Bethe's hypothesis and Feynman diagrams: Exact calculation of a three-body scattering amplitude by perturbation theory

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The perturbation series for three-body scattering is discussed and summed to all orders in a nonrelativistic one-dimensional theory with δ-function interaction. The summation is accomplished by analyzing three simultaneous integral equations satisfied by the three-body scattering amplitude and appropriately decomposing and recombining the kernels which appear in these equations. The homogeneous integral terms can be entirely eliminated from one equation in favor of the scattering amplitude itself, yielding an algebraic equation which is easily solved. The three-particle incoming wave function in configuration space is constructed from the scattering amplitude and shown to be that given by Bethe's hypothesis.

I. INTRODUCTION

The one-dimensional N-body problem described by the Hamiltonian

$$H=-\sum_{1}^{N}\frac{\partial^2}{\partial x_i^{\,2}}+2c\sum_{i<j}\delta(x_i-x_j) \tag{1}$$

has been considered by a number of authors.[1,2] The exact S matrix for any number of distinguishable particles has been given by Yang.[1] Such an exact result was made possible by Bethe's[3] hypothesis which provides a complete set of eigenfunctions of (1).[4] A notable feature of Yang's result is that it is analytic near $c=0$. It should therefore be possible to arrive at the same result by the more mundane procedure of summing up all Feynman graphs for N-body scattering. For $N=2$ all graphs are given by powers of the single-loop integral (Fig. 1). The resulting perturbation series of the form $1+B+B^2+\cdots$ is easily summed to give Yang's result. However, for the scattering of more than two particles the general Feynman graph is not easily characterized, and the combinatorical details involved in summing up all graphs are substantially more complex. In view of the extent to which intuition in particle theory is grounded in Feynman graphs, it would be of some interest to demonstrate explicitly the manner in which these graphs combine to yield an exact result, in spite of this model's obvious shortcomings of being nonrelativistic and one-dimensional. In this paper we provide such a demonstration for three-body scattering. In the body of the paper the particles are assumed to be identical bosons, the scattering amplitude being equal to an appropriate symmetrization of Yang's result.[1] The more general case of distinguishable particles is treated in an appendix.

II. EQUIVALENT GRAPHICAL RULES

For the problem of identical bosons, the second-quantized formalism may be employed. The Hamiltonian (1) is equivalent, in the N-particle sector, to

$$H=\int dx\{[\nabla\phi^*(x)][\nabla\phi(x)]+c\phi^*(x)\phi^*(x)\phi(x)\phi(x)\}, \tag{2}$$

where $\phi(x)$ is a quantized boson field. This Hamiltonian corresponds to a Lagrangian density

$$\mathcal{L}=i\phi^*\overleftrightarrow{\partial}_0\phi-(\nabla\phi^*)(\nabla\phi)-c\phi^*\phi^*\phi\phi\,, \tag{3}$$

from which the Feynman rules of the theory may be read off. The value of any graph is given by the rules (a):

(a.1) An energy-momentum integration $\int d\omega\, dq/(2\pi)^2$ for each closed loop.

(a.2) A propagator $i/(\omega-q^2+i\epsilon)$ for each internal line.

This integral is multiplied by a factor $(-4ic)$ for each vertex, the usual symmetry factors for identical lines, and a conventional factor of $(-i)$.

It will prove convenient to reduce the Feynman integral obtained from these rules to a standard form by carrying out all energy integrations. Since the integrand is always a product of simple poles, this can be done by contour integration. The result is always of a form which can be read off directly from the graph by the rules of old-fashioned time-independent perturbation theory. First draw the graph so that particles enter from the bottom and emerge from the top. Then cut the graph with horizontal lines separating each vertex, as, for example, in Fig. 2. An nth-order graph will have $n-1$ cuts. The integral given by the rules (a) is equal to that obtained from (b):

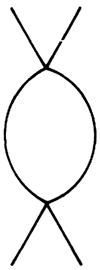

FIG. 1. One-loop diagram for two-body scattering.

(b.1) A momentum integration $\int dq/(2\pi)$ for each closed loop.

(b.2) A factor $i/(E-\sum_l E_l + i\epsilon)$ for each cut, where E is the initial total energy and $\sum_l E_l$ is the sum of energies of the cut lines, all internal lines being evaluated on the mass shell $(E_l = q_l^2)$.

(b.3) If there is more than one way to cut a particular graph, the value of the graph is the sum of all possible cuttings.

This last proviso is relevant to graphs in which the time ordering of the vertices is not fixed by the topology of the graph. This does not occur in a three-particle system, but does arise in four-particle graphs such as Fig. 3.

The rules (b) are just the graphical rules of old-fashioned perturbation theory and perhaps need no amplification. However, to make the discussion self-contained, a proof will be sketched for the three-particle case that the rules (a) and (b) are equivalent graph by graph. First, observe that the equivalence is trivially true for two-particle graphs since the two propagators in Fig. 1 have energy poles in opposite half-planes, and when one is evaluated at the pole of the other, it becomes an energy denominator of the form (b.2). By the same argument, equivalence is established for disconnected three-particle graphs. Now assume that equivalence of (a) and (b) has been established for graphs up to order $(n-1)$ and consider an nth-order connected graph $F^{(n)}$. Such a graph can be represented as in Fig. 4 where $F^{(n-1)}$ is some $(n-1)$th-order graph. If $F^{(n-1)}$ is disconnected, then $F^{(n)}$ is of the form shown in Fig. 5. In this case the equivalence of (a) and (b) is easily seen, since the extra propagator given by rules (a) and the extra energy denominator given by (b) are identical. Finally, if $F^{(n-1)}$ is connected, we must evaluate an integral of the form

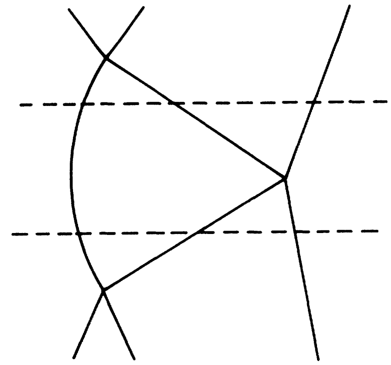

FIG. 2. Cutting scheme for a three-body scattering diagram.

