Electromagnetic radiation due to spacetime oscillations*

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Wave equations are derived in the Newman-Penrose formalism for mixed electromagnetic and gravitational perturbations on both a flat spacetime background and a slightly charged ($Q^2 \ll GM^2$) Reissner-Nordström background. The physical meaning of these equations is discussed and analytical results are derived for nonrelativistic sources and for ultrarelativistic particle motions. The relationship between even-parity (TM) electromagnetic radiation multipoles in the long-wavelength approximation and static multipoles is shown to be the same as for classical radiation, suggesting a simple picture for electromagnetic radiation induced by gravitational perturbations.

I. INTRODUCTION

Considerable interest has recently been shown in the problem of interacting gravitational and electromagnetic waves. Zerilli[1] has given equations in the Regge-Wheeler formalism for mixed electromagnetic and gravitational perturbations in a Reissner-Nordström background. His equations have been used to compute the two types of radiation generated by a particle falling into a Reissner-Nordström black hole.[2] Moncrief,[3] using a variational principle, has proved that the Reissner-Nordström geometry is stable against electromagnetic and gravitational perturbations and Gerlach[4] has used a WKB approach to study the interconversion of gravitational and electromagnetic radiation in a region of strong electromagnetic background fields.

We have previously given equations in the Newman-Penrose (NP) formalism[5] for mixed perturbations in both a flat spacetime background and a background with a slightly charged ($Q^2 \ll GM^2$) Reissner-Nordström black hole.[6] One of us (D.M.C.) has extended these results to a slightly charged Kerr-Newman black hole.[7] Though the equations in these papers lack the full generality of the Zerilli equations (in which Q^2 can be comparable to GM^2), they are somewhat easier to work with, allow a fairly straightforward interpretation, and—perhaps most important—can be extended to a Kerr background. In the present paper the details missing from the earlier brief paper are supplied and the equations are applied to several interesting physical situations.

Section II sets forth the theoretical framework. Perturbation equations are derived from the Bianchi identities and Maxwell's equations, and gauge transformations are discussed. In Section III several results are derived for flat spacetime backgrounds. Multipole radiation fields and static electric multipole fields are calculated and a simple physical picture is given to explain the nature of the radiation. Synchrotronlike electromagnetic spectra from uncharged particles in a Coulomb background are also derived. Section IV deals with the slightly charged black-hole background, especially with the mathematical techniques required for dealing with it.

We use here conventions which differ somewhat from the Newman-Penrose notation but are more convenient in a spherically symmetric background. The subscripts on the Ψ_i (Weyl tensor projections) and Φ_i (projections of $F^{\mu\nu}$) indicate the spin-weight of the fields. The notation for the Φ_{ij}, the Ricci tensor projections, are the same as those of Newman and Penrose. (The spin-weight of these fields is the second subscript minus the first.) A caret appearing over a quantity indicates that the quantity has been "despun," that is, the appropriate number of spin-weight-increasing ($-2^{-1/2}\eth$) or -decreasing ($-2^{-1/2}\bar{\eth}$) operators have been applied to change the spin-weight to zero. This not only simplifies the appearance of equations but helps to distinguish even- and odd-parity parts of perturbations; even and odd parts of despun quantities refer respectively to the real and imaginary parts of despun quantities.[8] This notation and others follow that of Ref. 8. Equations from Ref. 8, many of which are necessary to the present paper, will be indicated by square brackets, e.g. [B6i]. Equations from the Newman-Penrose paper[5] will be written, e.g., as (NP6.6).

II. WAVE EQUATIONS

A. Black-hole background

Here we consider the effects of gravitational perturbations in distorting the spherically symmetric electric field around a nonrotating black hole with a charge small enough that it does not significantly change the geometry. (The gravitational background then is essentially Schwarzschild rather than Reissner-Nordström.) The sources of stress-energy giving rise to the gravitational perturbations are assumed to be uncharged for simplicity; the modifications of the Maxwell equations for charged sources are straightforward.

In the NP formalism[5] the electromagnetic field is represented by the three complex fields Φ_{-1}, Φ_0, Φ_1. By the peeling theorem[5] Φ_0 and Φ_1 fall off too fast to represent outgoing radiation and

$$\Phi_{-1} \equiv F_{\mu\nu} m^{*\mu} n^{\nu}, \tag{2.1}$$

which falls off as $1/r$, contains all the information about outgoing radiation.

For the NP null tetrad fields $\vec{l}$, $\vec{n}$, $\vec{m}$, $\vec{m}^*$ we take a "special system," i.e., a system in which the tetrad is parallel propagated in the $\vec{l}$ direction. As in Ref. 8 we take the unperturbed tetrad in the Schwarzschild background to be

$$\begin{aligned} &\vec{l} = f^{-1}\vec{e}_t + \vec{e}_r, \quad \vec{n} = \tfrac{1}{2}\vec{e}_t - \tfrac{1}{2} f \vec{e}_r, \\ &\vec{m} = 2^{-1/2} r^{-1}(\vec{e}_\theta + i\vec{e}_\phi/\sin\theta), \quad f \equiv 1 - 2M/r, \end{aligned} \tag{2.2}$$

where $\vec{e}_t$, $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\phi$ are a coordinate basis in terms of the usual Schwarzschild coordinates. The nonvanishing spin coefficients, Ψ_i, and Φ_i for this tetrad system in the unperturbed geometry are

$$\begin{aligned} &\rho = -1/r, \quad \beta = -\alpha = \cot\theta / r 2\sqrt{2}, \\ &\gamma = M/2r^2, \quad \mu = -f/2r, \\ &\Psi_0 = -M/r^3, \quad \Phi_0 = -Q/2r^2. \end{aligned} \tag{2.3}$$

Equations in the NP formalism are greatly simplified if perturbation fields are despun and decomposed into spherical harmonics. (To simplify the notation, multipole indices will not explicitly appear on the fields. For example μ, not μ_{lm}, will be used since the order of the multipole will always be clear from the context.)

In despun NP form four of the eight Maxwell equations become the two complex equations

$$\Delta r^2 \hat{\Phi}_{0B} - r\hat{\Phi}_{-1} = Q(\hat{\mu}_B - \hat{U}_B/r), \tag{2.4}$$

$$r^2 D r\hat{\Phi}_{-1} + \tfrac{1}{2} l(l+1)\hat{\Phi}_{0B} r^2 = Q\hat{\omega}_B^*. \tag{2.5}$$

Here D and Δ are the differential operators

$$D \equiv f^{-1}\partial_t + \partial_r, \quad \Delta \equiv \tfrac{1}{2}\partial_t - \tfrac{1}{2} f\partial_r. \tag{2.6}$$

The quantities U_B and ω_B are perturbations in metric functions[5] and μ_B is the perturbation in the spin coefficient μ. The subscript B, indicating a perturbation, will often be omitted when omission introduces no ambiguities.

