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Gauge invariance in spontaneously broken symmetry

W. Fischler* and R. Brout

Faculté des Sciences, Université Libre de Bruxelles, Bruxelles, Belgium

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A gauge-invariant approach to an effective potential in the presence of spontaneously broken symmetry is formulated.

Recently it has been pointed out by Jackiw[1] that the conventional rules of construction for the effective potential used in field theories of spontaneous breakdown lead to a gauge-dependent quantity, and hence a theoretical entity of dubious physical significance. The issue raised is therefore quite serious.

In this note we study the question of *spontaneous breakdown* of gauge invariance of the second kind where the word *spontaneous* is to be taken quite literally. Namely, we work in the presence of no sources and hence by definition with a gauge-invariant theory at the outset. Such a theory is characterized by an order parameter φ_c, and we find that the rules for its calculation are the conventional ones for the evaluation of the effective potential, but with the proviso that no photon lines are singly connected to tadpoles. Thus one effectively works out the graphs in the Landau gauge (though the vacuum-to-vacuum S matrix is expressed in any gauge). We are at present investigating the relationship between Jackiw's formulation in the presence of sources and our own gauge-invariant consideration.

A convenient and natural formalism in which to express QED is that of Faddeev and Popov.[2] The vacuum-to-vacuum S matrix in a gauge in which the photon propagator is[3]

$$D_{\mu\nu}=(g_{\mu\nu}-q_\mu q_\nu/q^2)/q^2+\eta q_\mu q_\nu/q^4 \tag{1}$$

is shown by them to take the form

$$S=\int \mathfrak{D}\{\varphi\}\mathfrak{D}\{\varphi^*\}\mathfrak{D}\{A_\mu\}e^L\exp\left(-\int d^4x\frac{(\partial_\mu A_\mu)^2}{2\eta}\right), \tag{2}$$

where L has gauge invariance of the second kind. To show the gauge invariance, multiply this quantity by a constant in the form

$$\int \mathfrak{D}\{\xi(x)\}\prod_x \delta(\partial_\mu A_\mu-\Box\xi)$$

to give

$$S=\int \mathfrak{D}\{\xi\}\exp\left(-\frac{1}{2\eta}\int[\Box\xi(x)]^2d^4x\right)\int \mathfrak{D}\{\varphi\}\mathfrak{D}\{\varphi^*\}\mathfrak{D}\{A_\mu\}e^L\,\delta(\partial_\mu A_\mu-\Box\xi)\,. \tag{3}$$

In the second factor change the variable of integration to $\varphi\exp[ig\xi(x)]$, $\varphi^*\exp[-ig\xi(x)]$, $A_\mu+\partial_\mu\xi$. This leaves L the same and has Jacobian unity. The result is

$$S=\left[\int \mathfrak{D}\{\xi\}\exp\left(-\frac{1}{2\eta}\int d^4x[\Box\xi(x)]^2\right)\right]S_{\text{Landau}}\ , \tag{4}$$

where

$$S_{\text{Landau}}=\int \mathfrak{D}\{\varphi\}\mathfrak{D}\{\varphi^*\}\mathfrak{D}\{A_\mu\}e^L\,\delta(\partial_\mu A_\mu)\,; \tag{5}$$

we see that S/S_{Landau} =unity up to a numerical factor (which in fact may be normed to unity).

The same steps can be repeated to show how Green's functions transform. In an arbitrary gauge we have

$$G_{1\cdots 2n}\equiv\langle\varphi(x_1)\cdots\varphi(x_n)\varphi^*(x_{n+1})\cdots\varphi^*(x_{2n})\rangle$$

$$=\frac{\int \mathfrak{D}\{\varphi\}\mathfrak{D}\{\varphi^*\}\mathfrak{D}\{A_\mu\}e^L\exp\left(-\frac{1}{2\eta}\int d^4x(\partial_\mu A_\mu)^2\right)\varphi(x_1)\cdots\varphi(x_n)\varphi^*(x_{n+1})\cdots\varphi^*(x_{2n})}{\int \mathfrak{D}\{\varphi\}\mathfrak{D}\{\varphi^*\}\mathfrak{D}\{A_\mu\}e^L\exp\left(-\frac{1}{2\eta}\int d^4x(\partial_\mu A_\mu)^2\right)}\ . \tag{6}$$