$$\int \frac{d\omega}{(2\pi)}\left(\frac{i}{\omega-q^2+i\epsilon}\right)\left(\frac{i}{E_1+E_2-\omega-(k_1+k_2-q)^2+i\epsilon}\right)F^{(n-1)}(\omega,q;E_1+E_2-\omega,k_1+k_2-q;E_3,k_3)\,, \tag{4}$$

where the arguments of $F^{(n-1)}$ are the energies and momenta of the initial particles for that graph (the dependence of $F^{(n-1)}$ on the final-state variables is irrelevant to the discussion and is suppressed). By the induction hypothesis $F^{(n-1)}$ may be computed by the (b) rules. Since the initial total energy for $F^{(n-1)}$ is independent of ω, the only ω dependence enters when one of the cuts of $F^{(n-1)}$ crosses one of the initial lines labeled ω or $E_1+E_2-\omega$ (recall that all lines internal to the graph $F^{(n-1)}$ are evaluated on the mass shell and can therefore give rise to no ω dependence). But the cuts on $F^{(n-1)}$ can only cross one of these lines, not both, since one or both of them must terminate at the next vertex, beyond which only the momentum and not the energy carried by the line appears in the energy denominators. If either of the lines in Fig. 4 with energies ω and $E_1+E_2-\omega$ is crossed by the cuts of $F^{(n-1)}$, we can, without loss of generality, assume it is the one with energy ω. By looking at the generic energy denominator in (b.2) it is seen that the function $F^{(n-1)}$ in (4) has no singularities in the lower-half ω plane. The integration therefore places the particle with energy ω on the mass shell $(\omega = q^2)$. The line with energy $E_1+E_2-\omega$ was assumed to terminate at the next vertex, and the fact that it is not placed on the mass shell is of no consequence to the computation of $F^{(n-1)}$ by the (b) rules. The second propagator in (4) evaluated at $\omega=q^2$ is just the extra energy denominator which would appear in computing the whole graph $F^{(n)}$ by the (b) rules, and the induction is therefore complete.

There is a minor technicality involved in using the (b) rules to derive integral equations, regarding the difference between continuation of an am-

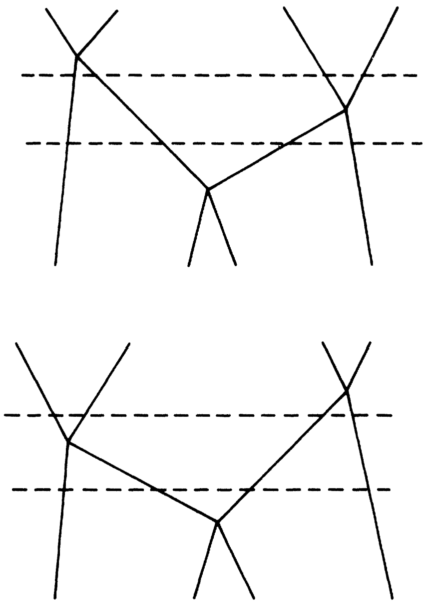

FIG. 3. Two possible cuttings for a four-body scattering diagram.

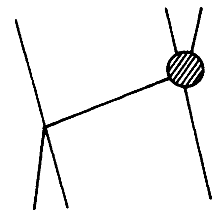

FIG. 5. Diagram for which $F^{(n-1)}$ is disconnected.

plitude off the mass shells of the individual particles with the total energy conserved between initial and final state and continuation off the total-energy shell with the individual particles remaining on mass shell. Strictly speaking, if we remain within the time-dependent formulation of scattering theory and consider integral equations of the Schwinger-Dyson or Bethe-Salpeter type, all amplitudes will conserve total energy and the integrations will take particles off mass shell. However, the integral equations considered in Sec. III, more in the spirit of the time-independent formulation, involve integrations over amplitudes for which the individual particles remain on mass shell, while the total energy goes off shell. Such amplitudes are unambiguously defined by the (b) rules, or equivalently, by

$$-(2\pi)\delta(P_f-P_i)T(f;i)=\langle f|H_1U(0,-\infty)|i\rangle\,, \qquad (5)$$

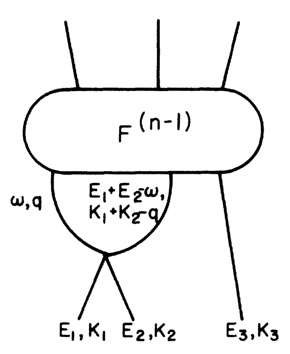

FIG. 4. Integral expression for an nth-order diagram.

where $|f\rangle$ and $|i\rangle$ are eigenstates of H_0, and U is the usual interaction-picture time-development operator. This approach to the integral equations is primarily a matter of convenience, and the connection with the Bethe-Salpeter-type equations is easily seen by considering the treatment of the energy integration in (4). There it was seen that, when the energy contour was closed, the energies of the three particles entering $F^{(n-1)}$ were evaluated at q^2, $E_1+E_2-q^2$, and E_3, and thus the q integration would not take $F^{(n-1)}$ off the total-energy shell. But it was shown that the line with energy $E_1+E_2-q^2$ was the one which terminated at the next vertex, and its energy was therefore irrelevant to the calculation of $F^{(n-1)}$. In deriving integral equations directly from the (b) rules, we are effectively replacing the energy of this particle by its on-shell value. This not only makes the notation easier to deal with, but also allows us to consider the problem of solving the integral equations for (5) even when $E_f\neq E_i$. A knowledge of the scattering amplitude (5) both on and off the total-energy shell is equivalent to a knowledge of the three-particle wave function, which will be calculated in Sec. IV. The corresponding problem in the time-dependent formulation, that of calculating the off-mass-shell Green's functions, appears to be much more difficult and remains unsolved.

