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Random electrodynamics: The theory of classical electrodynamics with classical electromagnetic zero-point radiation

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The theory of classical electrodynamics with classical electromagnetic zero-point radiation is outlined here under the title random electrodynamics. The work represents a reanalysis of the bounds of validity of classical electron theory which should sharpen the understanding of the connections and dinstinctions between classical and quantum theories. The new theory of random electrodynamics is a classical electron theory involving Newton's equations for particle motion due to the Lorentz force, and Maxwell's equations for the electromagnetic fields with point particles as sources. However, the theory departs from the classical electron theory of Lorentz in that it adopts a new boundary condition on Maxwell's equations. It is assumed that the homogeneous boundary condition involves random classical electromagnetic radiation with a Lorentz-invariant spectrum, classical electromagnetic zero-point radiation. The scale of the spectrum of random radiation is set by Planck's constant $\hbar$. In the limit $\hbar \to 0$, the theory of random electrodynamics becomes Lorentz's theory of electrons. Thus, random electrodynamics stands between two well-known theories—traditional classical electron theory with $\hbar = 0$ on the one hand and quantum electrodynamics with its noncommuting operators on the other. The paper discusses the role of boundary conditions in classical electrodynamics, the motivation for choosing a new boundary condition involving classical zero-point radiation, and the assumed random character of the radiation. Also, the implications of the theory of random electrodynamics are summarized, including the detection of zero-point radiation, the calculation of van der Waals forces, and the change of ideas in statistical thermodynamics. In these cases the summary accounts refer to published calculations which yield results in agreement with experiment. The implications of random electrodynamics for atomic structure, atomic spectra, and particle-interference effects are discussed on an order-of-magnitude or heuristic level. Some detailed mathematical connections and some merely heuristic connections are noted between random electrodynamics and quantum theory.

I. INTRODUCTION: CLASSICAL ELECTRON THEORY WITH A NEW BOUNDARY CONDITION

In this paper we present the theory of classical electrodynamics with classical electromagnetic zero-point radiation under the title "random electrodynamics." The basic grounding of the theory within classical electrodynamics is made explicit, along with indications of the connections with quantum theory and descriptions of physical phenomena.

Classical electrodynamics, or more specifically, classical electron theory, describes the behavior of charged point masses in electromagnetic fields. The theory consists of Newton's laws of motion for the point masses and Maxwell's differential equations for the electromagnetic fields, together with boundary conditions on the differential equations. Lorentz chose a particular boundary condition[1] for Maxwell's equations and so obtained a specific theory of electrons. In the early years of the twentieth century his theory successfully described a number of physical phenomena in connection with optical dispersion, the normal Zeeman effect, the Faraday effect, and electrical and magnetic birefringence, but seemed unable to account for new measurements of phenomena such as the blackbody radiation spectrum, molecular specific heats, atomic spectra, and intermolecular forces.

In this paper we wish to emphasize that Lorentz's choice of a boundary condition on Maxwell's equations is not the only choice, and indeed today it seems a poor choice. By changing the boundary condition in classical electron theory over to the presence of homogeneous random radiation with a Lorentz-invariant spectrum, this purely classical electromagnetic theory explains far more phenomena than did Lorentz's original electron theory. And indeed the results of the new theory[2-12] maintain very close connections with the results of quantum electrodynamics. A general program in theoretical physics has been undertaken[6-12] to determine the limits of validity for this new classical electron theory as a description of nature, and to determine exactly its areas of agreement and disagreement with quantum electrodynamics.

Although calculations within random electrodynamics[13] have been published for several years, it is clear that many readers are so steeped in traditional classical electrodynamics that they cannot conceive that Maxwell's equations require a choice of a boundary condition. Other readers apparently feel that the idea of zero-point radiation must involve quanta. Finally there are manuscripts coming to the author's attention which entirely misconstrue the author's ideas of random electrodynamics, adding extraordinary hypotheses on top of mixtures of classical and quantum constructs. It is in hopes of clarifying what is in its fundamentals a very simple theory that we will here describe the basic aspects of random electrodynamics. We first outline the role of the boundary condition on Maxwell's equations within classical electromagnetism. Next we describe the reasons for suspecting that traditional classical electrodynamics does not choose the best possible boundary condition for a theory of nature, and we then determine the new boundary condition to give random classical electrodynamics. Once this boundary condition is fixed, the theory is complete. There is no room for further hypotheses. Thus we proceed to outline the implications of the theory, noting the results of completed calculations. At a few places we have allowed ourselves to speculate on the results of calculations which have not yet been carried out; all speculations are clearly labeled as such. Finally we consider some striking parallels and departures from the results of quantum electrodynamics.

II. BOUNDARY CONDITIONS IN CLASSICAL ELECTRODYNAMICS

A. Basic aspects of classical electron theory

The classical electrodynamics of charged point particles consists of three essential items:

(i) Newton's laws of motion for the particles,

(ii) Maxwell's differential equations for the electromagnetic fields,

(iii) boundary conditions for the differential equations in (i) and (ii) above.

Although each of these three items forms an essential part of the theory, it is only the first two which are scrutinized in monographs on classical electrodynamics.[14] The boundary conditions on the theory go virtually unmentioned. In order to highlight the role of the boundary conditions on the differential equations, we will list the differential equations and the specific boundary conditions corresponding to each of several available classical theories.

Newton's second law involves the Lorentz force

$$m\ddot{\vec{r}} = e\left(\vec{E} + \frac{\dot{\vec{r}}}{c} \times \vec{B}\right) \tag{1}$$

for a particle of mass m and charge e in electric and magnetic fields $\vec{E}$ and $\vec{B}$. Maxwell's equations may be written in terms of scalar and vector potentials Φ and $\vec{A}$, giving the fields as

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\,, \quad \vec{B} = \nabla\times\vec{A}\,. \tag{2}$$

In the Coulomb gauge we have

$$\nabla^2\Phi = -4\pi\rho\,, \quad \Phi \to 0 \text{ as } r \to \infty \tag{3}$$

and

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}^{\perp}\,, \quad \nabla\cdot\vec{A} = 0\,. \tag{4}$$

The solutions for the potentials may be written for Cartesian coordinates in the form

$$\Phi(\vec{r}, t) = \int \frac{\rho(\vec{r}\,', t)}{|\vec{r} - \vec{r}\,'|}\, d^3r' \tag{5}$$

and

$$\vec{A}(\vec{r}, t) = \vec{A}_0(\vec{r}, t) + \int \vec{J}^{\perp}(\vec{r}\,', t')G(\vec{r}, t; \vec{r}\,', t')d^4x'\,, \tag{6}$$

where $G(\vec{r}, t; \vec{r}\,', t')$ is a Green's function for the scalar wave equation, and $\vec{A}_0$ is a transverse solution ($\nabla\cdot\vec{A}_0 = 0$), of the vector wave equation. When G is chosen as the retarded Green's function G_R, then $\vec{A}_0$ is denoted by $\vec{A}^{\text{in}}$; when G is chosen as the advanced Green's function G_A, $\vec{A}_0$ is denoted by $\vec{A}^{\text{out}}$. All the possible choices of Green's function differ from each other by solutions of the homogeneous scalar wave equation.

B. Boundary conditions in several available theories

In his concise summary, Coleman[15] lists three theories of classical electrodynamics corresponding to different choices of boundary condition imposed upon Maxwell's equations. These are (a) traditional electrodynamics, (b) time-reversed electrodynamics, (c) Wheeler-Feynman electrodynamics. In this paper we are proposing a fourth theory, (d) random electrodynamics.

(a) Traditional electrodynamics, $\vec{A}^{\text{in}} = 0$, $G = G_R$. This theory is usually the only one considered in textbooks on classical electrodynamics. The retarded Green's function insures causality. The boundary condition $\vec{A}^{\text{in}} = 0$ specifies that all radiation comes from somewhere at a finite time. Roughly speaking, as we go back to early times, $t \to -\infty$, the universe contains matter but no radiation. This is the form of classical electrodynamics accepted by Lorentz[1] in his *Theory of Electrons*. It seems so natural to most physicists that they are reluctant to consider any alternative. We will see that this classical theory fits nature less well than does random electrodynamics.

(b) Time-reversed electrodynamics, $\vec{A}^{out}=0$, $G=G_A$. This is the time-reversed theory from traditional electrodynamics. Now all radiation is absorbed somewhere at a finite time, and roughly at large times, $t\to+\infty$, the universe consists of matter but no radiation.

(c) Wheeler-Feynman electrodynamics, $\vec{A}^{in}+\vec{A}^{out}=0$, $G=\frac{1}{2}(G_R+G_A)$. Within this theory, radiation is emitted at a finite time only if it is absorbed at some other time. The Green's function involves radiation both forward and backward in time. In the presence of a perfect absorber when $\vec{A}^{out}=0$ (and only in this case), the theory gives the same results as traditional electrodynamics. Wheeler and Feynman[16] were at pains to show that the radiation both forward and backward in time did not violate elementary ideas of causality. The theory avoids problems of radiation reaction and mass renormalization.

(d) Random electrodynamics; $\vec{A}^{in}$ *is random radiation,* $G=G_R$. This theory assumes that the homogeneous solution to Maxwell's equations involves random classical radiation. In the analysis to follow, we will see that it is natural to choose a Lorentz-invariant spectrum of random radiation. The aspect of randomness in the theory is analogous to the randomness assumed in classical statistical mechanics involving the averaging over many microscopic but deterministic degrees of freedom. The theory describes accurately a number of phenomena usually thought to require a quantum description.

III. THEORY OF RANDOM ELECTRODYNAMICS

A. Choosing a boundary condition in classical electron theory

Having seen that a choice of boundary condition is required in classical electron theory, we clearly wish to make the choice which provides the best possible description of nature. Apparently most physicists react to this problem by suggesting that the choice of the retarded Green's function insures causality, and the choice $\vec{A}^{in}=0$ is obvious. However, if there is anything which twentieth century physics has taught us, it is that physical concepts which seem obvious should occasionally be reexamined in a critical fashion.

