Stringlike solution of the Higgs model with magnetic monopoles*

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We derive a static solution of the equations of motion following from a Higgs-type Lagrangian containing, in addition, static magnetic monopoles representing quarks. For this purpose, we use Zwanziger's approach to magnetic monopoles, and thus we are dealing with a local field theory for charged particles. We show that the solution has the form of a string of finite length for large coupling. We exhibit the dependence of the energy of the system (E) on interquark distance ($2a$), $E(2a) = -(g^2/8\pi a)\exp(-e|\Phi_0|2a) + C|\Phi_0|^2 a$, which is the form found by Nambu in his discussion of this type of model as a scheme which offers a mechanism for quark confinement. We therefore confirm that Nambu's results can be reached in a field-theoretic formulation.

I. INTRODUCTION

Recently, several authors[1-4] have suggested that Higgs-type Lagrangians containing couplings to "magnetic" monopoles may provide a scheme for quark confinement.

Nambu[1] treated the static problem in this scheme by introducing a static phenomenological Hamiltonian in which the Higgs mechanism is represented by a mass term for the vector field. He obtained a static stringlike solution and an expression for the energy of the system, which exhibits a Yukawa interaction for small interquark separations and a characteristic string potential responsible for quark confinement. In another paper[5] Nambu gave the full description of this scheme, using Dirac's formulation of magnetic-monopole theory and treating the Higgs effect by making an ansatz for the electric current known in the London theory of superconductivity. He derived the string Lagrangian density and discussed the quantization of the theory. In this approach, the unphysical Dirac string becomes physical due to the Higgs effect.

In this paper, we present the formulation of this model in terms of a local Lagrangian field theory. This is achieved by using Zwanziger's[6] Lagrangian approach to magnetic monopoles (instead of Dirac's approach) which can accommodate matter fields such as the Higgs scalar field. The "magnetic" monopoles (quarks) are treated as classical particles. The static solution of this model then reproduces the results of Nambu.[1] In this formulation one recovers at large distances the ansatz for the "electric" current used by Nambu, and one is also able to investigate the behavior of the Higgs field in the region of the string in a manner similar to that of Nielsen and Olesen.

We do not consider the nonstatic solutions and the quantization of this model, but since we are dealing with a local Lagrangian field theory, general arguments given by Nielsen and Olesen lead us to believe that one can deduce the Nambu-Goto Lagrangian for the string. This, together with quantization, can most probably be done by using the methods of Förster[7] and of Gervais and Sakita.[8] We intend to investigate this problem in the future.

II. REVIEW OF LAGRANGIAN FORMULATION OF MAGNETIC-MONOPOLE THEORY

The field theory of magnetic monopoles was formulated by Schwinger[9] and the Lagrangian formulation was given by Zwanziger. We shall now give a short review of Zwanziger's formalism. Maxwell's equation for the electric current

$$\partial_\mu F^{\mu\nu} = j_e^{\,\nu} \tag{1}$$

is satisfied by

$$F = -(\partial\wedge B)^d + (n\cdot\partial)^{-1}(n\wedge j_e)\,, \tag{2}$$

$$F^d = (\partial\wedge B) + (n\cdot\partial)^{-1}(n\wedge j_e)^d\,, \tag{3}$$

where B is a vector potential. The notation means that for arbitrary two-vectors C^μ and D^μ $(C\wedge D)^{\mu\nu} = C^\mu D^\nu - C^\nu D^\mu$ and our metric is $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The dual of an antisymmetric tensor $G^{\mu\nu}$ is defined as $G^{d\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu}{}_{\kappa\lambda}G^{\kappa\lambda}$.