III. SUMMING ALL GRAPHS

At this point all the energy integrations for an arbitrary graph can be carried out *ab initio* simply by using the (b) rules. The momentum integrations, on the other hand, involve quadratic denominators and, for individual graphs, become quite forbidding beyond the lowest few orders of perturbation theory. The important simplifications which allow an exact result to be obtained take place only after certain sets of graphs are added together. The necessary combinatoric analysis is most conveniently described by considering integral equations for the scattering amplitude. Here and throughout the discussion T will refer to the full three-body scattering amplitude (connected plus disconnected parts). In Fig. 6 and in subsequent graphical representations of integral equations, the graphs shown actually represent a sum

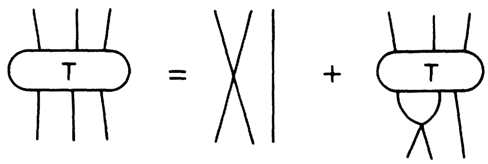

FIG. 6. First integral equation for T.

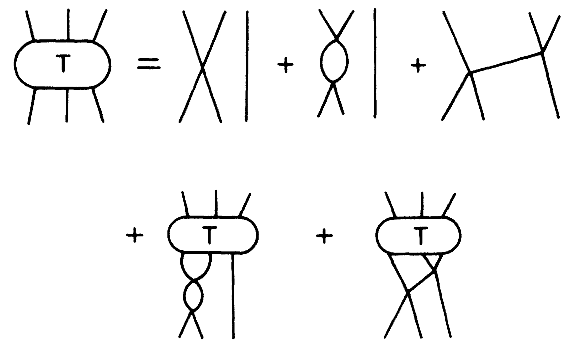

FIG. 7. Second integral equation for T.

of topologically similar graphs over the distinguishable permutations of the external momenta. Hereafter, it will be assumed that all external particle lines are evaluated on mass shell.

Before engaging the details of the argument, it may be worth outlining the essential steps. All together, three related integral equations, which are shown in Figs. 6, 7, and 8, will be used. Assuming all graphs to be calculated by the (b) rules, the kernels which appear in these integral equations are each in the form of a sum of on-mass-shell (but off-energy-shell) graphs multiplied by a single energy denominator arising from the cut which passes between the last explicit vertex and the nondescript blob. For the moment, let us refer to the part of the kernel without the last denominator as the subkernel. The subkernel in the first integral equation, Fig. 6, is the first-order scattering amplitude. Considering the homogeneous terms on the right-hand side of the second equation Fig. 7, as a single integral expression, its subkernel is the full second-order amplitude. These first two equations are eventually employed, not as integral equations, but as identities which allow us, when we encounter a certain integral over T, to substitute an expression involving T itself, without integrations. The denouement is supplied by the third equation, Fig. 8. The subkernel for this equation is the full third-order amplitude which, after some rather remarkable cancellations, can be expressed as a linear combination of the first- and second-order amplitudes, i.e., of the subkernels for the first two equations, plus an extra term which, when inserted in the integrand of Fig. 8, yields an expression that can be integrated out explicitly. Thus, using the first two equations as described, the third becomes a simple algebraic equation for T.

It will often be notationally convenient to regard the momenta of the three particles in the initial, intermediate, or final state as the components of a three-vector. Thus $T(\vec{p};\vec{k})$ denotes the scattering amplitude for a state of momenta (k_1,k_2,k_3) going to a state with (p_1,p_2,p_3) (all particles on mass shell). Many lengthy expressions can be avoided by introducing a symbolic symmetrization over the components of a particular momentum vector, defined on an arbitrary function $f(\vec{p})$ as

$$\mathcal{S}_p f(\vec{p}) = \frac{1}{3!}\sum_A f(A\vec{p}), \qquad (6)$$

where the vector $A\vec{p} = (p_{A_1}, p_{A_2}, p_{A_3})$ and the sum is over all permutations (A_1, A_2, A_3) of $(1,2,3)$. From the rules derived in the last section, the integral equation shown in Fig. 6 is

$$T(\vec{p};\vec{k}) = T^{(1)}(\vec{p};\vec{k}) + \tfrac{1}{2}(-4ic)3\mathcal{S}_k \int \frac{dq}{2\pi}\,\frac{i}{k_1{}^2+k_2{}^2-q^2-(k_1+k_2-q)^2+i\epsilon}\,T(\vec{p};q,k_1+k_2-q,k_3), \qquad (7)$$

where $T^{(1)}$ is the sum of first-order graphs. The factor of $\frac{1}{2}$ multiplying the integral term of (7) is obviously the required extra symmetry factor for those graphs in which the first two vertices form a closed bubble. But graphs for which this is not the case are counted twice when the equation is iterated, and, therefore, the $\frac{1}{2}$ appears as an over-all factor.

The sum of first-order graphs is

$$T^{(1)}(\vec{p};\vec{k}) = (12ic)(2\pi i)\mathcal{S}_p[\delta(p_1-k_1)+\delta(p_2-k_2)+\delta(p_3-k_3)]. \qquad (8)$$

Inserting some δ functions in (7) it can be written

$$T(\vec{p};\vec{k}) = T^{(1)}(\vec{p};\vec{k}) - \frac{1}{6(2\pi)^2}\int d^3q\,\delta(\textstyle\sum q - \sum k)T(\vec{p};\vec{q})\frac{1}{\sum k^2 - \sum q^2 + i\epsilon}\,T^{(1)}(\vec{q};\vec{k}), \qquad (9a)$$

where $\sum k \equiv \sum_{i=1}^{3} k_i$ and similarly for $\sum k^2$. The integral equations of Figs. 7 and 8 are obtained in a similar manner:

$$T(\vec{p};\vec{k})=T^{(1)}(\vec{p};\vec{k})+T^{(2)}(\vec{p};\vec{k})-\frac{1}{6(2\pi)^2}\int d^3q\ \delta(\textstyle\sum q-\sum k)T(\vec{p};\vec{q})\frac{1}{\sum k^2-\sum q^2+i\epsilon}T^{(2)}(\vec{q};\vec{k}), \tag{9b}$$

and

$$T(\vec{p};\vec{k})=T^{(1)}(\vec{p};\vec{k})+T^{(2)}(\vec{p};\vec{k})+T^{(3)}(\vec{p};\vec{k})-\frac{1}{6(2\pi)^2}\int d^3q\ \delta(\textstyle\sum q-\sum k)T(\vec{p};\vec{q})\frac{1}{\sum k^2-\sum q^2+i\epsilon}T^{(3)}(\vec{q};\vec{k}). \tag{9c}$$