The terms on the right in Eqs. (2.4) and (2.5), terms which are missing in Maxwell's equations for an unperturbed Schwarzschild background, represent effective charge currents due to gravitational distortion of the Coulomb field of the central charge. Equations (2.4) and (2.5) can be combined to form a single second-order equation for $r\hat{\Phi}_{-1}$:

$$W(\hat{\Phi}_{-1} r) = Q[l(l+1)(r^{-1}\hat{U}_B - \hat{\mu}_B) + 2\Delta\hat{\omega}_B^*], \tag{2.7}$$

where the d'Alembertian-type operator W is

$$\begin{aligned} W &\equiv 2\Delta r^2 D + l(l+1) \\ &= (\partial_t - f\partial_r) r^2 (f^{-1}\partial_t + \partial_r) + l(l+1). \end{aligned} \tag{2.8}$$

By itself Eq. (2.7) is of little value since the right-hand side is unknown. It is only when this result is combined with another similar wave equation that a useful result emerges. This second wave equation results from two of the equations, [B6c] and [B6g], expressing the Bianchi identities in NP form[9]:

$$D\hat{\Psi}_{-1} r^2 + \tfrac{1}{2} r l(l+1)\hat{\Psi}_{0B} = 3\hat{\omega}_B^* M/r^2 + S_{\text{I}}, \tag{2.9a}$$

$$\Delta\hat{\Psi}_{0B} r^3 - r^2\hat{\Psi}_{-1} = 3M(\hat{\mu}_B - \hat{U}_B/r) + S_{\text{II}}. \tag{2.9b}$$

The source terms S_{I}, S_{II} consisting of projections of the Ricci tensor or equivalently the stress-energy tensor are

$$\begin{aligned} S_{\text{I}} = \tfrac{1}{3} r^2 \{ &D\hat{\Phi}_{21} - r^{-1}\hat{\Phi}_{20} \\ &- 2[\tfrac{1}{2} l(l+1) r^{-1}\hat{\Phi}_{11} + f^{-1}\Delta f\hat{\Phi}_{10}]\}, \end{aligned} \tag{2.10a}$$

$$S_{\text{II}} = \tfrac{1}{3} r^2 [\hat{\Phi}_{21} - r^2 D r^{-1}\hat{\Phi}_{22} + 2(\Delta r\hat{\Phi}_{11} - \hat{\Phi}_{12})], \tag{2.10b}$$

where the stress-energy terms appearing here and in subsequent equations are defined[10] by

$$\begin{aligned} &\Phi_{00} = 4\pi T_{\mu\nu} l^\mu l^\nu, \quad \Phi_{22} = 4\pi T_{\mu\nu} n^\mu n^\nu, \\ &\Phi_{11} = 2\pi T_{\mu\nu}(n^\mu l^\nu + m^\mu m^{*\nu}), \\ &\Phi_{12} = 4\pi T_{\mu\nu} n^\mu m^\nu, \quad \Phi_{01} = 4\pi T_{\mu\nu} l^\mu m^\nu, \\ &\Phi_{02} = 4\pi T_{\mu\nu} m^\mu m^\nu, \quad \Lambda = \tfrac{1}{3}\pi T^\mu{}_\mu, \end{aligned} \tag{2.11}$$

and by

$$\Phi_{ij} = \Phi_{ji}^*.$$

Equations (2.9) can be combined to give a single wave equation, with the wave operator of Eq. (2.8),

$$\begin{aligned} W(\hat{\Psi}_{-1} r^2) = 3M[&l(l+1)(r^{-1}\hat{U}_B - \hat{\mu}_B) + 2\Delta\hat{\omega}_B^*] \\ &+ 2\Delta S_{\text{I}} r^2 - l(l+1) S_{\text{II}}. \end{aligned} \tag{2.12}$$

Since the same combination of gravitational perturbation terms U_B, μ_B, and ω_B^* appear here as in Eq. (2.7) the two wave equations can be combined to give a single wave equation:

$$\begin{aligned} W(\hat{\Theta}) &= (Q/3M)[l(l+1)S_{\text{II}} - 2\Delta r^2 S_{\text{I}}] \\ &\equiv (Q/3M) S_\Theta, \end{aligned} \tag{2.13}$$

for a field variable

$$\Theta \equiv r\Phi_{-1}-(Q/3M)r^2\Psi_{-1}, \qquad (2.14)$$

which contains information about both the electromagnetic and the Weyl tensors.

With the equations of motion ($T_\mu{}^\nu{}_{;\nu}=0$) in NP form[11]

$$Dr^4\hat{\Phi}_{11}+3r^4D\hat{\Lambda}+r^2f^{-2}\Delta f^2r^2\hat{\Phi}_{00}-r^3\hat{\Phi}_{10}-r^3\hat{\Phi}_{01}=0, \qquad (2.15a)$$

$$r^2Dr^2\hat{\Phi}_{22}+\Delta r^4\hat{\Phi}_{11}+3r^4\Delta\hat{\Lambda}-r^3\hat{\Phi}_{12}-r^3\hat{\Phi}_{21}=0, \qquad (2.15b)$$

$$Dr^3\hat{\Phi}_{12}+f^{-1}\Delta r^3f\hat{\Phi}_{01}+\tfrac{1}{2}r^2l(l+1)(\hat{\Phi}_{11}-3\hat{\Lambda})-r^2\hat{\Phi}_{02}=0, \qquad (2.15c)$$

the source in (2.13) can be put in the form

$$S_\Theta=-W(r^2\hat{\Phi}_{21})-l(l+1)r^2Dr\hat{\Phi}_{22}+2r^2\Delta r\hat{\Phi}_{20}-2fr^3D\hat{\Phi}_{21}+2[l(l+1)-3f]r^2\hat{\Phi}_{21}. \qquad (2.16)$$

This form is convenient since the $r^2\Phi_{21}$ term in the W operator can be combined with Θ so that the source terms have no second derivatives. In the radiation zone the source usually vanishes so that the $r^2\Phi_{21}$ term can usually be ignored in radiation calculations.

In order to extract Φ_{-1} from Θ we need an independent way of evaluating Ψ_{-1}. In the radiation zone this can be done with one of the sourceless (assuming no sources in the radiation zone) Bianchi identities [B6d]

$$Dr\hat{\Psi}_{-2}=-\tfrac{1}{2}(l-1)(l+2)\hat{\Psi}_{-1}+3\hat{\lambda}M/r^2 \qquad (2.17)$$

once Ψ_{-2} is known. The λ term can be ignored in the radiation zone since $\Psi_{-1}\sim r^{-2}$ and $\lambda\sim r^{-1}$ according to the peeling theorem.[5] To find Ψ_{-2} we can combine the Bianchi identities[9] [B6d] and [B6h],

$$Dr\hat{\Psi}_{-2}+\tfrac{1}{2}(l-1)(l+2)\hat{\Psi}_{-1}-3\hat{\lambda}M/r^2=-\tfrac{1}{2}(l-1)(l+2)\hat{\Phi}_{21}-\Delta r\hat{\Phi}_{20}, \qquad (2.18a)$$

$$f\Delta r^4f^{-1}\hat{\Psi}_{-1}-r^3\hat{\Psi}_{-2}+3\hat{\nu}Mr=\tfrac{1}{2}l(l+1)r^3\hat{\Phi}_{22}+r^2f\Delta r^2f^{-1}\hat{\Phi}_{21}, \qquad (2.18b)$$