Is the choice $\vec{A}^{in}=0$ really obvious? Most experiments in classical electrodynamics are conducted in a sea of thermal radiation at room temperature which goes completely unnoticed. A choice for $\vec{A}^{in}$ corresponding to this thermal radiation would be a better approximation to the actual electromagnetic field in such cases; yet usually the choice $\vec{A}^{in}=0$ does not provide any contradiction with experiment. Perhaps a nonzero $\vec{A}^{in}$ may have escaped experimental notice more generally.

The problems of the stability of matter against atomic collapse are a case where traditional classical electron theory, which assumes the boundary condition $\vec{A}^{in}=0$, seems to fail. Can we obtain agreement between theory and experiment by choosing $\vec{A}^{in}\neq 0$? At present the usual semiclassical description of an atom involves electrons fluctuating in position and velocity about a massive charged nucleus. Now a classical Newtonian particle does not fluctuate in velocity unless there is a random force on it. Random radiation would provide such a force, and the choice of $\vec{A}^{in}$ corresponding to random radiation is one of the possibilities available within classical electron theory. It is this possibility which we will continue to explore in this paper.

B. Properties required of random radiation

The heuristic notion of homogeneous radiation $\vec{A}^{in}$ corresponding to random radiation forms an interesting qualitative starting point. The crucial question remains whether there is any choice for such radiation which will provide a logical quantitative description of nature.

What properties should the random radiation possess in order to accord with presently accepted physical ideas? Since the random radiation is regarded as a fundamental property of the theory, it is natural that it should possess the fundamental aspects of what is presently regarded as empty space. It should be homogeneous since no position in space is preferred; it should be isotropic since no direction in space is preferred; and it should be Lorentz-invariant since no inertial frame is preferred. These properties for the random radiation are crucial for our understanding of the stability of the radiation spectrum, and of attempts to observe the radiation experimentally. In order to emphasize the implications of these uniformity hypotheses, we will go back and comment on them individually.

C. Homogeneity and isotropy

Assume for the moment that there is random radiation $\vec{A}^{in}$ present in the universe. When the random radiation falls on a charged dipole oscillator, the oscillator will respond with a random vibration. Now if the random radiation density is not homogeneous in space, then the changes in the spectrum of energy density can be measured by an observer who notes the amplitude of oscillator vibrations at different regions of space. Moreover, an inhomogeneous distribution of radiation would not be stable against scattering by a dipole oscillator; the oscillator vibrations would be larger, thus producing more scattering in regions of high-random radiation density.

An analogous situation would arise if the random radiation $\vec{A}^{in}$ were not isotropic. In this case, one expects forces due to radiation pressure in the particular directions associated with the failure of isotropy. Specifically, suppose $\vec{A}^{in}$ consisted of fluctuating radiation which could be expressed as a superposition of plane waves all moving in the positive x direction and polarized along the y axis. Then a dipole oscillator at the origin of coordinates, oriented along the y axis, would experience an average force pushing the oscillator in the positive x direction. Moreover, the radiation pattern would not be stable because the oscillator would be continually scattering radiation into directions other than that of the positive x axis. In contrast to this situation, if we assume that $\vec{A}^{in}$ is an isotropic spectrum of random radiation, then there will be no average force on an isolated dipole, and also the radiation pattern will be stable against scattering of radiation by an oscillator; that is, a dipole oscillator in isotropic random radiation will not redistribute the radiation so as to give any preferred direction for the radiation—even through the dipole itself has a preferred direction associated with its axis. The calculation proving this stability is given in Appendix B. We will see later that this is an important consideration in connection with atomic spectra.

D. Isotropy in all inertial frames: Lorentz invariance

It is quite possible for a radiation spectrum to be isotropic in one frame of reference K, but to be nonisotropic in a frame K' moving relative to the first. For example, let us assume that the random radiation is isotropic in a certain coordinate frame K, and, for convenience, we assume that the spectrum has a sharp cutoff at a certain frequency ω_c. But then in a coordinate frame K' moving along the x axis with constant velocity $\vec{v}=v\hat{i}$ relative to the first, the radiation is Doppler-shifted, and we will see that some of the random radiation with wave vector $\vec{k}'=-k'\hat{i}$ is at frequencies above ω_c, whereas none of the radiation with wave vector $\vec{k}'=k'\hat{i}$ extends even up to frequencies ω_c. In the moving frame K' the radiation pattern is clearly nonisotropic. In order to avoid the presence of a preferred frame of reference associated with the random radiation $\vec{A}^{in}$, we will hypothesize that the radiation spectrum is isotropic in all inertial frames. This assumption is equivalent to the requirement that the radiation pattern has a Lorentz-invariant spectrum.

In an earlier paper,[6] it was shown that if the random radiation spectrum gives an energy density u,

$$u=\frac{2}{8\pi}\int d^3k\, f(\omega_{\vec{k}}),\quad \omega_{\vec{k}}=c(k_x{}^2+k_y{}^2+k_z{}^2)^{1/2}, \tag{7}$$

so that $f(\omega_{\vec{k}})$ can be taken as the spectral density at wave vector $\vec{k}$ in a given frame K, then in a frame K' moving along the x axis of the first with a velocity $\vec{v}=v\hat{i}$, the energy density u' is given by

$$u'=\frac{2}{8\pi}\int d^3k'\, f'(c\vec{k}'), \tag{8}$$

where the spectral density

$$f'(c\vec{k}')=f(\gamma[\omega_{\vec{k}'}+vk_x'])\gamma^{-1}(1+vk_x'/\omega_{\vec{k}'})^{-1}, \tag{9}$$

with

$$\gamma=\left(1-\frac{v^2}{c^2}\right)^{-1/2}. \tag{10}$$

Thus if the spectral density $f'(c\vec{k}')$ in the moving frame is to be isotropic, we must have $f(\omega_{\vec{k}})$ linear in frequency

$$f(\omega_{\vec{k}})=\text{const}\times\omega_{\vec{k}}, \tag{11}$$

$$\begin{aligned} f'(c\vec{k}') &= \text{const}\times\gamma(\omega_{\vec{k}'}+vk_x')\gamma^{-1}\left(1+\frac{vk_x'}{\omega_{\vec{k}'}}\right)^{-1} \\ &= \text{const}\times\omega_{\vec{k}'}\,. \end{aligned} \tag{12}$$

In this case, and only in this case, the directional dependence introduced by the appearance of vk_x' is removed. Also we notice that the spectrum in the moving frame K' takes the same functional form

$$f'(c\vec{k}')=f(\omega_{\vec{k}'}) \tag{13}$$

as in the initial frame K, corresponding to invariance of the spectrum under a Lorentz transformation.

E. Appearance of Planck's constant $\hbar$

We thus find that the requirement of Lorentz invariance implies a random radiation spectrum for $\vec{A}^{in}$ which is unique up to a multiplicative constant

$$f(\omega_{\vec{k}})=\text{const}\times\omega_{\vec{k}}\,.$$

We will denote this constant for convenience as $\hbar/2\pi^2$,

$$f(\omega_{\vec{k}})=\frac{1}{2\pi^2}\,\hbar\omega_{\vec{k}}, \tag{14}$$

corresponding to an energy of $\frac{1}{2}\hbar\omega$ per normal mode. It turns out that a number of experimental results are adequately represented by a value

$$\hbar=1.0545\times10^{-27}\text{ erg sec}.$$

In other words, the value for the one undetermined constant appearing in the boundary conditions on the theory is related to Planck's constant $h=2\pi\hbar$.

Planck's constant thus appears in our theory as the number determining the scale of the incoming radiation. This is the one place that the constant $\hbar$ is put into the theory. There is no other. Every further appearance of Planck's constant is derived from this role through classical electromagnetic calculations.

F. Traditional classical limit $\hbar \to 0$

From the development given here, it is clear that when Planck's constant vanishes, $\hbar \to 0$, we recapture traditional classical electromagnetic theory. The limit $\hbar \to 0$ means that the random radiation in the universe vanishes, $\vec{A}^{\text{in}} = 0$, corresponding exactly to the boundary condition on Maxwell's equations assumed in traditional classical electromagnetism. [See theory (a) of Sec. II B.] Moreover, since the theory with random radiation, $\vec{A}^{\text{in}} \neq 0$, and also the traditional theory, $\vec{A}^{\text{in}} = 0$, are both classical theories involving the same notions for particles, forces, and fields, there is no change in the conceptual framework at $\hbar = 0$. When $\hbar = 0$, our theory becomes the classical electron theory of Lorentz.

G. Formula for the random radiation

The assumptions regarding the random radiation $\vec{A}^{\text{in}}$ can now be incorporated in an explicit mathematical expression. In the Coulomb gauge, the random radiation is a transverse solution of the homogeneous vector wave equation. As such, it can be expressed as a linear superposition of plane waves. Including the assumptions of homogeneity, isotropy, and Lorentz invariance,

$$\vec{A}^{\text{in}}(\vec{r}, t) = \sum_{\lambda=1}^{2} \int d^3k \, \hat{\epsilon}(\vec{k}, \lambda) \frac{c}{\omega} \mathfrak{h}(\vec{k}, \lambda) \times \sin[\vec{k}\cdot\vec{r} - \omega t + \theta(\vec{k}, \lambda)] , \quad (15)$$

where $\hat{\epsilon}(\vec{k}, \lambda)$ is a unit polarization vector referring to wave vector $\vec{k}$ and polarization λ,

$$\hat{\epsilon}(\vec{k}, \lambda) \cdot \hat{\epsilon}(\vec{k}, \lambda') = \delta_{\lambda\lambda'} , \quad \vec{k} \cdot \hat{\epsilon}(\vec{k}, \lambda) = 0 \quad (16)$$

the number $\mathfrak{h}(\vec{k}, \lambda)$ sets the scale as

$$\pi^2 \mathfrak{h}^2(\vec{k}, \lambda) = \tfrac{1}{2} \hbar \omega_{\vec{k}} , \quad (17)$$

and $\theta(\vec{k}, \lambda)$ is a random phase which will be discussed below. Rather than working with the vector potential, it is just as convenient for most calculations to deal with the electric and magnetic fields

$$\vec{E}^{\text{in}} = -\frac{1}{c} \frac{\partial \vec{A}^{\text{in}}}{\partial t} , \quad \vec{B}^{\text{in}} = \nabla \times \vec{A}^{\text{in}} , \quad (18)$$

$$\vec{E}^{\text{in}}(\vec{r}, t) = \sum_{\lambda=1}^{2} \int d^3k \, \hat{\epsilon}(\vec{k}, \lambda) \mathfrak{h}(\vec{k}, \lambda) \times \cos[\vec{k}\cdot\vec{r} - \omega t + \theta(\vec{k}, \lambda)] , \quad (19)$$

$$\vec{B}^{\text{in}}(\vec{r}, t) = \sum_{\lambda=1}^{2} \int d^3k \, \frac{\vec{k} \times \hat{\epsilon}(\vec{k}, \lambda)}{k} \mathfrak{h}(\vec{k}, \lambda) \times \cos[\vec{k}\cdot\vec{r} - \omega t + \theta(\vec{k}, \lambda)] . \quad (20)$$

H. Meaning of randomness of the classical radiation

In the analysis above we have dealt with only one of the two points which contemporary physicists find hard to accept in a classical electron theory with random classical radiation. We have emphasized that classical electrodynamics indeed requires a choice of homogeneous boundary condition for Maxwell's equations, and that this boundary condition should be chosen so as to best fit the physical world. The second troublesome point to which we will now turn involves the random character of the radiation $\vec{A}^{\text{in}}$.