Let us consider the connected and disconnected parts of $T^{(2)}$ and $T^{(3)}$ separately, writing $T^{(n)}=T_C^{(n)}+T_D^{(n)}$. The second-order disconnected graphs are calculated by carrying out the loop integration. If the initial momenta are labeled such that

$$k_1<k_2<k_3, \tag{10}$$

this gives

$$T_D^{(2)}(\vec{p},\vec{k})=(12ic)(2\pi i)\mathcal{S}_p[x_{23}\delta(p_1-k_1)+x_{13}\delta(p_2-k_2)+x_{12}\delta(p_3-k_3)], \tag{11}$$

where

$$x_{ij}\equiv ic/(k_i-k_j). \tag{12}$$

The second-order connected graphs are all like the third graph on the right-hand side of Fig. 7. Their sum is

$$T_C^{(2)}(\vec{p};\vec{k})=(-i)(-4ic)^2 9\mathcal{S}_p\mathcal{S}_k\frac{1}{k_1^2+k_2^2-p_1^2-(k_1+k_2-p_1)^2+i\epsilon}. \tag{13}$$

For the particular ordering of initial momenta (10), the energy denominator in (13) factorizes, i.e.,

$$\frac{ic}{k_1^2+k_2^2-p_1^2-(k_1+k_2-p_1)^2+i\epsilon}=\tfrac{1}{2}x_{12}\left(\frac{1}{p_1-k_2-i\epsilon}+\frac{1}{k_1-p_1-i\epsilon}\right). \tag{14}$$

Writing out all the permutations of k in (13) and factorizing each term as in (14) gives

$$T_C^{(2)}(\vec{p};\vec{k})=24ic\mathcal{S}_p\left[x_{12}\left(\frac{1}{p_1-k_2-i\epsilon}+\frac{1}{k_1-p_1-i\epsilon}\right)\right.$$
$$\left.+x_{13}\left(\frac{1}{p_1-k_3-i\epsilon}+\frac{1}{k_1-p_1-i\epsilon}\right)+x_{23}\left(\frac{1}{p_1-k_3-i\epsilon}+\frac{1}{k_2-p_1-i\epsilon}\right)\right]. \tag{15}$$

Finally, we will need the third-order amplitude, represented by the fourth, fifth, sixth, and seventh graphs on the right-hand side of Fig. 8. The disconnected part is easily calculated from the fourth graph, giving

$$T_D^{(3)}(\vec{p};\vec{k})=(12ic)(2\pi i)\mathcal{S}_p[x_{23}{}^2\delta(p_1-k_1)+x_{13}{}^2\delta(p_2-k_2)+x_{12}{}^2\delta(p_3-k_3)]. \tag{16}$$

The third-order connected graphs are of three types,

$$T_C^{(3)}=T_{C_1}^{(3)}+T_{C_2}^{(3)}+T_{C_3}^{(3)}, \tag{17}$$

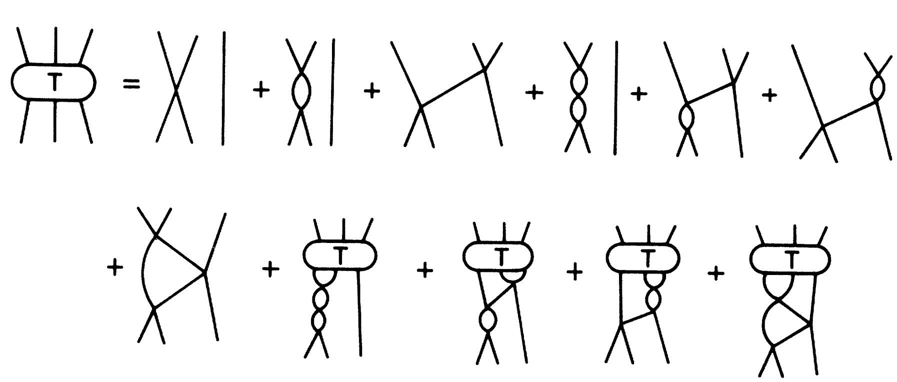

FIG. 8. Third integral equation for T.

where the three terms refer to the three sets of graphs represented by the fifth, sixth, and seventh graphs in Fig. 8, respectively. The calculation of $T_{C_1}^{(3)}$ proceeds in essentially the same way as that of $T_C^{(2)}$, with the bubble contributing an extra factor of x_{ij} and the other energy denominator factorizing as in (14). This gives

$$T_{C_1}^{(3)}(\vec{p};\vec{k}) = 24ic\mathcal{S}_p\left[x_{12}{}^2\left(\frac{1}{p_1-k_2-i\epsilon}+\frac{1}{k_1-p_1-i\epsilon}\right)\right.$$
$$\left.+x_{13}{}^2\left(\frac{1}{p_1-k_3-i\epsilon}+\frac{1}{k_1-p_1-i\epsilon}\right)+x_{23}{}^2\left(\frac{1}{p_1-k_3-i\epsilon}+\frac{1}{k_2-p_1-i\epsilon}\right)\right]. \tag{18}$$

