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Lorentz covariance and Matthews's theorem for derivative-coupled field theories*

Claude Bernard†

Jefferson Physical Laboratories, Harvard University, Cambridge, Massachusetts 02138

Anthony Duncan

Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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The canonical quantization of scalar-field Lagrangians involving at most first derivatives of the fields ("first-order" Lagrangian) or second derivatives ("second-order") is discussed. A direct, but necessarily perturbative, quantization procedure for a general first-order Lagrangian is used to show that such theories yield a Lorentz-invariant S matrix to low orders of perturbation theory provided a covariant regularization scheme (e.g., dimensional or Pauli-Villars—but *not* a high-momentum cutoff) is employed. Matthews's theorem is verified in this context—the naive Feynman rules are valid. Second-order Lagrangians, quadratic in second but arbitrary in first derivatives, are shown to satisfy Matthews's theorem to all orders of perturbation theory and to be equivalent to first-order theories with Pauli-Villars regularization, thereby yielding a proof of Matthews's theorem for an arbitrary Pauli-Villars–regulated first-order theory. It is shown that for spectral reasons second- (and presumably, higher-) order theories are unacceptable physically. Finally, the canonical quantization of a second-order gauge theory is performed explicitly; the results show that (a) the naive Faddeev-Popov prescription remains valid in the presence of higher derivatives, and (b) the spectral pathology of second-order theories persists in gauge theories.

I. INTRODUCTION

The question of the Lorentz invariance of the scattering matrix in field theories with arbitrarily many derivatives has always been somewhat obscure. In addition, the form of the correct Feynman rules and their relation to the naive rules which come directly from the Lagrangian remains unclear in such theories. The confusion stems from the fact that when the Lagrangian contains arbitrary powers of field derivatives, the Hamiltonian is a complicated, nonpolynomial function of conjugate momenta and can rarely be written in closed form. Further, when a term in the Lagrangian contains more derivatives than fields, there is the additional problem that the Hamiltonian does not even exist until auxiliary fields are introduced. The difficulty in proving Lorentz invariance and deriving covariant Feynman rules by canonical quantization of such Hamiltonians is obvious. Further, the Feynman functional integral formulation[1] seems not to help since one cannot perform the momentum integration explicitly when the integrand is not a quadratic function of momenta.

The need to understand the properties of theories with many derivatives has acquired an added sense of urgency because of recent attempts[2] to devise a quantum theory of gravitation which has some hope of being renormalizable. The proposed Lagrangians for general relativity, which have added terms of the form $R^{\mu\nu}R_{\mu\nu}$ and R^2, contain both terms quartic in first derivatives of fields and terms quadratic in second derivatives of fields.

There is, of course, a strictly formal proof of the invariance of the S matrix which is dependent merely on the existence of the generators of the Poincaré algebra with the proper commutation relations.[3] However, the existence of the generators is a highly nontrivial question for theories with gauge freedom.[4] Even for purely scalar theories, the formal proof might fail in perturbation theory; moreover, the proof gives no hint as to the actual form of the covariant Feynman rules.

In this paper, we provide some solutions to the difficulties encountered with derivative-coupled field theories. We deal first with theories with arbitrary derivative interactions, subject only to the restriction that the Lagrangian can, by sufficient partial integrations, be put in a form where no more than one derivative operator acts on each field. We call such Lagrangians "first-order." In Sec. II we show to low orders in perturbation theory that with a suitable regularization scheme the S matrix is invariant for any first-order scalar Lagrangian. In fact, the use of dimensional regularization makes the naive version of "Matthews's theorem"[5] correct: The Feynman rules are just those obtained by using the interaction Lagrangian to determine the vertices and the covariant T^* product to determine the propagators. We prove this by starting with the "stage one" functional formalism with integrations over both fields and conjugate momenta and then carrying out the momentum integration perturbatively. We conjecture that Matthews's theorem is, in fact, true to all orders with this regularization scheme, but we have not completed the general proof. At the end of Sec. II we work out a specific example where we perform the canonical quantization explicitly and show how Matthews's theorem conspires to work out for a simple process in low order.

In Sec. III, we consider what we call "second-order" scalar Lagrangians where we allow two derivatives to act on each field in the quadratic part of the action. Such theories are interesting because they have a greater resemblance than first-order theories to the Lagrangians proposed for general relativity. In this case we are able to give a nonperturbative proof of Matthews's theorem, again by starting with the "stage one" functional formalism. Theories of this type also have the interesting property that they are equivalent to those discussed in Sec. II, with the additions of negative metric regulator fields of the Pauli-Villars variety.[6] Thus, we recover the results of Sec. II, but now to all orders of perturbation theory, and in the context of a Pauli-Villars regularization. However, these theories are inherently "sick" for a reason that has nothing to do with Lorentz invariance: With finite regulator mass, they are plagued by either negative-metric states, a Hamiltonian unbounded from below, or a lack of unitarity.

In Sec. IV we discuss second-order Abelian gauge field theories, and prove the logical extension of Matthews's theorem to theories with spurious (gauge) degrees of freedom, that is, the Faddeev-Popov orbit volume prescription.[7] These theories have the same undesirable properties as the second-order scalar theories: The extra gauge freedom does not allow for the elimination of (in fact, has nothing to do with) the negative-metric ghosts. This is somewhat disappointing for the new theories of general relativity, and leads us to guess that they, too, will have similar irreparable defects.[8] Section V contains some further comments and conclusions.

II. FIRST-ORDER SCALAR THEORIES

In this section we deal with scalar Lagrangians of the form

$$\mathcal{L}(\varphi, \partial_\mu\varphi) = \mathcal{L}_0(\varphi, \partial_\mu\varphi) + \lambda\mathcal{L}_I(\varphi, \partial_\mu\varphi) , \qquad (2.1)$$

where $\mathcal{L}_0$ is at most quadratic in the field or its derivative, but $\mathcal{L}_I$ is an arbitrary polynomial. Since such theories will in general be highly divergent and nonrenormalizable, we are in fact only able to verify Matthews's theorem in a suitably regularized version of them. Clearly, if we wish to investigate the problem of Lorentz invariance, we should choose a regularization scheme which manifestly preserves Lorentz invariance at all stages of our proof. In this section we adopt dimensional regularization as the most convenient cutoff. In Secs. III and IV we proceed by adding an additional term to the Lagrangian, which turns out to be equivalent to the use of Pauli-Villars regulator fields. Another possible choice would seem to be a symmetrical high momentum ($k^2=\Lambda^2$) cutoff in Euclidean space. However, if a Minkowski-space integral is divergent by power counting, its Wick-rotated replacement in Euclidean space does not have the same value when a $k^2=\Lambda^2$ cutoff is used. The difference is simply due to the nonvanishing (and sometimes noncovariant) contributions from the part of the Wick contour at infinity. At the end of this section we will show in a specific example how the naive use of a $k^2=\Lambda^2$ cutoff leads to a noninvariant result.

The chief feature of the 't Hooft-Veltman dimensional regularization scheme that we will make use of is the prescription that $\delta^4(0)$ is set to zero. This follows from the identity[9]

$$\int d^\kappa p\, \frac{1}{(p^2+m^2)^\alpha} = \frac{i\pi^{\kappa/2}\Gamma(\alpha-\frac{1}{2}\kappa)}{\Gamma(\alpha)}(m^2)^{\kappa/2-\alpha}$$

$$\Rightarrow \int d^\kappa p\, \frac{1}{(p^2)^\alpha} = 0\ ,\ \ \tfrac{1}{2}\kappa > \alpha$$

$$\Rightarrow \int d^\kappa p = 0\ . \qquad (2.2)$$

In addition, the ordering problems which have been worrisome in previous discussions of derivative-coupling theories can be neatly avoided in this scheme. For example, the relative order of the canonical momenta and fields is irrelevant since $[\pi(x), \varphi(x)] \to i\delta^3(0)$ and $\delta^3(0)$ must be set to zero in a mass-independent regularization scheme.