Today many physicists apparently feel that if radiation involves fluctuations, then it must be quantum mechanical in nature. It is interesting that this difficulty did not hold for the researchers on thermal radiation before 1900. To Planck and other nineteenth century researchers, thermal radiation was classical electromagnetic radiation; it was also random radiation. The situation was analogous to that for mechanical energy. The thermal energy of a gas was classical particle motion; it was also random motion when viewed on a macroscopic scale. To be sure, on a microscopic scale the particle motions were completely deterministic.

In the present paper we are dealing with random classical electromagnetic radiation, in the same manner as the nineteenth century researchers dealt with thermal radiation. The treatment outlined below follows that of Planck,[17] of Einstein and Hopf,[18] and of Rice.[19] Suppose we take a long but finite time interval $[0, T]$ and consider during this interval a random function such as the x component of the electric field $E_x(\vec{r}, t)$ at some given point in space, for example, $\vec{r} = 0$. The electric field can be expressed as a Fourier series in time

$$E_x(0, t) = \sum_n (a_n \cos\omega_n t + b_n \sin\omega_n t) , \quad (21)$$

where the frequencies are

$$\omega_n = \frac{2\pi}{T} n = n \Delta\omega .$$

If we chose other intervals of time $[t_1, t_1 + T]$, $[t_2, t_2 + T]$, etc., all of the length T, then we expect to find not unique values for a_n and b_n but rather a distribution of values. We assume that the a_n and b_n are independent and are distributed normally about zero with a standard deviation $[f(\omega_n)\Delta\omega]^{1/2}$, where $f(\omega_n)$ is associated with the energy spectrum of the radiation. The sine and cosine terms are related by a shift in the origin of time t, and hence we expect a_n and b_n to have the same distributions. An alternative description involves a random phase notation. Here the Fourier expansion is written

$$E_x(0, t) = \sum_n c_n \cos(\omega_n t - \phi_n), \tag{22}$$

where c_n is a number equal to $\sqrt{2}$ times the standard deviation of a_n and b_n,

$$c_n = [2f(\omega_n)\Delta\omega]^{1/2},$$

and ϕ_n **is a random phase distributed uniformly** over $(0, 2\pi)$. We are interested in the case $\Delta\omega \to 0$, $T \to \infty$ where the frequency spectrum becomes continuous, and the Fourier series become Fourier integrals.

In the case of random radiation, we carry out not a time Fourier series but an expansion in plane waves in a cubic region of side L,

$$\vec{E}(\vec{r}, t) = \sum_{\lambda=1}^{2} \sum_{\vec{k}} [a(\vec{k}, \lambda)\hat{\epsilon}(\vec{k}, \lambda)\cos(\vec{k}\cdot\vec{r} - \omega t) + b(\vec{k}, \lambda)\hat{\epsilon}(\vec{k}, \lambda)\sin(\vec{k}\cdot\vec{r} - \omega t)], \tag{23}$$

$$\vec{k} = \hat{\imath}\,\frac{2\pi}{L}\,l + \hat{\jmath}\,\frac{2\pi}{L}\,m + \hat{K}\,\frac{2\pi}{L}\,n, \quad l, m, n = 0, \pm 1, \pm 2, \ldots \tag{24}$$

where $a(\vec{k}, \lambda)$ and $b(\vec{k}, \lambda)$ are independent random variables normally distributed about zero with the same standard deviation. Alternatively we write

$$\vec{E}(\vec{r}, t) = \sum_{\lambda=1}^{2} \sum_{\vec{k}} c(\vec{k}, \lambda)\hat{\epsilon}(\vec{k}, \lambda) \times \cos[\vec{k}\cdot\vec{r} - \omega t + \theta(\vec{k}, \lambda)], \tag{25}$$

with $\vec{k}$ again given as in (24), and the $\theta(\vec{k}, \lambda)$ are independent random variables uniformly distributed on $(0, 2\pi)$. Here again $c(\vec{k}, \lambda)$ is a number equal to $\sqrt{2}$ times the standard deviation of $a(\vec{k}, \lambda)$ and of $b(\vec{k}, \lambda)$.

In order to obtain average values for the fields $\vec{E}$ and $\vec{B}$, we average over the distribution of the random variables. For example, in the notation of Eq. (25) we have

$$\langle \vec{E}(\vec{r}, t)\rangle = \sum_{\lambda=1}^{2} \sum_{\vec{k}} c(\vec{k}, \lambda)\hat{\epsilon}(\vec{k}, \lambda) \times \langle \cos[\vec{k}\cdot\vec{r} - \omega t + \theta(\vec{k}, \lambda)]\rangle = 0. \tag{26}$$

Here the average over the uniformly distributed random phase $\theta(\vec{k}, \lambda)$ is simply

$$\langle \cos[\vec{k}\cdot\vec{r} - \omega t + \theta(\vec{k}, \lambda)]\rangle = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \cos(\vec{k}\cdot\vec{r} - \omega t + \theta)\,d\theta = 0. \tag{27}$$

The correlation function $\langle \vec{E}(\vec{r}_1, t_1)\vec{E}(\vec{r}_2, t_2)\rangle$ can be evaluated in analogous fashion,

$$\langle E_i(\vec{r}_1, t_1)E_j(\vec{r}_2, t_2)\rangle = \sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\sum_{\vec{k}_1}\sum_{\vec{k}_2} c(\vec{k}_1, \lambda_1)c(\vec{k}_2, \lambda_2) \times \epsilon_i(\vec{k}_1, \lambda_1)\epsilon_j(\vec{k}_2, \lambda_2)\langle \cos[\vec{k}_1\cdot\vec{r}_1 - \omega_1 t_1 - \theta(\vec{k}_1, \lambda_1)]\cos[\vec{k}_2\cdot\vec{r}_2 - \omega_2 t_2 - \theta(\vec{k}_2, \lambda_2)]\rangle. \tag{28}$$

Now if $\vec{k}_1 \neq \vec{k}_2$ or $\lambda_1 \neq \lambda_2$, then $\theta(\vec{k}_1, \lambda_1)$ and $\theta(\vec{k}_2, \lambda_2)$ are distributed independently and

$$\begin{aligned}\langle \cos[\vec{k}_1\cdot\vec{r}_1 - \omega_1 t_1 - \theta(\vec{k}_1, \lambda_1)]\cos[\vec{k}_2\cdot\vec{r}_2 - \omega_2 t_2 - \theta(\vec{k}_2, \lambda_2)]\rangle &= \frac{1}{2\pi}\int_{\theta_1=0}^{2\pi} d\theta_1 \cos(\vec{k}_1\cdot\vec{r}_1 - \omega_1 t_1 - \theta_1)\,\frac{1}{2\pi}\int_{\theta_2=0}^{2\pi} d\theta_2 \cos(\vec{k}_2\cdot\vec{r}_2 - \omega_2 t_2 - \theta_2)\\ &= 0.\end{aligned} \tag{29}$$

On the other hand, if $\vec{k}_1 = \vec{k}_2$ and $\lambda_1 = \lambda_2$, then the average over the random phase involves

$$\begin{aligned}\langle \cos[\vec{k}_1\cdot\vec{r}_1 - \omega_1 t_1 - \theta(\vec{k}_1, \lambda_1)]\cos[\vec{k}_1\cdot\vec{r}_2 - \omega_1 t_2 - \theta(\vec{k}_1, \lambda_1)]\rangle &= \frac{1}{2\pi}\int_{\theta_1=0}^{2\pi} d\theta_1 \cos(\vec{k}_1\cdot\vec{r}_1 - \omega_1 t_1 - \theta_1)\cos(\vec{k}_1\cdot\vec{r}_2 - \omega_1 t_2 - \theta_1)\\ &= \tfrac{1}{2}\cos[\vec{k}_1\cdot(\vec{r}_1 - \vec{r}_2) - \omega_1(t_1 - t_2)].\end{aligned} \tag{30}$$

Thus the correlation of the random electric field becomes

$$\langle E_i(\vec{r}_1, t_1)E_j(\vec{r}_2, t_2)\rangle = \sum_{\lambda=1}^{2}\sum_{\vec{k}} \tfrac{1}{2}c^2(\vec{k}, \lambda)\epsilon_i(\vec{k}, \lambda)\epsilon_j(\vec{k}, \lambda)\cos[\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) - \omega(t_1 - t_2)]. \tag{31}$$

All further averages over random phases can be carried out in the same style indicated here. In general we are interested in the limit of a Fourier transform where the sum over discrete wave vectors $\sum_{\vec{k}}$ is taken to the limiting integral $\int d^3k$. Any ambiguities in the integral expression can be removed by carrying out the averages in the discrete summation before proceeding to the continuous limit.