The other Maxwell's equation

$$\partial_\mu F^{d\mu\nu} = j_g^{\,\nu}\,, \tag{4}$$

where j_g^ν is the magnetic current, is satisfied by

$$F^d = (\partial\wedge A)^d + (n\cdot\partial)^{-1}(n\wedge j_g)\,, \tag{5}$$

$$F = (\partial\wedge A) - (n\cdot\partial)^{-1}(n\wedge j_g)^d\,. \tag{6}$$

Now one may express $F^{\mu\nu}$ locally in terms of the potentials

$$F = \frac{1}{n^2}\left\{(n\wedge[n\cdot(\partial\wedge A)]) - (n\wedge[n\cdot(\partial\wedge B)])^d\right\}\,, \tag{7}$$

so that after substitution into (1) and (4) we have Maxwell's equations in terms of the potentials. As Zwanziger has shown, these equations can be derived from the Lagrangian

$$\mathcal{L}_Z = -\frac{1}{2n^2}[n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d] + \frac{1}{2n^2}[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^d] - \frac{1}{2n^2}[n\cdot(\partial\wedge A)]^2 - \frac{1}{2n^2}[n\cdot(\partial\wedge B)]^2 - j_e\cdot A - j_g\cdot B\ . \quad (8)$$

In the static case with the choice $n^\mu = (0, \vec{n})$ Eqs. (5) and (2) imply

$$H_i \equiv -F^{d0i} = \vec{\nabla}\times\vec{A} + \vec{n}(\vec{n}\cdot\vec{\nabla})^{-1} j_g^0\ , \quad (9)$$

$$E_i \equiv -F^{0i} = -\vec{\nabla}\times\vec{B} + \vec{n}(\vec{n}\cdot\vec{\nabla})^{-1} j_e^0\ , \quad (10)$$

while (1) and (4) yield the equations of motion

$$\vec{\nabla}\times\vec{H} = \vec{j}_e\ , \quad (11)$$

$$\vec{\nabla}\times\vec{E} = -\vec{j}_g \quad (12)$$

III. THE MODEL AND ITS STATIC SOLUTION

In our model we couple the potential A^μ to a complex scalar field Φ, which carries only "electric" charge, in order to produce the Higgs effect. The quarks are represented by static point monopoles. The total Lagrangian of the system is therefore

$$\mathcal{L} = \mathcal{L}_Z^0(A, B) + \mathcal{L}'(\phi, A) - j_g^0 B^0\ , \quad (13)$$

where $\mathcal{L}_Z^0$ is the free part of Zwanziger's Lagrangian (8) and

$$\mathcal{L}'(\phi, A) = \tfrac{1}{2}|(\partial_\mu - ieA_\mu)\phi|^2 - \tfrac{1}{2}m^2|\phi|^2 - \tfrac{1}{4}h|\phi|^4\ , \quad (14)$$

with $m^2 < 0$ to allow for constant asymptotic behavior $\Phi_0 = (-m^2/h)^{1/2}$. This yields the electric current

$$j_e^\mu = -e^2|\phi|^2 A^\mu + e|\phi|^2\partial^\mu\chi\ , \quad (15)$$

where χ is the phase of the complex scalar field Φ. The magnetic charge distribution is

$$j_g^0 = g[\delta(z-a) - \delta(z+a)]\delta(x)\delta(y)\ , \quad (16)$$

corresponding to two static magnetic monopoles (or, more precisely a monopole and an antimonopole) located at points $\vec{x} = \pm a\vec{\epsilon}_3$. We shall be looking for a cylindrically symmetric solution and, with this in mind, we set $\vec{n} = \vec{\epsilon}_3$.

Combining (9) and (11) we find

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + (\vec{\nabla}\times\vec{\epsilon}_3)\partial_3^{-1} j_g^0 = \vec{j}_e\ . \quad (17)$$

The standard discussion of flux quantization implies $\chi = \mu\theta$, where θ is the azimuthal angle and μ is the flux quantum number. We choose $\mu = 0$ in what follows. This choice is different from Nielsen and Olesen's choice $\mu \neq 0$, and we wish to stress that we are looking for a static solution due solely to the presence of the monopoles and with no superimposed vertex line of the Nielsen-Olesen type. The choice $\mu = 0$ does not imply that the flux is zero everywhere. Indeed (9) with the choice $\vec{n} = \vec{\epsilon}_3$ implies for the magnetic flux

$$\Phi_H = \int \vec{H}\cdot\vec{\epsilon}_3 ds = \oint \vec{A}\cdot d\vec{l} + \tfrac{1}{2}g[\epsilon(z-a) - \epsilon(z+a)]\ . \quad (18)$$