A direct calculation of the other two sets of third-order connected graphs, $T_{C_2}^{(3)}$ and $T_{C_3}^{(3)}$, is fairly straightforward since each involves only a single-loop integration. However, calculating each graph separately only makes the task of combining them more difficult. In particular, the simple relationship between $T_C^{(3)}$ and $T_C^{(2)}$, which is the key to solving the integral equations, is obscured by this approach. The more judicious procedure is to combine the graphs before carrying out the loop integration. For $T_{C_2}^{(3)}$, represented by the sixth graph on the right-hand side of Fig. 8, one of the energy denominators factorizes as in (14) and the other appears in the loop integral

$$T_{C_2}^{(3)}(\vec{p};\vec{k}) = (16ic^2)3\mathcal{S}_p\left\{\left[x_{12}\left(\frac{1}{p_1-k_2-i\epsilon}+\frac{1}{k_1-p_1-i\epsilon}\right)\right.\right.$$
$$\left.+x_{13}\left(\frac{1}{p_1-k_3-i\epsilon}+\frac{1}{k_1-p_1-i\epsilon}\right)+x_{23}\left(\frac{1}{p_1-k_3-i\epsilon}+\frac{1}{k_2-p_1-i\epsilon}\right)\right]$$
$$\left.\times\int\frac{dq}{2\pi}\left(\frac{1}{\sum k^2-p_1{}^2-q^2-(p_2+p_3-q)^2+i\epsilon}\right)\right\}. \tag{19}$$

For $T_{C_3}^{(3)}$, both energy denominators appear under the integral sign, but it is again advantageous to factorize one of them, giving

$$T_{C_3}^{(3)}(\vec{p};\vec{k}) = (32ic^2)3\mathcal{S}_p\left\{\int\frac{dq}{2\pi}\left(\frac{1}{\sum k^2-p_1{}^2-q^2-(p_2+p_3-q)^2+i\epsilon}\right)\right.$$
$$\times\left[x_{12}\left(\frac{1}{q-k_2-i\epsilon}+\frac{1}{k_1-q-i\epsilon}\right)\right.$$
$$\left.\left.+x_{13}\left(\frac{1}{q-k_3-i\epsilon}+\frac{1}{k_1-q-i\epsilon}\right)+x_{23}\left(\frac{1}{q-k_3-i\epsilon}+\frac{1}{k_2-q-i\epsilon}\right)\right]\right\}. \tag{20}$$

Adding (19) and (20) together, we observe that the sum of the terms in the square brackets is just a symmetrization over the object $q=(p_1, q, p_2+p_3-q)$. Thus,

$$T_{C_2}^{(3)}(\vec{p};\vec{k})+T_{C_3}^{(3)}(\vec{p};\vec{k}) = 9(16ic^2)\mathcal{S}_p\left\{\int\frac{d^3q}{2\pi}\left(\frac{1}{\sum k^2-\sum q^2+i\epsilon}\right)\delta(q_1-p_1)\delta\left(\sum q-\sum p\right)\right.$$
$$\times\mathcal{S}_q\left[x_{12}\left(\frac{1}{q_1-k_2-i\epsilon}+\frac{1}{k_1-q_1-i\epsilon}\right)+x_{13}\left(\frac{1}{q_1-k_3-i\epsilon}+\frac{1}{k_1-q_1-i\epsilon}\right)\right.$$
$$\left.\left.+x_{23}\left(\frac{1}{q_1-k_3-i\epsilon}+\frac{1}{k_2-q_1-i\epsilon}\right)\right]\right\}. \tag{21}$$

The remaining energy denominator in (21) cannot be factorized in the simple manner of (14). However, because $\sum p=\sum k$ by over-all momentum conservation and $\sum q=\sum p$ by the δ function in (21), we can write

$$\sum k^2-\sum q^2=(k_1-q_1)[(k_1-k_2)+(q_1-q_2)]+(q_3-k_3)[(k_2-k_3)+(q_2-q_3)], \tag{22}$$

which suggests the very useful identity

$$\mathcal{S}_q\left(\sum k^2-\sum q^2\right)\left[\frac{1}{(k_1-q_1-i\epsilon)(q_3-k_3-i\epsilon)}\right]=\mathcal{S}_q\left[\frac{k_1-k_2}{q_3-k_3-i\epsilon}+\frac{k_2-k_3}{k_1-q_1-i\epsilon}\right]. \tag{23}$$

In (23), the symmetrization over q has eliminated the terms proportional to (q_1-q_2) and (q_2-q_3). In order to make use of this identity in (21), the first factor on the left-hand side must be written as an energy denominator on the right-hand side, and care must be taken that it has an infinitesimal imaginary part of the proper sign. It is easy to see that this sign is determined by the sign of (k_1-k_3), specifically

$$\frac{1}{\sum k^2-\sum q^2+i\epsilon(k_3-k_1)}\mathcal{S}_q\left[\frac{k_1-k_2}{q_3-k_3-i\epsilon}+\frac{k_2-k_3}{k_1-q_1-i\epsilon}\right]=\mathcal{S}_q\left[\frac{1}{(k_1-q_1-i\epsilon)(q_3-k_3-i\epsilon)}\right]. \tag{24}$$

Other identities are obtained by permuting the k's on both sides, the useful ones being those with positive imaginary parts in the energy denominator. Now, the integrand in (21) has the seemingly fortuitous property that it can be written entirely in terms of expressions like the left-hand side of (24), which are then profitably replaced by the more readily integrated right-hand side. Using the fact that

$$\int\frac{d^3q}{2\pi}\,\delta(q_1-p_1)\delta(\textstyle\sum q-\sum p)\mathcal{S}_q\left[\frac{1}{(k_1-q_1-i\epsilon)(q_3-k_3-i\epsilon)}\right]=\frac{i}{6}\left(\frac{1}{k_1-p_1-i\epsilon}+\frac{1}{p_1-k_3-i\epsilon}\right), \tag{25}$$

and permutations thereof, (21) reduces to

$$\begin{aligned}T_{C_2}^{(3)}(\vec{p};\vec{k})+T_{C_3}^{(3)}(\vec{p};\vec{k})=24ic\mathcal{S}_p\bigg[&x_{12}x_{13}\left(\frac{2}{k_1-p_1-i\epsilon}+\frac{1}{p_1-k_2-i\epsilon}+\frac{1}{p_1-k_3-i\epsilon}\right)\\&+x_{13}x_{23}\left(\frac{1}{k_1-p_1-i\epsilon}+\frac{1}{k_2-p_1-i\epsilon}+\frac{2}{p_3-k_3-i\epsilon}\right)\\&+x_{12}x_{23}\left(\frac{1}{k_1-p_1-i\epsilon}+\frac{1}{p_1-k_3-i\epsilon}\right)\bigg].\end{aligned} \tag{26}$$