We should emphasize again that we are employing the traditional mathematical techniques for handling random classical electromagnetic radiation. Up to this point, our mathematical analysis could be interpreted as a description of nineteenth century ideas of classical thermal radiation. It is only the choice of a Lorentz-invariant spectrum of radiation as in Eq. (17) which distinguishes our work from the nineteenth century calculations. Physically, the randomness in our work is purely formal, analogous to that of traditional classical statistical mechanics where the microscopic behavior is regarded as deterministic.

I. Origin and stability of the zero-point radiation spectrum

We have now discussed at some length the random radiation which we proposed to introduce into classical electron theory as a boundary condition on Maxwell's equations. We refer to this fluctuating radiation as "classical electromagnetic zero-point radiation," emphasizing that the radiation is not thermal radiation but persists at zero temperature.

Readers of this series of papers[6-12] involving classical electrodynamics with classical electromagnetic zero-point radiation sometimes demand to know the origin of this random radiation. On one level this question needs no answer. The assumption of classical zero-point radiation is a postulate of the theory which stands or falls with the predictions derived from the theory. Asking the origin of the zero-point radiation in the universe is a teleological question comparable to inquiring after the origin of the matter in the universe.

However, on another level the question involves significant physical ideas connected with the stability of a collection of charged particles and radiation. Suppose a jumble of charged particles were moving under mutual electromagnetic interactions. If the particles were present inside a finite container, then one might expect the particles to interact, to radiate, and eventually to come to an equilibrium situation of maximum disorder involving a distribution of kinetic energy and radiation described by some temperature T. However, if the particles were not confined to a finite container, they would presumably continue indefinitely in an evolution toward a system of increasing disorder. It seems natural that zero-point radiation should be connected with ideas of maximum disorder for electromagnetic systems not confined to a container. Clearly a random radiation spectrum which is homogeneous, isotropic, and Lorentz-invariant conforms to our ideas of radiation at the greatest entropy in the sense of greatest disorder. Moreover, as shown in Appendix B, the zero-point radiation spectrum is stable against scattering by a dipole oscillator. An oscillator radiates as much energy into waves characterized by a wave vector $\vec{k}$ as it absorbs from waves at wave vector $\vec{k}$. This stability is not found for other distributions of random radiation.

IV. PHYSICAL IMPLICATIONS OF RANDOM ELECTRODYNAMICS

A. Direct observation of classical zero-point radiation

The theory of random electrodynamics proposed here seems markedly different from presently accepted theories, and one might expect it to be easy to obtain experimental evidence contradicting the idea of classical electromagnetic zero-point radiation. Two immediate suggestions for observing zero-point radiation involve being able to see the radiation with one's eyes or to detect it with some instrument designed to measure thermal radiation. The failure to detect such temperature-independent radiation forms the basis for the usual statement that $\vec{A}^{\text{in}} = 0$ is obvious in classical electrodynamics.

However, the situation is seen to be vastly more complicated when one realizes that random electrodynamics is intended as a theory of atomic structure. Thus as outlined below in Sec. IV D, zero-point radiation is expected to provide the random forces which prevent electrons from falling into nuclei. One does not observe the zero-point radiation which maintains the structure of the molecules in the eye or of a mechanical detector, but only the radiation above the zero-point background.

When dealing with mechanical detectors, we should differentiate between those intended for observation of coherent radiation and those for incoherent radiation. Any mechanical detector which depends upon the coherent motion of many electrons will not detect zero-point radiation because of the random nature of the radiation. Thus a very small amount of coherent radiation will cause an observable oscillating current in a circuit, whereas the random radiation will not. Indeed, it has been shown[2,6] that dipole-oscillator systems pick up energy from zero-point radiation giving oscillations of exactly the magnitude which are usually assigned to quantum fluctuations. The second type of mechanical detector, that intended for the observation of incoherent radiation such as thermal radiation, depends upon a change in density of random radiation at different points in space. Zero-point radiation is homogeneous, isotropic, and Lorentz-invariant, and hence one cannot obtain a net transfer of zero-point radiation between two points in space. One cannot expect to observe zero-point radiation with this type of detector.

Thus it turns out that the direct observation of classical electromagnetic zero-point radiation is not a trivial task. The assumption $\vec{A}^{\text{in}}=0$ of traditional classical electron theory is not obvious. Indeed, in the sections to follow we will find that classical electromagnetic zero-point radiation provides the basis for an explanation of a number of phenomena not covered by Lorentz's theory of electrons.

B. Van der Waals forces in random electrodynamics

1. Forces between macroscopic objects

The zero-point radiation spectrum (19) holds in vacuum corresponding to a homogeneous Lorentz-invariant situation. If matter is present the zero-point radiation will interact with the matter and hence give rise to a changed random radiation pattern. Now the changed zero-point radiation pattern causes average forces between objects. These forces can be calculated in classical electromagnetism,[4,5,10-13] and turn out to agree with the Van der Waals force calculations of quantum theory.

As an elementary illustration of the modification of the zero-point radiation pattern by matter and the consequent forces, we will consider the presence of some conducting parallel plates. If one plate lies in the xy plane, then the zero-point radiation pattern is not that of (19), but rather must include the wave reflected from the conductor, giving a zero-point electric field $\vec{E}_{\text{ZPR}}$ for $z>0$,

$$\vec{E}_{\text{ZPR}}(\vec{r},t)=\sum_{\lambda=1}^{2}\int_{k_z<0} d^3k\,\mathfrak{h}(\vec{k},\lambda)2[\hat{K}\epsilon_z\cos k_z z\cos(k_x x+k_y y-\omega t+\theta) -(\hat{i}\epsilon_x+\hat{j}\epsilon_y)\sin k_z z\sin(k_x x+k_y y-\omega t+\theta)]\,, \tag{32}$$

and a corresponding magnetic field $\vec{B}_{\text{ZPR}}$. In the region $z<0$, we take $k_z>0$ in the integral over wave vectors. If a second conductor occupies the plane $z=d$, then in order to meet the boundary conditions on the fields at both planes, we must have a restriction to the values

$$k_z=\frac{\pi n}{d}\,,\quad n=0,1,\ldots \tag{33}$$

in Eq. (32). However, now we can imagine taking a surface surrounding one of the two plates and evaluating Maxwell's stress-energy tensor T over this surface.[20] Thus the average force on the plate is

$$\langle\vec{F}\rangle=\left\langle\oint\vec{T}\cdot\hat{n}\,da\right\rangle, \tag{34}$$

where we average over the random phases as described in Sec. III H.

This integral has been evaluated[4] and gives

$$F_z=-\frac{\pi^2\hbar c\mathfrak{A}}{240d^4}\,, \tag{35}$$

where $\mathfrak{A}$ is the area of each of the plates. Thus, because the plates modify the zero-point radiation pattern, there is a force on the plates. The force is known as the Casimir effect after Casimir, who first predicted and calculated the effect[21] using ideas of quantum electromagnetic zero-point energy. The force (35) has been measured experimentally.[22]

The basic idea here, that zero-point radiation interacts with matter and so leads to forces on macroscopic objects, has been extended by Henry and Marshall[5] to forces between dielectric plates. Also the calculation of the stresses on a spherical conducting shell has been carried out.[23] The present calculation in the literature is phrased in terms of quantum electromagnetic zero-point energy, but the procedure for using classical electromagnetic zero-point radiation is immediate and provides the same answer. The calculations of Van der Waals forces between other macroscopic objects are limited only by the difficulty of the mathematics.

2. Forces between microscopic objects

Microscopic objects with electromagnetic interactions also modify the pattern of zero-point radiation and hence also experience average forces. Indeed, the physical description of forces between microscopic polarizable objects in random electrodynamics comes very close to the semiclassical descriptions[24] of Van der Waals forces presented in quantum mechanics. Moreover, the classical calculations[10-12] for these forces in random electrodynamics are in exact agreement with those of quantum electrodynamics.

In the simplest Drude-Lorentz approximation, an atom or molecule is represented by a charged harmonic oscillator of damping constant

$$\Gamma = \frac{2}{3}\frac{e^2}{mc^3} \tag{36}$$

and natural frequency ω_0. In traditional classical electrodynamics, the oscillator would have radiated away any initial energy, leaving the dipole moment zero. However, the semiclassical description[24] following from quantum mechanics suggests that spontaneous quantum fluctuations set the oscillator into random vibration. The vibrating oscillations for different particles interact and so produce forces.

The description given by random electrodynamics fits this semiclassical view precisely, and further allows exact calculations of the forces. Thus the oscillator equation of motion in random electrodynamics is

$$m\ddot{x} = -m\omega_0^2 x + \frac{2}{3}\frac{e^2}{c^3}\dddot{x} + e(E_{\mathrm{other}\,x} + E_{\mathrm{ZP}\,x}), \tag{37}$$

where classical renormalization[15,25] has been used to include the self-interaction of the oscillator. Here m is the (renormalized) physical mass, $-m\omega_0^2 x$ is the elastic restoring force, $\frac{2}{3}(e^2/c^3)\dddot{x}$ is the radiation-reaction force appearing from the renormalization, $\vec{E}_{\mathrm{other}}$ is the retarded electric field due to all the other oscillators, and $\vec{E}_{\mathrm{ZP}}$ is the random zero-point radiation field (19). The motions of several oscillators are coupled together through the fields $\vec{E}_{\mathrm{other}}$ and through the correlations of the fluctuations of $\vec{E}_{\mathrm{ZP}}$. The classical force on the dipole oscillator is

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E} + \frac{1}{c}\dot{\vec{p}}\times\vec{B}, \tag{38}$$

where $\vec{p} = ex\hat{i}$ is the electric dipole moment. The force $\vec{F}$ is clearly a fluctuating force with an average value $\langle\vec{F}\rangle$ found by averaging over the random phases in the zero-point radiation. The set of coupled linear differential equations (37) for the oscillators can be evaluated exactly, and the average force $\langle\vec{F}\rangle$ calculated in random electrodynamics.[12] The results agree precisely with those of nonrelativistic quantum electrodynamics where an exact solution is also possible.[25]

It is interesting that within classical theory it is natural to work directly with the force $\vec{F}$, whereas in quantum theory the potential U is first evaluated and the the force obtained as

$$\vec{F} = -\nabla U.$$

Although a number of authors have suggested that the concept of force is disappearing from physics because of quantization, this concept retains its traditional classical role within random electrodynamics.