We are now ready to begin our proof that Matthews's theorem is satisfied (at least to low orders in perturbation theory) for theories like (2.1). The first step is to calculate the Hamiltonian. It will not be possible, in general, to write $\mathcal{H}$ in closed form, but the first few terms in a perturbation expansion in λ will suffice for our purposes. For simplicity, we assume that the field has been redefined so that

$$\mathcal{L}_0(\varphi, \partial_\mu\varphi) = -\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - L(\varphi) , \qquad (2.3)$$

where $L(\varphi)$ is a quadratic polynomial containing no derivatives. The conjugate momentum is

$$\pi=\frac{\delta\mathcal{L}}{\delta(\partial_0\varphi)}$$

$$=\frac{\delta\mathcal{L}_0}{\delta(\partial_0\varphi)}+\lambda\frac{\delta\mathcal{L}_I}{\delta(\partial_0\varphi)}$$

$$=\partial_0\varphi+\lambda\frac{\delta\mathcal{L}_I}{\delta(\partial_0\varphi)}\ , \tag{2.4}$$

$$\dot\varphi\equiv\partial_0\varphi=f(\varphi,\vec\nabla\varphi,\pi;\lambda)\ , \tag{2.5}$$

where f, in general, will be some complicated, nonpolynomial function of the expansion parameter λ. It is useful to define

$$f(\varphi,\vec\nabla\varphi,\pi;\lambda)\equiv\sum_{n=0}^{\infty}\lambda^n f^{(n)}(\varphi,\vec\nabla\varphi,\pi)\ . \tag{2.6}$$

$f^{(0)}$ is of course just π. Similarly, the Hamiltonian density will be written

$$\mathcal{H}=\pi f-\mathcal{L}(\varphi,\vec\nabla\varphi,f(\varphi,\vec\nabla\varphi,\pi))$$

$$\equiv\sum_{n=0}^{\infty}\lambda^n\mathcal{H}^{(n)}(\varphi,\vec\nabla\varphi,\pi)\ . \tag{2.7}$$

Using (2.4), a short calculation yields

$$f^{(1)}=-\frac{\delta\mathcal{L}_I}{\delta\dot\varphi}(\varphi,\vec\nabla\varphi,f^{(0)})\ , \tag{2.8}$$

$$f^{(2)}=\frac{\delta^2\mathcal{L}_I}{\delta\dot\varphi^2}(\varphi,\vec\nabla\varphi,f^{(0)})\frac{\delta\mathcal{L}_I}{\delta\dot\varphi}(\varphi,\vec\nabla\varphi,f^{(0)})\ . \tag{2.9}$$

Similarly, expanding $\mathcal{L}_0(\varphi,\vec\nabla\varphi,f)$ in powers of λ, we find, correct to $O(\lambda^2)$,

$$\mathcal{L}_0(\varphi,\vec\nabla\varphi,f)=\mathcal{L}_0-\lambda\frac{\delta\mathcal{L}_0}{\delta\dot\varphi}\frac{\delta\mathcal{L}_I}{\delta\dot\varphi}+\lambda^2\frac{\delta\mathcal{L}_0}{\delta\dot\varphi}\frac{\delta\mathcal{L}_I}{\delta\dot\varphi}\frac{\delta^2\mathcal{L}_I}{\delta\dot\varphi^2}+\frac{\lambda^2}{2}\frac{\delta^2\mathcal{L}_0}{\delta\dot\varphi^2}\left(\frac{\delta\mathcal{L}_I}{\delta\dot\varphi}\right)^2+O(\lambda^3)\ , \tag{2.10}$$

where all the functions on the right-hand side are evaluated at $(\varphi,\vec\nabla\varphi,f^{(0)})$. A similar result holds for the expansion of $\mathcal{L}_I(\varphi,\vec\nabla\varphi,f)$. Substituting (2.8), (2.9), and (2.10) into (2.7) and using $f^0=\pi$, we obtain the desired results

$$\mathcal{H}^{(0)}(\varphi,\vec\nabla\varphi,\pi)=\tfrac{1}{2}(\pi^2+|\vec\nabla\varphi|^2)+L(\varphi)\ , \tag{2.11}$$

$$\mathcal{H}^{(1)}(\varphi,\vec\nabla\varphi,\pi)=-\mathcal{L}_I(\varphi,\vec\nabla\varphi,\pi)\ , \tag{2.12}$$

$$\mathcal{H}^{(2)}(\varphi,\vec\nabla\varphi,\pi)=\frac{1}{2}\left(\frac{\delta\mathcal{L}_I}{\delta\dot\varphi}(\varphi,\vec\nabla\varphi,\pi)\right)^2\ . \tag{2.13}$$

It is clear that the procedure outlined above can be extended in principle to an arbitrarily high order of perturbation theory; of course, the computations involved become progressively more tedious.

It is now possible to demonstrate the invariance of the S matrix to low orders of perturbation theory. We have verified the validity of Matthews's theorem through fourth order for these theories, dimensionally regulated; for the sake of brevity, the proof is given below through second order. We take as our starting point the Feynman path integral formula[1] for the generating functional $Z[J]$, namely,

$$Z[J]=\int[d\varphi]\int[d\pi]\exp\left\{i\int d^4x[\pi\dot\varphi-\mathcal{H}(\varphi,\vec\nabla\varphi,\pi)+J\varphi]\right\}\ . \tag{2.14}$$

Equation (2.14) can be shown to be equivalent to the generating functional obtained by a direct canonical quantization of the theory (after due attention has been paid to ordering ambiguities). Defining $\mathcal{H}=\mathcal{H}^{(0)}+\mathcal{H}_{\text{int}}$, $\mathcal{H}_{\text{int}}=\lambda\mathcal{H}^{(1)}+\lambda^2\mathcal{H}^{(2)}+O(\lambda^3)$, using (2.11), and introducing a source K coupled to π, we can rewrite (2.14) as

$$Z[J]=\int[d\varphi]\exp\left\{i\int d^4x[-L(\varphi)-\tfrac{1}{2}|\vec\nabla\varphi|^2+J\varphi]\right\}\exp\left[-i\int d^4x\mathcal{H}_{\text{int}}\left(\varphi,\vec\nabla\varphi,\frac{\delta}{\delta iK}\right)\right]\times\int[d\pi]\exp\left\{-i\int d^4x\left[\tfrac{1}{2}\pi^2-\pi(K+\dot\varphi)\right]\right\}\Bigg|_{K=0}\ . \tag{2.15}$$

(2.15) is to be evaluated at $K=0$. The integration over π can now be done by completing the square. Using (2.3), we have

$$Z[J]=\int[d\varphi]\exp\left[i\int d^4x(\mathcal{L}_0+J\varphi)\right]\exp\left[-i\int d^4x\mathcal{H}_{\text{int}}\left(\varphi,\vec\nabla\varphi,\frac{\delta}{\delta iK}\right)\right]\exp\left[i\int d^4x(\tfrac{1}{2}K^2+K\dot\varphi)\right]\Bigg|_{K=0}\ . \tag{2.16}$$

We now employ the functional identity[10]

$$F\left[\frac{\delta}{\delta iK}\right]G[K]=G\left[\frac{\delta}{\delta i\rho}\right]F[\rho]\exp\left(i\int d^4xK\rho\right)\Bigg|_{\rho=0} \tag{2.17}$$

to put (2.16) in the form

$$Z[J]=\int[d\varphi]\exp\left[i\int d^4x(\mathcal{L}_0+J\varphi)\right]\exp\left[\int d^4x\,\dot\varphi(x)\frac{\delta}{\delta\rho(x)}\right]\exp\left[-\tfrac{1}{2}i\int d^4x\left(\frac{\delta}{\delta\rho(x)}\right)^2\right]\times\exp\left[-i\int d^4x\mathcal{H}_{\rm int}(\varphi,\vec\nabla\varphi,\rho)\right]\Bigg|_{\rho=0}. \quad (2.18)$$

