-(\rho+\tfrac{1}{2}t)A_4^{ij}]d\nu = \frac{\rho}{2\pi}\,\mathrm{P}\int_{-\infty}^\infty F_{23}^{ij}\frac{d\omega}{\omega^2}\,. \quad (4.16)$$

These sum rules follow from our causality sum rules (2.36)–(2.38) on using (2.15)–(2.17), provided that $F_L^{ij}=F_4^{ij}=0$. The vanishing of these scaling functions is obtained by the authors of Ref. 3 as a consequence of their use of unsubtracted dispersion relations for the structure functions.

Thus we have demonstrated our assertion that all the light-cone sum rules (4.1)–(4.6) and their counterparts follow from causality, scaling, and current algebra. In fact, all these sum rules, except (4.1), (4.2), and their counterparts, are consequences of causality and scaling.

In addition to the sum rules (4.1)–(4.6), Dicus and Teplitz[3] also obtain the following sum rule, involving only the scaling functions,

$$\int_0^1 F_3^{(ij)}(\omega, t, \rho)\frac{d\omega}{\omega} = \tfrac{1}{2}\int_0^1 F_2^{(ij)}(\omega, t, \rho)\frac{d\omega}{\omega^2} . \tag{4.17}$$

This sum rule follows from our causality sum rule (B8). On setting $F_L^{ij}=0$, we also obtain [see Eq. (B10)] a similar sum rule involving F_5^{ij},

$$\int_0^1 (F_3^{[ij]} + \omega F_5^{[ij]})\frac{d\omega}{\omega^2} = \tfrac{1}{2}\int_0^1 F_2^{[ij]}\frac{d\omega}{\omega^3}, \tag{4.18}$$

which is not obtained in the work of Ref. 3. In fact, as noted by the authors of that work, their analysis, which is based on the light-cone commutators $(+, \nu)$, does not give rise to sum rules involving the structure function W_5^{ij}. Thus none of our causality sum rules (B5), (B6), (B7), and (B10) containing W_5^{ij} or F_5^{ij} appears in their paper.

With

$$F_L^{ij} = F_4^{ij} = 0 ,$$

and

$$W_L^{ij} \sim -\frac{1}{2\omega Q^2} G_L^{ij}(\omega, t, \rho) , \tag{4.19}$$

one finds [using Eq. (A1) of Appendix A] that $\nu^2 A_1^{ij} \sim G_1^{ij}$ in the scaling limit where

$$G_1^{ij}(\omega, t, \rho) = \frac{1}{8\pi\omega^2}\left[tF_{23}^{ij}(\omega, t, \rho) - \frac{1}{2\omega} G_L^{ij}(\omega, t, \rho)\right] . \tag{4.20}$$

The sum rule (3.33) for G_1^{ij} then gives

$$\int G_L^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^3} = 2t\int F_{23}^{ij}(\omega, t, \rho)\frac{d\omega}{\omega^2} , \tag{4.21}$$

which also follows from the analysis of Ref. 3. Using our sum rule (4.18) one may also write from (4.21) the result

$$\int_0^1 G_L^{[ij]}(\omega, t, \rho)\frac{d\omega}{\omega^3} = 2t\int_0^1 F_5^{[ij]}(\omega, t, \rho)\frac{d\omega}{\omega} . \tag{4.22}$$

In conclusion we state that we have demonstrated that all of the fixed-mass light-cone sum rules so far obtained for the spinless single-particle matrix element of the commutator of conserved vector currents can be obtained either from causality and scaling alone or from causality, scaling, and equal-time algebra (for time-components) on using the refined infinite-momentum procedure.

