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Instability of intermediate singularities in general relativity

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It has been shown that "intermediate" singularities, where all Riemann tensor invariants are finite, occur in certain cosmological models. Associated with the singularities are Cauchy horizons, across which the matter flows into a stationary region of space-time. We investigate scalar-wave propagation in these spaces. Our results suggest that the intermediate singularities become localized curvature singularities while the Cauchy horizons are a stable feature of the models.

I. INTRODUCTION

In a recent paper[1] it was shown that the spatial homogeneity of certain perfect-fluid relativistic cosmological models comes to an end in a region where the hypersurfaces of homogeneity are tending to become null. An intermediate singularity and associated Cauchy horizon result, the matter flowing across the horizon in a nonsingular way into a stationary inhomogeneous region of spacetime. At the intermediate singularity all Riemann tensor components are finite in an orthonormal frame intrinsically defined by the fluid, but any orthonormal frame parallel propagated along a curve leading into the singularity suffers an infinite Lorentz rotation relative to this fluid frame, so Riemann tensor components in this basis diverge. In Ref. 1 it was conjectured that intermediate singularities are unstable against developing into curvature singularities when small perturbations of the gravitational field are applied. This is because a photon projected along a null generator of the horizon into the intermediate singularity gets infinitely blue-shifted as it approaches the singularity; thus it arrives with infinite energy and would tend to cause a curvature singularity. One would also like to know if the horizon is stable against perturbations.

To investigate these problems further we shall consider perturbations of the type-V LRS (locally rotationally symmetric) spacetimes, the simplest members of our class. A complete calculation might use electromagnetic or gravitational perturbations, but as is often done (e.g., Ref. 2) we shall use the massless scalar field as a model. This will give essentially the same results while reducing irrelevant technical details. We shall assume that the field amplitude $\varphi(x^i)$ is so small that it does not couple to the gravitational field through the Einstein field equations. Of course if at any point φ is tending to become infinite, this assumption breaks down and we would need to to perform the complete self-consistent calculation, allowing φ to couple to the gravitational field in order to see what is really happening. This is a formidable task, and it seems reasonable to conjecture instead that $\varphi \to \infty$ implies a curvature singularity. On the other hand, where φ remains bounded this would seem to indicate the stability of that region against developing into a curvature singularity.

II. THE SPACETIME GEOMETRY AND THE SCALAR WAVE EQUATION

We shall consider for simplicity only *dust-filled* type-V LRS spacetimes. Using a tetrad system intrinsically defined by the matter, associated global coordinates can be found so that the metric takes the form

$$ds^2 = -(dt + a\,dz)^2 + a^2 Z^2 dz^2 + X^2 e^{-2z}(dx^2 + dy^2)\,, \tag{2.1}$$

where X and Z are positive functions of t only, and a is a nonzero constant. The matter density $\rho > 0$ is equal to $\rho_0 X^{-2} Z^{-1}$, where $\rho_0 > 0$ is a constant. ρ, X, and Z satisfy the "constraint equations"

$$\rho = \frac{2}{a}\frac{\dot X}{X}\left(1 - \frac{1}{Z^2}\right) - \frac{2\dot Z}{aZ} - \frac{2}{a^2 Z^2}\,, \tag{2.2}$$

$$\rho = \frac{2\dot X}{XZ^2}\left(\frac{\dot X}{X} + \frac{1}{a}\right) + \frac{\dot X}{X}\left(\frac{\dot X}{X} + \frac{2\dot Z}{Z}\right) - \frac{3}{Z^2}\left(\frac{\dot X}{X} + \frac{1}{a}\right)^2\,, \tag{2.3}$$

with $\dot X = dX/dt$, etc., while X and Z obey (with m a constant)