C. Statistical thermodynamics within random electrodynamics

1. *Failure of traditional ideas of classical statistical mechanics*

Although the ideas of random electrodynamics fit beautifully with semiclassical descriptions usually advanced for obtaining a heuristic picture of Van der Waals forces, such a meshing with presently accepted views does not occur in statistical thermodynamics. Rather, random electrodynamics requires an entirely new perspective on the problems of thermal energy.

The traditional classical statistical mechanics developed before 1900 assumes that the heat energy of mechanical systems can be treated while ignoring the interaction with radiation. The thermal behavior of radiation is to be added as an afterthought. Today every student of elementary physics knows that this attempt to add classical electromagnetic interactions to statistical mechanics was an abysmal failure, leading directly to Planck's introduction of the idea of quanta in connection with the blackbody radiation spectrum.

Within the context of random electrodynamics it is obvious that any such attempt to deal separately with radiation and with mechanical systems is doomed to failure because there is temperature-independent random radiation at the absolute zero of temperature. All of the physical systems considered in random electrodynamics are assumed to have classical electromagnetic interactions, and hence all are in random motion at $T=0$. We thus find that random motion is not necessarily thermal motion, and it follows that all of the traditional proofs of the energy equipartition theorem in classical statistical mechanics collapse in random electrodynamics.

2. *New ideas of statistical thermodynamics*

The development[6-9] of the ideas of statistical thermodynamics within random electrodynamics involves some subtlety. However, the main point to be emphasized is just that familiar from pure thermodynamics—that only changes in internal energy and changes in entropy are important. Thus, for example, a zero-temperature array of harmonic oscillators in random electrodynamics already has zero-point energy, and also a random distribution in phase space which can be associated with disorder and entropy. Thermodynamics then involves heat energy, which is random energy above the zero-point oscillator energy, and involves (caloric) entropy, which is disorder above the zero-point disorder.

It is easy to see where the results of statistical thermodynamics within random electrodynamics go over to those of traditional statistical mechanics. In any case where the particle zero-point energy makes a negligible contribution to the total random energy, the results approach the traditional classical values. Thus if the group of oscillators mentioned above is at high temperatures, the zero-point energy is negligible and the energy distribution becomes that of traditional equipartition. Also for massive particles, the zero-point energy of a particle in a box can be shown to be small, and hence the mechanical energy distribution becomes that of equipartition. Indeed, one finds that the conditions giving agreement and departure with traditional statistical mechanics are just those where quantum theory gives agreement and departure.

3. *Derivation of the blackbody radiation spectrum, etc.*

The task of working out the ideas of statistical thermodynamics within random electrodynamics has only been begun. Several results have been published and further calculations have been undertaken.

The blackbody radiation spectrum was obtained[6] as one of the first results within random electrodynamics. Here we can make use of a model proposed by Einstein and Hopf.[18] We think of a massive particle which contains a dipole oscillator. The massive nature of the particle leads us to anticipate equipartition for the kinetic energy, while the oscillator provides a connection with the random radiation field. Einstein and Hopf analyzed this system within traditional classical electrodynamics, $\vec{A}^{\,\text{in}} = 0$, and found the thermal radiation spectrum was that of the Rayleigh-Jeans law. When analyzed within random electrodynamics, $\vec{A}^{\,\text{in}} \neq 0$, the calculation yields the Planck spectrum.

As we proceed further, it is possible to analyze the fluctuations in thermal radiation. Einstein[26] concluded that there were additional fluctuations above those derived from traditional classical electrodynamics, and he suggested that the basis of these fluctuations was light quanta. However, analyzed within random electrodynamics,[7] these additional fluctuations are natural and quite classical. They are associated with the zero-point radiation. These same additional zero-point fluctuations of random electrodynamics provide a basis for understanding the third law of thermodynamics,[8] lead to a decrease in rotational specific heats at low temperatures,[9] and explain the temperature dependence of the diamagnetic behavior of some systems.[2]

D. Atomic structure in random electrodynamics

1. *Puzzle of atomic collapse*

The nuclear model for atomic structure was introduced into classical electron theory in connection with Rutherford's scattering of alpha particles from thin foils. In this original description an atom was pictured as a massive charged nucleus about which electrons moved in orbits owing to Coulomb attraction. The atom was a miniature solar system involving electromagnetic forces.

This simple model stumbled over the problem of atomic collapse. The planetary electrons were accelerating, and hence, according to classical electromagnetism, were radiating away electromagnetic energy. Ideas of energy conservation required that the electrons should lose energy and spiral into the nucleus within a fraction of a second.

The first answer to the problem of atomic collapse was the quantum postulate of Niels Bohr. Within certain preferred stationary orbits, the laws of classical electromagnetism are suspended. Electrons in stationary orbits do not radiate. It is only in the transition between stationary states that radiation is emitted. However, even here classical electromagnetism does not apply. The frequency of radiation is not determined by the frequency of particle motion, but rather is given by the energy difference $\mathcal{E}_f - \mathcal{E}_i$ between the initial and final states

$$\nu = \frac{\mathcal{E}_f - \mathcal{E}_i}{h} . \tag{39}$$

With the advent of wave mechanics, the planetary model has receded into the background, but the notion of radiationless states has remained.

Random electrodynamics is classical electron theory with a new boundary condition. As such, the planetary model for an atom must be reconsidered. The electrons moving about the nucleus are indeed radiating away energy according to classical electromagnetic calculations. However, a new element enters. The random zero-point radiation acts to produce random motions of the electrons, in effect transferring energy to the electrons by random classical electromagnetic forces. It is the balance between the energy loss by radiation and the energy pickup from zero-point radiation which must account for the stability of matter in random electrodynamics.

2. Energy balance in the ground state of hydrogen

The author is in only the early stages of analysis of atomic structure within random electrodynamics. However, it is possible to perform calculations to obtain the rough size of the stable hydrogen atom.

The hydrogen atom is pictured as a massive positive nucleus of charge $+e$ about which an electron of charge $-e$ and mass m is moving. In the elementary considerations of this present paper, we will consider only approximately circular orbits. In lowest approximation the orbit is given by the Coulomb forces

$$\frac{e^2}{r^2} = \frac{mv^2}{r} . \tag{40}$$

Thus the electron revolves around the nucleus at an angular frequency

$$\omega = \frac{v}{r} = \left(\frac{e^2}{mr^3}\right)^{1/2} . \tag{41}$$

In the dipole approximation, the radiation emitted is that of a point dipole

$$\vec{p} = er(\hat{i}\cos\omega t + \hat{j}\sin\omega t) . \tag{42}$$

The energy loss due to radiation is thus

$$\frac{d\mathcal{E}_{\text{loss}}}{dt} = \frac{2}{3}\frac{e^2}{c^3}\omega^4 r^2 . \tag{43}$$

However, the electron also picks up energy from the zero-point radiation at a rate

$$\frac{d\mathcal{E}_{\text{gain}}}{dt} = \frac{e^2\hbar\omega^3}{2mc^3} . \tag{44}$$

The rate of energy pickup for an oscillator in zero-point radiation is given in Appendix A. The pick-up rate for the revolving electron may be obtained in dipole approximation from this calculation or from the result for the average impulse $\langle R\rangle$ in an earlier paper[12] on rotators in zero-point radiation. In the stable equilibrium situation, we expect that these two rates should balance, giving

$$\frac{d\mathcal{E}_{\text{loss}}}{dt} = \frac{d\mathcal{E}_{\text{gain}}}{dt} , \tag{45}$$

$$\frac{2}{3}\frac{e^2}{c^3}\omega^4 r^2 = \frac{e^2\hbar\omega^3}{2mc^3} , \tag{46}$$

or

$$m\omega r^2 = \tfrac{3}{4}\hbar . \tag{47}$$

Except for the factor of $\frac{3}{4}$, which cannot be taken seriously because of the unphysical restriction to cricular orbits, this agrees with the Bohr condition for stationary states

$$m\omega r^2 = n\hbar , \tag{48}$$

where $n=1$ for the ground state. Thus for the stable hydrogen atom, we suspect that the Bohr condition for a stationary state is a condition of energy balance involving classical electromagnetic zero-point radiation.

3. Observation of radiation spectra

The rate of energy loss in the circular orbits in hydrogen can be rewritten from (41) and (43) as

$$\frac{d\mathcal{E}_{\text{loss}}}{dt} = \frac{2}{3}\frac{e^6}{m^2c^3r^4} . \tag{49}$$

The energy gain from zero-point radiation from (41) and (44) is

$$\frac{d\mathcal{E}_{\text{gain}}}{dt} = \frac{e^5\hbar}{2m^{5/2}c^3r^{9/2}} . \tag{50}$$

Thus we see that for a large radius r, the energy loss predominates, and at very small radius the energy gain predominates. The curves for the rates cross at the single radius corresponding to a stable ground state.

On a heuristic level, these ideas accord in part with ideas of radiation emission by the excited hydrogen atom. Just as the Bohr correspondence principle regains the classical radiation frequencies at large radii, so here we find that the classical radiation loss is far above the rate of energy gain for large radii. Thus at large radii the radiation emission goes over to that of traditional classical electron theory which neglects the zero-point radiation of random electrodynamics. For radii near the equilibrium radius, the radiation pattern in this classical theory seems enormously complicated.

Although we have noted that the presence of classical electromagnetic zero-point radiation will lead to a stable hydrogen ground state, it does not follow that the classical radiation emitted by the ground-state electron will be unobservable. This must be derived from classical electromagnetic calculations.

The classical electromagnetic zero-point radiation in random electrodynamics is present throughout the universe in an essentially homogeneous pattern, deviations from homogeneity being caused by nearby matter. In particular, all electromagnetic detectors are bathed in classical electromagnetic zero-point radiation. An observer will interpret a signal as arriving only when the detector records a value above the background reading.

In line with this notion of detectability, we will say that the radiation emitted by an atom is observable provided that at some frequency a net transfer of radiation occurs across some surface at a large distance from the atom. Of course zero-point radiation is crossing this surface all the time. But the homogeneity and isotropy of zero-point radiation are such that, on the average, there is no net transfer of radiation across the surface.

