Restoration of dynamically broken symmetries at finite temperature*

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(Received 5 August 1974)

A field-theoretic model is constructed which is asymptotically free for all temperatures and which possesses a phase transition in which masses are dynamically generated. The critical temperature above which the particle masses vanish is calculated in terms of the zero-temperature fermion mass. Our results may be of interest not only in theories of the early universe but also in the study of the formative stages of neutron stars.

I. INTRODUCTION

Within the context of a general renormalizable field theory, Weinberg[1] has shown that a spontaneously broken symmetry can be restored by finite-temperature effects. Critical temperatures were determined from the vanishing of an effective (temperature-dependent) scalar-boson mass. The presence of canonical scalar fields in the original Lagrangian was thus essential, both to break and to restore the symmetry. However, such scalar fields have not been experimentally observed, nor are they theoretically desirable from the viewpoint of asymptotic freedom.[2] These shortcomings have recently spurred much interest[3] in an alternative mass-generating mechanism: dynamical symmetry breaking. In particular, Gross and Neveu[4] have constructed a nontrivial asymptotically free model in which dynamical symmetry breaking occurs for any value of the coupling constant. If we wish to adopt the attitude that the presently observable broken symmetries are the relics of a pristine epoch, then the following question is of paramount importance: Can a symmetry which is dynamically broken at zero temperature be restored by raising the temperature?

To answer this question, we consider in Sec. II a model[4] which contains N-component fermion fields with a quartic self-interaction in two space-time dimensions. This model reflects many of the desirable features of a realistic field theory, such as nontrivial scattering, renormalizability, asymptotic freedom, and dynamical mass generation. In Sec. III we calculate to all orders in the coupling constant and to leading order in $1/N$ the temperature-dependent equilibrium value of the dynamically generated fermion mass. A critical temperature is determined at which the full chiral symmetry of the Lagrangian is restored. The renormalization group is utilized in Sec. IV to demonstrate that the critical temperature is "physical" in the sense that it is independent of our choice of renormalization point. It is also found that finite-temperature effects do not alter the underlying asymptotic freedom. A limitation of our result due to the peculiarities of the $1/N$ expansion in two dimensions is considered in Sec. V. We conclude in Sec. VI with a short discussion of the possible cosmological relevance of our work.

II. THE MODEL

The only field-theoretic models which are presently known[4] to possess dynamical symmetry breakdown, and to be nontrivial in the sense of having an S matrix not equal to 1, are variants of the two-dimensional N-component Thirring model. We consider the simple prototype described by the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/)\psi + \frac{1}{2}\frac{\lambda}{N}(\bar{\psi}\psi)^2 , \tag{1}$$

where ψ is the massless fermion field with N components. Note that this Lagrangian is chirally invariant, so that the discrete γ_5 invariance would forbid the generation of a fermion mass to any order in perturbation theory. The $1/N$ expansion will allow us to go beyond a perturbative result.

The Green's functions appropriate to the Lagrangian of (1) are generated by functional derivatives of

$$Z[n,\bar{n}] = c\int[d\psi][d\bar{\psi}]\exp\left[i\int d^2x(\mathcal{L}+\bar{n}\psi+\bar{\psi}n)\right], \tag{2}$$

where c is a normalizing constant. By introducing a constraint field $\sigma(x)$, we may write the generating functional of (2) equivalently as

$$Z[n,\bar{n}] = c'\int[d\psi][d\bar{\psi}][d\sigma]\exp\left[i\int d^2x(\mathcal{L}_\sigma+\bar{n}\psi+\bar{\psi}n)\right], \tag{3}$$

where

$$\mathcal{L}_\sigma = \bar{\psi}(i\partial\!\!\!/)\psi - \frac{1}{2}\sigma^2 - \left(\frac{\lambda}{N}\right)^{1/2}\bar{\psi}\psi\sigma , \tag{4}$$

and c' is another constant. The constraint field reduces the counting of powers of N to the topologically trivial result[5] that each fermion loop produces a factor of N while each $\bar{\psi}\psi\sigma$ vertex produces a factor of $N^{-1/2}$. In addition, the constraint field produces a criterion for determining the existence of a fermion mass. If, in equilibrium, the vacuum expectation value of σ, denoted by σ_M, is nonvanishing, then a fermion mass M_F will be generated with