with the auxiliary equation [B5h]

$$\Delta\hat{\lambda}r^2f^{-1}+\tfrac{1}{2}rf^{-1}(l-1)(l+2)\hat{\nu}=-r^2f^{-1}\hat{\Psi}_{-2}, \qquad (2.19)$$

to arrive at the single wave equation

$$r^{-4}f\Delta[r^4f^{-1}D(\hat{\Psi}_{-2}r)]+[\tfrac{1}{2}r^{-1}(l-1)(l+2)+3Mr^{-2}]\hat{\Psi}_{-2}=S_\Psi$$
$$=-\tfrac{1}{2}(l-1)(l+2)[\tfrac{1}{2}l(l+1)r^{-1}\hat{\Phi}_{22}+fr^{-2}\Delta(r^2f^{-1}\hat{\Phi}_{21})+r^{-4}f\Delta(r^4f^{-1}\hat{\Phi}_{21})]-r^{-4}f\Delta[r^4f^{-1}\Delta(r\hat{\Phi}_{20})]. \qquad (2.20)$$

The set of equations (2.13), (2.17), (2.20) gives us, at least in principle, a way of determining Φ_{-1} and Ψ_{-2} and therefore (as will be shown) all information about outgoing radiation. This procedure fails, however, for the dipole case $l=1$. Equation (2.13) remains valid but Eqs. (2.17) and (2.20) are vacuous for $l=1$. In Appendix A it is shown that for the dipole case Ψ_{-1} has only a stationary odd-parity part which falls off as r^{-4}, so that in evaluating dipole radiation the Ψ_{-1} can be ignored in Θ, a consequence, of course, of the fact that there can be no dipole gravitational radiation.

B. Flat spacetime background

Mixed electromagnetic and gravitational waves can also be considered when no strong background curvature is present. We investigate here the generation of electromagnetic waves by small spacetime distortions near a point charge. The point charge of course will be associated with a mass, but we need not concern ourselves whether it is a black hole, an electron, or whatever, only that the gravitational field of the object is not strong. More precisely, the gravitational radius of the object must be much smaller than all other length scales in the configuration.

The emphasis here on a point charge rather than, say, a dipole as the source of the background electric field is motivated only partly by simplicity. If a more complicated source, a source with "structure," were considered, the gravitational perturbations would distort the source generating real charge currents. For a structureless point charge, however, a change in the electric field is more naturally viewed as a consequence of the interaction of the gravitational perturbations and the background *field*.

Formally it would seem that the flat spacetime background equations correspond to setting $M=0$ in the equations of Sec. II A, but this clearly invalidates the derivations and gives no useful results. Rather it is necessary to start with Eq. (2.7) again. In Appendix B it is shown that with the Newman-Penrose equations Eq. (2.7) can be put in the form

$$W\{r\hat{\Phi}_{-1}+[2Q/(l-1)(l+2)]\,[-\hat{\lambda}-r\hat{\Psi}_{-1}+\tfrac{1}{4}l(l+1)r\hat{\Psi}_{0B}]\}=W(S_1)+S_2+S_3. \qquad (2.21)$$

Here S_2, which is pure imaginary (odd parity), S_3, which is pure real (even parity), and S_1 are source terms given in Appendix B and W is the wave operator (with $M=0$, $f=1$) given by Eq. (2.8). The S_1 term can be moved to the left-hand side of Eq. (2.21) and, as for the $r^2\Phi_{21}$ term in Eq. (2.16), can usually be ignored for radiation calculations. The λ, $r\Psi_{-1}$, and $r\Psi_0$ terms can be ignored as well for outgoing radiation according to the peeling theorem, since they fall off more quickly at large radius than $r\Phi_{-1}$. Exact solutions to Eq. (2.21) via Green's-function techniques are given in Sec. III.

Equation (2.21) clearly does not apply to dipole modes, but for odd parity it can be transformed into the form

$$W[r\hat{\Phi}_{-1}+\tfrac{1}{2}Q(\hat{\tau}+\hat{\mu})]=S_2\,, \qquad (2.22)$$

which remains valid for dipole modes. No even-parity (electric) dipole equation can be derived, but this is to be expected. An even-parity $l=1$ source in linearized gravitation theory would correspond to motions of the center of mass of the total source. Since the equations of motion are incompatible with such a source there can be no electric dipole induced by gravitational perturbations of a Coulomb field, and hence no wave equation like Eq. (2.21). For odd parity, dipole motions correspond to a (stationary) rotation of the stress-energy source and give rise not to radiation but to a stationary magnetic moment. In the black-hole background even-parity dipole motions *are* allowed [and are described by Eq. (2.13)]. Since the "source" for the black-hole case does not include the mass of the black hole itself the center of mass of the source can move. The electric dipole radiation generated this way is in fact the classically expected radiation due to the small motions of the charged black hole about the center of mass of the black hole plus source system.

C. Gauge transformations and physical interpretation

In interpreting solutions to our wave equations we must be sure that any physical conclusions are not the result of using the wrong coordinate or tetrad system. We have defined the unperturbed coordinate system and tetrad system, but in the presence of perturbations, ambiguity arises in the meaning of coordinates and tetrads. Because of this we can make two types of gauge transformations: (i) small (the order of the perturbations) changes in the definitions of the coordinates of the form $x^\mu \to x^\mu+\xi^\mu$, and (ii) small rotations of the null tetrad.

The first type of gauge transformation is very common in perturbation analysis and is particularly simple here as we are dealing exclusively with scalar quantities. Such transformations, however, are of secondary importance since most of these scalars—except those in Eq. (2.3)—vanish in the unperturbed geometry and hence are invariant with respect to coordinate gauge transformations.

The second type of gauge transformation is more interesting. It gives, in principle, six degrees of freedom at a point corresponding to the six degrees of freedom of the Lorentz group.[12] We cannot, however, exploit this full gauge group since we have already constrained the tetrad system by choosing a geometrically defined "special" tetrad system. Since $\vec{l}$ is geometrically defined we must only consider transformations in the little group of $\vec{l}$. If a is an expansion parameter, the transformations, to first order in a, which leave $\vec{l}$ and the scalar products of $\vec{l}$, $\vec{n}$, and $\vec{m}$ invariant are

$$\vec{l}\to\vec{l},\quad \vec{m}\to\vec{m}+a\vec{l},\quad \vec{n}\to\vec{n}+a\vec{m}^*+a^*\vec{m}\,. \qquad (2.23)$$

Since we also demand that the tetrad be parallel transported in the $\vec{l}$ direction, a is further constrained by the relation

$$Da=0\,, \qquad (2.24)$$

and therefore can be a function only of retarded time. We shall call transformations specified by Eqs. (2.23) and (2.24), *restricted gauge transformations*. The parameter a has spin-weight unity and like other perturbation fields can be despun and analyzed into spherical harmonics.