$$\dot X^2 = \frac{\rho_0 a}{3X} + m^2\,, \tag{2.4}$$

$$Z = \frac{1}{m}\left(\dot X + \frac{X}{a}\right)\,. \tag{2.5}$$

Equations (2.4) and (2.5) may be integrated exactly,[4,5] but the general solution is rather unwieldy and we shall not use it here. In the metric (2.1) the surfaces $\{t = \text{constant}\}$ are the surfaces of transitivity of a four-parameter group of motions, the coordinates x and y being ignorable and the spacetime having rotational symmetry at each point about an axis parallel to the z axis. We choose t so that as $t \to -\infty$, X and $Z \to \infty$. Here the surfaces $\{t = \text{constant}\}$ are spacelike. If $a < 0$, it can be shown[1] that the dust encounters a curvature singularity at a finite value of t where the surfaces $\{t = \text{constant}\}$ are still spacelike. However, if

$$a > 0 , \qquad (2.6)$$

it can be shown[1] that these surfaces become null and then timelike before the curvature singularity is reached. We can take $t = 0$ to be the value of t at which the surfaces become null, so $Z > 1$ for $t < 0$, $Z = 1$ at $t = 0$ and $Z < 1$ for $t > 0$. (At the curvature singularity we have $X = 0$, but $1 > Z \neq 0$.) We shall henceforth assume that (2.6) holds. The fluid flow vector $\vec{u}$ has a $\partial/\partial z$ component which is negative,[3] so that the dust moves in the $-z$ direction across the homogeneous surfaces $\{t = \text{constant}\}$. The null surface $t = 0$ has two parts, one being the Cauchy horizon across which the matter flows into a stationary inhomogeneous region, the other being the intermediate singularity (cf. Fig. 1). (Note that our arrow of time is the opposite of the more usual one.)

The field amplitude $\varphi(x^i)$ satisfies the massless scalar wave equation

$$\Box\varphi \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i}\left(\sqrt{-g}\; g^{ij} \frac{\partial \varphi}{\partial x^j}\right) = 0 ,$$

where g_{ij} is the metric tensor and $g = \det g_{ij}$. The x, y part of this equation for the metric (2.1) is

$$\frac{e^{2z}}{X^2}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) . \qquad (2.7)$$

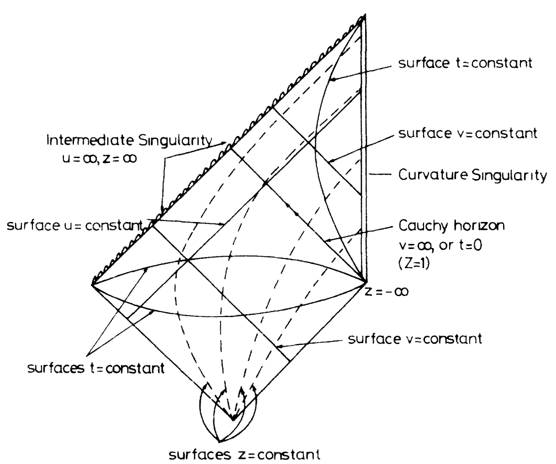

FIG. 1. A conformally rescaled diagram of a 2-surface x = constant, y = constant showing the global structure of the solutions and the (t, z) and (u, v) coordinate systems.

Hence because of the LRS symmetry we can separate variables, the ignorable coordinates x and y contributing only multiplicative components of the form

$$\sin(bx + x_0)\sin(cy + y_0)$$

and

$$\sinh(bx + x_0)\sinh(cy + y_0)$$

to φ. Thus the eigenvalues of (2.7) are of the form $EX^{-2}e^{2z}$, where $E \geq 0$ for the hyperbolic functions and $E \leq 0$ for the circular functions of x, y, and φ has the form

$$\sum_{b,c} \bar{\varphi}_{bc}(t, z)\sin(bx + x_0)\sin(cy + y_0) + \sum_{b,c} \bar{\varphi}'_{bc}(t, z)\sinh(bx + x_0)\sinh(cy + y_0) , \qquad (2.8)$$

where the values of b, c, x_0, y_0 and the choice of circular or hyperbolic functions are determined by the x, y boundary conditions on φ. We shall return to the question of circular versus hyperbolic functions later, but the values of b, c, x_0, and y_0 are of no interest in the situation we are considering, and without any essential loss of generality we shall simply consider one term only of this series. Then $\bar{\varphi}$ satisfies the equation