$$M_F = \left(\frac{\lambda}{N}\right)^{1/2} \sigma_M \,. \tag{5}$$

To determine the equilibrium value of σ, we define first the generator $W[J]$ of the connected Green's functions for the σ field[6]

$$e^{iW[J]} \equiv \int [d\psi][d\bar{\psi}][d\sigma] \exp\left[i\int d^2x (\mathcal{L}_\sigma + J_\sigma)\right] \tag{6}$$

and then the Legendre transform of $W[J]$

$$\Gamma[\sigma_c(x)] = \int d^2x\, \sigma_c(x) J(x) - W[J] \,, \tag{7}$$

where the classical field $\sigma_c(x)$ is defined by

$$\sigma_c(x) = \frac{\delta W[J]}{\delta J(x)} \,. \tag{8}$$

When the current vanishes, the classical field becomes the vacuum expectation value σ_M. The stability of the theory defined by Eqs. (4)–(8) can be studied by considering diagrams where all the external $\sigma_c(x)$ lines carry zero two-momentum. Using translational invariance, we can then extract the effective potential V from the effective action Γ[7]:

$$V(\sigma_c) = -\sum_{n=2}^{\infty} \frac{1}{n!} \tilde{\Gamma}^{(n)}(0, \ldots, 0)(\sigma_c - \sigma_M)^n \,, \tag{9}$$

where $\tilde{\Gamma}^{(n)}$ is the proper momentum-space Green's function with n zero-two-momentum external $\sigma_c(x)$ lines and σ_c is independent of space-time. The position of the absolute minimum of $V(\sigma_c)$ determines σ_M. The assumption that the equilibrium value of $\sigma(x)$ is independent of coordinates will be further considered in Sec. V.

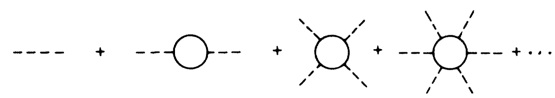

FIG. 1. Leading-order (in $1/N$) graphs which contribute to $V(\sigma_c)$. Dashed lines refer to σ while solid lines refer to ψ.

III. EFFECTIVE POTENTIAL

The effective potential is calculated in the large-N limit where the leading terms (exact when $N\to\infty$) are given by the tree graph and all one-fermion-loop graphs[4] (see Fig. 1):

$$V(\sigma_c) = \tfrac{1}{2}\sigma_c^{\,2} - Ni \sum_{n=1}^{\infty} \int^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{n} \left(\frac{\lambda}{N}\frac{\sigma_c^{\,2}}{k^2}\right)^n \,, \tag{10}$$

where Λ is an ultraviolet cutoff. This potential can be translated away from zero temperature by the usual substitutions,[1,8] namely

$$k_0 \to i\omega_n \,, \qquad \int \frac{dk_0}{2\pi} \to i\theta \sum_{n=-\infty}^{+\infty} \,, \tag{11}$$

where

$$\omega_n = (2n+1)\pi\theta \,, \tag{12}$$

and θ is the temperature (multiplied by the Boltzmann constant k). The effective potential at finite temperature $V(\sigma_c, \theta)$ becomes

$$V(\sigma_c, \theta) = \tfrac{1}{2}\sigma_c^{\,2} - 2N\theta \sum_{n=-\infty}^{+\infty} \int_0^{\infty} \frac{dk}{2\pi}\left[\ln\left(k^2 + \omega_n^{\,2} + \frac{\lambda}{N}\sigma_c^{\,2}\right) - \ln(k^2 + \omega_n^{\,2})\right] \,, \tag{13}$$

where we have summed over the external σ_c lines. The summation over the energy modes can be performed by using the identity[9]

$$\theta \sum_n \ln(\omega_n^{\,2} + a^2) = a + 2\theta \ln(1 + e^{-a/\theta}) + c \,, \tag{14}$$

where c is an infinite constant independent of a. We find that the temperature-independent part of the effective potential can be isolated, i.e.,