Our choice $\mu = 0$ only means that

$$\oint \vec{A}\cdot d\vec{l} = 0\ , \quad (19)$$

so that

$$\Phi_H = -g, \quad |z| < a$$
$$\Phi_H = 0, \quad |z| > a\ . \quad (20)$$

Thus

$$j_e^\mu = -e^2|\phi|^2 A^\mu\ , \quad (21)$$

and Eq. (17) becomes

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} + e^2|\phi|^2\vec{A} = (\vec{\epsilon}_3\times\vec{\nabla})\partial_3^{-1} j_g^0\ . \quad (22)$$

Let us discuss the behavior of the solution for large ρ. We shall be interested in solutions for which $|\Phi| \to |\Phi_0| = \text{const} \neq 0$ as $\rho\to\infty$. Thus, in this asymptotic region the vector field has a mass and

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} + M^2\vec{A} = (\vec{\epsilon}_3\times\vec{\nabla})\partial_3^{-1} j_g^0\ , \quad (23)$$

where $M = e|\Phi_0|$. Taking the divergence of (23) we find

$$M^2\vec{\nabla}\cdot\vec{A} = 0\ , \quad (24)$$

so that (23) becomes

$$(-\nabla^2 + M^2)\vec{A} = (\vec{\epsilon}_3\times\vec{\nabla})\partial_3^{-1} j_g^0\ . \quad (25)$$

This equation agrees with Eq. (4) in Nambu's paper.[1] This is seen by identifying Nambu's source $\vec{\Phi}$ with $\vec{\epsilon}_3\partial_3^{-1} j_g^0$ and using

$$\partial_3^{-1} f(x,y,z) = \tfrac{1}{2}\int_{-\infty}^{+\infty} \epsilon(z-s) f(x,y,s)\, ds\ . \quad (26)$$

The solution for $\vec{A}$ is now

$$\vec{A}(\vec{x}) = \int G(\vec{x}-\vec{x}')(\vec{\epsilon}_3\times\vec{\nabla}')(\partial_3')^{-1} j_g^0(\vec{x}')\, d\vec{x}'\ , \quad (27)$$

where the Green's function for the massive vector field is

$$G(\vec{x}-\vec{x}')=\frac{1}{4\pi}\frac{e^{-M|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\ . \tag{28}$$

In order to find the magnetic field, we use the equation

$$\vec{H}=\vec{\nabla}\times\vec{A}+\vec{\epsilon}_3\partial_3{}^{-1}j_g^0\ , \tag{29}$$

which appears also in Nambu's paper. Now

$$\vec{H}(\vec{x})=\int G(\vec{x}-\vec{x}')[-\vec{\nabla}'(\vec{\nabla}'\cdot\vec{\epsilon}_3)+\vec{\epsilon}_3\vec{\nabla}'^2](\partial_3')^{-1}j_g^0(\vec{x}')d\vec{x}'+\vec{\epsilon}_3\partial_3{}^{-1}j_g^0$$

$$=-\vec{\nabla}\int G(\vec{x}-\vec{x}')j_g^0(\vec{x}')d\vec{x}'+M^2\vec{\epsilon}_3\int G(\vec{x}-\vec{x}')(\partial_3')^{-1}j_g^0(\vec{x}')d\vec{x}'\ . \tag{30}$$

If M is large, the first term is seen to contribute only in the immediate vicinity of the points $\vec{x}=\pm a\vec{\epsilon}_3$. The second term can be written as

$$\vec{H}_2=-gM^2\vec{\epsilon}_3\int_{-a}^{a}dz'\int G(\vec{x}-\vec{x}'')\,\delta^{(3)}(\vec{x}''-z'\vec{\epsilon}_3)d\vec{x}''\ , \tag{31}$$

and thus is an integral sum of terms, each of which is non-negligible (for large M) only in the vicinity of the point $\vec{x}=z'\vec{\epsilon}_3$. One therefore obtains a stringlike entity extending only from $\vec{x}=-a\vec{\epsilon}_3$ to $\vec{x}=a\vec{\epsilon}_3$ (that is, from one quark to another). The magnetic field is essentially zero outside the string, with a characteristic exponential cutoff of the order of $1/M$.