One finally remarks that, for $\rho = 0$, all the fixed-mass sum rules we derived, including the new ones (B5), (B6), (B7), and (B10), are trivially satisfied in the free-quark-model Born approximation of the amplitudes[3] where

$$W_2^{ij} = \frac{i\pi}{2\nu(1-\frac{1}{4}t)} f^{ijk}\lambda_k\left[\delta\left(1-\omega-\frac{t}{8\nu}\right)+\delta\left(1+\omega+\frac{t}{8\nu}\right)\right] + \frac{\pi}{2\nu(1-\frac{1}{4}t)} d^{ijk}\lambda_k\left[\delta\left(1-\omega-\frac{t}{8\nu}\right)-\delta\left(1+\omega+\frac{t}{8\nu}\right)\right] , \tag{4.23}$$

$$W_3^{ij} = \frac{i\pi}{4\nu(1-\frac{1}{4}t)} f^{ijk}\lambda_k\left[\delta\left(1-\omega-\frac{t}{8\nu}\right)-\delta\left(1+\omega+\frac{t}{8\nu}\right)\right] + \frac{\pi}{4\nu(1-\frac{1}{4}t)} d^{ijk}\lambda_k\left[\delta\left(1-\omega-\frac{t}{8\nu}\right)+\delta\left(1+\omega+\frac{t}{8\nu}\right)\right] , \tag{4.24}$$

and

$$W_L^{ij} = W_4^{ij} = W_5^{ij} = 0 .$$

It might also be interesting to test our new fixed-mass sum rules in realistic perturbation-theoretic models as well as in, for example, the nonperturbative parton model of Landshoff, Polkinghorne, and Short.[16] We hope to report on this in a future paper.

As a last remark we mention that the spin-dependent fixed-mass sum rules of Ref. 2 were recently considered by one of us[17] and it was shown, on using the refined infinite-momentum technique, that these follow from causality and scaling alone. Consequently, in this case, our methods yield, from causality, the modified form of the Bég sum rule[18] obtained in Ref. 2 from light-cone commutators.

APPENDIX A: THE RELATIONS BETWEEN THE AMPLITUDES A_k^{ij} AND W_k^{ij}

The amplitudes A_k^{ij} of Eq. (2.3) are related to the conventional amplitudes W_k^{ij} of Eq. (2.9) by

$$A_1^{ij} = \frac{1}{2\pi(Q^2+\frac{1}{4}t+\rho)(Q^2+\frac{1}{4}t-\rho)}\left[(Q^2+\tfrac{3}{4}t)W_L^{ij} - \frac{\nu^2 t}{Q^2-\frac{1}{4}t}W_2^{ij} - \nu t W_3^{ij} + 2\nu\rho W_4^{ij} + (\rho+\tfrac{1}{2}t)(\rho-\tfrac{1}{2}t)W_5^{ij}\right] , \tag{A1}$$

$$A_2^{ij}=\frac{1}{2\pi(Q^2-\frac{1}{4}t)}W_2^{ij}\ ,\tag{A2}$$

$$A_3^{ij}=\frac{1}{2\pi(Q^2+\frac{1}{4}t-\rho)}\left[\frac{\nu W_2^{ij}}{Q^2-\frac{1}{4}t}+W_3^{ij}+W_4^{ij}\right]\ ,\tag{A3}$$

$$A_4^{ij}=\frac{-1}{2\pi(Q^2+\frac{1}{4}t+\rho)}\left[\frac{\nu W_2^{ij}}{Q^2-\frac{1}{4}t}+W_3^{ij}-W_4^{ij}\right]\ ,\tag{A4}$$

$$A_5^{ij}=\frac{1}{2\pi(Q^2+\frac{1}{4}t+\rho)(Q^2+\frac{1}{4}t-\rho)}\left[W_L^{ij}-\frac{2\nu^2}{Q^2-\frac{1}{4}t}W_2^{ij}-2\nu W_3^{ij}+(Q^2-\frac{1}{4}t)W_5^{ij}\right]\ .\tag{A5}$$

For the sake of completeness we also give the expressions for the W_k^{ij} in terms of the A_k^{ij}. These are

$$W_L^{ij}=2\pi(Q^2-\tfrac{1}{4}t)A_1^{ij}-2\pi\nu(\rho-\tfrac{1}{2}t)A_3^{ij}-2\pi\nu(\rho+\tfrac{1}{2}t)A_4^{ij}-2\pi(\rho^2-\tfrac{1}{4}t^2)A_5^{ij}\ ,\tag{A6}$$