This is now easily combined with (18) to yield the third-order connected amplitude

$$\begin{aligned}T_C^{(3)}(\vec{p};\vec{k})=24ic\mathcal{S}_p\bigg[&(x_{12}+x_{13})(x_{12}+x_{13}+x_{23})\left(\frac{1}{k_1-p_1-i\epsilon}\right)+x_{12}(x_{12}+x_{13})\left(\frac{1}{p_1-k_2-i\epsilon}\right)\\&+x_{23}(x_{13}+x_{23})\left(\frac{1}{k_2-p_1-i\epsilon}\right)+(x_{13}+x_{23})(x_{12}+x_{13}+x_{23})\left(\frac{1}{p_1-k_3-i\epsilon}\right)\bigg].\end{aligned} \tag{27}$$

Comparing this expression with (15) reveals a striking similarity between the second- and third-order connected amplitudes

$$T_C^{(3)}(\vec{p};\vec{k})=(x_{12}+x_{13}+x_{23})T_C^{(2)}(\vec{p};\vec{k})-24ic\,x_{12}x_{23}\mathcal{S}_p(2\pi i)\delta(p_2-k_2). \tag{28}$$

Inspection of (8), (11), and (16) supplies a relationship among the disconnected amplitudes

$$\begin{aligned}T_D^{(3)}(\vec{p};\vec{k})=&(x_{12}+x_{13}+x_{23})T_D^{(2)}(\vec{p};\vec{k})-(x_{12}x_{13}+x_{12}x_{23}+x_{13}x_{23})T^{(1)}(\vec{p};\vec{k})\\&+(12ic)\mathcal{S}_p(2\pi i)[x_{12}x_{13}\delta(p_1-k_1)+x_{12}x_{23}\delta(p_2-k_2)+x_{13}x_{23}\delta(p_3-k_3)].\end{aligned} \tag{29}$$

Finally, adding (28) and (29) together, we get

$$\begin{aligned}T^{(3)}(\vec{p};\vec{k})=&(x_{12}+x_{13}+x_{23})T^{(2)}(\vec{p};\vec{k})-(x_{12}x_{13}+x_{12}x_{23}+x_{13}x_{23})T^{(1)}(\vec{p};\vec{k})\\&+(12ic)\mathcal{S}_p(2\pi i)[x_{12}x_{13}\delta(p_1-k_1)-x_{12}x_{23}\delta(p_2-k_2)+x_{13}x_{23}\delta(p_3-k_3)].\end{aligned} \tag{30}$$

This equation, along with (9a) and (9b) are sufficient to eliminate the integral term in (9c). Substituting (30) into the integrand of (9c), the first two terms give the same integrals which appear in (9a) and (9b).

Collecting the needed equations in abbreviated form, we write the integral equations (9) as

$$T=T^{(1)}+I_1, \tag{9a$'$}$$

$$T=T^{(1)}+T^{(2)}+I_2, \tag{9b$'$}$$

$$T=T^{(1)}+T^{(2)}+T^{(3)}+I_3, \tag{9c$'$}$$

and (30) is written

$$I_3=\phi_1(x)I_2-\phi_2(x)I_1+J_3, \tag{31}$$

where the ϕ's are the symmetric polynomials in the x variables

$$\phi_1(x)=x_{12}+x_{13}+x_{23}, \tag{32a}$$

$$\phi_2(x)=x_{12}x_{13}+x_{12}x_{23}+x_{13}x_{23}, \tag{32b}$$

$$\phi_3(x)=x_{12}x_{13}x_{23}, \tag{32c}$$

and J_3 is the integral obtained by substituting the last term of (31) into the integrand of (9c).

The only task now remaining is to calculate J_3, which can be done explicitly. Inserting the last term of (31) into the integrand of (9c) and factorizing the energy denominator as in (13) gives

$$J_3 = \frac{-i}{\pi}\,\phi_3(x)\int d^3q\,\delta(\sum q-\sum k)T(\vec{p};\vec{q})\mathcal{S}_q\left[\left(\frac{1}{q_2-k_3-i\epsilon}\right)\delta(q_1-k_1)-\left(\frac{1}{q_3-k_3-i\epsilon}\right)\delta(q_2-k_2)+\left(\frac{1}{q_1-k_2-i\epsilon}\right)\delta(q_3-k_3)\right]$$

$$\equiv \frac{-i}{\pi}\,\phi_3(x)\int d^3q\,\delta(\sum q-\sum k)T(\vec{p};\vec{q})h(\vec{q};\vec{k})\,. \qquad (33)$$

The evaluation of this integral is possible because of a property of $T(\vec{p};\vec{q})$ which is exhibited by a generalized form of the integral equation (9a).

First define a continuation of the scattering amplitude in the initial energy variable, $T_z(\vec{p};\vec{q})$, which is calculated by the (b) rules with the energy denominator in (b.2) replaced by

$$i/(z-\sum_l E_l)\,. \qquad (34)$$

In the time-independent formalism this corresponds to introducing the resolvent operator. The scattering amplitude is obtained by approaching the real axis from above

$$T(\vec{p};\vec{q}) = T_z(\vec{p};\vec{q})|_{z=\Sigma k^2+i\epsilon}\,. \qquad (35)$$

By repeating the steps which led up to the integral equations (9), it is easy to show that T_z satisfies equations of an identical form. In particular, (9a) becomes

$$T_z(\vec{p};\vec{q}) = -\frac{1}{6(2\pi)^2}\int d^3r\left[\frac{1}{z-\sum r^2+i\epsilon}\,T_z(\vec{p};\vec{r})-6(2\pi)^2\delta(p_1-r_1)\delta(p_2-r_2)\right]\delta(\sum r-\sum q)T^{(1)}(\vec{r};\vec{q})$$

$$\equiv \int d^3r f_z(\vec{p};\vec{r})\delta(\sum r-\sum q)T^{(1)}(\vec{r};\vec{q})\,. \qquad (36)$$