We shall also be interested in the behavior of the solution for $|\Phi|$ for small ρ $[\rho=(x^2+y^2)^{1/2}]$. For this purpose one uses Eq. (22) together with the equation for the scalar field

$$2ie\vec{A}\cdot\vec{\nabla}|\phi|-\vec{\nabla}^2|\phi|+(e^2\vec{A}^2+m^2+h|\phi|^2)|\phi|=0\ , \tag{32}$$

which is derivable from the Lagrangian density of Eq. (13). We first note that we can look for the solution such that $A_\rho=A_z=0$ so that $\vec{\nabla}\cdot\vec{A}=(1/\rho)\partial A_\theta/\partial\theta=0$ because of the cylindrical symmetry of the problem. Let us define a potential $\vec{A}'$ by the relation

$$\vec{A}(\vec{x})=\vec{A}'(\vec{x})+\int\vec{a}_{\vec{\epsilon}_3}(\vec{x}-\vec{x}')j_g^0(\vec{x}')d\vec{x}'\ . \tag{33}$$

Here, the vector function $\vec{a}_{\vec{\epsilon}_3}$ will be written down in the next section and we shall need only its properties

$$-\vec{\nabla}^2\vec{a}_{\vec{\epsilon}_3}(\vec{x})=\vec{\epsilon}_3\times\vec{\nabla}\partial_3{}^{-1}\delta(\vec{x})\ , \tag{34}$$

$$\lim\oint d\vec{l}\cdot\vec{a}_{\vec{\epsilon}_3}(\vec{x})=-\tfrac{1}{2}\epsilon(z)\ , \tag{35}$$

where the limit sign indicates that the curve is infinitesimal. The curve is drawn positively about the z axis. These relations are given in Ref. 9. One then finds that Eqs. (22) and (32) can be written in an equivalent form

$$-\vec{\nabla}^2\vec{A}'+e^2|\phi|^2\left(\vec{A}'-\frac{1}{e}\vec{\nabla}\chi\right)=0\ , \tag{36}$$

$$2ie\left(\vec{A}'-\frac{1}{e}\vec{\nabla}\chi\right)\cdot\vec{\nabla}|\phi|-\vec{\nabla}^2|\phi|+\left[e^2\left(\vec{A}'-\frac{1}{e}\vec{\nabla}\chi\right)^2+m^2+h|\phi|^2\right]|\phi|=0\ , \tag{37}$$

where χ is given by

$$\chi(\vec{x})=-e\int_{\infty}^{\vec{x}}d\vec{l}'\cdot\int\vec{a}_{\vec{\epsilon}_3}(\vec{x}'-\vec{x}'')j_g^0(\vec{x}'')d\vec{x}''\ . \tag{38}$$

These equations can be understood as sourceless equations for the potential $\vec{A}'$ coupled to a scalar field with a phase given by χ. Now, for an infinitesimal closed curve about the z axis, the change of the phase of the scalar field after going around the curve once is

$$\nabla\chi(z)=-eg\oint d\vec{l}\cdot[\vec{a}_{\vec{\epsilon}_3}(\vec{x}-a\vec{\epsilon}_3)-\vec{a}_{\vec{\epsilon}_3}(\vec{x}+a\vec{\epsilon}_3)]$$

$$=\tfrac{1}{2}eg[\epsilon(z-a)-\epsilon(z+a)]\ , \tag{39}$$

where we have used Eq. (35). Therefore,

$$\nabla\chi=-4\pi n\ \text{for}\ |z|<a\ , \tag{40}$$

$$\nabla\chi=0\ \text{for}\ |z|>a\ , \tag{41}$$

with the use of Schwinger's charge quantization condition[10] $eg/4\pi=n$. The first equation, together with the continuity of the field $\Phi=e^{i\chi}|\Phi|$, implies that $|\Phi|\to 0$ as $\rho\to 0$ when $|z|<a$. Thus $|\Phi|\to 0$ in the string region and it therefore carries energy, since its value in this region is different from its vacuum value. Also, in this region there is no Higgs effect. We still have to find the small-ρ behavior for $|z|>a$. We use Eqs. (22) and (32) rewritten in cylindrical coordinates

$$-\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\theta)\right]-\frac{\partial^2}{\partial z^2}A_\theta+e^2|\phi|^2A_\theta=0\ , \tag{42}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}|\phi|\right)-\frac{\partial^2}{\partial z^2}|\phi|+(e^2A_\theta{}^2+m^2+h|\phi|^2)|\phi|=0\ . \tag{43}$$