$$V(\sigma_c, \theta) = V_1(\sigma_c) + V_2(\sigma_c, \theta) \,, \tag{15}$$

where

$$V_1(\sigma_c) = \tfrac{1}{2}\sigma_c^{\,2} - 2N\int_0^{\Lambda} \frac{dk}{2\pi}\left[\left(k^2 + \frac{\lambda}{N}\sigma_c^{\,2}\right)^{1/2} - k\right] \tag{16}$$

and

$$V_2(\sigma_c, \theta) = -4N\theta \int_0^\infty \frac{dk}{2\pi} \left(\ln\left\{ 1 + \exp\left[-\frac{1}{\theta}\left(k^2 + \frac{\lambda}{N}\sigma_c^2 \right)^{1/2} \right] \right\} - \ln(1 + e^{-k/\theta}) \right) . \tag{17}$$

This explicit form for the effective potential possesses a number of interesting features:

(i) Since $\lambda \neq 0$, the effective potential contains an interaction-induced gap parameter $(\lambda/N)\sigma_c^2$. It will be shown [see (25)] that, in equilibrium, for temperatures $\theta < \theta_{crit}$ the gap persists, while for $\theta > \theta_{crit}$ the gap disappears. From the superconducting analogy,[10] we expect this gap in the energy spectrum to generate a mass which depends on the coupling strength in a nonanalytic way. This will also be demonstrated [see (22) and Ref. 24].

(ii) Since $\lim_{\theta \to 0} V_2(\sigma_c, \theta) = 0$ and V_2 is finite, all zero-temperature renormalizations of the effective potential will not be affected by the temperature. In particular, we find by imposing the Coleman-Weinberg renormalization condition[11] at zero temperature

$$\left. \frac{\partial^2 V_1(\sigma_c)}{\partial \sigma_c^2} \right|_{\sigma_c = \sigma_0} = 1 \tag{18}$$

that

$$V_1(\sigma_c, \sigma_0) = \tfrac{1}{2}\sigma_c^2 + \frac{\lambda}{4\pi}\sigma_c^2 \left[\ln\left(\frac{\sigma_c}{\sigma_0} \right)^2 - 3 \right] , \tag{19}$$

where σ_c is an arbitrary renormalization point. Equation (19) is the result found in Ref. 4.

(iii) Since $\lim_{\sigma_c \to \infty} V_2(\sigma_c, \theta)$ is finite, the temperature effects will not alter the stability of the zero-temperature theory, i.e., for all θ we know that the effective potential monotonically increases for sufficiently large values of σ_c.

The last observation implies that in order to determine whether the finite-temperature effects can alter the symmetry, we need only study the effective potential for σ_c in the neighborhood of the origin. When $\theta < \theta_{crit}$, the effective potential will decrease as we go away from the origin so that the minimum must occur away from zero and a fermion mass is necessarily generated (Fig. 2). When $\theta > \theta_{crit}$, the effective potential will increase as we go away from the origin so that $\sigma_c = 0$ becomes a minimum (though possibly not an absolute minimum) point in a stable theory and the fermions probably remain massless (Fig. 3).

Thus we must study the extremum of the renormalized $V(\sigma_c, \theta)$ in the neighborhood of the origin in classical field space, i.e., for $(\lambda/N)\sigma_c^2 \ll \theta^2$. From (17) and (19) we have for any size σ_c

$$\frac{\partial V(\sigma_c, \sigma_0, \theta)}{\partial \sigma_c} = \frac{\partial V(\sigma_c, \sigma_0, 0)}{\partial \sigma_c} + \frac{2\lambda\sigma_c}{\pi} \int_0^\infty dk \left(k^2 + \frac{\lambda}{N}\sigma_c^2 \right)^{-1/2} \left\{ \exp\left[\frac{1}{\theta}\left(k^2 + \frac{\lambda}{N}\sigma_c^2 \right)^{1/2} \right] + 1 \right\}^{-1} , \tag{20}$$

where the zero-temperature contribution is given by

$$\frac{\partial V(\sigma_c, \sigma_0, 0)}{\partial \sigma_c} = \sigma_c + \frac{\lambda}{2\pi}\sigma_c \left[\ln\left(\frac{\sigma_c}{\sigma_0} \right)^2 - 2 \right] \tag{21}$$