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^A}\left(\sqrt{-g}\, g^{AB} \frac{\partial \bar{\varphi}}{\partial x^B}\right) + \frac{E e^{2z}}{X^2}\, \bar{\varphi} = 0 ,$$

where A and B label t and z. Since $\sqrt{-g} = aX^2 Z e^{-2z}$ is a function of t, z only, this is a second-order hyperbolic equation in two variables only. As is usual in dealing with such equations, it is convenient to choose characteristic coordinates, which are here simply double null coordinates in the t, z surface. We set

$$u = az + \int^t \frac{dt'}{Z(t') + 1} ,$$
$$v = -az + \int^t \frac{dt'}{Z(t') - 1} . \qquad (2.9)$$

Then metric is then

$$ds^2 = (1 - Z^2)\, du\, dv + X^2 e^{-2z}(dx^2 + dy^2) . \qquad (2.10)$$

To determine the run of the null coordinates u, v we need the behavior of $Z(t)$ near $t = 0$. From (2.4) and (2.5) it follows straightforwardly that

$$Z = 1 - Z_1 t + O(t^2) , \qquad (2.11)$$

where Z_1 is a strictly positive constant, so as $t \to 0$

$$v \sim -az - \frac{1}{Z_1}\ln|t| + \text{const.} \qquad (2.12)$$

Thus for bounded z (or equivalently, bounded u) $v \to +\infty$ as $t \to 0$. If we keep v bounded as $t \to 0$, (2.12) shows that

$$z \sim -\frac{1}{aZ_1}\ln|t| \to +\infty$$

as $t \to 0$, so (2.9) implies

$$u \sim az \sim -\frac{1}{Z_1}\ln|t| \to \infty \text{ as } t \to 0\,. \qquad (2.13)$$

We assert now that the null surface $v = \infty$ is the Cauchy horizon while the intermediate singularity is at $u = z = \infty$. [Note that from (2.1) the hypersurfaces (z = constant) are always timelike for finite z.] This follows from the fact that the matter (which crosses the horizon) moves in the direction of decreasing z when t increases, as does v; put another way, (2.9) implies

$$-\frac{\partial}{\partial z} = a\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right).$$

As v decreases from $+\infty$ through positive values we move away from the horizon on either side. Figure 1 gives a diagrammatic representation of this coordinate system in a 2-surface x = constant, y = constant; we have conformally rescaled so that infinity appears on the picture.

Transformation (2.9) puts our wave equation into the form

$$\frac{2}{\Delta}\frac{\partial^2\varphi}{\partial u\,\partial v} + \frac{1}{\Delta Q}\left(\frac{\partial Q}{\partial u}\frac{\partial \varphi}{\partial v} + \frac{\partial Q}{\partial v}\frac{\partial \varphi}{\partial u}\right) + \frac{E}{Q}\varphi = 0\,,$$

where we have dropped the overbar from $\bar{\varphi}$ and written $\Delta = Z^2 - 1$, $Q = X^2 e^{-2z}$. The substitution

$$\varphi = \frac{e^z}{X}\psi \qquad (2.14)$$

reduces the equation to the self-adjoint form

$$\frac{\partial^2\psi}{\partial u\,\partial v} + A(u, v)\psi = 0\,, \qquad (2.15)$$

where

$$A(u, v) = \frac{\Delta E}{2Q} - \frac{Q^{-3/2}\partial^2(Q^{1/2})}{\partial u\,\partial v}$$
$$= \frac{e^{2z}}{X^2}\left[(Z^2 - 1)\frac{E}{2} - \frac{X_{uv}}{X} + \frac{X_v z_u}{X} + \frac{X_u z_v}{X} + z_{uv}\right]$$

and we have written $X_u = \partial X/\partial u$, etc. Using (2.9), a little algebra shows that

$$A = (Z - 1)e^{2z}B(t)\,, \qquad (2.16)$$

where

$$B(t) = \frac{Z+1}{X^2 Z}\left[E + \frac{\dot{X}\dot{Z}}{2XZ} + \frac{\dot{X}}{2aX} - \frac{\dot{Z}}{4aZ}\right].$$