Since the magnetic flux is zero for $|z|>a$ and H_z is essentially zero for reasonably large ρ in this region, it must also be negligibly small for small ρ. The resulting allowed behavior for A_θ implies that $A_\theta \to 0$ for small ρ. The behavior $A_\theta \sim \text{const}/\rho$ is ruled out by noting that in the region $|z|>a$

$$\begin{aligned} 0 &= \Phi_H \\ &= 2\pi \int_0^\infty \frac{1}{\rho} \left[\frac{\partial}{\partial\rho}(\rho A_\theta)\right] \rho\, d\rho \\ &= 2\pi(\rho A_\theta)_{\rho=\infty} - 2\pi(\rho A_\theta)_{\rho=0} \\ &= -2\pi(\rho A_\theta)_{\rho=0}\ , \end{aligned} \tag{44}$$

where we have used the fact that A_θ falls off exponentially for large ρ.

One can then solve Eq. (43) by the ansatz

$$|\phi| \underset{\rho\to 0}{\longrightarrow} C(z)\ , \tag{45}$$

which implies

$$-\frac{\partial^2}{\partial z^2} C(z) + [m^2 + hC^2(z)]C(z) = 0\ , \tag{46}$$

with the following approximate solution for large z ($|z|>a$):

$$C(z) = \left(\frac{-m^2}{h}\right)^{1/2} + \text{const}\times\exp[-(-2m^2)^{1/2}(z-a)]\ . \tag{47}$$

Thus, if m is large

$$|\phi| \underset{\rho\to 0}{\longrightarrow} \phi_0 = \left(\frac{-m^2}{h}\right)^{1/2} \tag{48}$$

essentially everywhere for $|z|>a$. The scalar field therefore carries no energy in the region $|z|>a$, since it is equal to its vacuum value practically everywhere in this region.

We have already seen that H_z must be negligibly small for small ρ. Since also $A_\theta \to 0$ for small ρ, $H_\rho = -\partial A_\theta/\partial z$ also tends to zero. Thus, the magnetic field also carries a negligible energy in the region $|z|>a$.

To summarize the results of this section: First, H_ρ is appreciable in the immediate vicinity of the points $\vec{x} = \pm a\vec{\epsilon}_3$, so essentially only the z component of the magnetic field is nonzero. As seen from Eq. (31), H_z is appreciable only for $|z|<a$ and falls off exponentially in the ρ direction with a characteristic length of the order of the inverse mass of the vector field $(1/M)$. Therefore, the solution for the magnetic field has a string structure. The scalar field Φ is zero in the string region, and equal to its vacuum value $\Phi_0 = (-m^2/h)^{1/2}$ essentially everywhere else.

We have thus confirmed that the static solution of our problem is a stringlike entity with a very small width $(1/M)$ and finite length $(2a)$. It extends only from one quark to another, as expected.

IV. INDEPENDENCE OF THE SOLUTION OF THE ARBITRARY VECTOR $\vec{n}$

In the field-theoretic approach to magnetic-monopole theory, which we use, there appears an arbitrary vector $\vec{n}$, as seen in Sec. II.

In the preceding section we chose $\vec{n} = \vec{\epsilon}_3$, and one may ask the question whether the solution depends on the choice of $\vec{n}$. We will show that it does not, in the sense that one can obtain the same solution for the magnetic field, and the absolute value of the scalar field for arbitrary $\vec{n}$, by making a judicious choice of the phase of the scalar field.