$$W_2^{ij}=2\pi(Q^2-\tfrac{1}{4}t)A_2^{ij}\ ,\tag{A7}$$

$$W_3^{ij}=\pi(Q^2+\tfrac{1}{4}t-\rho)A_3^{ij}-\pi(Q^2+\tfrac{1}{4}t+\rho)A_4^{ij}-2\pi\nu A_2^{ij}\ ,\tag{A8}$$

$$W_4^{ij}=\pi(Q^2+\tfrac{1}{4}t-\rho)A_3^{ij}+\pi(Q^2+\tfrac{1}{4}t+\rho)A_4^{ij}\ ,\tag{A9}$$

$$W_5^{ij}=-2\pi A_1^{ij}+2\pi\nu A_3^{ij}-2\pi\nu A_4^{ij}+2\pi(Q^2+\tfrac{3}{4}t)A_5^{ij}\ .\tag{A10}$$

APPENDIX B: SUMMARY OF FIXED-MASS SUM RULES

1. Sum rules following from causality and scaling

$$\int W_4^{ij}\,d\nu=0\ ,\tag{B1}$$

from Eq. (2.39);

$$\frac{1}{Q^2-\frac{1}{4}t}\int\nu W_2^{ij}d\nu+\int W_3^{ij}d\nu=0\ ,\tag{B2}$$

from Eq. (2.40);

$$\frac{1}{Q^2-\frac{1}{4}t}\int\nu^2W_2^{ij}d\nu+\int\nu W_3^{ij}d\nu=\tfrac{1}{2}(Q^2+\tfrac{1}{4}t)\mathrm{P}\int F_{23}^{ij}\frac{d\omega}{\omega^2}+\tfrac{1}{2}\rho\,\mathrm{P}\int F_4^{ij}\frac{d\omega}{\omega^2}\ ,\tag{B3}$$

from Eq. (2.41);

$$-\int\nu W_4^{ij}d\nu=\tfrac{1}{2}(Q^2+\tfrac{1}{4}t)\mathrm{P}\int F_4^{ij}\frac{d\omega}{\omega^2}+\tfrac{1}{2}\rho\,\mathrm{P}\int F_{23}^{ij}\frac{d\omega}{\omega^2}\ ,\tag{B4}$$

from Eq. (2.42);

$$\int W_L^{ij}d\nu+(Q^2-\tfrac{1}{4}t)\int W_5^{ij}d\nu=(Q^2+\tfrac{1}{4}t)\mathrm{P}\int F_{23}^{ij}\frac{d\omega}{\omega^2}+\rho\,\mathrm{P}\int F_4^{ij}\frac{d\omega}{\omega^2}\ ,\tag{B5}$$

from Eq. (2.43);

$$(Q^2+\tfrac{3}{4}t)\int W_L^{ij}d\nu+(\rho^2-\tfrac{1}{4}t^2)\int W_5^{ij}d\nu=\tfrac{1}{4}[(Q^2+\tfrac{1}{4}t)^2-\rho^2]\,\mathrm{P}\int F_L^{ij}\frac{d\omega}{\omega^3}+\tfrac{1}{2}t(Q^2+\tfrac{1}{4}t)\mathrm{P}\int F_{23}^{ij}\frac{d\omega}{\omega^2}+\tfrac{1}{2}\rho t\,\mathrm{P}\int F_4^{ij}\frac{d\omega}{\omega^2}\ ,\tag{B6}$$

from Eq. (2.44);

$$\int\nu W_L^{ij}d\nu-\frac{2}{Q^2-\frac{1}{4}t}\int\nu^3W_2^{ij}d\nu-2\int\nu^2W_3^{ij}d\nu+(Q^2-\tfrac{1}{4}t)\int\nu W_5^{ij}d\nu=\tfrac{1}{4}[(Q^2+\tfrac{1}{4}t)^2-\rho^2]\,\mathrm{P}\int\left[F_{23}^{ij}-2\omega F_5^{ij}-\frac{1}{2\omega}F_L^{ij}\right]\frac{d\omega}{\omega^3}\ ,\tag{B7}$$

from Eq. (2.45);

$$\int F_{23}^{ij}\frac{d\omega}{\omega}=0\ ,\tag{B8}$$

from Eq. (2.47);