Using (2.12) and (2.13), expanding the exponential through $O(\lambda^2)$, we have

$$\exp\left[-i\int d^4x\mathcal{H}_{\rm int}(\varphi,\vec\nabla\varphi,\rho)\right]=1+i\lambda\int d^4x\mathcal{L}_I(\varphi,\vec\nabla\varphi,\rho)-i\tfrac{1}{2}\lambda^2\int d^4x\left(\frac{\delta\mathcal{L}_I}{\delta\rho}\right)^2-\tfrac{1}{2}\lambda^2\left(\int d^4x\mathcal{L}_I\right)^2+O(\lambda^3). \quad (2.19)$$

We must now apply the operator $\exp\{-\frac{1}{2}i\int d^4x[\delta/\delta\rho(x)]^2\}$ to (2.19). Note that with the prescription $\delta^4(0)\to 0$ [cf. (2.2)], the linear term in the expansion of the exponential in this operator gives a nonvanishing result only when acting on the term $-\frac{1}{2}\lambda^2(\int d^4x\mathcal{L}_I)^2$ in (2.19). Similarly terms quadratic or higher in the expansion of the operator always give zero when acting on (2.19). The result is

$$\begin{aligned}\exp\left[-\tfrac{1}{2}i\int d^4x\left(\frac{\delta}{\delta\rho(x)}\right)^2\right]&\exp\left[-i\int d^4x\mathcal{H}_{\rm int}(\varphi,\vec\nabla\varphi,\rho)\right]\\ &=1+i\lambda\int d^4x\mathcal{L}_I-\tfrac{1}{2}i\lambda^2\int d^4x\left(\frac{\delta\mathcal{L}_I}{\delta\rho}\right)^2\\ &\quad-\tfrac{1}{2}\lambda^2\left(\int d^4x\mathcal{L}_I\right)^2+\tfrac{1}{2}i\lambda^2\int d^4x\left(\frac{\delta\mathcal{L}_I}{\delta\rho}\right)^2+O(\lambda^3)\\ &=\exp\left[i\int d^4x\lambda\mathcal{L}_I(\varphi,\vec\nabla\varphi,\rho)\right]+O(\lambda^3). \end{aligned}\quad (2.20)$$

Looking back at (2.18) we see that we must now operate on (2.20) with $\exp\{\int d^4x\,\dot\varphi(x)[\delta/\delta\rho(x)]\}$, and then set $\rho=0$. But this procedure merely replaces ρ everywhere by $\dot\varphi$. So we have

$$Z[J]=\int[d\varphi]\exp\left\{i\int d^4x[\mathcal{L}_0(\varphi,\vec\nabla\varphi,\dot\varphi)+\lambda\mathcal{L}_I(\varphi,\vec\nabla\varphi,\dot\varphi)+J\varphi]\right\}+O(\lambda^3). \quad (2.21)$$

But this is just Matthews's theorem, here verified to second order. It says that the correct Feynman rules are the covariant ones obtained by using the quadratic part of $\mathcal{L}$ to determine the covariant (T^*) propagators and the rest of $\mathcal{L}$ to determine the vertices. The procedure for checking the theorem in higher orders is exactly parallel; however, we have been unable to complete the proof to all orders with the above methods.

We now present a simple example which illustrates some of the subtle points underlying our proof of Lorentz invariance. In particular, we wish to point out (a) the necessity for including external-line corrections in obtaining invariant S-matrix elements, and (b) the failure of Lorentz invariance with a $k^2=\Lambda^2$ cutoff in Euclidean space—as mentioned previously, this cutoff is simply not an invariant one when applied to Minkowski space integrals.

Consider 2-2 scattering of scalar particles in a theory described by the Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\tfrac{1}{2}m^2\varphi^2-\tfrac{1}{4}\lambda(\partial_\mu\varphi\partial^\mu\varphi)^2. \quad (2.22)$$

Equations (2.11)–(2.13) here yield

$$\begin{aligned}\mathcal{H}^{(0)}&=\tfrac{1}{2}(\pi^2+|\nabla\varphi|^2+m^2\varphi^2),\\ \mathcal{H}^{(1)}&=\tfrac{1}{4}\lambda(\pi^2-|\nabla\varphi|^2)^2,\\ \mathcal{H}^{(2)}&=\tfrac{1}{2}\lambda^2\pi^2(\pi^2-|\nabla\varphi|^2)^2.\end{aligned}\quad (2.23)$$

Going to the interaction picture defined by $\mathcal{H}^{(0)}$, we see that 2-2 scattering is described to $O(\lambda^2)$ by three types of graphs (see Figs. 1–3).

The graph in Fig. 2 arises from a single $\mathcal{H}^{(2)}$ vertex. The graphs in Fig. 3, though involving external-line corrections, must be included as they give rise to noncovariant contributions which are not canceled by additional graphs involving insertions of mass and wave-function counterterms on the external lines—in fact, these noncovariant contributions are essential in canceling similar terms in Figs. 1 and 2.

We can now proceed to calculate the graphs. The propagator of two field derivatives is given by

$$\langle 0|T(\partial_\mu\varphi(x)\partial_\nu\varphi(y))|0\rangle=\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\langle 0|T(\varphi(x)\varphi(y))|0\rangle-ig_{\mu 0}g_{\nu 0}\delta^4(x-y) \quad (2.24)$$

or, in momentum space, by

$$\Delta^{\mu\nu}(k)=\frac{-ik^\mu k^\nu}{k^2+m^2-i\epsilon}-ig^\mu_0 g^\nu_0. \quad (2.25)$$

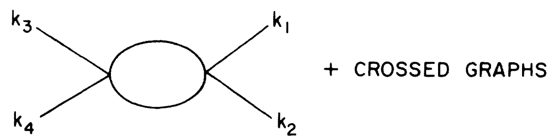

FIG. 1. One-particle-irreducible graphs for 2-2 scattering arising from two $\mathcal{H}^{(1)}$ vertices.

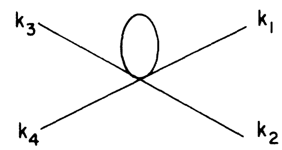

FIG. 2. Contribution from a single $\mathcal{H}^{(2)}$ vertex.

We must keep in mind that (2.25) does not apply to a line which begins and ends at the same vertex. Each interaction term in the Lagrangian must be regarded as a simple product of fields at the same space-time point (*not* time ordered), so that contractions within field products are just the covariant vacuum expectation values of field products, which are easily seen to coincide with the covariant part of (2.25). That is,

$$\Delta^{\mu\nu}_{\text{same vertex}}(k) = -\frac{ik^\mu k^\nu}{k^2+m^2-i\epsilon}\ . \tag{2.26}$$

We can now write down the values of each of the diagrams, regularized in dimension κ.