The minimum at zero temperature is determined from (21) to be

$$\sigma_M = \left(\frac{\lambda}{N} \right)^{1/2} \sigma_0 \exp\left(1 - \frac{\pi}{\lambda} \right) , \tag{22}$$

so that, from (5), a fermion mass is dynamically generated. Note that this result is essentially nonperturbative in the coupling constant since the mass depends on λ in a nonanalytic way. The evaluation of the finite-temperature correction requires an approximate evaluation of the integral in (20) for $(\lambda/N)(\sigma_c/\theta)^2 \ll 1$. Following Dolan and Jackiw,[9] we find

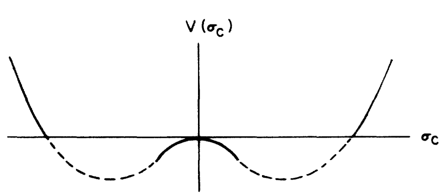

FIG. 2. Form of $V(\sigma_c)$ for $\theta < \theta_{crit}$. Dashed lines are postulated extrapolations.

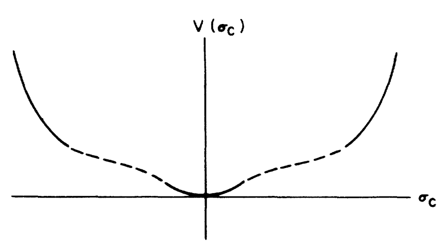

FIG. 3. Form of $V(\sigma_c)$ for $\theta > \theta_{crit}$. Dashed lines are postulated extrapolations.

$$\int_0^\infty dx \frac{1}{(x^2+a^2)^{1/2}} \frac{1}{e^{(x^2+a^2)^{1/2}}+1} = -\tfrac{1}{4}\ln\left(\frac{a}{\pi}\right)^2 - \tfrac{1}{2}\gamma_E + O(a^2) , \quad (23)$$

where $\gamma_E = 0.577\ldots$ is Euler's constant. Using (23) in (20), we find

$$\frac{\partial V(\sigma_c, \sigma_0, \theta)}{\partial \sigma_c} = \sigma_c \left[1 - \frac{\lambda}{\pi}(1+\gamma_E) + \frac{\lambda}{2\pi} \ln \frac{N}{\lambda}\left(\frac{\pi\theta}{\sigma_0}\right)^2 + O\left(\frac{\lambda}{N}\frac{\sigma_c^{\,2}}{\theta^2}\right)\right] . \quad (24)$$

The origin is then always an extremum whose nature is determined by the sign of the sum of the terms in the square brackets of (24). When $\theta < \theta_{\rm crit}$ the terms in the brackets will sum to a negative number and the extremum at the origin will be a maximum, while for $\theta > \theta_{\rm crit}$ the sum will be positive and the extremum becomes a minimum. The critical temperature at which the symmetry is altered is given by

$$\theta_{\rm crit} = \frac{1}{\pi}\left(\frac{\lambda}{N}\right)^{1/2} \sigma_0 \exp\left(1 - \frac{\pi}{\lambda} + \gamma_E\right) \quad (25)$$

or, if we use (22) and (5),

$$\theta_{\rm crit} = a M_F(\sigma_0, \lambda) , \quad (26)$$

where $a = \pi^{-1}\exp(\gamma_E) \simeq 0.567$. Thus, the critical temperature is of the order of the mass dynamically generated at zero temperature.

IV. RENORMALIZATION GROUP

In imposing the Coleman-Weinberg renormalization condition (18), we have introduced an arbitrary renormalization parameter σ_0. We will now show that our result for the critical temperature is actually independent of σ_0, as any physical quantity should be.