The restricted gauge transformations[13] of greatest interest in the black-hole background are

$$\hat{\Phi}_{-1}\to\hat{\Phi}_{-1}+2\hat{a}^*\Phi_0=\hat{\Phi}_{-1}-\hat{a}^*Q/r^2\,, \qquad (2.25a)$$

$$\hat{\Psi}_{-1}\to\hat{\Psi}_{-1}+3\hat{a}^*\Psi_0=\hat{\Psi}_{-1}-3\hat{a}^*M/r^3\,, \qquad (2.25b)$$

$$\hat{\lambda}\to\hat{\lambda}\ -\tfrac{1}{2}r^{-1}(l-1)(l+2)\hat{a}^*\,, \qquad (2.25c)$$

$$\hat{\mu}\to\hat{\mu}+r^{-1}\hat{a}^*\,, \qquad (2.25d)$$

$$\hat{\tau}\to\hat{\tau}-r^{-1}\hat{a}\,, \qquad (2.25e)$$

$$\hat{\Psi}_{0B}\to\hat{\Psi}_{0B},\quad \hat{\Psi}_{-2}\to\hat{\Psi}_{-2}\,. \qquad (2.25f)$$

As a consequence of Eqs. (2.25a) and (2.25b) the Θ field defined in Eq. (2.14) is invariant with respect to such transformations. Since it vanishes in the unperturbed geometry it is also invariant under coordinate gauge transformations. This complete gauge invariance also holds for the field (Ψ_{-2}) in the differential wave operator in Eq. (2.2), for the field in the wave operator in Eq. (2.21) in the case of a flat background, and for the field in the wave operator in Eq. (2.22) in the case of a flat background and odd parity.

This invariance is to be expected. Source terms as those in Eqs. (2.13), (2.20), (2.21), and (2.22) are all invariant and all these equations can, at least in principle, be solved to give solutions as integrals over the sources, solutions with no gauge arbitrariness. This would be incompatible with any gauge arbitrariness in the fields in the wave operators.

Of crucial importance to the interpretation of electromagnetic and gravitational radiation is the radial dependence of the gauge terms in Eqs. (2.25a) and (2.25b). By the peeling theorem, for outgoing radiation $\Phi_{-1}\sim r^{-1}$ and $\Psi_{-1}\sim r^{-2}$, *the radiative parts of* Φ_{-1} *and* Ψ_{-1} *are therefore gauge-invariant.* It is physically clear why this is so. For a purely radial Coulomb $\vec{E}$ field a slight "tilt" of the tetrad gives the $\vec{E}$ field a tangential component as seen by the tetrad, and therefore a nonzero Φ_{-1}. Since the tilt must be independent of r, by Eq. (2.24), the induced contributions to tangential fields and Φ_{-1} fall off as a Coulomb field, i.e., as r^{-2}. The same considerations explain the gauge term for gravitational perturbations.

In an unperturbed Schwarzschild spacetime (or in flat spacetime with $M=0$) Φ_{-1} has the form [36c],

$$\Phi_{-1}=-2^{-3/2}f^{1/2}\left[E^{[\theta]}+B^{[\phi]}-i(E^{[\phi]}-B^{[\theta]})\right], \tag{2.26}$$

in terms of the physical components of $\vec{E}$ and $\vec{B}$. All the information about the intensity and polarization of outgoing radiation is contained in this form of Φ_{-1}. In particular, the time-averaged power per unit solid angle is

$$dP/d\Omega=r^2T^{tr}=(r^2/2\pi)|\Phi_{-1}|^2. \tag{2.27}$$

Since the radiative part of Φ_{-1} is gauge-invariant we can continue to use Eqs. (2.26) and (2.27) in the perturbed geometry; perturbations introduce terms in these equations which we can ignore at large radii.

III. FLAT SPACETIME BACKGROUND

In a flat background Eq. (2.21) can be solved exactly with a Green's-function approach. If we assume a dependence $e^{-i\omega t}$ for all fields, then Eq. (2.21) becomes

$$d^2\Phi/dr^2+[\omega^2-2i\omega/r-l(l+1)/r^2]\Phi=-(S_2+S_3)/r, \tag{3.1}$$

$$\Phi(r)e^{-i\omega t}=r^2\hat{\Phi}_{-1}+[2Q/(l-1)(l+2)]\times[-r\hat{\lambda}-r^2\hat{\Psi}_{-1}+\tfrac{1}{4}l(l+1)r^2\hat{\Psi}_{0B}]-rS_1. \tag{3.2}$$

In terms of spherical Bessel and Hankel functions, homogeneous solutions of Eq. (3.1) are

$$\mathcal{J}_l\equiv(i\omega^2r^2-l\omega r)j_l(\omega r)+\omega^2r^2j_{l-1}(\omega r)\sim[(l+1)/(2l+1)!!](\omega r)^{l+1}\text{ at }\omega r\to 0, \tag{3.3a}$$

$$\mathcal{H}_l\equiv(i\omega^2r^2-l\omega r)h_l^{(1)}(\omega r)+\omega^2r^2h_{l-1}^{(1)}\sim 2(-i)^l\omega re^{i\omega r}\text{ at }\omega r\to\infty, \tag{3.3b}$$

and their Wronskian is

$$[\mathcal{J}_l,\mathcal{H}_l]=-i\omega l(l+1). \tag{3.4}$$

At large r the solution to Eq. (2.21) in terms of these functions is

$$\hat{\Phi}_{-1}\approx\frac{e^{-i\omega t}}{r^2}\Phi=\frac{e^{-i\omega t}\mathcal{H}_l(\omega r)\int r^{-1}(S_2+S_3)\mathcal{J}_l(\omega r)dr\,Y_{lm}(\theta,\phi)}{i\omega r^2l(l+1)}. \tag{3.5}$$

To calculate Φ_{-1} we operate on this equation with $2^{1/2}\eth/l(l+1)$ and find

$$\Phi_{-1}=\frac{2^{3/2}(-i)^{l-1}\exp(i\omega(r-t))}{r[l(l+1)]^{3/2}}\times\left[\int r^{-1}(S_2+S_3)\mathcal{J}_l(\omega r)dr\right]{}_{-1}Y_{lm}(\theta,\phi), \tag{3.6}$$

where we have used the limit in Eq. (3.3b). The time-averaged power contained in these waves according to Eq. (2.27) is

$$dP/d\Omega=2\pi^{-1}[l(l+1)]^{-3}\left|\int r^{-1}(S_2+S_3)\mathcal{J}_l(\omega r)dr\right|^2\times|{}_{-1}Y_{lm}|^2, \tag{3.7a}$$

$$P=2\pi^{-1}[l(l+1)]^{-3}\left|\int r^{-1}(S_2+S_3)\mathcal{J}_l(\omega r)dr\right|^2 \tag{3.7b}$$

It is interesting to compare this electromagnetic radiation generated by gravitational perturbations with electromagnetic radiation generated in the usual way by a charge current $\vec{J}$. From two of the Maxwell equations in NP form

$$Dr\hat{\Phi}_{-1}=-\tfrac{1}{2}l(l+1)\hat{\Phi}_0-2\pi r\hat{J}_4, \tag{3.8a}$$

$$\Delta r^2\hat{\Phi}_0=\hat{\Phi}_{-1}r+2\pi r^2\hat{J}_2, \tag{3.8b}$$

$$J_4\equiv\vec{J}\cdot\vec{m}^*,\quad J_2\equiv\vec{J}\cdot\vec{n}, \tag{3.8c}$$

we can form a wave equation

$$W(r\hat{\Phi}_{-1})=S_J=-2\pi[l(l+1)r^2\hat{J}_2+2\Delta r^3\hat{J}_4]. \tag{3.9}$$

By comparing this with Eq. (2.21) we see that for gravitationally induced radiation around a point charge, the charge source S_J is replaced by the stress-energy sources S_2+S_3. The stress-energy perturbations coupled to the Coulomb field act as an effective charge current. If both gravitational perturbations and charge currents are present, we need only replace S_2+S_3 by $S_2+S_3+S_J$ in Eqs. (3.5), (3.6), and (3.7).