Now rewrite (33) with T replaced by T_z,

$$J_{3z} \equiv \frac{-i}{\pi}\,\phi_3(x)\int d^3q\,\delta(\sum q-\sum k)T_z(\vec{p};\vec{q})h(\vec{q};\vec{k})\,, \qquad (37)$$

where

$$J_3 = J_{3z}|_{z=\Sigma k^2+i\epsilon}\,. \qquad (38)$$

When (36) is substituted into (37), the q integrations can be carried out explicitly using (8) for $T^{(1)}$. The result is simple:

$$\int d^3q\,\delta(\sum q-\sum k)T^{(1)}(\vec{r};\vec{q})h(\vec{q};\vec{k}) = i\pi T^{(1)}(\vec{r};\vec{k})\,. \qquad (39)$$

Thus (37) becomes

$$J_{3z} = \phi_3(x)\int d^3r\,\delta(\sum r-\sum q)f_z(\vec{p};\vec{r})T^{(1)}(\vec{r};\vec{k})\,, \qquad (40)$$

and upon letting $z\to\sum k^2+i\epsilon$, we get

$$J_3 = \phi_3(x)T(\vec{p};\vec{k})\,. \qquad (41)$$

Using (9a), (9b), (31), and (41), the integral equation (9c) reduces to an algebraic one

$$[1-\phi_1(x)+\phi_2(x)-\phi_3(x)]T(\vec{p};\vec{k}) = [1-\phi_1(x)+\phi_2(x)]T^{(1)}(\vec{p};\vec{k}) + [1-\phi_1(x)]T^{(2)}(\vec{p};\vec{k})+T^{(3)}(\vec{p};\vec{k})\,. \qquad (42)$$

The solution (42) for $T(\vec{p};\vec{k})$ is valid both on and off the total-energy shell. However, the corresponding expression for the S matrix

$$S(\vec{p};\vec{k}) = \langle p|k\rangle + i(2\pi)^2\delta(\sum p^2-\sum k^2)\times\delta(\sum p-\sum k)T(\vec{p};\vec{k}) \qquad (43)$$

is particularly simple. Here,

$$\langle p|k\rangle = 3!(2\pi)^3\mathcal{S}_p\delta(p_1-k_1)\delta(p_2-k_2)\delta(p_3-k_3)\,. \qquad (44)$$

The calculation of S from T is straightforward, the only subtlety arising in demonstrating that $T_C^{(2)}(\vec{p};\vec{k})$ reduces to a sum of δ functions when $\sum p^2=\sum k^2$. This is most easily seen by first showing that (15) vanishes on the energy shell providing that $p_i\neq k_j$. Thus, the only contribution of $T_C^{(2)}$ to (43) comes from the δ-function parts of the denominators in (15). $T_C^{(2)}$ is therefore purely imaginary on the energy shell and may be replaced by $\frac{1}{2}[T_C^{(2)}-T_C^{(2)*}]$, reducing it to a sum of δ functions. It is then easy to show that (43) becomes

$$S(\vec{p};\vec{k}) = \left(\frac{1+x_{12}}{1-x_{12}}\right)\left(\frac{1+x_{13}}{1-x_{13}}\right)\left(\frac{1+x_{23}}{1-x_{23}}\right)\langle p|k\rangle\,. \qquad (45)$$

Finally, it should be remarked that although (45) was obtained for the particular ordering of initial momenta (10), its validity is easily extended to all orderings by making the replacement

$$x_{ij}\to ic/|k_i-k_j|\,. \qquad (46)$$

Since the scattering amplitude defined by (5) satisfies the integral equations (9) even if $\sum p^2 \neq \sum k^2$, the result (42) is also valid off the total-energy shell. Whereas the S matrix reflects only the asymptotic part of the three-particle wave function, we can use (42) to reconstruct the exact incoming wave function. Using some standard formulas from scattering theory, the wave function can be written in momentum space as

$$\langle p|\Psi^{(+)}(k)\rangle = \langle p|U(0,-\infty)|k\rangle$$
$$= \langle p|k\rangle - \frac{1}{\sum k^2 - \sum p^2 + i\epsilon} \times (2\pi)\delta(\textstyle\sum p - \sum k) T(\vec{p};\vec{k}) \ . \qquad (47)$$

This expression can be reduced to a compact form by introducing the six functions

$$\Phi(A) = (-12\pi)\delta(\textstyle\sum p - \sum k)\mathcal{S}_p \dfrac{1}{(p_1 - k_{A_1} - i\epsilon)(k_{A_3} - p_3 - i\epsilon)} \ , \qquad (48)$$

where (A) is some permutation of (123). The first term in (47) can be written in terms of these functions by the identity

$$\langle p|k\rangle = \sum_A \Phi(A) \ , \qquad (49)$$

where the sum is over all permutations. By taking the solution (42) for the amplitude and again using identities like (24), the second term of (47) can also be written as a linear combination of the functions (48). Combining this result with (49) yields the three-particle incoming wave function in momentum space

$$\langle p|\Psi^{(+)}(k)\rangle = \sum_A C(A)\Phi(A) \ , \qquad (50)$$

where

$$C(123) = 1 \ , \qquad (51a)$$

$$C(132) = \frac{1 + x_{23}}{1 - x_{23}} \ , \qquad (51b)$$

$$C(213) = \frac{1 + x_{12}}{1 - x_{12}} \ , \qquad (51c)$$

$$C(231) = \left(\frac{1 + x_{12}}{1 - x_{12}}\right)\left(\frac{1 + x_{13}}{1 - x_{13}}\right) \ , \qquad (51d)$$

$$C(312) = \left(\frac{1 + x_{13}}{1 - x_{13}}\right)\left(\frac{1 + x_{23}}{1 - x_{23}}\right) \ , \qquad (51e)$$

$$C(321) = \left(\frac{1 + x_{12}}{1 - x_{12}}\right)\left(\frac{1 + x_{13}}{1 - x_{13}}\right)\left(\frac{1 + x_{23}}{1 - x_{23}}\right) \ . \qquad (51f)$$

By noting that the functions $\Phi(A)$ are just the Fourier transforms of a plane wave multiplied by a spatial ordering function

$$\theta(x_{A_1} - x_{A_2})\theta(x_{A_2} - x_{A_3}) \ ,$$

it can be seen that (50) is the Fourier transform of the familiar Bethe wave function.