For simplicity in this discussion, we will consider a harmonically bound electron with a natural frequency ω_0 and a damping constant Γ as in Eqs. (36) and (37). Then, as shown in Appendix A, the electron dipole moment ex is given by

$$ex=\sum_{\lambda=1}^{2}\int d^3k\,\frac{3c^3}{2\omega^3}\,\epsilon_x\,\mathfrak{h}\sin\alpha\cos[-\omega t+\alpha+\theta]\,, \tag{51}$$

where

$$\cot\alpha=\frac{\omega_0^{\,2}-\omega^2}{\Gamma\omega^3}\quad . \tag{52}$$

Again there is a balance between the energy loss by classical radiation and the energy pickup from the random zero-point radiation. The dipole radiation $\vec{E}'$, $\vec{B}'$ due to the oscillating dipole in (48) can be calculated in classical electromagnetic theory. Clearly if we calculate the Poynting vector

$$\vec{S}'=\frac{c}{4\pi}\,\vec{E}'\times\vec{B}'\,, \tag{53}$$

then the average $\langle\vec{S}'\rangle\propto 1/r^2$ is nonzero. There is an average emission of radiation by the dipole. However, if we compute the net radiation in space where we include the zero-point radiation field which originally set the oscillator into random oscillation in the first place,

$$\vec{S}=\frac{c}{4\pi}\,(\vec{E}_{ZP}+\vec{E}')\times(\vec{B}_{ZP}+\vec{B}')\,, \tag{54}$$

then we find that the average Poynting vector vanishes:

$$\langle\vec{S}\rangle=0\,. \tag{55}$$

There is no net radiation emerging from the dipole. A detector which responds to radiation which differs from the zero-point radiation pattern will not observe the radiation emitted by the ground state of a dipole oscillator.

We conjecture that this same interference between the zero-point radiation and the emitted radiation will make the ground-state radiation unobservable for all atomic systems in random electrodynamics. The extraction of the observable radiation pattern for an excited classical system is trivial for the case of a harmonic oscillator. The spectrum of the hydrogen atom in random electrodynamics is presently being investigated by the author. Detailed results on spectral emission for all systems in random electrodynamics will be provided in a future publication devoted to this question.

E. Connections between random electrodynamics and quantum theory

1. Random mechanics

The theory of random electrodynamics outlined in this paper is a theory of electromagnetic interactions between point particles intended to describe atomic physics—at least in some approximation. However, an apparently acceptable theory of atomic physics already exists in the form of quantum mechanics and more generally quantum electrodynamics. Hence it is of interest, at the very least of curiosity value, to see how the present random theory fits with the quantum predictions. The understanding of the random-quantum connections is incomplete. In Sec. V E 2 we will describe the first results giving the general connection between the theories for free fields and for harmonic-oscillator systems. In subsequent parts of Sec. V, we will speculate on some further connections between the theories.

In quantum theory one speaks separately of quantum mechanics and quantum electrodynamics, whereas we have described only a theory of random electrodynamics. Thus as our next step here, we must remark that there is a theory of random mechanics which forms a natural parallel to quantum mechanics.

The theory of random electrodynamics predicts a random motion for all particles which are coupled to the random classical zero-point radiation. For example, a charged dipole oscillator acquires a random oscillation involving a balance between the absorption of energy from the radiation field and the emission of energy as a dipole oscillator. However, in the limit $e\to 0$ when the coupling to the radiation field is removed, the random motion of the oscillator persists. Both the rate of energy absorption and the rate of energy emission vanish as $e\to 0$, but the ratio between the two, which determines the scale of oscillator vibrations, is maintained independent of the magnitude of e. Thus the expectation value for the mean-square displacement of a charged dipole oscillator of mass m and natural frequency ω_0 in random electrodynamics is

$$\langle x^2\rangle=\frac{1}{2}\,\frac{\hbar}{m\omega_0}+O(e^2)\,, \tag{56}$$

and in the limit $e \to 0$ the radiative corrections vanish giving

$$\langle x^2 \rangle = \frac{1}{2} \frac{\hbar}{m\omega_0} \quad . \tag{57}$$

Thus in the limit that the particles in the theory are decoupled from the radiation field, we derive a new theory which we will call random mechanics.

It is to be emphasized that random mechanics is a derived theory. It seems unphysical from a classical perspective only because zero-point radiation and the limit $e \to 0$ are unfamiliar to most physicists. As taught in elementary physics, quantum mechanics involves a new kinematics which is an inherent, not a derived theory. Thus the textbooks expect that quantum mechanics is applicable not only to electromagnetic forces but also to nuclear forces. This view may or may not be in accord with nature. However, in order to obtain a comparison between our ideas arising from classical electromagnetic zero-point radiation and those of familiar quantum physics, we will regard quantum mechanics as a derived theory, derived from quantum electrodynamics in the limit $e \to 0$. This view may be unorthodox, but it should be consistent. In our work with harmonic-oscillator systems described below, we find it easiest to obtain a connection between the complete electrodynamic theories, quantum and random electrodynamics. We then take the limit $e \to 0$ in order to obtain the results for random mechanics. Finally we compare these results with those of familiar nonrelativistic quantum mechanics where the development makes no reference to quantum electrodynamics.

For general physical systems the connections between random and quantum theories are not known. In the future we plan to investigate the stochastic process for charged particles implied by the coupling to random zero-point radiation but taken in the limit $e \to 0$. It may be that the stochastic process is close to that involved in Nelson's derivation[27] of the Schrödinger equation from ideas of Brownian motion superimposed upon Newtonian mechanics.

2. *General random-quantum connections proved for harmonic-oscillator systems*

There are two cases where it is possible to obtain exact solutions in quantum electrodynamics and in random electrodynamics. These are free-field electromagnetism and point harmonic-oscillator systems with electromagnetic interactions. In these situations a general comparison of the theories has been carried out recently.[28] Here we will merely summarize some of the results.

The physically meaningful quantities for free electromagnetic fields involve the expectation values for the correlation functions

$$\langle E_{i_1}(\vec{r}_1, t_1) B_{i_2}(\vec{r}_2, t_2) \cdots E_{i_n}(\vec{r}_n, t_n) \rangle$$

of products of electric and magnetic fields evaluated for the equilibrium situation in vacuum or at temperature T. In general these correlation functions do not agree between random electrodynamics and quantum electrodynamics because a change in operator order alters the values of the quantum expressions. For example, $\langle E_{i_1}(\vec{r}_1, t_1) E_{i_2}(\vec{r}_2, t_2) \rangle$ is not equal to $\langle E_{i_2}(\vec{r}_2, t_2) E_{i_1}(\vec{r}_1, t_1) \rangle$ in the quantum theory. What physical distinction is represented by the dependence upon operator order is not clear. However, if within quantum electrodynamics the dependence upon operator order is removed by completely symmetrizing all products of operators, averaging over all possible permutations of the operator order, then these quantum correlation functions agree exactly with those of random electrodynamics.

The same situation holds for harmonic-oscillator systems with electromagnetic interactions. Again the average values $\langle xp \cdots x \rangle$ involving products of the position and momentum of the oscillators agree exactly between the random and quantum theories provided the quantum operator order is completely symmetrized in all products. Agreement persists in the limit as the coupling of the oscillators to the electromagnetic field vanishes and quantum electrodynamics becomes quantum mechanics and random electrodynamics becomes random mechanics.

3. *Heisenberg uncertainty principle*

At this point we turn away from the mathematical calculations in random electrodynamics and attempt to comment upon some qualitative features of the theory. At least on a heuristic level, the theory involves a number of the same surprising ideas which first arose in connection with quantum mechanics—an uncertainty principle, apparent wave aspects for particles, and disturbances due to measuring apparatus.

The Heisenberg uncertainty principle relates the uncertainties in position and momentum of a particle as

$$\Delta x \Delta p \gtrsim \hbar \; . \tag{58}$$

The principle actually takes several guises in quantum mechanics being associated with wave packets and also with measurement processes. Here we point out that an uncertainty principle is also expected in random electrodynamics and in random mechanics. The fluctuations in the electromagnetic field cause fluctuations in the positions of particles with electromagnetic interactions. In the case of the harmonic oscillator, exact calculations are possible. Thus we find in random mechanics[2,11] for equilibrium at zero temperature

$$\langle x^2\rangle\langle p^2\rangle=\left(\frac{1}{2}\,\frac{\hbar}{m\omega_0}\right)\left(\tfrac{1}{2}\hbar m\omega_0\right)$$
$$=\tfrac{1}{4}\hbar^2, \tag{59}$$

which is in exact agreement with quantum theory.

4. *Speculations on the apparent wave nature of particles in random electrodynamics*

Wave-particle duality has become one of the foundation stones for the philosophical interpretation of quantum mechanics. According to this view light shows wavelike characteristics in interference experiments and photonlike characteristics in Compton scattering and the photoelectric effect. Analogously, electrons show their particlelike properties in bubble chambers and photographic emulsions while revealing wavelike interference patterns in scattering from crystals.

The theory of random electrodynamics includes no such duality. The fundamental dichotomy between point-particle singularities and electromagnetic waves is as absolute as in Lorentz's classical electrodynamics. Nevertheless, the theory suggests the possibility of understanding wave-particle duality as an apparent effect due to classical electromagnetic zero-point radiation. Random electrodynamics has already provided a purely wave explanation[7] of the fluctuations in thermal radiation which are usually assigned to photon statistics. Moreover, it includes the heuristic suggestion that particles might show interference pattern effects.

The possibility of wavelike effects for particles becomes clear when we recall that the spectrum of fluctuations in zero-point radiation is modified by the presence of matter which interacts with the radiation. Indeed, it is precisely this change in the fluctuation pattern of the zero-point radiation which forms the basis for the random electromagnetic calculations of Van der Waals forces between two conducting walls or between combinations of walls and particles. In all these cases, the classical results agree exactly with the quantum results.

Suppose now that a slit is cut into a wall. Then the pattern of classical electromagnetic zero-point radiation will be changed so as to reflect the presence of this slit. A particle passing through will be influenced by the radiation fluctuations and conceivably might arrive at a distant screen with a probability given by a single-slit diffraction pattern characterized by the de Broglie wavelength associated with the velocity of the particle. In this case the wavelike properties of the particle are only apparent and actually reflect the fluctuation pattern of the random radiation. At present the crucial calculations to test for particle-interference patterns have not been carried out.