As the clearest example of this comparison we can easily calculate the radiation when the source is nonrelativistic ($T^{tt} \gg T^{\mu i}$ and all velocities $\ll c$). In this case the source is predominately even-parity ($S_2 \ll S_3$) and

$$\Phi_{11} \approx \Phi_{22} \approx 3\Lambda \approx \pi T^{tt} \tag{3.10}$$

so that

$$S_3 \approx -Ql(l+1)\pi r T^{tt}\,, \tag{3.11}$$

while

$$S_J \approx -l(l+1)\pi r^2 \rho\,. \tag{3.12}$$

For nonrelativistic sources then, the mass-energy density times the central charge acts like charge density times distance to the central charge.

We can exploit this equivalence to calculate the static multipole moments induced by gravitational distortions of the Coulomb field of the point charge. Since the multipole moments[14] are

$$q_{lm} = \int r^{l+2}\rho_{lm}\,dr \tag{3.13}$$

for a classical electrostatic configuration, for a static gravitationally induced electrostatic multipole they must be

$$q_{lm} = Q\int r^{l+2} T^{tt}_{lm}\,dr\quad. \tag{3.14}$$

Note that mass-energy perturbations are relatively more effective at small radii in creating multipole moments. This is consistent with the picture that spacetime distortions closer to the point charge are in a stronger-field region and should produce greater distortions of the field.

In the long-wavelength limit ($\omega^{-1} \gg$ source dimensions) integrals such as those in Eqs. (3.7) are particularly simple. For radiation due *either* to charged sources *or* to gravitational perturbations

$$P = 2\pi\omega^{2l+2}|q_{lm}|^2(l+1)/l[(2l+1)!!]^2\,, \tag{3.15}$$

where q_{lm} is the appropriate expression from Eq. (3.13) or Eq. (3.14). This offers an interesting way of visualizing the manner in which electromagnetic radiation is generated by gravitational perturbations. Consider, e.g., a static quadrupole stress-energy perturbation and an equivalent classical electrostatic quadrupole. If both quadrupoles are now given the same time dependence, the radiation generated will be the same, at least in the long-wavelength approximation.

This can be stated more generally and precisely. The field in the near zone [$\omega r \ll 1$ but $r \gg$ source dimensions so that the Φ_{-1} term dominates in Eq. (2.21)] will be an electrostatic one (i.e., a solution to Maxwell's equations with $\omega = 0$) modified by a time-varying factor. The radiation in the radiation zone ($\omega r \gg 1$) will be related to the near-zone fields in a manner independent of whether the source is classical or gravitationally induced. Once one knows the static (or near-field) influence of a stress-energy distribution, one can ignore the fact that it originates as a spacetime distortion and can treat the radiation problem as if it were classical. *One can therefore view the radiation both qualitatively and (in the long-wavelength case) quantitatively as resulting from the time dependence of gravitational distortions of the Coulomb field.*

The similarity between Eqs. (2.22) and (3.9) can also be exploited in order to calculate the magnetic moment and gyromagnetic ratio of a mass-energy distribution rotating about a point charge. Since this involves only odd-parity sources we need only consider the J_4 term in Eq. (3.9). With the aid of the equations of motion, Eq. (2.22) for odd parity stationary sources, and for $r \gg$ source dimensions, becomes[15]

$$\begin{aligned} W(r\Phi_{-1}) &= Qr\partial_r(r\Phi_{10}) \\ &= 4\pi Q r\partial_r(r\vec{T}\cdot\vec{m}^*)\,, \end{aligned} \tag{3.16}$$

where $\vec{T}$ is the three vector with components T^{tj}. The analogous equation for the magnetic fields generated classically by charge currents is

$$W(r\Phi_{-1}) = 2\pi\partial_r(r^3\vec{J}\cdot\vec{m}^*)\,. \tag{3.17}$$

In solving these equations Φ_{-1} would be found in terms of integrals of the form $\int rS_J\,dr$ and $\int rS_2\,dr$. Since we can integrate the derivatives in the source by parts, in these integrals S_2 and S_J are equivalent to

$$S_2 = -8\pi Q r^2\vec{T}\cdot\vec{m}^*,\quad S_J = -2\pi r^3\vec{J}\cdot\vec{m}^* \tag{3.18}$$

and we see that $4Q\vec{T}/r$ takes the place of $\vec{J}$ for a gravitationally induced magnetic moment.

From the usual expression for magnetic moment

$$\vec{\mu} = \tfrac{1}{2}\int \vec{r}\times\vec{J}\,dV\,, \tag{3.19}$$

it follows that the gravitationally induced moment is

$$\vec{\mu} = 2Q\int(\vec{r}\times\vec{T})r^{-1}\,dV\,. \tag{3.20}$$

This integral and the integral for the angular momentum of the stress-energy source,

$$\vec{L}=\int(\vec{r}\times\vec{T})dV\,, \tag{3.21}$$

give us the gyromagnetic ratio for the point charge plus stress-energy configuration

$$g\equiv 2(\text{total mass/total charge})\vec{\mu}/\vec{L}$$
$$=4(\text{total mass})\int(\vec{r}\times\vec{T})r^{-1}dV\Big/\int(\vec{r}\times\vec{T})dV\,. \tag{3.22}$$

The gyromagnetic ratio then depends on the distribution of the mass-energy sources. In the simplest case, in which the mass-energy is totally contained in a shell of radius a, the gyromagnetic ratio is

$$g=4(\text{total mass})/a\,. \tag{3.23}$$

(This result was also derived, with a totally different approach by Cohen *et al.*[16]) Since the rotating matter is more efficient in producing $\vec{\mu}$ at small radii than in producing $\vec{L}$, Eq. (3.23) does not hold for distributed sources, but in fact is a minimum for sources of radius a provided all the stress-energy in the source rotates in the same sense. If the sources contain stress-energy rotating in opposite directions, even negative gyromagnetic ratio are possible.