Multiply numerator and denominator by

$$\int \mathfrak{D}\{\xi(x)\}\prod_x \delta(\partial_\mu A_\mu - \Box\xi)$$

to give

$$G_{1\cdots 2n} = \frac{\int \mathfrak{D}\{\xi\}\exp\left(-\frac{1}{2\eta}\int d^4x(\Box\xi)^2\right)\int \mathfrak{D}\{\varphi\}\mathfrak{D}\{\varphi^*\}\mathfrak{D}\{A_\mu\}e^L\,\delta(\partial_\mu A_\mu - \Box\xi)\varphi(x_1)\cdots\varphi(x_n)\varphi^*(x_{n+1})\cdots\varphi^*(x_{2n})}{\mathfrak{D}\{\xi\}\exp\left(-\frac{1}{2\eta}\int d^4x(\Box\xi)^2\right)\int \mathfrak{D}\{\varphi\}\mathfrak{D}\{\varphi^*\}\mathfrak{D}\{A_\mu\}e^L\,\delta(\partial_\mu A_\mu - \Box\xi)} . \tag{7}$$

Once more, change the variables as above to give

$$G_{1\cdots 2n} = \langle \exp\{ig[\xi(x_1) + \cdots + \xi(x_n) - \xi(x_{n+1}) - \cdots - \xi(x_{2n})]\}\rangle_\eta G^{\text{Landau}}_{1\cdots 2n} . \tag{8}$$

$G^{\text{Landau}}_{1\cdots 2n}$ means the Green's function calculated in the Landau gauge and the symbol $\langle\,\rangle_\eta$ means the Gaussian average over the ξ field with weight $\exp\{-(1/2\eta)[\Box\xi(x)]^2\}$ for each space-time point.[4]

The Gaussian average can be performed by completing squares, most conveniently in momentum space. The result is

$$\langle \exp\{ig[\xi(x_1)+\cdots+\xi(x_n) - \xi(x_{n+1}) - \cdots - \xi(x_{2n})]\}\rangle_\eta = \left[\exp\left(-\frac{\eta g^2}{2(2\pi)^4}\int \frac{d^4q}{q^4}\right)\right]^{2n}\exp\left(\sum_{i\neq j}\frac{\eta g^2}{2(2\pi)^4}\int\frac{d^4q}{q^4}e^{iq(x_i - x_j)}\right) . \tag{9}$$

The first of these factors is a wave-function renormalization factor; the second corresponds graphically to gauge photons which are exchanged between each pair of external lines. (This factor even converts Green's functions which are disconnected in the Landau gauge to connected ones in other gauges.)

Equations (4), (8), and (9) give the totality of the effect of a change in gauge and these properties must be respected when the symmetry is spontaneously broken as well. Namely, in transforming from the Landau to an arbitrary gauge one multiplies $\varphi(x)$ by $\exp[-ig\xi(x)]$ and averages point by point with weight

$$\exp\{-(1/2\eta)[\Box\xi(x)]^2\} .$$

In view of the above considerations we notice a complication which arises when one transforms Green's functions by gauge transformations in the presence of spontaneous broken symmetry (sbs). Suppose that in a certain gauge one has the usual asymptotic decomposition in the presence of sbs and translational symmetry

$$\lim_{|x-x'|\to\infty}\langle\varphi(x)\varphi^*(x')\rangle = \varphi_c^{\,2} , \tag{10}$$

where

$$\varphi_c = \langle\varphi(x)\rangle . \tag{11}$$

In some other gauge Eq. (10) becomes

$$\lim_{|x-x'|\to\infty}\langle\varphi(x)\varphi^*(x')\rangle_\eta = \varphi_c^{\,2}\lim_{|x-x'|\to\infty}\langle\exp\{ig[\xi(x)-\xi(x')]\}\rangle_\eta = (Z_\eta\varphi_c^{\,2})\lim_{|x-x'|\to\infty}\frac{\eta g^2}{2(2\pi)^4}\int\frac{d^4q}{q^4}e^{iq(x-x')} .$$