Fig. 1:

$$24\lambda^2\delta^\kappa(0)k_1^0k_2^0k_3^0k_4^0 - 4\lambda^2\delta^\kappa(0)(k_1^0k_2^0k_3\cdot k_4+\text{perms}) + \frac{8i\lambda^2}{(2\pi)^4}\pi^{\kappa/2}(m^2)^{2+\kappa/2}\Gamma(-\tfrac{1}{2}\kappa)(k_1^0k_2^0k_3\cdot k_4+\text{perms})+\text{invariant combinations of } k_1, k_2, k_3, k_4; \tag{2.27}$$

Fig. 2:

$$-\frac{20i}{(2\pi)^4}\lambda^2\pi^{\kappa/2}(m^2)^{2+\kappa/2}\Gamma(-\tfrac{1}{2}\kappa)(k_1^0k_2^0k_3\cdot k_4+\text{perms})\ ; \tag{2.28}$$

Fig. 3:

$$\frac{12i\lambda^2}{(2\pi)^4}\pi^{\kappa/2}(m^2)^{2+\kappa/2}\Gamma(-\tfrac{1}{2}\kappa)(k_1^0k_2^0k_3\cdot k_4+\text{perms})+\text{invariant combinations.} \tag{2.29}$$

$\delta^\kappa(0)$ is defined to be $\int d^\kappa p/(2\pi)^4$, which is of course equal to zero in this scheme; we leave it in merely for comparison with other regularization methods. The first two terms in (2.27) come from taking the noncovariant, momentum-independent parts of both internal propagators; the third term comes from one covariant part and one noncovariant part. The first term in (2.29) comes from the noncovariant part of the nonloop propagator.

Adding (2.27)–(2.29) one finds that the total 2-2 S-matrix element is invariant after letting $\delta^\kappa(0) = 0$. In addition, Matthews's theorem is satisfied to this order: Contributions of the noninvariant $\mathcal{H}^{(2)}$ completely cancel contributions due to the noncovariant piece of the propagator, leaving the Feynman rules one would get naively from $\mathcal{L}$. The same results hold with a Pauli-Villars cutoff. In that case, the noncovariant piece of the propagator, which produces the noncovariant terms in Figs. 1 and 3, is canceled by the corresponding piece of the regulator particle proagator, while Fig. 2 is entirely absent, since the contributions to $\mathcal{H}^{(2)}$ from the positive and negative metric particles exactly cancel. On the other hand, with a naive Λ^2 cutoff, the noncovariant $\delta^4(0)$ terms which are left over after summing the diagrams cannot be discarded. In particular, the tensor structure of the first term in (2.27) cannot arise in any of the other diagrams; we are left with a net contribution from all graphs proportional to $\Lambda^4 k_1^0k_2^0k_3^0k_4^0$. This is not surprising, however, since this cutoff does not regulate in Minkowski space but only in Euclidean space, where it confines the integration to a bounded region. By a Wick rotation, the Minkowski integral can be set equal to the Euclidean integral plus the contribution from the

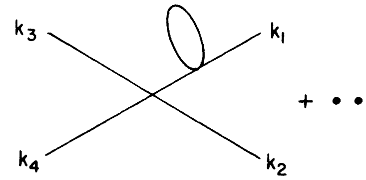

FIG. 3. External-leg corrections arising from two $\mathcal{H}^{(1)}$ vertices.

contour at infinity, but this contribution does not vanish for a divergent integral. In fact, the contribution is noncovariant in this case, which we have discovered by explicitly evaluating the integral over the contour for $\int d^4p[p^\mu p^\nu/(p^2-i\epsilon)]$, with the cutoff $p_0^2+p_1^2+p_2^2+p_3^2<\Lambda^2$.

III. SECOND-ORDER SCALAR THEORIES: PROOF OF MATTHEWS'S THEOREM

In this section we will discuss the quantization of scalar field theories in which the Lagrangian is quadratic in second derivatives of the field. For simplicity, we consider theories involving a single scalar field φ. We will eventually show that the quantization of the second-order theory leads directly to (a) a proof of Matthews's theorem for an arbitrary *first-order* theory regularized by the Pauli-Villars technique. Our treatment includes (b) a discussion of the spectral problems of these theories (which were first pointed out some time ago by Pais and Uhlenbeck[11]).

(a) We begin with a Lagrangian of the form

$$\mathcal{L}=-\tfrac{1}{2}\alpha(\partial_\mu\partial_\nu\varphi)(\partial^\mu\partial^\nu\varphi)-\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\tfrac{1}{2}\beta\varphi^2+\mathcal{L}_I(\varphi,\partial_\mu\varphi)\ . \tag{3.1}$$

It is assumed in (3.1) that the interaction density is an arbitrary scalar density built from at most first derivatives of the field. The Lagrangian (3.1) defines an action in the usual fashion. Extremization of this action leads to the generalized Euler-Lagrange equation

$$\frac{\delta\mathcal{L}}{\delta\varphi}-\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}+\partial_\mu\partial_\nu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\partial_\nu\varphi)}=0\ , \tag{3.2}$$

i.e.,

$$(\alpha\Box^2-\Box+\beta)\varphi=\frac{\delta\mathcal{L}_I}{\delta\varphi}-\partial_\mu\frac{\delta\mathcal{L}_I}{\delta\partial_\mu\varphi}\ . \tag{3.3}$$

The canonical procedure is not directly applicable to the Lagrangian (3.1) as it stands, however, as a result of the second derivatives $\partial_\mu\partial_\nu\varphi$. The canonical methods must therefore be implemented indirectly by finding an *equivalent* Lagrangian $\hat{\mathcal{L}}$ which contains only first derivatives but yields the same equations of motion. The standard method for effecting this, due originally to Ostrogradski,[12] requires the introduction of an auxiliary field ψ as follows:

$$\hat{\mathcal{L}}=\frac{1}{2\alpha}\psi^2-\partial^\mu\psi\partial_\mu\varphi-\tfrac{1}{2}\beta\varphi^2+\mathcal{L}_I(\varphi,\partial_\mu\varphi)\ . \tag{3.4}$$

One easily sees that the Euler-Lagrange equations for $\hat{\mathcal{L}}$ are equivalent to (3.3), after eliminating the field ψ, which is found by the field equation to be just $-\alpha\Box\varphi$. Thus (3.4) amounts to a restatement of the dynamical content of the original Lagrangian in a framework amenable to canonical methods. It is now possible to compute the Hamiltonian and the generating functional in the standard fashion. At the end of the calculation, we will see that $\hat{\mathcal{L}}$ defines a Lorentz-covariant theory and that, in fact, Matthews's theorem holds to all orders of perturbation theory. We will then turn to a study of the spectral pathology of the theory based on (3.4).

From (3.4), we find for the canonical momenta

$$\pi_\varphi=\dot\psi+\dot\varphi+\frac{\delta\mathcal{L}_I}{\delta\dot\varphi}\ ,\qquad \pi_\psi=\dot\varphi\ . \tag{3.5}$$

In contrast to the situation in Sec. II, one can now solve explicitly for the time derivatives $\dot\varphi$, $\dot\psi$, in terms of canonical fields and momenta. Namely,

$$\dot\varphi=\pi_\psi\ ,\qquad \dot\psi=\pi_\varphi-\pi_\psi-\frac{\delta\mathcal{L}_I}{\delta\dot\varphi}(\varphi,\vec\nabla\varphi,\pi_\psi)\ . \tag{3.6}$$

For the Hamiltonian, one thus obtains [note that there are no ordering ambiguities in $\mathcal{L}_I(\varphi,\vec\nabla\varphi,\pi_\psi)$, since π_ψ and φ commute]

$$\begin{aligned}\mathcal{H}&=\pi_\varphi\dot\varphi+\pi_\psi\dot\psi-\hat{\mathcal{L}}\\&=\pi_\varphi\pi_\psi-\tfrac{1}{2}\pi_\psi^{\,2}-\frac{\psi^2}{2\alpha}+\vec\nabla\psi\cdot\vec\nabla\varphi+\tfrac{1}{2}|\vec\nabla\varphi|^2+\tfrac{1}{2}\beta\varphi^2\\&\quad-\mathcal{L}_I(\varphi,\vec\nabla\varphi,\pi_\psi)\ .\end{aligned} \tag{3.7}$$