Schwinger[9] has discussed the problem of $\vec{n}$ dependence in his pioneering work on the quantum field theory of magnetic monopoles. He showed that the theory is independent of this vector $\vec{n}$ (and the corresponding singularity line), if the change quantization condition[10] $eg/4\pi = n$ is satisfied. We shall use his method adapted to our problem and for arbitrary $\vec{n}$ choose the phase of the scalar field as follows:

$$\chi(\vec{x}) = e\int_\infty^{\vec{x}} d\vec{l}\,'\cdot \int [\vec{a}_{\vec{n}}(\vec{x}\,'-\vec{x}\,'') - \vec{a}_{\vec{\epsilon}_3}(\vec{x}\,'-\vec{x}\,'')] \times j_g^0(\vec{x}\,'')d\vec{x}\,''\ . \tag{49}$$

Here the function $\vec{a}_{\vec{n}}$ is defined[9] as

$$\vec{a}_{\vec{n}}(\vec{x}) = \frac{\vec{n}\times\vec{x}}{8\pi|\vec{x}|}\left(\frac{1}{|\vec{x}|+\vec{n}\cdot\vec{x}} - \frac{1}{|\vec{x}|-\vec{n}\cdot\vec{x}}\right)\ . \tag{50}$$

It can be shown with the use of Schwinger's charge quantization condition that the change of the phase of the scalar field

$$\Delta\chi = e\oint_C d\vec{l}\,'\cdot\int [\vec{a}_{\vec{n}}(\vec{x}\,'-\vec{x}\,'') - \vec{a}_{\vec{\epsilon}_3}(\vec{x}\,'-\vec{x}\,'')]\ j_g^0(\vec{x}\,'')d\vec{x}\,'' = 2\pi m\ , \tag{51}$$

where m is an integer, for an arbitrary closed curve C. For arbitrary $\vec{n}$, we have the equation for $\vec{A}$, which follows from (9) and (11):

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + e^2|\phi|^2\vec{A} - e|\phi|^2\vec{\psi} = (\vec{n}\times\vec{\nabla})(\vec{n}\cdot\vec{\nabla})^{-1}j_g^0\ , \tag{52}$$

where we denoted

$$\vec{\psi}(\vec{x}) = \frac{1}{e}\vec{\nabla}\chi(\vec{x}) = \int [\vec{a}_{\vec{n}}(\vec{x}-\vec{x}\,') - \vec{a}_{\vec{\epsilon}_3}(\vec{x}-\vec{x}\,')]\, j_g^0(\vec{x}\,')d\vec{x}\,'\ . \tag{53}$$

This is actually the choice for the gradient and it is important to note that one always has to take the gradient first, and then act with subsequent differential operators. One also has the equation for the Higgs field

$$2ie\vec{A}\cdot\vec{\nabla}\phi - \vec{\nabla}^2\phi + (m^2 + h|\phi|^2 + e^2\vec{A}^2)\phi = 0\ . \qquad (54)$$

Substituting $\Phi = e^{i\chi}|\Phi|$ with the choice of phase (49) and using

$$\vec{\nabla}^2\chi(\vec{x}) = 0\ , \qquad (55)$$

which follows from a property of $\vec{a}_{\vec{n}}(\vec{x})$,

$$\vec{\nabla}\cdot\vec{a}_{\vec{n}}(\vec{x}) = 0\ , \qquad (56)$$

one obtains

$$2ie(\vec{A} - \vec{\psi})\cdot\vec{\nabla}|\phi| - \vec{\nabla}^2|\phi| + [m^2 + h|\phi|^2 + e^2(\vec{A} - \vec{\psi})^2]|\phi| = 0\ . \qquad (57)$$

With the use of another property of $\vec{a}_{\vec{n}}(\vec{x})$

$$\vec{\nabla}^2\vec{a}_{\vec{n}}(\vec{x}) = (\vec{\nabla}\times\vec{n})(\vec{n}\cdot\vec{\nabla})^{-1}\delta^{(3)}(\vec{x})\ , \qquad (58)$$

the equation for the vector field becomes

$$\vec{\nabla}\times[\vec{\nabla}\times(\vec{A} - \vec{\psi})] + e^2|\phi|^2(\vec{A} - \vec{\psi}) = (\vec{\epsilon}_3\times\vec{\nabla})\partial_3^{-1}j_g^0\ . \qquad (59)$$