But if the longitudinal photon has vanishing mass (and it is intrinsic to the Faddeev-Popov formalism as well as all conventional formalism that it does) the above limit is undefined. We expect that this difficulty is purely fictitious and corresponds to a poor formulation of the physics. We show below that there is always a good order parameter; namely $\langle\varphi(x)\exp[-ig\xi(x)]\rangle$ $(=\varphi_c)$ (see Ref. 5) and that sbs is naturally expressed in terms of this quantity in any gauge. Only in the Landau gauge is it true that $\langle\varphi(x)\rangle = \varphi_c$.

In Eq. (3), first perform all the integrations of φ, φ^*, A_μ at fixed $\xi(x)$. The result is

$$S = \int\mathfrak{D}\{\xi\}\exp\left(-\frac{1}{2\eta}(\Box\xi)^2\right)\exp[-F(\langle\varphi(x)\rangle_\xi;\xi)]\Big/\int\mathfrak{D}\{\xi\}\exp\left[-\frac{1}{2\eta}(\Box\xi)^2\right], \tag{12}$$

where we have supposed that for each set of effective external fields $\partial_\mu \xi$, the integral is possessed of spontaneously broken symmetry; the field average is designated by $\langle \varphi(x)\rangle_\xi$ for each set ξ.[6]

Gauge and Lorentz invariance tell us that F is of the form

$$F \sim \int [|(\partial_\mu - igA_\mu^{\text{ext}})\langle\varphi(x)\rangle_\xi|^2 + P(|\langle\varphi(x)\rangle|_\xi^2)]\, d^4x, \tag{13}$$

where P is a polynomial in $|\langle\varphi\rangle_\xi|^2$ having a minimum at $|\langle\varphi\rangle_\xi| \neq 0$ (our assumption of sbs). Higher-order powers in $|(\partial_\mu - igA_\mu^{\text{ext}})\langle\varphi\rangle_\xi|^2$ may also arise, but the only property we appeal to is that the existence of field gradients in the absence of A_μ^{ext} necessarily increases the "free energy" F. The form of (13) is of course the Ginzburg-Landau[7] expression for the free energy of a superconductor in the presence of an external field. F is proportional to the "volume of space-time." Hence one needs to consider F only at the values of the field $\varphi(x)$ which minimize it, whence the appearance of $\langle\varphi(x)\rangle_\xi$ in Eq. (12).

In our case, $A_\mu^{\text{ext}} = \partial_\mu \xi$ and we see that F is minimized by the conditions

$$\langle\varphi(x)\rangle_\xi = \varphi_c e^{ig\xi(x)},$$
$$\frac{\partial P}{\partial \varphi_c} = 0. \tag{14}$$

At the minimum we see that $F = P(|\varphi_c|^2)$ is independent of ξ. The integrand on ξ is independent of ξ, as it should be since in the original formula (3) the coefficient of

$$\exp\left(-\frac{1}{2\eta}\int d^4x[\Box\,\xi(x)]^2\right)$$

is indeed independent of ξ.

The result is that

$$-\ln S = P(|\varphi_c|^2),$$

where $\partial P/\partial\varphi_c = 0$. By construction, $P(|\varphi_c|^2)$ is the value of $-\ln S$ calculated in the Landau gauge. Namely, $P(|\varphi_c|^2)$ is calculated as if $\xi = 0$, and this is the only admitted ξ when $\eta = 0$. Equation (14) also tells us that the true order parameter φ_c is $\langle\varphi(x)e^{-ig\xi(x)}\rangle$ as asserted. Indeed, Eq. (14) may be rewritten $\varphi_c = \langle\varphi(x)e^{-ig\xi(x)}\rangle_\xi$ since ξ is a fixed external field. Averaging over ξ then gives back the constant φ_c.