The generating functional for the Green's functions $\langle 0|T(\varphi(x_1)\cdots\varphi(x_n))|0\rangle$ is given by the "stage one" Feynman formula,[1]

$$\begin{aligned}Z_F[J]&=\int[d\varphi\,d\psi\,d\pi_\varphi d\pi_\psi]\exp\left\{i\int[J(x)\varphi(x)+\pi_\varphi(x)\dot\varphi(x)+\pi_\psi(x)\dot\psi(x)-\mathcal{H}(\varphi,\psi,\pi_\varphi,\pi_\psi,\vec\nabla\varphi,\vec\nabla\psi)]\,dx\right\}\\&=\int[d\varphi\,d\psi]\exp\left[i\int\left(J\varphi+\frac{\psi^2}{2\alpha}-\vec\nabla\psi\cdot\vec\nabla\varphi-\tfrac{1}{2}|\nabla\varphi|^2-\tfrac{1}{2}\beta\varphi^2\right)dx\right]\mathfrak{z}[\vec\nabla\varphi,\dot\varphi,\dot\psi]\ ,\end{aligned} \tag{3.8}$$

where

$$\mathfrak{z}[\vec{\nabla}\varphi, \dot{\varphi}, \dot{\psi}] = \int [d\pi_\varphi d\pi_\psi] \exp\left\{ i \int [\pi_\varphi \dot{\varphi} + \pi_\psi \dot{\psi} + \tfrac{1}{2}\pi_\psi{}^2 - \pi_\varphi \pi_\psi + \mathfrak{L}_I(\varphi, \vec{\nabla}\varphi, \pi_\psi)]\, dx \right\}$$

$$= \int [d\pi_\psi] \delta[\dot{\varphi} - \pi_\psi] \exp\left\{ i \int [\pi_\psi \dot{\psi} + \tfrac{1}{2}\pi_\psi{}^2 + \mathfrak{L}_I(\varphi, \vec{\nabla}\varphi, \pi_\psi)]\, dx \right\}$$

$$= \exp\left\{ i \int [\dot{\varphi}\dot{\psi} + \tfrac{1}{2}\dot{\varphi}^2 + \mathfrak{L}_I(\varphi, \partial_\mu \varphi)]\, dx \right\} . \tag{3.9}$$

A divergent multiplicative factor has been discarded in the usual way. Substituting (3.9) into (3.8), and performing the Gaussian integration over the ψ field, one finds

$$Z_F[J] = \int [d\varphi] \exp\left\{ i \int [J\varphi - \tfrac{1}{2}\alpha(\partial_\mu \partial_\nu \varphi)(\partial^\mu \partial^\nu \varphi) - \tfrac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \tfrac{1}{2}\beta\varphi^2 + \mathfrak{L}_I(\varphi, \partial_\mu \varphi)]\, dx \right\}$$

$$= \int [d\varphi] \exp\left\{ i \int [J\varphi + \mathfrak{L}(\varphi, \partial_\mu \varphi, \partial_\mu \partial_\nu \varphi)]\, dx \right\} . \tag{3.10}$$

This completes the proof of Matthews's theorem for the second-order theory. The Feynman rules are obtained by deriving vertices and propagators from the equivalent, explicitly covariant, second-order Lagrangian $\mathfrak{L}$.

(b) Although the naive canonical procedure for $\hat{\mathfrak{L}}$ seems to apply, a careful study of the spectral properties of this theory[11] leads unfortunately to the conclusion that *no sensible, unitary interpretation of the theory defined by the generating functional (3.10) is possible.* In particular, we now show that the theory defined by the Lagrangian (3.4) is exactly equivalent to a Pauli-Villars regularization of the theory based on $\mathfrak{L}(\alpha = 0)$, where the regulator particles actually appear in the physical spectrum for α finite and nonzero.

To study the spectrum of the theory, we must look at the Fourier components of the fields $\varphi(x)$, $\psi(x)$ in the interaction picture defined by the free Hamiltonian density

$$\mathcal{H}_0 = \pi_\varphi \pi_\psi - \tfrac{1}{2}\pi_\psi{}^2 - \frac{\psi^2}{2\alpha} + \vec{\nabla}\psi \cdot \vec{\nabla}\varphi + \tfrac{1}{2}|\vec{\nabla}\varphi|^2 + \tfrac{1}{2}\beta\varphi^2 . \tag{3.11}$$

In this picture, the fields satisfy the free equations of motion

$$(\alpha\Box^2 - \Box + \beta)\varphi = 0 , \qquad \psi = -\alpha\Box\varphi . \tag{3.12}$$

Consequently, φ has the Fourier decomposition

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} (1 - 4\alpha\beta)^{-1/4} \times \int dk \{ e^{ik\cdot x} \theta(k^0)[\delta(k^2 + m^2) a(\vec{k}) + \delta(k^2 + M^2) A(\vec{k})] + \text{H.c.} \} , \tag{3.13}$$

where m (M) is the lower (upper) real positive root of

$$\alpha\mu^4 - \mu^2 + \beta = 0 . \tag{3.14}$$

To exclude tachyons, we insist that

$$\alpha, \beta > 0, \quad \alpha\beta < \tfrac{1}{4} . \tag{3.15}$$

Using (3.12), the dependent field $\psi(x)$ has the decomposition

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} (1 - 4\alpha\beta)^{-1/4} \times \int dk \{ e^{ik\cdot x} \theta(k^0)[-\alpha m^2 \delta(k^2 + m^2) a(\vec{k}) - \alpha M^2 \delta(k^2 + M^2) A(\vec{k})] + \text{H.c.} \} . \tag{3.16}$$

After some algebra, the inversion formulas for the destruction operators $a(k), A(k)$ are found to be

$$a(\vec{k}) = \frac{i}{(2\pi)^{3/2}} (1 - 4\alpha\beta)^{-1/4} \times \int d\vec{x}\, e^{-i(\vec{k}\cdot\vec{x} - \omega_k t)} \overleftrightarrow{\frac{\partial}{\partial t}} (\alpha M^2 \varphi + \psi) , \tag{3.17}$$

$$A(\vec{k}) = -\frac{i}{(2\pi)^{3/2}} (1 - 4\alpha\beta)^{-1/4} \times \int d\vec{x}\, e^{-i(\vec{k}\cdot\vec{x} - \Omega_k t)} \overleftrightarrow{\frac{\partial}{\partial t}} (\alpha m^2 \varphi + \psi) , \tag{3.18}$$

where $\omega_k \equiv (|\vec{k}|^2 + m^2)^{1/2}$, $\Omega_k = (|\vec{k}|^2 + M^2)^{1/2}$. To derive the algebra satisfied by $a(\vec{k}), a^\dagger(\vec{k}), A(\vec{k}), A^\dagger(\vec{k})$, we must refer to the equal-time canonical commutation relations:

$$[\varphi(\vec{x}, t), \varphi(\vec{y}, t)] = [\psi(\vec{x}, t), \psi(\vec{y}, t)] = [\varphi(\vec{x}, t), \psi(\vec{y}, t)] = 0 , \tag{3.19}$$

$$\begin{aligned}[\dot{\varphi}(\vec{x},t),\dot{\varphi}(\vec{y},t)] &= [\dot{\psi}(\vec{x},t),\dot{\psi}(\vec{y},t)] \\ &= [\dot{\varphi}(\vec{x},t),\dot{\psi}(\vec{y},t)] \\ &= 0\,, \\ [\varphi(\vec{x},t),\dot{\varphi}(\vec{y},t)] &= 0\,, \\ [\psi(\vec{x},t),\dot{\psi}(\vec{y},t)] &= -i\delta^3(\vec{x}-\vec{y})\,, \\ [\varphi(\vec{x},t),\dot{\psi}(\vec{y},t)] &= i\delta^3(\vec{x}-\vec{y})\,, \\ [\psi(\vec{x},t),\dot{\varphi}(\vec{y},t)] &= i\delta^3(\vec{x}-\vec{y})\,. \end{aligned} \tag{3.20}$$