Since all renormalizations are done at $\theta = 0$, $V_1(\sigma_c, \sigma_0) \equiv V(\sigma_c, \sigma_0, 0)$ obeys the renormalization-group equation

$$\left[\sigma_0 \frac{\partial}{\partial\sigma_0} + \beta(\lambda)\frac{\partial}{\partial\lambda} - \gamma(\lambda)\sigma_c\frac{\partial}{\partial\sigma_c}\right] V(\sigma_c, \sigma_0, 0) = 0 , \quad (27)$$

so that, using (19), we have

$$\beta(\lambda) = 2\lambda\gamma(\lambda) \quad (28)$$

and

$$\gamma(\lambda) = -\frac{\lambda/2\pi}{1+\lambda/2\pi} . \quad (29)$$

Note that at zero temperature the theory is asymptotically free. The finite-temperature contribution to the effective potential $V_2(\sigma_c, \theta)$ is independent of σ_0 and only depends on λ and σ_c in the combination $\lambda\sigma_c^{\,2}$ [see (17)]. Thus V_2 also satisfies (27) provided (28) holds, and we have

$$\left[\sigma_0 \frac{\partial}{\partial\sigma_0} + \beta(\lambda)\frac{\partial}{\partial\lambda} - \gamma(\lambda)\sigma_c\frac{\partial}{\partial\sigma_c}\right] V(\sigma_c, \sigma_0, \theta) = 0 , \quad (30)$$

with $\beta(\lambda)$ and $\gamma(\lambda)$ unaltered. In particular, the asymptotic freedom is not affected by the temperature.

To see that M_F is independent of σ_0, we use (5), (22), (28), and (29) to find that

$$\left[\sigma_0 \frac{\partial}{\partial\sigma_0} + \beta(\lambda)\frac{\partial}{\partial\lambda}\right] M_F(\sigma_0, \lambda) = 0 , \quad (31)$$

i.e., M_F satisfies the homogeneous renormalization-group equation and is thus independent of σ_0. But the critical temperature is given in terms of M_F (26), so $\theta_{\rm crit}$ is also independent of σ_0.

V. THE 1/N EXPANSION

We now consider the possibility that, in equilibrium, $\sigma(x)$ depends on coordinates. Within the context of the perturbative $1/N$ expansion utilized in the previous sections this difficulty does not arise. However, the leading terms in the $1/N$ expansion do not include σ loops, so that this particular perturbative approach ignores fluctuations of the σ "particle." It is possible that due to these fluctuations the minimum value of σ will be $+\sigma_M$ for some values of the coordinates and $-\sigma_M$ for others, so that, on the average over an arbitrarily long system, σ will vanish. Such nonperturbative effects in N would be difficult to evaluate within a purely field-theoretic scheme.

However, there are arguments[12] in the framework of statistical mechanics that call the $1/N$ expansion into question in one space dimension. In particular, though it takes energy E to break up a linear chain or ordered one-dimensional system, entropy is gained so that the minimum of $E - TS$ for $T \neq 0$ may occur for the position-dependent σ_M. Results similar to ours have been found[13] in the mean-field-theory approximation to the one-dimensional Ginzburg-Landau equations, though the exact numerical evaluations do not confirm the mean-field solutions near the critical temperature.[14]

VI. DISCUSSION

Using the $1/N$ expansion, we have found to all orders in the coupling that the fermion mass dynamically generated at zero temperature will vanish as the temperature is raised. The critical temperature at which chiral symmetry is restored is explicitly calculated (26) in terms of the zero-temperature mass.

It should be noted that the critical temperatures found here are much more accessible than those arising from theories in which scalar fields are used to break the symmetry.[1] Rather than having $\theta_{crit} \sim 10^{15}$ °K, we have for electrons $\theta_{crit} \sim 3\times10^{9}$ °K, while for protons $\theta_{crit} \sim 6\times10^{12}$ °K. Thus, in addition to the study of the early universe, our results may be of some interest in the study of the formative stages of neutron stars and pulsars, where temperatures greater than 10^{10} °K can invariably be found.[15]

Subsequent to the completion of our work, we became aware of the papers[16] of Dashen, Ma, and Rajaraman, Jacobs, and Hiro-O-Wada, who find the critical temperature of Eq. (26).

ACKNOWLEDGMENTS

We are grateful to Alan Luther and especially Paul Martin for extended and informative discussions of the physical limitations of the O(N) model and the problems with phase transitions in one space dimension. We also thank Soo Yong Park, who unselfishly provided his time and his ideas to us during the course of this work.

*Work supported in part by the National Science Foundation under Grant No. NSF GP-40397X.

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