IV. CONCLUSION

In this paper we have discussed the structure of the perturbation series for three-body scattering in a model described by the Lagrangian (3). While this structure is of interest in itself, it is hoped that the approach initiated here can be extended to consideration of some very interesting and unexplored aspects of this and related models.[4] Some features of the present calculation, such as the appearance of symmetric polynomials in the x_{ij} variables, seem to suggest a natural extension to the N-body problem. Perhaps of greatest interest is the dynamical behavior of finite-density systems.[5,6] The excitations above the ground state of such a system can be described in terms of two kinds of "quasiparticle," and Yang has pointed out that consideration of the thermodynamics of this model reveals that the quasiparticles interact.[4] A detailed description of the quasiparticle dynamics and its relationship to the underlying Lagrangian would be of great interest and might shed light on the behavior of field theories in an abnormal vacuum.

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APPENDIX

The three-body scattering amplitude for distinguishable particles (but still equal masses and coupling constants) can be calculated in a manner entirely similar to the boson case. In this Appendix, $T(\vec{p};\vec{k})$ will denote the amplitude for, e.g., three particles which are distinguished by the labels "red," "white," and "blue" of momenta k_1, k_2, and k_3, respectively, to scatter into a state with momenta p_1, p_2, and p_3, respectively. The Feynman rules are similar except that each graph is constructed out of red, white, and blue lines, and only graphs in which each color forms a continuous line through the graph are allowed. It is not difficult to see that $T(\vec{p};\vec{k})$ satisfies integral equations identical to (9). The structure of the solution to these integral equations is similar to that leading up to Eq. (42), and the result is that $T(\vec{p};\vec{k})$ for distinguishable particles satisfies exactly the same equation, the difference in the result arising from the different form of $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$. The calculation of these latter quantities is straightforward except for one detail which may be worth mentioning. In the calculation of $T_{C_2}^{(3)}+T_{C_3}^{(3)}$ one encounters an integral like Eq. (21) except that the expression in the square brackets (which is just $T_C^{(2)}$) is no longer symmetric in q. The identity (23) is not sufficient to eliminate the extra energy denominator. However, by using identities like

$$(k_1-q_2)(k_1-q_3)-(q_1-k_2)(q_1-k_3)=\tfrac{1}{2}(\sum k^2-\sum q^2) \tag{A1}$$

an over-all $(\sum k^2-\sum q^2)$ can be factored out of $T_C^{(2)}$, canceling the energy denominator and leaving only linear momentum denominators. Some care is required here in the treatment of infinitesimal imaginary parts.

For notational convenience, we introduce the following symbol:

$$(ij;m)\equiv\frac{1}{k_i-p_m-i\epsilon}+\frac{1}{p_m-k_j-i\epsilon}\ , \tag{A2}$$

which has the properties

$$(ij;m)+(kl;m)=(il;m)+(kj;m)\ , \tag{A3}$$

$$(ii;m)=2\pi i\delta(p_m-k_i)\ . \tag{A4}$$

The first- and second-order disconnected amplitudes are

$$T^{(1)}=(2ic)[(11;1)+(22;2)+(33;3)]\ , \tag{A5}$$

and

$$T_D^{(2)}=(2ic)[x_{23}(11;1)+x_{13}(22;2)+x_{12}(33;3)]\ . \tag{A6}$$

The second-order connected amplitude is

$$T_C^{(2)}=(2ic)\{x_{12}[(12;1)+(12;2)]+x_{13}[(13;1)+(13;3)]+x_{23}[(23;2)+(23;3)]\}\ . \tag{A7}$$

After considerable calculation, the third-order amplitude can be written as

$$T^{(3)}=\phi_1T^{(2)}-\phi_2T^{(1)}+(2ic)\{x_{12}x_{13}[(11;1)+(13;3)]-x_{12}x_{23}[(12;1)+(23;3)]+x_{13}x_{23}[(13;1)+(33;3)]\}\ , \tag{A8}$$

where ϕ_i are again the symmetric polynomials (32). The full amplitude T is given in terms of these expressions by Eq. (42).

Once again, the on-shell S matrix assumes a particularly simple form. By using identities like (A1), it is easy to show that both $T^{(2)}$ and $T^{(3)}$ vanish when $\sum k^2=\sum p^2$ unless $p_i=k_j$. Hence, the S matrix is a sum of products of δ functions. Writing

$$\langle p|S|k\rangle=(2\pi)^3\sum_A S(A)\delta(p_{A_1}-k_1)\times\delta(p_{A_2}-k_2)\delta(P_{A_3}-k_3)\ , \tag{A9}$$

the coefficients $S(A)$ are easily computed from (A5)–(A8) and (42). Taking out a common factor

$$S(A)=[(1-x_{12})(1-x_{13})(1-x_{23})]^{-1}\tilde{S}(A)\ , \tag{A10}$$

we get

$$\tilde{S}(123)=1\ , \tag{A11}$$

$$\tilde{S}(132)=x_{23}\ , \tag{A12}$$

$$\tilde{S}(213)=x_{12}\ , \tag{A13}$$

$$\tilde{S}(312)=S(231)=x_{12}x_{23}\ , \tag{A14}$$

$$\tilde{S}(321)=x_{13}+x_{12}x_{13}x_{23}\ , \tag{A15}$$

in agreement with Yang's[1] result for the case of three distinguishable particles.

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