We see now that the solution of the set of equations (57) and (59) is given by

$$\vec{A}(\vec{x}) = \vec{A}_{\vec{\epsilon}_3}(\vec{x}) + \int [\vec{a}_{\vec{n}}(\vec{x} - \vec{x}') - \vec{a}_{\vec{\epsilon}_3}(\vec{x} - \vec{x}')]j_g^0(\vec{x}')d\vec{x}'\ ,$$

$$|\phi| = |\phi|_{\vec{\epsilon}_3}\ , \qquad (60)$$

where $\vec{A}_{\vec{\epsilon}}$ and $|\phi|_{\vec{\epsilon}}$ are the solutions for the choice $\vec{n} = \vec{e}_3$. One may note that the transformation made is actually a gauge transformation almost everywhere, as explained in Schwinger's paper.[9]

Finally, to calculate the magnetic field, we use

$$\vec{\nabla}\times\vec{a}_{\vec{n}}(\vec{x}) = -\vec{\nabla}\frac{1}{4\pi|\vec{x}|} - \vec{n}(\vec{n}\cdot\vec{\nabla})^{-1}\delta(\vec{x}) \qquad (61)$$

and find

$$\begin{aligned}\vec{H} &= \vec{\nabla}\times\vec{A} + \vec{n}(\vec{n}\cdot\vec{\nabla})^{-1}j_g^0\\ &= \nabla\times\vec{A}_{\vec{\epsilon}_3} + \vec{\epsilon}_3\partial_3^{-1}j_g^0\\ &= \vec{H}_{\vec{\epsilon}_3}\ .\end{aligned} \qquad (62)$$

We have thus shown that for arbitrary $\vec{n}$ one can obtain the same solution for the magnetic field and for the absolute value of the scalar field as for the choice $\vec{n} = \vec{\epsilon}_3$. Only the phase of the scalar field is different.

V. ENERGY OF THE SYSTEM

In order to calculate the energy of the system one has to derive first the Hamiltonian density corresponding to the Lagrangian density:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_Z^0(A, B)\\ &\quad + \tfrac{1}{2}|(\partial_\mu - ieA_\mu)\phi|^2 - \tfrac{1}{2}m^2|\phi|^2 - \tfrac{1}{4}h|\phi|^4\ .\end{aligned} \qquad (63)$$

Using the definition for the electric and the magnetic fields $E_i \equiv -F^{0i}$ and $H_i \equiv -F^{d0i}$ as well as the form for the free Zwanziger's Lagrangian given by Eq. (8) one finds after some calculation

$$\begin{aligned}\mathcal{H} &= \tfrac{1}{2}(\vec{E}^2 + \vec{H}^2)\\ &\quad + \tfrac{1}{2}|(\vec{\nabla} - ie\vec{A})\phi|^2 + \tfrac{1}{2}m^2|\phi|^2 + \tfrac{1}{4}h|\phi|^4\ .\end{aligned} \qquad (64)$$

There is no contribution to the energy from the term $-j_0^g B_0$, as witnessed by the fact that the additional term in the Lagrangian for nonstatic monopoles should actually be written as

$$\begin{aligned}\mathcal{L} &= \sum_{i=1}^{2}[-m_i(-\dot{x}_i^{\,2})^{1/2}] - g\dot{x}_1\cdot B(x_1)\\ &\quad + g\dot{x}_2\cdot B(x_2)\ ,\end{aligned} \qquad (65)$$

where $x_1^\mu(\tau)$ and $x_2^\mu(\tau)$ are the positions of the monopoles in Minkowski space. With the choice of the phase of the scalar field $\chi = 0$ (corresponding to the choice $\vec{n} = \vec{\epsilon}_3$) one gets the energy of the system

$$\begin{aligned}E &= \tfrac{1}{2}\int d\vec{x}(\vec{H}^2 + e^2|\phi|^2\vec{A}^2)\\ &\quad + \int d\vec{x}[\tfrac{1}{2}(\vec{\nabla}|\phi|)^2 + \tfrac{1}{2}m^2|\phi|^2 + \tfrac{1}{4}h|\phi|^4]\ .\end{aligned} \qquad (66)$$