From (2.2) and (2.3) it follows that $B(t)$ is bounded except at the curvature singularity and at $t = -\infty$. To investigate the behavior of ψ and φ near the horizon ($v = \infty$, u bounded) and the intermediate singularity ($u = \infty$, v bounded) we shall need the asymptotic forms of $A(u, v)$ there. From (2.11) and (2.12) we have that as $v \to \infty$ for bounded u,

$$A \sim B' e^{-v}\,, \qquad (2.17)$$

where B' is a bounded function of t which may be taken as constant near $t = 0$. Similarly, from (2.11) and (2.13) we find that as $u \to \infty$ with v bounded,

$$A \sim B'' e^{u}\,, \qquad (2.18)$$

with $B'' = B''(t)$ constant near $t = 0$.

III. CAUCHY PROBLEMS

To investigate the stability of the spacetime we consider various Cauchy problems for φ—that is, we give bounded initial data for it on some noncharacteristic (spacelike) initial surface. What particularly concerns us is whether φ has any singularities in the Cauchy development of the initial surface. From the form of (2.14), (2.15), and (2.16) it is clear that such singularities could only occur at either the curvature singularity, the intermediate singularity, or the horizon $v = \infty$. We must therefore choose initial surfaces whose Cauchy developments contain these regions. In the spatially homogeneous region $t < 0$ an obvious candidate for an initial surface is a $\{t = \text{constant}\}$ hypersurface; this, however, has the disadvantage that its Cauchy development is limited by the null surface $v = \infty$ which is one of the regions of interest. If, however, the initial data for φ is only

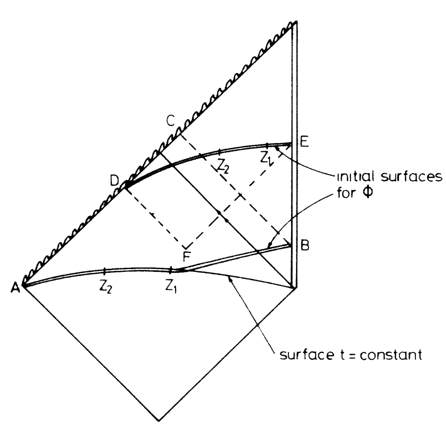

FIG. 2. Initial surfaces for the Cauchy problem. AC, BC represent the boundary of the Cauchy development of the surface AB, while DF, EF show the boundary of the Cauchy development of DE.

nonzero for a finite range of values of z, $z_1 \leq z \leq z_2$, on this hypersurface we can obtain a suitable initial surface by choosing a spacelike hypersurface which coincides with the $\{t = \text{constant}\}$ surface for $z_1 \leq z \leq z_2$, but crosses the horizon $v = \infty$ as shown in Fig. 2. (The author is indebted to Dr. S. W. Hawking for this suggestion.) The Cauchy development ABC of this surface then contains that portion of the horizon where φ may be nonzero.

In the above we have been investigating the situation when φ propagates in the direction of increasing t; we might wish to consider what happens if if propagates in the direction of *decreasing* t —indeed such a direction of time gives an expanding rather than contracting spatially homogeneous region and so corresponds better to the observed universe. We therefore need an initial surface in the stationary inhomogeneous region. However, because of the timelike character of the curvature singularity in this region there are no global Cauchy surfaces there: "past"-directed timelike curves (i.e., those directed in the t-increasing direction) can avoid any spacelike hypersurface in this region by hitting the singularity instead. The best we can do is to choose a partial Cauchy surface whose Cauchy development includes a part of the horizon. For initial data of compact support in z (i.e., which is nonzero only for a finite range of z) we may choose the surface DE as shown in Fig. 2. In this case the intermediate singularity does not even lie on the boundary of the Cauchy development as in the earlier case.

We have restricted the initial data for φ to have compact support in z on the initial surfaces; if we do not do the same for the x and y variation of φ, we must choose the circular functions in (2.8) to ensure that φ is bounded initially, so $E < 0$. If we assume also compact x and y support for φ on the initial surfaces, there is no restriction on the sign of E.