A further interesting application of the equations in this section is the evaluation of the synchrotron-like radiation which would be generated by (uncharged) point particles moving at ultrarelativistic velocities near a point charge. Assume that two equally massive particles move in the same circular orbit 180° out of phase. The radiation coming out at high frequencies ($\omega\gg$ orbital frequency $=\omega_0$), and hence high mode numbers (l and $m\gg 1$), will be predominately even-parity, so by Eqs. (3.7) the power at high frequencies will be

$$P=(2/\pi m^6)\left|\int r^{-1}S_3\mathcal{J}_l(m\omega_0 r)dr\right|^2\,. \tag{3.24}$$

For the two point particles of mass μ moving with angular velocity ω_0 and velocity V in a circular orbit of radius $a=V/\omega_0$ the stress-energy is

$$T^{\mu\nu}=(U^\mu U^\nu/U^t r^2)\delta(r-a)\delta(\cos\theta)$$
$$\times[\delta(\phi-\omega_0 t)+\delta(\phi-\omega_0 t-\pi)]\,,$$
$$U^t=\gamma=(1-V^2)^{-1/2}\,,\quad U^\phi=\omega_0\gamma\,. \tag{3.25}$$

If this expression is used in the definiton of S_3 and only the highest-order terms in m are kept, we find that $S_3=0$ for m odd, that

$$S_3=-2Q\mu\pi\gamma^{-1}a^{-1}m^2 Y_{lm}(\tfrac{1}{2}\pi,0) \tag{3.26}$$

for m even, and that

$$P_{l,\omega}=\frac{32Q^2\mu^2\pi}{a^4m^2\gamma^2}\,|\,Y_{lm}(\tfrac{1}{2}\pi,0)|^2|\mathcal{J}_l(mv)|^2 \tag{3.27}$$

for a given frequency $\omega=|m|\omega_0\gg\omega_0$ and a given multipole index l.

Equation (3.27) can be simplified with high-frequency approximations: (i) For frequencies much greater than the critical frequency $\omega_c\equiv 3\gamma^3\omega_0$, the Bessel functions in $\mathcal{J}_l(mV)$ can be evaluated with the Debye approximation;[17] and (ii) $Y_{lm}(\frac{1}{2}\pi,0)$ can be estimated with Stirling's approximation, for $l\gg 1$. If the simplified $P_{l,\omega}$ is then summed over l up to $l=\omega/\omega_0$, the result is

$$P(\omega)=2(3/\pi)^{1/2}Q^2\mu^2\gamma\omega_0{}^4(\omega/\omega_c)^{1/2}e^{-2\omega/\omega_c}$$
$$\text{for }\omega\gg\omega_c\,. \tag{3.28}$$

For $\omega\gg\omega_0$ but $\omega\ll\omega_c$ the expression in Eq. (3.27) can also be roughly simplified:

$$P(\omega)\sim Q^2\mu^2\omega_0{}^4\gamma(\omega/\omega_c)\ \text{ for }\omega_0\ll\omega\ll\omega_c\,. \tag{3.29}$$

The total power radiated is therefore of order

$$P_{\text{total}}\sim Q^2\mu^2\omega_0{}^4\gamma\int_0^{\omega=\omega_c}(\omega/\omega_c)d(\omega/\omega_0)$$
$$\sim Q^2\mu^2\omega_0{}^4\gamma^4\,. \tag{3.30}$$

In the case of ordinary synchrotron radiation—for which the charge resides on the ultrarelativistic particles—the spectrum is qualitatively similar, but

$$P_{\text{total}}\sim Q^2\omega_0{}^2\gamma^4\,. \tag{3.31}$$

The gravitationally induced synchrotron radiation is then smaller than ordinary synchrotron radiation by μ^2/a^2, a factor which must be very small for the assumption of flat spacetime background to be valid.

(In the above analysis we have ignored the distortions of spacetime due to whatever forces are holding the particle in orbit. If the constraining forces are massless ropes, they generate electromagnetic power with the same spectrum $[\sim Q^2\mu^2\omega_0{}^4\gamma(\omega/\omega_c)]$ as the point particles, but the power is expected to be somewhat smaller than that generated[18] by the point particles. For real massive "ropes" the power generated would exceed that of the particles, but *real* ropes would in any case break.)

IV. BLACK-HOLE BACKGROUND

A. Green's-function solution

We have seen that a radiation calculation in the Schwarzschild background requires a solution of both Eqs. (2.13) and (2.20). To deal with these equations we assume a time dependence $e^{-i\omega t}$ and define

$$\vartheta e^{-i\omega t} \equiv r\hat{\Theta}, \tag{4.1a}$$

$$\psi e^{-i\omega t} \equiv r^3 f^{-1/2}\hat{\Psi}_{-2} \tag{4.1b}$$

so that the wave equations become

$$\frac{d^2\vartheta}{dr^2} + \left[\frac{\omega^2 - i\omega(2-6M/r)/r}{f^2} - \frac{l(l+1)}{fr^2}\right]\vartheta = -\frac{QS_\Theta}{3Mfr}, \tag{4.2}$$

$$\frac{d^2\psi}{dr^2} + \left[\frac{\omega^2 r^2 - 4i\omega(r-3M)}{f^2 r^2} + \frac{4-(l-1)(l+2)-16M/r}{fr^2}\right]\psi = -\frac{2r^2 S_\Psi}{f^{3/2}}. \tag{4.3}$$

To form a Green's-function solution we need the homogeneous solutions corresponding to ingoing waves at the event horizon and those corresponding to outgoing waves at spatial infinity:

$$\psi_{\rm in} \sim f^{3/2} e^{-i\omega r^*}, \quad \vartheta_{\rm in} \sim f e^{-i\omega r^*} \quad \text{at } r\to 2M,\ r^*\to -\infty \tag{4.4a}$$

$$\psi_{\rm out}, \vartheta_{\rm out} \sim e^{i\omega r} \text{ at } r\to\infty. \tag{4.4b}$$

Here $r^* = r - 2M\ln(r/2M-1)$, and $t\pm r^*$ represents advanced (+) and retarded(−) time.[19]

As in Sec. III, formal solutions can be constructed from the homogeneous solutions:

$$\hat{\Theta} = \frac{\vartheta e^{-i\omega t}}{r} = \frac{Qe^{-i\omega t}}{3Mr}\vartheta_{\rm out}\frac{\int(-S_\Theta/fr)\vartheta_{\rm in}\,dr}{[\vartheta_{\rm in},\vartheta_{\rm out}]}, \tag{4.5a}$$

$$\hat{\Psi}_{-2} = \frac{\psi e^{-i\omega t} f^{1/2}}{r^3} = \frac{f^{1/2}e^{-i\omega t}}{r^3}\psi_{\rm out}\frac{\int(-2r^2S_\Psi/f^{3/2})\psi_{\rm in}\,dr}{[\psi_{\rm in},\psi_{\rm out}]}. \tag{4.5b}$$

If M/r is very small, Eq. (4.2) becomes Eq. (3.1) so in the limit $r\gg M$, approximate homogeneous solutions for ϑ are $\mathcal{J}_l$ and $\mathcal{H}_l$. The analogous $r\gg M$ solutions for ψ are $\bar{\mathcal{J}}_l$ and $\bar{\mathcal{H}}_l$ given in Table I. Since we are only interested in $\vartheta_{\rm out}$ and $\psi_{\rm out}$ in the radiation zone we can choose the normalization of these functions so that $\vartheta_{\rm out}\approx\mathcal{H}_l$ and $\psi_{\rm out}\approx\bar{\mathcal{H}}_l$, in Eqs. (4.5), for the radiation zone.