From the formulas (3.17)–(3.20) one now obtains the momentum-sapce algebra by completely straightforward computations:

$$\begin{aligned}[a(\vec{k}),a(\vec{k}')] &= [a(\vec{k}),A(\vec{k}')] \\ &= [A(\vec{k}),A(\vec{k}')] \\ &= 0\,, \end{aligned} \tag{3.21}$$

$$\begin{aligned}[a(\vec{k}),A^\dagger(\vec{k}')] &= 0\,, \\ [a(\vec{k}),a^\dagger(\vec{k}')] &= 2\omega_k\,\delta^3(\vec{k}-\vec{k}')\,, \\ [A(\vec{k}),A^\dagger(\vec{k}')] &= -2\Omega_k\,\delta^3(\vec{k}-\vec{k}')\,. \end{aligned} \tag{3.22}$$

The interpretation of $a^\dagger(\vec{k}), A^\dagger(\vec{k})$ as creation operators is fixed by the requirement of energy positivity: One easily finds that

$$\begin{aligned} Ha^\dagger(\vec{k})|0\rangle &= \omega_k a^\dagger(\vec{k})|0\rangle\,, \\ HA^\dagger(\vec{k})|0\rangle &= \Omega_k A^\dagger(\vec{k})|0\rangle\,. \end{aligned} \tag{3.23}$$

Thus the negative sign in the commutation relation of $A(\vec{k})$ with $A^\dagger(\vec{k}')$ implies *a negative metric for the particle with the larger mass*. This result follows inexorably whenever the quadratic part of the Lagrangian has the form assumed in (3.1). Furthermore, exclusion of the nonphysical states from unitarity sums destroys unitarity, as can easily be verified by explicit calculation in some simple models [say, in 2-2 scattering with $\mathcal{L}_I(\varphi) = (\lambda/4!)\varphi^4$]. We conclude that theories based on Lagrangians of the form (3.1) cannot simultaneously yield (a) a consistent probability interpretation, (b) a positive (semi-) definite energy, and (c) physical unitarity.

These results may alternatively be summarized by stating that the theory obtained by canonical quantization of a second-order Lagrangian is completely equivalent to the Pauli-Villars regularization of the corresponding first-order theory (obtained by setting $\alpha=0$). However, the regulator mass M is not here taken to infinity, hence the spectral problem.

We are finally in a position to state Matthews's theorem for an arbitrary *first-order* theory of a single scalar field (the generalization to N independent scalar fields is trivial). We have shown that an arbitrary such first-order theory, after inclusion of regulator fields (and, say, dimensional regularization to control any remaining infinities) can be rewritten in a completely equivalent form as a (dimensionally regularized) second-order theory. Our proof of Matthews's theorem in the latter context, therefore, automatically establishes it for an arbitrary Pauli-Villars regularized first-order theory.

We suspect that a dimensional regularization of an arbitrary first-order theory is sufficient (i.e., without the inclusion of regulators) to ensure the validity of Matthews's theorem. As we have seen in the preceding section, this conjecture is certainly verified in low orders of perturbation theory. We have, however, been unable to find a direct proof of the theorem to all orders of perturbation theory in the context of a purely dimensional regularization.

IV. CANONICAL QUANTIZATION AND SPECTRUM OF A SECOND-ORDER GAUGE THEORY

For theories with local gauge symmetries, Matthews's theorem (i.e., the instruction to form propagators from $\mathcal{L}_{\rm quad}$ and vertices from $\mathcal{L}_{\rm int}$) cannot apply, since the quadratic part of the Lagrangian is singular, and the propagator, prior to the imposition of a gauge condition, ill defined. We may, however, regard the Faddeev-Popov (FP) orbit volume prescription[7] as the natural extension of Matthews's theorem to such cases. In this section we shall consider a simple second-order Abelian gauge theory ("QED with higher derivatives") and study (a) the validity of the naive FP prescription, and (b) the spectral properties of the theory.

(a) We start from the Lagrangian (second order)

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\alpha\partial_\mu F^{\mu\rho}\partial_\nu F^\nu{}_\rho + J_\mu A^\mu\,. \tag{4.1}$$

Here J_μ is a conserved current (possibly involving spinor fields, although these will not appear explicitly in our discussion), and the higher-derivative term is the most general gauge-invariant quantity quadratic in the gauge field. In the usual fashion, we reduce (4.1) to the equivalent first-order Lagrangian $\hat{\mathcal{L}}$:

$$\hat{\mathcal{L}} = \frac{\psi_\mu\psi^\mu}{2\alpha} - \tfrac{1}{2}F_{\mu\nu}G^{\mu\nu} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu\,, \tag{4.2}$$

$$G_{\mu\nu} = \partial_\mu\psi_\nu - \partial_\nu\psi_\mu\,. \tag{4.3}$$

The Euler-Lagrange equations for $\hat{\mathcal{L}}$ are

$$\psi^\mu = -\alpha\partial_\nu F^{\nu\mu}\,, \quad \partial_\nu(F^{\nu\mu}+G^{\nu\mu}) + J^\mu = 0\,. \tag{4.4}$$

The canonical procedure will be carried out in Coulomb gauge for the A_μ field:

$$\partial_a A_a = 0 \quad (a = 1, 2, 3)$$
$$\Rightarrow A_3 = -\partial_3^{-1}\partial_i A_i \quad (i = 1, 2) \,. \tag{4.5}$$

At the outset, we eliminate the dependent fields A_3, A_0, ψ_0 from the Lagrangian. The equations of motion imply

$$A_0 = (\alpha\Delta - 1)^{-1}\Delta^{-1}(J_0 - \partial_a \partial_0 \psi_a) \,, \tag{4.6}$$

$$\psi_0 = -\alpha\Delta A_0$$
$$= -\alpha(\alpha\Delta - 1)^{-1}(J_0 - \partial_a\partial_0\psi_a) \,. \tag{4.7}$$

Thus, the Lagrangian may be written

$$\hat{\mathcal{L}} = -\frac{1}{2\alpha}\,\psi_0^{\,2} + F_{0a}G_{0a} + \tfrac{1}{2}F_{0a}F_{0a} + \frac{1}{2\alpha}\,\psi_a\psi_a$$
$$-\tfrac{1}{2}F_{ab}G_{ab} - \tfrac{1}{4}F_{ab}F_{ab} + J_a A_a \,, \tag{4.8}$$

where A_3, A_0, ψ_0 are given by (4.5), (4.6), (4.7) and

$$F_{i3} \equiv -\partial_3\left(\delta_{ij} + \frac{\partial_i\partial_j}{\partial_3^{\,2}}\right) A_j \,,$$
$$F_{0i} \equiv \partial_0 A_i - (\alpha\Delta - 1)^{-1}\Delta^{-1}\partial_i(J_0 - \partial_a\partial_0\psi_a) \,,$$
$$F_{03} \equiv -\partial_0\partial_3^{-1}\partial_i A_i \tag{4.9}$$
$$-\partial_3(\alpha\Delta - 1)^{-1}\Delta^{-1}(J_0 - \partial_a\partial_0\psi_a) \,,$$
$$G_{0a} \equiv \partial_0\psi_a + \alpha(\alpha\Delta - 1)^{-1}\partial_a(J_0 - \partial_b\partial_0\psi_b) \,.$$