Continuing the same heuristic argument, we note that if two slits are cut in the wall, the zero-point radiation pattern reflects this fact. In the quantum terminology one sometimes speaks of a particle as passing through both slits so as to form a double-slit particle diffraction pattern on a distant screen. However, in our classical view, the particle passes through only one slit, but the pattern of zero-point fluctuations reflects the presence of both slits. Covering one of the slits of course changes the radiation fluctuation pattern, and accordingly changes the influence on the passing particle. At least qualitatively, the mysterious aspects of the patricle-wave behavior are removed by the theory of random electrodynamics.

5. *Influence of the measuring apparatus*

Quantum theory, especially in its philosophical interpretations, is full of ideas about the influence of measuring apparatus upon the system being measured. This notion of unavoidable influence of the measuring apparatus is a natural deduction in our classical theory with zero-point radiation. The measuring apparatus will involve matter with electromagnetic interactions. This matter changes the pattern of zero-point radiation near the apparatus and so alters the system being observed. We notice that in random electrodynamics, the physical situation is entirely comprehensible in classical terms involving electromagnetic interactions of radiation and matter.

6. *Relativistic theory*

Random electrodynamics consists of classical electron theory with a change in boundary condition corresponding to classical electromagnetic zero-point radiation. All of the calculations which have been published to date involve nonrelativistic equations of motion for the particles. However, the theory itself is a relativistic theory. The zero-point radiation involves a Lorentz-invariant spectrum of electromagnetic radiation, and the equations of motion of traditional classical electron theory can be treated in relativistic form.[15] One may speculate that the relativistic theory may bring us close to what is presently regarded as relativistic quantum mechanics and relativistic quantum field theory.[29]

V. EARLIER WORK ON ZERO-POINT RADIATION

The present writer is hardly the first to be interested in ideas of zero-point energy or zero-point radiation. The notion of zero-point energy for material systems appeared in 1911 in Planck's second theory[30] for blackbody radiation, and was then introduced in calculations of oscillator specific heats and of blackbody radiation by Einstein and Stern.[31] However, the idea of zero-point radiation, of assigning zero-point energy to the electromagnetic field, seems to have been taken up first by Nernst[32] in 1916. A historical survey of some of the ideas of zero-point energy has been prepared recently by Mehra and Rechenberg.[33] Within quantum theory, zero-point energy and zero-point radiation have generally played the role of curious and occasionally troublesome hangers-on.

The idea of zero-point energy in the electromagnetic field was first exploited in a calculation by Casimir[21] in 1948 for the force between two conducting parallel plates. In 1949 Casimir[34] showed that ideas of electromagnetic zero-point energy could be used to calculate the asymptotic form of the Van der Waals force between two polarizable particles.

More recently a number of writers[35] have entertained ideas of zero-point radiation with varying degrees of lucidity. In general the calculations have been informal, often mixed with ideas outside classical electromagnetism. The first careful and also extensive work on a theory of classical electromagnetism with classical electromagnetic zero-point radiation is that of Marshall[2-5] between 1963 and 1965. Marshall determined the spectrum of zero-point radiation[2] by requiring that a classical charged harmonic oscillator take up a mean-square displacement $\langle x^2\rangle = \frac{1}{2}\hbar/(m\omega_0)$, no matter where the oscillator was located in space. This assumption required the presence of random radiation. Marshall investigated the spectral properties of the radiation, derived the Lorentz invariance of the spectrum,[3] and also derived the Van der Waals forces between parallel slabs of conducting or dielectric material.[4,5]

The present author became interested in the ideas of zero-point radiation entirely through pursuit of Casimir's ideas on quantum electromagnetic zero-point energy. After performing a number of calculations on Van der Waals forces between macroscopic objects[36] and becoming increasingly impressed by the essentially classical character of Casimir's ideas, the author independently realized and proved[7] the Lorentz-invariant nature of the spectrum of classical electromagnetic zero-point radiation with energy $\frac{1}{2}\hbar\omega$ per normal mode. This Lorentz invariance seemed so fundamental that it prompted the line of development summarized in the present paper.

VI. CLOSING SUMMARY

The classical electron theory of Lorentz was regarded as a successful theory of atomic phenomena in the earliest years of the twentieth century. Subsequent developments in theoretical physics led to the introduction of quantum theory and the collapse of classical electron theory as a serious description of nature. It survives merely as a qualitative description of some phenomena such as optical dispersion and the Zeeman effect.

In this article we have pointed out that classical electron theory can be modified by the change of the homogeneous boundary condition on Maxwell's equations. If the homogeneous boundary condition is chosen to correspond to random classical electromagnetic radiation with a Lorentz-invariant spectrum, then we obtain a theory which in the past we have termed classical electrodynamics with classical electromagnetic zero-point radiation and which in the future we propose to call random electrodynamics. The theory involves classical ideas of particle position, force, and measurement. It is an extension of Lorentz's theory. The new theory makes possible a classical understanding of a number of phenomena which are usually regarded as requiring quantum explanations. The predictions of random electrodynamics have close connections with those of quantum electrodynamics for free-field systems and harmonic-oscillator systems. However, a general understanding of the areas of agreement and disagreement for quantum and random electrodynamics awaits further mathematical analysis of the new theory.

APPENDIX A: ENERGY GAIN FOR A DIPOLE OSCILLATOR IN ZERO-POINT RADIATION

In this appendix we will calculate the average rate of energy pickup for a point dipole oscillator in classical electromagnetic zero-point radiation. We consider a nonrelativistic dipole oscillator of natural frequency ω_0 oriented along the x axis and located at the origin,

$$m\ddot{x} = -m\omega_0{}^2 x + \frac{2}{3}\frac{e^2}{c^3}\dddot{x} + eE_{\mathrm{ZP}\,x}(0,t)\,. \qquad \text{(A1)}$$

If we read off the zero-point radiation field $\vec{E}_{\mathrm{ZP}}$ from (19), the steady-state solution is

$$x=\frac{1}{e}\sum_{\lambda=1}^{2}\int d^3k\,\frac{3c^3}{2\omega^3}\,\epsilon_x(\vec{k},\lambda)\mathfrak{h}(\vec{k},\lambda)\sin\alpha(\omega)\cos[-\omega t+\alpha(\omega)+\theta(\vec{k},\lambda)]\;, \tag{A2}$$

where

$$\cot\alpha(\omega)=\frac{\omega_0{}^2-\omega^2}{\Gamma\omega^3}\;,\quad \Gamma=\frac{2}{3}\,\frac{e^2}{mc^3}\;. \tag{A3}$$

Now the work performed on the oscillator during a time τ is

$$\begin{aligned}\Delta W&=\int_0^\tau dt\,\dot{x}\,(eE_{\mathrm{ZP}\,x})\\&=\sum_{\lambda=1}^{2}\sum_{\lambda'=1}^{2}\int d^3k\int d^3k'\,\frac{3c^3}{2\omega^2}\,\epsilon_x\epsilon_x'\,\mathfrak{h}\mathfrak{h}'\sin\alpha(\omega)\int_0^\tau dt\,\sin[-\omega t+\alpha(\omega)+\theta(\vec{k},\lambda)]\cos[-\omega' t+\theta(\vec{k}',\lambda')]\;.\end{aligned} \tag{A4}$$

In the present situation we may average over the random phases before integrating in time:

$$\begin{aligned}\left\langle\int_0^\tau dt\,\sin[-\omega t+\alpha(\omega)+\theta(\vec{k},\lambda)]\cos[-\omega' t+\theta(\vec{k}',\lambda')]\right\rangle&=\int_0^\tau dt\,\sin\alpha(\omega)\tfrac{1}{2}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\\&=\sin\alpha(\omega)\tfrac{1}{2}\tau\,\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\;.\end{aligned} \tag{A5}$$

Thus the average work performed on the oscillator by the zero-point field is

$$\begin{aligned}\langle\Delta W\rangle&=\sum_{\lambda=1}^{2}\int d^3k\,\frac{3c^2}{2\omega^2}\,\sin^2\alpha(\omega)\epsilon_x{}^2\mathfrak{h}^2(\vec{k},\lambda)\tfrac{1}{2}\tau\\&=\sum_{\lambda=1}^{2}\int d^3k\,\frac{2}{3}\,\frac{\omega^4}{2c^3}\left(\frac{e^2}{m}\right)^2\frac{\epsilon_x{}^2}{(\omega_0{}^2-\omega^2)^2+(\Gamma\omega^3)^2}\,\frac{\hbar\omega}{2\pi^2}\,\frac{\tau}{2}\;.\end{aligned} \tag{A6}$$

The integral can be evaluated to lowest order in e by standard techniques[6,11] giving the rate of gain of energy by the oscillator

$$\begin{aligned}\frac{d\mathcal{E}_{\mathrm{gain}}}{dt}&=\frac{1}{\tau}\,\langle\Delta W\rangle\\&=\frac{2}{3}\,\frac{\omega_0{}^4}{c^3}\left(\frac{1}{2}\,\frac{\hbar e^2}{m\omega_0}\right)\;.\end{aligned} \tag{A7}$$

As a further comment upon the dipole oscillator, we note that the rate of energy loss due to radiation is

$$\frac{d\mathcal{E}_{\mathrm{loss}}}{dt}=\frac{2}{3}\,\frac{\omega_0{}^4}{c^3}\,\langle e^2x^2\rangle\;. \tag{A8}$$

Thus the energy gain balances the energy loss on the average for a mean-square displacement

$$\langle x^2\rangle=\frac{1}{2}\,\frac{\hbar}{m\omega_0}\quad. \tag{A9}$$

Actually here we may calculate $\langle x^2\rangle$ directly from the solution (A2) above.

APPENDIX B. RADIATION EMERGING FROM A DIPOLE OSCILLATOR IN ZERO-POINT RADIATION: STABILITY OF THE SPECTRUM

In this appendix we wish to show that the zero-point radiation spectrum is stable against scattering by a dipole oscillator, and hence that the radiation emitted by a dipole oscillator in its ground state cannot be detected above the zero-point radiation pattern.