A necessary step toward the solution is the calculation of Ψ_{-1} from Ψ_{-2} according to Eq. (2.17), for which we will need

$$D(r^{-2}e^{-i\omega t}\bar{\mathcal{H}}_l) = \omega r^{-2}(l-1)(l+2)(-i)^l \times\exp[i\omega(r-t)], \quad \omega r\gg 1. \tag{4.6}$$

With this and the large-ωr approximation for $\mathcal{H}_l$ the solutions in the radiation zone are

$$\hat{\Theta} = \frac{2Q(-i)^l\omega\exp[i\omega(r-t)]}{3M[\vartheta_{\rm in},\vartheta_{\rm out}]}\times\int\left(-\frac{S_\Theta}{fr}\vartheta_{\rm in}\right)dr, \tag{4.7a}$$

$$\hat{\Psi}_{-1} = -\frac{2(-i)^l\omega\exp[i\omega(r-t)]}{r^2[\psi_{\rm in},\psi_{\rm out}]}\times\int\left(-\frac{2r^2S_\Psi}{f^{3/2}}\psi_{\rm in}\right)dr, \tag{4.7b}$$

so that

$$\hat{\Phi}_{-1} = \hat{\Theta}/r + Qr\hat{\Psi}_{-1}/3M = \frac{2Q\omega(-i)^l\exp[i\omega(r-t)]}{3Mr}\times\left(\frac{\int(-S_\Theta/fr)\vartheta_{\rm in}\,dr}{[\vartheta_{\rm in},\vartheta_{\rm out}]} - \frac{\int(-2r^2S_\Psi/f^{3/2})\psi_{\rm in}\,dr}{[\psi_{\rm in},\psi_{\rm out}]}\right). \tag{4.8}$$

Once this solution is known, e.g., by numerical integration for $\psi_{\rm in}$ and $\vartheta_{\rm in}$, the time-averaged power can be computed from Eq. (2.27).

B. Mathematical methods for approximate solutions

The solution in Eq. (4.8) requires in general a numerical computation since the homogeneous solutions of Eqs. (4.2) and (4.3) are not know in terms of tabulated functions. We can, however, use a scheme similar to one described by Fackerell[20] to find solutions in the nonrelativistic limit (specifically in the limit $\omega M\ll 1$) as well as low-order relativistic corrections.

This method involves splitting the range of r from event horizon to infinity into three zones:

Region I	$1-2M/r\ll 1$	no restriction on ωr
Region II	$\omega r\ll 1-2M/r$	no restriction on M/r
Region III	$M/r\ll 1$	no restriction on ωr.

Region I is the neighborhood of the even horizon. If only the dominant terms in f^{-1} are kept, the homogeneous differential equations for ψ and ϑ are readily solvable. In Region II the ω-dependent terms in the differential equations (4.2) and (4.3) can be ignored so that we have in effect the static equations. The homogeneous solutions to these static equations can be written in terms of associated Legendre functions. In Region III we can ignore M/r terms and we are left with the flat spacetime equations, which can be solved in terms of spherical Bessel functions. Since the three regions overlap (e.g., regions I and II overlap for $\omega r \ll 1-2M/r\ll 1$), solutions can be matched from one

TABLE I. Properties of the homogeneous solutions.[a,b]

Region of validity	ϑ Homogeneous solutions of Eq. (4.2)	ψ Homogeneous Solution of Eq. (4.3)
Region I $1-2M/r \ll 1$	$\vartheta_{\rm I}^A = e^{i\omega r^*}$ $\vartheta_{\rm I}^B = f e^{-i\omega r^*}$	$\psi_{\rm I}^A = f^{-1/2} e^{i\omega r^*}$ $\psi_{\rm I}^B = f^{3/2} e^{-i\omega r^*}$
Region II $\omega r \ll 1-2M/r$	$\vartheta_{\rm II}^A = (r/M) f^{1/2} P_l^{(1)}(r/M-1)$ $\vartheta_{\rm II}^B = (r/M) f^{1/2} Q_l^{(1)}(r/M-1)$	$\psi_{\rm II}^A = (r/M) f^{1/2} P_l^{(2)}(r/M-1)$ $\psi_{\rm II}^B = (r/M) f^{1/2} Q_l^{(2)}(r/M-1)$
Region III $M/r \ll 1$	$\vartheta_{\rm III}^A = (i\omega^2 r^2 - l\omega r) j_l(\omega r) + \omega^2 r^2 j_{l-1}(\omega r)$ $\equiv \mathcal{J}_l(\omega r) \sim \left[\frac{(l+1)}{(2l+1)!!}\right](\omega r)^{l+1}$ as $\omega r \to 0$	$\psi_{\rm III}^A = [\frac{1}{2} l(l-1)\omega r - i(l-1)\omega^2 r^2 - \omega^3 r^3] j_l(\omega r)$ $+ (\omega^2 r^2 + i\omega^3 r^3) j_{l-1}(\omega r)$ $\equiv \bar{\mathcal{J}}_l(\omega r) \sim \frac{(l+1)(l+2)}{2(2l+1)!!}(\omega r)^{l+1}$ as $\omega r \to 0$
	$\vartheta_{\rm III}^B = (i\omega^2 r^2 - l\omega r) h_l^{(1)}(\omega r) + \omega^2 r^2 h_{l-1}^{(1)}(\omega r)$ $\equiv \mathcal{H}_l(\omega r) \sim 2(-i)^l \omega r e^{i\omega r}$ as $\omega r \to \infty$	$\psi_{\rm III}^B = [\frac{1}{2} l(l-1)\omega r - i(l-1)\omega^2 r^2 - \omega^3 r^3] h_l^{(1)}(\omega r)$ $+ (\omega^2 r^2 + i\omega^3 r^3) h_{l-1}^{(1)}(\omega r)$ $\equiv \bar{\mathcal{H}}_l(\omega r) \sim -2(-i)^{l+1}\omega^2 r^2 e^{i\omega r}$ as $\omega r \to \infty$
	Ingoing and outgoing homogeneous solutions[c]	
	Ingoing solutions at $r=2M$	
Region I	$\vartheta_{\rm in} = \frac{2(l+1)(l+1)!}{(2l-1)!!(2l+1)!!}(\omega M)^{l+1}\vartheta_{\rm I}^B$	$\psi_{\rm in} = \frac{(l+1)(l+2)(l+2)!}{2(2l-1)!!(2l+1)!!}(\omega M)^{l+2}\psi_{\rm I}^B$
Region II	$\approx \frac{(l+1)(l-1)!}{(2l-1)!!(2l+1)!!}(\omega M)^{l+1}\vartheta_{\rm II}^A$	$\approx \frac{(l+1)(l+2)(l-2)!}{2(2l-1)!!(2l+1)!!}(\omega M)^{l+2}\psi_{\rm II}^A$
Region III	$\simeq \mathcal{J}_l(\omega r)$	$\approx \bar{\mathcal{J}}_l(\omega r)$
	Outgoing solutions at $r \to \infty$	
Region I	$\vartheta_{\rm out} \approx \frac{2il(2l-1)!!(2l+1)!!}{(l+1)!}(\omega M)^{-l}\vartheta_{\rm I}^A$	$\psi_{\rm out} \approx -\frac{l(l-1)(2l-1)!!(2l+1)!!}{2(l+2)!}(\omega M)^{-l}\psi_{\rm I}^A$
Region II	$\approx \frac{-il(2l-1)!!(2l+1)!!}{(l+1)!}(\omega M)^{-l}\vartheta_{\rm II}^B$	$\approx -\frac{l(l-1)(2l-1)!!(2l+1)!!}{2(l+2)!}(\omega M)^{-l}\psi_{\rm II}^B$
Region III	$= \mathcal{H}_l(\omega r)$	$= \bar{\mathcal{H}}_l(\omega r)$
	Wronskians	
	$W[\vartheta_{\rm in}, \vartheta_{\rm out}] = -i\omega l(l+1)$	$W[\psi_{\rm in}, \psi_{\rm out}] = \frac{i}{4}\frac{(l+2)!}{(l-2)!}$