IV. STABILITY

We can now investigate the behavior of φ at the intermediate singularity and at the Cauchy horizon. We shall do this by considering Eq. (2.15) for ψ as $u \to \infty$ and $v \to \infty$. Since the initial data for φ has compact support in z, the scaling (2.14) is finite on those portions of the initial surfaces where $\varphi \neq 0$. Thus ψ also has bounded initial data with compact support in z. We assert now that *given bounded initial data for φ with compact support in z, φ remains bounded on the horizon $v = \infty$, regardless of the direction of time assumed.*

Because of the scaling (2.14), which is finite on $v = \infty$, the result for φ will follow if we can prove it for ψ. From (2.17) we know that $A(u, v) \to 0$ on $v = \infty$. The question is whether A becomes zero rapidly enough. To answer this we bring the horizon in to a finite distance in both the spatially homogeneous and stationary regions by replacing v by $v' = \tan^{-1} v$. Then the horizon $v = \infty$ is $v' = \frac{1}{2}\pi$, and (2.15) becomes

$$\frac{\partial^2 \psi}{\partial u \partial v'} + \frac{A(u, v')}{\cos^2 v'} \psi = 0 . \tag{4.1}$$

The coefficient of ψ in this equation can be rewritten as $A(u, v)(1 + v^2)$, which by (2.17) becomes $B' e^{-v}(1 + v^2)$, which tends to zero as $v \to \infty$. Thus (4.1) remains perfectly regular at the finite points (u bounded, $v' = \frac{1}{2}\pi$), and by the known regularity of the Cauchy problem for the hyperbolic equation in two variables (see e.g. Refs. 6 and 7) ψ is bounded there. Thus ψ and hence φ is bounded on the horizon.

The situation near the intermediate singularity is somewhat more complicated as $A(u, v)$ may diverge either positively or negatively as $u \to \infty$. The sign of this divergence is determined by that of

$$\frac{E}{2} + \frac{\dot{X}\dot{Z}}{2XZ} + \frac{\dot{X}}{2aX} - \frac{\dot{Z}}{4aZ} .$$

Using (2.2) and (2.3) it follows easily that

$$\frac{\dot{X}\dot{Z}}{2XZ} + \frac{\dot{X}}{2aX} - \frac{\dot{Z}}{4aZ} \equiv -\tfrac{1}{2} E_0 > 0 \text{ on } t = 0 .$$

Thus for $E > E_0$, $A(u, v) \to \infty$ as $u \to \infty$, while for $E < E_0$, $A(u, v) \to -\infty$ as $u \to \infty$; hence in (2.18) $B'' > 0$ (< 0) accordingly as $E > E_0$ ($< E_0$).

With the asymptotic form (2.18), Eq. (2.15) for ψ becomes

$$\frac{\partial^2 \psi}{\partial u \partial v} + B'' e^u \psi = 0 ,$$

which is separable, having a solution of the form

$$\psi = \sum_k \psi_k \exp\left(k B'' e^u - \frac{v}{k}\right) ,$$

where the k's are separation constants and the ψ_k are constants. By (2.13), the scaling (2.14) gives

$$\varphi = \sum_k \frac{\psi_k}{X} \exp\left(k B'' e^u + \frac{u}{a} - \frac{v}{k}\right) .$$

The terms of this series tend to zero or infinity a according as kB'' is negative or positive, so we see that *for $E > E_0$ the terms with positive k diverge, while for $E < E_0$ those with negative k diverge.* Since the terms diverge at different rates there is no possibility of cancellation, so that φ *itself must diverge in these cases.*

V. CONCLUSION

We have considered various Cauchy problems for the massless scalar wave equation where the initial data were bounded and had compact support (in z). We may conclude from the work of the previous section that the picture of the perturbed situation given in Ref. 1 is broadly correct: That is, the intermediate singularity becomes a "little bang" (a localized curvature singularity) which most of the matter avoids. In fact, some special types of perturbation (e.g., those with $E < E_0$ and all $k > 0$) remain at the intermediate singularity, but in the generic case the field amplitude diverges. The horizon is stable against all scalar-wave perturbations except possibly those not having compact support in z on some initial surface.

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