One now finds for the canonical momenta

$$\pi_i^A = \frac{\delta\hat{\mathcal{L}}}{\delta\partial_0 A_i}$$
$$= G_{0i} + F_{0i} - \partial_3^{-1}\partial_i(G_{03} + F_{03}) \,, \tag{4.10}$$

$$\pi_a^\psi = \frac{\delta\hat{\mathcal{L}}}{\delta\partial_0\psi_a} = F_{0a} \,, \tag{4.11}$$

which may be inverted to solve for the time derivatives of the independent fields

$$\partial_0 A_i = \pi_i^\psi - \Delta^{-1}\partial_i\,\partial_a\pi_a^\psi \,, \tag{4.12}$$

$$\partial_0\psi_a = \left(\delta_{aj} - \frac{\partial_a\partial_j}{\Delta}\right)\pi_j^A - (\delta_{ab} - \alpha\partial_a\partial_b)\pi_b^\psi + \frac{\partial_a}{\Delta}J_0 \,. \tag{4.13}$$

The dependent fields A_0, ψ_0 may also be expressed more simply in terms of canonical momenta:

$$A_0 = -\Delta^{-1}\partial_a\pi_a^\psi \,,$$
$$\psi_0 = \alpha\partial_a\pi_a^\psi \,. \tag{4.14}$$

The calculation of the Hamiltonian is straightforward; the result is

$$\mathcal{H} = -\tfrac{1}{2}\pi_a^\psi(\delta_{ab} - \alpha\partial_a\partial_b)\pi_b^\psi + \pi_i^A\left(\delta_{ia} - \frac{\partial_i\partial_a}{\Delta}\right)\pi_a^\psi - J_0\Delta^{-1}\partial_a\pi_a^\psi - \frac{1}{2\alpha}\psi_a\psi_a + \tfrac{1}{2}F_{ij}G_{ij} + F_{i3}G_{i3}$$
$$+ \tfrac{1}{4}F_{ij}F_{ij} + \tfrac{1}{2}F_{i3}F_{i3} - J_a A_a \,. \tag{4.15}$$

One can now check that the Heisenberg equations generated by this Hamiltonian (together with canonical commutation relations) are equivalent to the Euler-Lagrange equations (4.4), thereby establishing the correctness of our canonical procedure.

The calculation of the generating functional presents no special difficulties. We begin with

$$Z[\mathcal{J}] = \int [dA_i][d\psi_a]\exp\left[i\int\left(\mathcal{J}_i A_i + J_a A_a - \tfrac{1}{4}F_{ab}F_{ab} - \tfrac{1}{2}F_{ab}G_{ab} + \frac{\psi_a\psi_a}{2\alpha}\right)d^4x\right]\,\mathfrak{z}[\partial_0 A_i, \partial_0\psi_a, J_0] \,, \tag{4.16}$$

where

$$\mathfrak{z} \equiv \int [d\pi_i^A][d\pi_a^\psi]\exp\left(i\int\left\{\pi_a^\psi\partial_0\psi_a + \pi_i^A\partial_0 A_i + \tfrac{1}{2}\pi_a^\psi(\delta_{ab} - \alpha\partial_a\partial_b)\pi_b^\psi - \pi_a^\psi\left[\left(\delta_{aj} - \frac{\partial_a\partial_j}{\Delta}\right)\pi_j^A + \frac{\partial_a}{\Delta}J_0\right]\right\}d^4x\right) \,. \tag{4.17}$$

In calculating $\mathfrak{z}$, we perform the π_i^ψ integrations using δ functions arising from the π_i^A integrals; the remaining π_3^ψ integral is Gaussian and can be readily evaluated by completing the square. Thus, we obtain

$$\mathfrak{z} = \exp\left\{i\int\left[\tfrac{1}{2}\partial_0 A_i\left(\delta_{ij} + \frac{\partial_i\partial_j}{\partial_3^{\,2}}\right)\partial_0 A_j + \partial_0\psi_a\,\partial_0 A_a + \tfrac{1}{2}(\partial_a\partial_0\psi_a - J_0)\Delta^{-1}(1 - \alpha\Delta)^{-1}(\partial_b\,\partial_0\psi_b - J_0)\right]d^4x\right\} \,. \tag{4.18}$$

Substituting (4.18) into (4.16) and performing the ψ_a integrations (again Gaussian), we obtain finally

$$Z = \int [dA_i]\exp\left\{i\int[\mathcal{J}_i A_i + \vec{\mathrm{J}}\cdot\vec{\mathrm{A}} + \tfrac{1}{2}A_a\Box(1 - \alpha\Box)A_a + \tfrac{1}{2}J_0\Delta^{-1}(1 - \alpha\Box)^{-1}J_0]\,d^4x\right\} \,, \tag{4.19}$$

where $A_3 \equiv -\partial_3^{-1}\partial_i A_i$. This result can be written in the more familiar form

$$Z=\int [dA_\mu]\delta[\partial_a A_a]\exp\left[i\int (\mathcal{J}_a A_a+J_\mu A^\mu-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}\alpha\partial_\mu F^{\mu\rho}\partial_\nu F^\nu{}_\rho)d^4x\right]$$
$$=\int [dA_\mu]\delta[\partial_a A_a]\exp\left[i\int (\mathcal{J}_a A_a+\mathcal{L})d^4x\right] . \tag{4.20}$$

To restore rotational symmetry we have introduced a source $\mathcal{J}_3$ for A_3. This result corresponds exactly to the FP prescription and establishes the Lorentz and gauge invariance of our theory. We strongly suspect that the validity of the naive FP prescription extends also to second-order *non-Abelian* gauge theories. For example, in gravitation, the addition of terms involving $R_{\mu\nu}R^{\mu\nu}$ and R^2 to the Lagrangian presumably does not alter the Faddeev-Popov ghost structure of the theory with a given gauge-fixing term. This is reasonable since the Faddeev-Popov ghosts arise in the functional formalism from a purely kinematical condition which is independent of the precise form of the generally covariant action.[13]

(b) We turn now to the study of the spectral characteristics of the second-order gauge theory defined by (4.1). It will be seen that the presence of a gauge symmetry does not prevent (and basically has nothing to do with) the development of a spectral pathology completely analogous to that exhibited in the previous section for second-order scalar field theories. One should note that in more complicated second-order gauge theories such as gravity not all the fields need appear with second derivatives. One might then hope to confine the pathology to the unphysical degrees of freedom.[13]

To study the spectrum, we go to the interaction picture in which the field equations are determined by the quadratic part of the Hamiltonian. Specifically,

$$\Box(-\alpha\Box+1)A_a=0 ,$$
$$(-\alpha\Box+1)A_0=0 , \tag{4.21}$$
$$\psi_a=-\alpha(\Box A_a+\partial_a\partial_0 A_0) . \tag{4.22}$$

(4.21) and (4.22) imply the spectral decompositions

$$A_a(x)=\frac{1}{(2\pi)^{3/2}}\int dk\left\{e^{ik\cdot x}\theta(k^0)\sum_{\lambda=1}^{2}\epsilon_a^\lambda(\vec{k})[\delta(k^2)a_\lambda(\vec{k})+\delta(k^2+\alpha^{-1})\mathcal{A}_\lambda(\vec{k})]+\text{H.c.}\right\} , \tag{4.23}$$

where $\epsilon_a^\lambda(\vec{k})$ are the usual polarization vectors satisfying $\epsilon_a^\lambda(\vec{k})k_a=0$,

$$A_0(\vec{x})=\frac{1}{(2\pi)^{3/2}}\int dk\{e^{ik\cdot x}\theta(k^0)\delta(k^2+\alpha^{-1})\mathcal{A}_0(\vec{k})+\text{H.c.}\} , \tag{4.24}$$

$$\psi_a(x)=-\frac{1}{(2\pi)^{3/2}}\int dk\left\{e^{ik\cdot x}\theta(k^0)\left[\sum_{\lambda=1}^{2}\epsilon_a^\lambda(\vec{k})\delta(k^2+\alpha^{-1})\mathcal{A}_\lambda(\vec{k})+\alpha k_a k^0\delta(k^2+\alpha^{-1})\mathcal{A}_0(\vec{k})\right]+\text{H.c.}\right\} . \tag{4.25}$$