We consider a single dipole oscillator located at the origin of coordinates and oriented along the direction $\hat{p}$. The oscillator both absorbs and emits radiation in all directions. Now the dipole oscillator has a preferred direction given by its axis $\hat{p}$, and hence it is possible that it might absorb energy at some wave vectors and then emit this radiation at other wave vectors so as to alter the pattern of zero-point radiation. We can test whether or not the radiation pattern is stable by computing Poynting's vector

$$\vec{S}=\frac{c}{4\pi}\,\vec{E}\times\vec{B} \tag{B1}$$

at every point in space. If there is a net outflow of energy $\langle\vec{S}\rangle\neq 0$ in some direction at some frequency, then the radiation pattern is unstable and this scattered radiation can be detected above the zero-point background.

We will denote by $\vec{E}'$ and $\vec{B}'$ the (retarded) electromagnetic fields caused by the oscillator. Thus the total fields in space are

$$\vec{E}=\vec{E}_{ZP}+\vec{E}', \quad \vec{B}=\vec{B}_{ZP}+\vec{B}'. \tag{B2}$$

The steady-state dipole moment for the oscillator in zero-point radiation is given in (A2). The (retarded) fields of an oscillating dipole at frequency ω can then be obtained from standard textbooks. Thus we have in condensed notation

$$\vec{E}_{ZP}(\vec{r},t)=\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\mathfrak{h}\exp[-i\omega t+i\theta][\hat{\epsilon}\exp(i\vec{k}\cdot\vec{r})], \tag{B3}$$

$$\vec{B}_{ZP}(\vec{r},t)=\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\mathfrak{h}\exp[-i\omega t+i\theta]\times\left(\frac{\vec{k}\times\hat{\epsilon}}{k}\exp[i\vec{k}\cdot\vec{r}]\right), \tag{B4}$$

$$\vec{E}'(\vec{r},t)=\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\frac{\beta}{C}\,\mathfrak{h}\exp(-i\omega t+i\theta)\vec{G}, \tag{B5}$$

$$\vec{B}'(\vec{r},t)=\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\frac{\beta}{C}\,\mathfrak{h}\exp(-i\omega t+i\theta)\vec{H}, \tag{B6}$$

where

$$\beta=\tfrac{3}{2}\Gamma c^3=\frac{e^2}{m}, \tag{B7}$$

$$C=-\omega^2+\omega_0^2-i\Gamma\omega^3, \tag{B8}$$

$$\vec{G}=k^3\exp(ikr)\left[(\hat{n}\times\hat{p})\times\hat{n}\left(\frac{1}{kr}\right)+[3\hat{n}(\hat{n}\cdot\hat{p})-\hat{p}]\left(\frac{1}{(kr)^3}-\frac{i}{(kr)^2}\right)\right], \tag{B9}$$

$$\vec{H}=k^3\exp(ikr)\hat{n}\times\hat{p}\left(\frac{1}{kr}+\frac{i}{(kr)^2}\right). \tag{B10}$$

The average of Poynting's vector is

$$\langle\vec{S}\rangle=\frac{c}{4\pi}\langle\vec{E}\times\vec{B}\rangle =\tfrac{1}{2}\mathrm{Re}\,\frac{c}{4\pi}\{\langle\vec{E}^*_{ZP}\times\vec{B}_{ZP}\rangle+\langle\vec{E}'^*\times\vec{B}'\rangle+\langle\vec{E}^*_{ZP}\times\vec{B}'\rangle+\langle\vec{E}'^*\times\vec{B}_{ZP}\rangle\}. \tag{B11}$$

After we average over the random phase $\theta(\vec{k},\lambda)$ in the zero-point radiation, the terms in (B11) become

$$\langle\vec{E}^*_{ZP}\times\vec{B}_{ZP}\rangle=\int d^3k\,\mathfrak{h}^2\left[\sum_{\lambda=1}^{2}\hat{\epsilon}\times\left(\frac{\vec{k}\times\hat{\epsilon}}{k}\right)\right], \tag{B12}$$

$$\langle\vec{E}'^*\times\vec{B}'\rangle=\int d^3k\,\mathfrak{h}^2\,\frac{\beta^2}{|C|^2}\,\vec{G}^*\times\vec{H}\left[\sum_{\lambda=1}^{2}(\hat{\epsilon}\cdot\hat{p})^2\right], \tag{B13}$$

$$\langle\vec{E}^*_{ZP}\times\vec{B}'\rangle=\int d^3k\,\mathfrak{h}^2\,\frac{\beta}{|C|^2}\,(-C^*\vec{H})\times\left[\sum_{\lambda=1}^{2}\hat{\epsilon}(\hat{\epsilon}\cdot\hat{p})\exp(-i\vec{k}\cdot\vec{r})\right], \tag{B14}$$

$$\langle\vec{E}'^*\times\vec{B}_{ZP}\rangle=\int d^3k\,\mathfrak{h}^2\,\frac{\beta}{|C|^2}\,C\vec{G}^*\times\left[\sum_{\lambda=1}^{2}\frac{\vec{k}\times\hat{\epsilon}}{k}(\hat{\epsilon}\cdot\hat{p})\exp(i\vec{k}\cdot\vec{r})\right]. \tag{B15}$$

The sum over polarizations gives

$$\sum_{\lambda=1}^{2}\epsilon_i(\vec{k},\lambda)\epsilon_j(\vec{k},\lambda)=\delta_{ij}-\frac{k_ik_j}{k^2}, \tag{B16}$$

while we note that

$$\vec{k}\cdot\hat{\epsilon}(\vec{k},\lambda)=0. \tag{B17}$$

Thus the angular integrations in k take the form

$$\int d\Omega_{\vec{k}}\left[\sum_{\lambda=1}^{2}\hat{\epsilon}\times\left(\frac{\vec{k}\times\hat{\epsilon}}{k}\right)\right]=\int d\Omega_{\vec{k}}\,\frac{2\vec{k}}{k}$$

$$=0\,, \tag{B18}$$

$$\int d\Omega_{\vec{k}}\left[\sum_{\lambda=1}^{2}(\hat{\epsilon}\cdot\hat{p})^2\right]=\int d\Omega_{\vec{k}}\left[1-\left(\frac{\vec{k}\cdot\hat{p}}{k}\right)^2\right]$$

$$=\tfrac{8}{3}\pi\,, \tag{B19}$$

$$\int d\Omega_{k}\left[\sum_{\lambda=1}^{2}\hat{\epsilon}(\hat{\epsilon}\cdot\hat{p})\exp(-i\vec{k}\cdot\vec{r})\right]=\int d\Omega_{\vec{k}}\left[\hat{p}-\frac{\vec{k}(\vec{k}\cdot\hat{p})}{k^2}\right]\exp(-i\vec{k}\cdot\vec{r})$$

$$=\frac{4\pi}{k^3}\,\mathrm{Im}\vec{G}\,, \tag{B20}$$

$$\int d\Omega_{\vec{k}}\left[\sum_{\lambda=1}^{2}\frac{\vec{k}\times\hat{\epsilon}}{k}(\hat{\epsilon}\cdot\hat{p})\exp(i\vec{k}\cdot\vec{r})\right]=\int d\Omega_{\vec{k}}\left(\frac{\vec{k}\times\hat{p}}{k}\right)\exp(i\vec{k}\cdot\vec{r})$$

$$=\frac{-4\pi i}{k^3}\,\mathrm{Re}\vec{H}\,. \tag{B21}$$

These angular integrations are easily carried out one component at a time, choosing $\vec{r}$ along the z axis and using spherical polar coordinates.

Substituting the results for the angular integrations into the previous expressions, we have

$$\langle\vec{S}\rangle=\frac{c}{4\pi}\int_0^\infty dk\,k^2\mathfrak{h}^2\tfrac{1}{2}\mathrm{Re}\left[0+\frac{\beta^2}{|C|^2}\vec{G}^*\times\vec{H}\left(\tfrac{8}{3}\pi\right)-\frac{\beta}{|C|^2}C^*\vec{H}\times\left(\frac{4\pi}{k^3}\mathrm{Im}\vec{G}\right)+\frac{\beta}{|C|^2}C\vec{G}^*\times\left(\frac{-4\pi i}{k^3}\mathrm{Re}\vec{H}\right)\right]. \tag{B22}$$

Now

$$\mathrm{Re}(C^*\vec{H})=\mathrm{Re}C\,\mathrm{Re}\vec{H}+\mathrm{Im}C\,\mathrm{Im}\vec{H}\,, \tag{B23}$$

while

$$\mathrm{Re}(iC\vec{G}^*)=\mathrm{Re}C\,\mathrm{Im}\vec{G}-\mathrm{Im}C\,\mathrm{Re}\vec{G}\,. \tag{B24}$$

Introducing these expressions, we obtain some cancellation in the last two terms of (B22), leaving

$$\langle\vec{S}\rangle=\frac{c}{4\pi}\int_0^\infty dk\,k^2\mathfrak{h}^2\left[\tfrac{8}{3}\pi\,\frac{\beta^2}{|C|^2}(\mathrm{Re}\vec{G}\times\mathrm{Re}\vec{H}+\mathrm{Im}\vec{G}\times\mathrm{Im}\vec{H})+\frac{4\pi}{k^3}\frac{\beta}{|C|^2}(\mathrm{Im}C\,\mathrm{Im}\vec{G}\times\mathrm{Im}\vec{H}+\mathrm{Im}C\,\mathrm{Re}\vec{G}\times\mathrm{Re}\vec{H})\right]. \tag{B25}$$

But now from (B8)

$$\mathrm{Im}C=-\Gamma\omega^3=-\tfrac{2}{3}\beta k^3\,. \tag{B26}$$

Thus the integrand in (B25) vanishes exactly,

$$\langle\vec{S}\rangle=0\,. \tag{B27}$$

This means that despite the presence of the dipole which is absorbing radiation from the zero-point radiation field and then remitting radiation, there is no average transfer of energy in any direction at any frequency. The properties of the random radiation spectrum which enter this proof are homogeneity and isotropy relative to the rest frame of the oscillator. The random radiation pattern is stable despite the presence of the oscillator.

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