$$\int F_4^{ij}\frac{d\omega}{\omega}=0\ ,\tag{B9}$$

from Eq. (2.48);

$$\int\left[F_{23}^{ij}-\omega F_5^{ij}-\frac{1}{4\omega}F_L^{ij}\right]\frac{d\omega}{\omega^2}=0\ ,\tag{B10}$$

from Eq. (2.49); where

$$F_{23}^{ij}=\frac{1}{2\omega}F_2^{ij}-F_3^{ij}\ .$$

2. Sum rules following from causality, scaling, and time-time algebra

$$\frac{1}{2\pi}\int W_2^{ij}d\nu=if^{ijk}F_k(t)\ ,\tag{B11}$$

from (3.27) and (3.29).

$$\frac{1}{\frac{1}{4}t-Q^2}\int \nu W_2^{ij}d\nu=\frac{1}{2}\mathrm{P}\int F_2^{ij}\frac{d\omega}{\omega^2}\ , \qquad \text{(B12)}$$

from Eq. (3.28);

$$\frac{1}{2\pi}\int F_2^{[ij]}\frac{d\omega}{\omega}=if^{ijk}F_k(t)\ , \qquad \text{(B13)}$$

from Eq. (3.26). In all the equations (B1)–(B13), $W_k^{ij}=W_k^{ij}(\nu,t,Q^2,\rho)$, $F_k^{ij}=F_k^{ij}(\omega,t,\rho)$ and $t, Q^2<0$.

3. Schwinger-term sum rule

From the assumed structure for E_{00}^{ij} one also obtains, using causality and scaling, the Schwinger term sum rule[12]

$$S^{ij}=\frac{1}{4\pi}\ \mathrm{P}\int F_L^{(ij)}(\omega,t,\rho)\frac{d\omega}{\omega^2}\ . \qquad \text{(B14)}$$

The form (3.31) for E_{0r}^{ij} then implies (3.32).

[1] J. M. Cornwall and R. Jackiw, Phys. Rev. D 4, 367 (1971).

[2] D. A. Dicus, R. Jackiw, and V. L. Teplitz, Phys. Rev. D 4, 1733 (1971).

[3] D. A. Dicus and V. L. Teplitz, Phys. Rev. D 6, 2262 (1972); 8, 1942(E) (1973).

[4] S. Fubini and G. Furlan, Physics 1, 229 (1965).

[5] A refined infinite-momentum limit was earlier considered by G. Segrè, Phys. Rev. Lett. 26, 796 (1971). See also M. O. Taha, Nucl. Phys. B66, 245 (1973).

[6] M. O. Taha, Nuovo Cimento 18A, 663 (1973).

[7] A-M. M. Abdel-Rahman, M. A. Ahmed, and M. O. Taha, Phys. Rev. D 9, 2349 (1974).

[8] D. J. Gross, Phys. Rev. D 4, 1130 (1971).

[9] M. A. Keppel-Jones, Phys. Rev. D 6, 1140 (1972).

[10] B. F. L. Ward, Nucl. Phys. B61, 1 (1973); we thank Professor Ward for bringing this work and that of Ref. 13 to our attention.

[11] B. F. L. Ward, SLAC Report No. PUB-1371 (1974) (unpublished).

[12] R. Jackiw, R. Van Royen, and G. West, Phys. Rev. D 2, 2473 (1970); H. Leutwyler and J. Stern, Nucl. Phys. B20, 77 (1970).

[13] N. M. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. 157, 1376 (1967).

[14] V. De Alfaro, S. Fubini, G. Furlan, and C. Rossetti, *Currents in Hadron Physics* (North-Holland, Amsterdam, 1973), p. 146.

[15] See, for example, C. G. Callan and D. Gross, Phys. Rev. Lett. 22, 156 (1969).

[16] P. V. Landshoff, J. C. Polkinghorne, and R. D. Short, Nucl. Phys. B28, 225 (1971); P. V. Landshoff and J. C. Polkinghorne, Phys. Rep. 5C, 1 (1972).

[17] A-M. M. Abdel-Rahman, Phys. Rev. D (to be published).

[18] M. A. B. Bég, Phys. Rev. Lett. 17, 333 (1966).