The final rules for the construction of the effective potential to be minimized are then summarized by the gauge-invariant statement: Calculate V according to the Feynman graph rules of the theory as formulated with sources[8] but connect no photons singly to tadpoles. The value of the tadpole is φ_c, but in an arbitrary gauge this φ_c is not $\langle\varphi(x)\rangle$; rather

$$\langle\varphi(x)\rangle = \int \mathfrak{D}\{\xi\}\, e^{ig\xi(x)} \exp\left(-\frac{[\Box\,\xi(x)]^2}{2\eta}\right)\varphi_c \Big/ \int \mathfrak{D}\{\xi\} \exp\left(-\frac{(\Box\xi)^2}{2\eta}\right)$$
$$= \exp\left(-\frac{\eta g^2}{2(2\pi)^4}\int\frac{d^4q}{q^4}\right)\varphi_c .$$

The ideas which are being expressed here lead to the concept of a vacuum degeneracy which is far richer than in broken invariance of the first kind. There is a phase degeneracy at each point x which must be averaged over. We expect that a deeper appreciation of this point will lead to more profound insights of the phenomena in question.

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*Boursier auprès de l'IRSIA (Institut pour l'Encouragement de la Recherche Scientifique dans l'Industrie et l'Agriculture).

[1]R. Jackiw, Phys. Rev. D 9, 1686 (1974).

[2]V. Popov and L. D. Faddeev, Kiev Report No. ITP-67-36 (unpublished) [NAL Report No. NAL-Thy-57 (unpublished)]; L. D. Faddeev and V. Popov, Phys. Lett. 25B, 29 (1967).

[3]In what follows, we are restricting ourselves to a subclass of all possible gauges, viz., the covariant gauges summarized by Eq. (1).

[4]L. D. Landau and I. M. Khalatnikov, Zh. Eksp. Theor. Fiz. 29, 89 (1955) [Sov. Phys.—JETP 2, 69 (1956)]. See also K. Johnson and B. Zumino, Phys. Rev. Lett. 3, 351 (1959).

[5]It might be instructive to point out the analogy with an antiferromagnet. The order parameter in this case is certainly not

$$\int \langle\varphi(x)\rangle\, dx \;\; (=0)$$

but rather

$$\int \langle\varphi(x)e^{iq_0x}\rangle\, dx,$$

where q_0 is the reciprocal lattice vector corresponding to the antiferromagnetic array.

[6]Some of our colleagues in particle physics seem to find it difficult to understand the reduction of Eq. (3) to Eq. (12), though this procedure follows a method well known in the statistical mechanics of sbs. The problem posed is how to express $\ln S(\xi)$ in terms of $\langle\varphi(x)\rangle_\xi$ in spite of the fact that strictly speaking the latter vanishes (owing to the gauge symmetry). Here $\langle\varphi(x)\rangle_\xi$ is used to designate the average of $\varphi(x)$ in the external (longitudinal) field, $\partial_\mu\xi$. One proceeds as follows: First fix the phase $\psi(x_0)$ of $\varphi(x_0)$ where x_0 is some arbitrary point. Then sbs fixes the phase of $\langle\varphi(x)\rangle$ at all other points $x\neq x_0$ with respect to this initially chosen phase. This gives rise to an expression for $\ln S(\xi)$ which is a function of $\langle\varphi(x)\rangle_\xi$ and ξ but independent of the initial choice of phase $\psi(x_0)$ (by gauge invariance). Integration over the latter simply adds an irrelevant constant, $\ln 2\pi$ to $\ln S$. In our present case the form of the function $F(\langle\varphi\rangle_\xi;\xi)$ is dictated by gauge and Lorentz invariance as stipulated by Eq. (13).

[7]V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. 20, 1064 (1950).

[8]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).