The inversion formulas are here

$$a_\lambda(\vec{k})=\frac{i}{(2\pi)^{3/2}}\int d\vec{x}\,e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\times\frac{\overleftrightarrow{\partial}}{\partial t}[A_a(\vec{x},t)+\psi_a(\vec{x},t)]\epsilon_a^\lambda(\vec{k}) , \tag{4.26}$$

$$\mathcal{A}_\lambda(\vec{k})=-\frac{i}{(2\pi)^{3/2}}\int d\vec{x}\,e^{-i(\vec{k}\cdot\vec{x}-\Omega_k t)}\times\frac{\overleftrightarrow{\partial}}{\partial t}\psi_a(\vec{x},t)\epsilon_a^\lambda(\vec{k}) , \tag{4.27}$$

$$\mathcal{A}_0(\vec{k})=\frac{i}{(2\pi)^{3/2}}\frac{1}{|\vec{k}|^2}\int d\vec{x}\,e^{-i(\vec{k}\cdot\vec{x}-\Omega_k t)}\times\frac{\overleftrightarrow{\partial}}{\partial t}\partial_a\pi_a^\psi(\vec{x},t) . \tag{4.28}$$

Here, we have defined $\omega_k\equiv|\vec{k}|$, $\Omega_k\equiv(|\vec{k}|^2+\alpha^{-1})^{1/2}$. In (4.28), the time derivative acting on $\partial_a\pi_a^\psi$ is found from the Heisenberg equations of motion to be

$$\frac{\partial}{\partial t}\partial_a\pi_a^\psi=\alpha^{-1}\partial_a\psi_a . \tag{4.29}$$

The commutation algebra of the $\{a_\lambda, \mathcal{A}_\lambda, \mathcal{A}_0\}$ is now fully determined by the CCR's (canonical commutation relations)

$$[\psi_a(\vec{x},t),\psi_b(\vec{y},t)]=[\psi_a(\vec{x},t),A_b(\vec{y},t)]=[A_a(\vec{x},t),A_b(\vec{y},t)]=0 , \tag{4.30}$$

$$[A_a(\vec{x},t),\partial_0 A_b(\vec{y},t)]=0 ,$$
$$[A_a(\vec{x},t),\partial_0\psi_b(\vec{y},t)]=i\left(\delta_{ab}-\frac{\partial_a\partial_b}{\Delta}\right)\delta^3(\vec{x}-\vec{y}) ,$$
$$[\psi_a(\vec{x},t),\partial_0 A_b(\vec{y},t)]=i\left(\delta_{ab}-\frac{\partial_a\partial_b}{\Delta}\right)\delta^3(\vec{x},-\vec{y}) ,$$
$$[\psi_a(\vec{x},t),\partial_0\psi_b(\vec{y},t)]=-i(\delta_{ab}-\alpha\partial_a\partial_b)\delta^3(\vec{x}-\vec{y}) . \tag{4.31}$$

Use of the inversion formulas in conjunction with these commutation relations yields the following momentum-space algebra:

$$\begin{aligned}[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] &= [a_\lambda(\vec{k}), \mathfrak{a}_{\lambda'}(\vec{k}')] \\ &= [a_\lambda(\vec{k}), \mathfrak{a}_0(\vec{k}')] \\ &= 0\,, \\ [\mathfrak{a}_\lambda(\vec{k}), \mathfrak{a}_{\lambda'}(\vec{k}')] &= [\mathfrak{a}_0(\vec{k}), \mathfrak{a}_0(\vec{k}')] \\ &= [\mathfrak{a}_\lambda(\vec{k}), \mathfrak{a}_0(\vec{k}')] \\ &= 0\,, \\ [a_\lambda(\vec{k}), \mathfrak{a}^\dagger_{\lambda'}(\vec{k}')] &= [\mathfrak{a}_\lambda(\vec{k}), \mathfrak{a}^\dagger_0(\vec{k}')] \\ &= [\mathfrak{a}_0(\vec{k}), a^\dagger_\lambda(\vec{k}')] \\ &= 0\,,\end{aligned} \tag{4.32}$$

$$[a_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{k}')] = 2\omega_k \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\,, \tag{4.33}$$

$$\begin{aligned}[\mathfrak{a}_\lambda(\vec{k}), \mathfrak{a}^\dagger_{\lambda'}(\vec{k}')] &= -2\Omega_k \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\,, \\ [\mathfrak{a}_0(\vec{k}), \mathfrak{a}^\dagger_0(\vec{k}')] &= -2\Omega_k \frac{1}{\alpha|\vec{k}|^2}\delta^3(\vec{k}-\vec{k}')\,.\end{aligned} \tag{4.34}$$

The appearance of the negative sign in (4.34) signals, as before, the loss of at least one of the three crucial requirements, (a) a positive-energy spectrum, (b) a consistent probability interpretation, and (c) unitarity. Apparently, the gauge freedom is of no use in transforming away the troublesome degrees of freedom, in contrast to the situation in certain covariant quantization procedures (e.g., Gupta-Bleuler quantization of QED) for gauge theories.

V. SUMMARY AND CONCLUSIONS

Our study of derivative-coupled field theories has led us to some fairly definite conclusions concerning the restrictions imposed on local Lagrangian field theories by the physical constraints of Lorentz covariance and unitarity.

In the first place, scalar field theories based on first-order Lagrangians (i.e., in which at most first derivatives of fields necessarily appear) will in general generate noncovariant contributions proportional to the divergent quantity $\delta^4(0)$. However, in explicitly covariant regularization schemes, such as Pauli-Villars or dimensional regularization, these contributions (at least to low order in perturbation theory) are seen to vanish, yielding a result in consonance with Matthews's theorem (namely, the statement that vertices and propagators may be read off directly from the Lagrangian).

A study of second-order scalar Lagrangians, quadratic in second derivatives of the fields, enables us to establish Matthews's theorem (and *ipso facto,* Lorentz covariance) for arbitrary *first-order, Pauli-Villars–regulated* theories. This is done (a) by proving Matthews's theorem for a Lagrangian quadratic in second derivatives of the field(s), and (b) by noting the equivalence of this theory to a Pauli-Villars regularization of the corresponding first-order theory obtained by omitting the term quadratic in second derivatives.

On the basis of our work it seems at least plausible that Matthews's theorem actually holds for an arbitrary local, Lorentz scalar Lagrangian built from canonically independent fields (excluding, for example, gauge-symmetric theories), subject only to the restriction on regularization discussed above.

We have also presented a detailed canonical analysis of a second-order Abelian gauge theory—basically, QED with higher derivatives. Here, the appropriate question is clearly the validity of the naive Faddeev-Popov orbit volume prescription for constructing the generating functional, in the presence of higher derivatives. We have shown that this prescription does hold. This suggests that, in more complicated cases (such as quantum gravity) where the canonical procedure is difficult to implement, the Feynman rules for a gauge-invariant theory with higher field derivatives are obtained by modifying the action in the obvious fashion (thereby including additional vertices); the ghost structure persists unaltered.

In the course of our study of second-order theories (both with and without gauge symmetries) quadratic in second derivatives of the fields, we have shown that such theories possess an unfortunate spectral pathology. Specifically, it appears to be impossible to simultaneously maintain (a) unitarity, (b) a positive-energy spectrum, and (c) a consistent probability interpretation. This fundamental spectral restriction on the derivative structure of local Lagrangian field theory appears to have nothing to do with the possible presence of gauge symmetries. It strongly suggests that the recently proposed renormalizable modification of the Einstein Lagrangian for quantum gravity[2] lacks, by virtue of its second-order structure, a sensible physical interpretation. There is still the hope, however, that the apparent violation of unitarity is an artifact of perturbation theory, and that renormalization-group methods might be used to prove the unitarity of the theory nonperturbatively.[14]

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†NSF predoctoral fellow.

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