We now calculate the first part of the energy:

$$\begin{aligned}E_1 &= \tfrac{1}{2}\int(\vec{H}^2 + e^2|\phi|^2\vec{A}^2)d\vec{x}\\ &= \tfrac{1}{2}\int[\vec{H}\cdot(\vec{\nabla}\times\vec{A} + \vec{\epsilon}_3\partial_3^{-1}j_g^0) + e^2|\phi|^2\vec{A}^2]d\vec{x}\\ &= \tfrac{1}{2}\int(\vec{j}_e\cdot\vec{A} + H_3\partial_3^{-1}j_g^0 + e^2|\phi|^2\vec{A}^2)d\vec{x}\\ &= \tfrac{1}{2}\int(H_3\partial_3^{-1}j_g^0)d\vec{x}\ .\end{aligned} \qquad (67)$$

We can estimate this part by extrapolating the solution for large ρ into the small-ρ region. Using Eq. (30) we find

$$\begin{aligned}E_1 &= \tfrac{1}{2}\int j_g^0(\vec{x})G(\vec{x} - \vec{x}')j_g^0(\vec{x}')d\vec{x}\,d\vec{x}'\\ &\quad + \tfrac{1}{2}M^2\int\partial_3^{-1}j_g^0(\vec{x})G(\vec{x} - \vec{x}')(\partial_3')^{-1}j_g^0(\vec{x}')d\vec{x}\,d\vec{x}'\ ,\end{aligned} \qquad (68)$$

which is the form written down by Nambu if we remember the identification for his source $\vec{\Phi} = \vec{\epsilon}_3\partial_3^{-1}j_g^0$. The first term contributes the Yukawa potential while the second term is proportional to $2a$ for $2a \gg 1/M$. The second part of the energy which is due to the scalar field receives the contribution only from the string region, since Φ is essentially equal to its vacuum value everywhere else. It can be estimated by the argument similar to that of Nielsen and Olesen and it is found to be proportional to $2a|\Phi_0|^2$. The total energy is

$$E=-\frac{g^2}{4\pi}\,\frac{e^{-2Ma}}{2a}+C|\phi_0|^2a\ . \tag{69}$$

Since the energy grows linearly with quark separation for large a, one has a mechanism for quark confinement. For implications of such a scheme to hadron physics, the reader is referred to Nambu's paper.[5]

As a final comment, we note that since Eq. (64) for the energy density contains a term explicitly dependent on the vector field $\vec{A}$, the question of $\vec{n}$ dependence of the energy is relevant. It is resolved by noting that this term is gauge-invariant and that the passage from one $\vec{n}$ to another is actually a gauge transformation. Thus the energy is $\vec{n}$-independent.

Note added in proof. When this paper was finished, we received a report by A. P. Balachandran, H. Rupertsberger, and J. Schechter,[11] where Nambu's static phenomenological Hamiltonian was also derived. These authors use the formulation of magnetic monopole theory with an explicit mass term for the vector field,[12] while we generate the mass term by the Higgs mechanism. We would like to point out that an explicit mass term breaks the Lorentz invariance of the theory, because gauge invariance is crucial for Lorentz invariance in magnetic monopole theories.

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[1]Y. Nambu, in *Proceedings of the Johns Hopkins Workshop on Current Problems in High Energy Particle Theory, 1974,* edited by G. Domokos *et al.* (Johns Hopkins Univ., Baltimore, 1974).

[2]G. Parisi, this issue, Phys. Rev. D 11, 970 (1975).

[3]M. Creutz, Phys. Rev. D 10, 2696 (1974).

[4]H. B. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1973).

[5]Y. Nambu, Phys. Rev. D 10, 4262 (1974).

[6]D. Zwanziger, Phys. Rev. D 3, 880 (1971).

[7]D. Forster, Nucl. Phys. B81, 84 (1974).

[8]J. L. Gervais and B. Sakita (private communication).

[9]J. Schwinger, Phys. Rev. 144, 1087 (1966).

[10]In the field-theoretic approach to magnetic monopoles, one has to use the Schwinger charge quantization condition, which differs by a factor of 2 from Dirac's condition $eg/4\pi=\frac{1}{2}n$.

[11]A. P. Balachandran, H. Rupertsberger, and J. Schechter, Syracuse Univ. Report No. SU-4205-41, 1974 (unpublished).

[12]A. P. Balachandran, H. Rupertsberger, and J. Schechter, Syracuse Univ. Report No. SU-4205-37, 1974 (unpublished).