[a] Here $f = 1 - 2M/r$.
[b] $r^* = r + 2M\ln(r/2M - 1)$.
[c] Wavy signs $\approx$ indicate that the omitted terms are smaller by a factor of order $(\omega M)^n$, e.g., in $\vartheta_{\rm in}$ the coefficient of $\vartheta_{\rm II}^B$ is of the order $(\omega M)^{3l+2}$.

region to another and approximate ingoing and outgoing solutions can be constructed over the full range of r^*. The details of this process are summarized in Table I.

The approximate ingoing solutions and Wronskians can be used in Eq. (4.8) to calculate nonrelativistic radiation. In the approximate solutions higher-order terms can also be kept, e.g., terms of order M/r, $M^2/r^2, \ldots,$ in Region III. If such solutions are used, low-order relativistic corrections to the radiation can be computed. Such calculations are very tedious but can be somewhat simplified by the following observation: Suppose in the integrands of Eq. (4.8) we keep all corrections of order ωr, but none of order M/r. [This means setting $M=0$ in the sources and taking

$\vartheta_{\rm in} = \mathcal{J}_l(\omega r)$ and $\psi_{\rm in} = \bar{\mathcal{J}}_l(\omega r)$.] The right-hand side of Eq. (4.8) vanishes in this case, for any source, to all orders of ωr. This, of course, is not surprising. Ignoring M/r corrections is tantamount to using Eqs. (2.12), (2.17), and (2.20) in the flat background ($M=0$) limit. In this limit they merely provide two equivalent ways of calculating ψ_{-1}, and Eq. (4.8) just represents the difference. No information is gained about the perturbations of the geometry on the right-hand side of Eq. (2.7) and hence Φ_{-1} cannot be determined. [A slight subtlety in this is that the result of using $M=0$ in Eq. (4.8) need not *a priori* give a vanishing result without the use of the equations of motion. The forms of S_Θ and S_Ψ have carefully been chosen so that in fact the equations of motion do *not* have to be introduced; the right-hand side of Eq. (4.8) vanishes directly once M is set to zero.]

With this observation it is clear that nonvanishing results come only from terms containing M. The lowest-order nonvanishing results are those in which only M/r correction terms are kept, so that

$$\vartheta_{\rm in} \approx \frac{(l-1)!(l+1)}{(2l-1)!!(2l+1)!!} \times (\omega M)^{l+1}\left(\frac{r}{M}\right) f^{1/2} P_l^{(1)}\left(\frac{r}{M}-1\right) \approx \frac{(l+1)}{(2l+1)!!}(\omega r)^{l+1}\left[1-(l+1)\frac{M}{r}\right] \tag{4.9a}$$

and similarly

$$\psi_{\rm in} \approx \frac{1}{2}\,\frac{(l+1)(l+2)}{(2l+1)!!}(\omega r)^{l+1}\left[1-(l+1)\frac{M}{r}\right]. \tag{4.9b}$$

The resulting integrals will be proportional to M and this M factor will cancel with the leading M^{-1} factor in Eq. (4.8), so that the result for Φ_{-1} is independent of M. This M-independent lowest-order solution is of course identical to the long-wavelength ($\omega r \to 0$) limit of Sec. III, so that the radiation to this order is precisely that described by Eqs. (3.14) and (3.15).

To get relativistic corrections we can use the techniques of this section to calculate a correction factor to the source integrals of order $M(1 + M/R_{\rm source} + \omega M + \cdots)$ and therefore corrections to the radiation of order $1 + M/R_{\rm source} + \omega M + \cdots$. These then must be combined with the correction factors of order $(\omega R_{\rm source})$, $(\omega R_{\rm source})^2, \ldots$ from Sec. III.

C. Electric dipole radiation

The one case in which the equations in the Schwarzschild background are simple to use is that of even parity and $l=1$, the electric dipole case. In this case Ψ_{-1} vanishes because there is no gravitational even-parity dipole. (More precisely, Ψ_{-1} is removable by a restricted gauge transformation, as demonstrated in the beginning of Appendix A.) We can therefore take $\Theta = r\Phi_{-1}$.

The simplest and most interesting case is that in which the dipole is nonrelativistic. For this case

$$\begin{aligned} \vartheta_{\rm in} &\approx \tfrac{2}{3}(\omega r)^2, \\ \Phi_{22} &\approx \pi T^{tt}_{l=1,m}, \\ S_\Theta &\approx -l(l+1) r^2 D r \hat{\Phi}_{22} \\ &\approx -2\pi r^2 \partial_r (r T^{tt}_{l=1,m}). \end{aligned} \tag{4.10}$$

Since S_Θ is to be used in an integral [Eq. (4.7a)] with $\vartheta_{\rm in}/r \propto r$, then by integration by parts it is equivalent to

$$S_\Theta \approx 6\pi r^2 T^{tt}. \tag{4.11}$$

The electromagnetic fields can be found, of course, from Eq. (4.7a), but it is more instructive to compare the source for $W(\hat{\Theta})$ in Eq. (2.13) with the classical source in Eq. (3.9) and to notice the equivalence of

$$\begin{aligned} S_J &\approx -2\pi r^2 \rho, \\ (Q/3M) S_\Theta &\approx 2\pi (Q/M) r^2 T^{tt}. \end{aligned} \tag{4.12}$$

Thus $-(Q/M)T^{tt}$ is an effective charge density in the nonrelativistic electric dipole case and we can immediately conclude that the electromagnetic fields generated by the gravitational perturbations are those of an induced electric dipole:

$$\vec{P}_{\rm induced} = \int \vec{r} \rho_{\rm effective}\, dV = -(Q/M)\int \vec{r} T^{tt}\, dV. \tag{4.13}$$

Now let us define a "classical" dipole moment for the black hole plus source system. The center of mass of the system relative to the center of the black hole is at

$$\vec{R}_{\rm c.m} = \int \vec{r} T^{tt}\, dV/M. \tag{4.14}$$

Since the charge Q, residing on the black hole, is located at $-\vec{R}_{\rm c.m}$ relative to this center of mass, a classical electric dipole

$$\vec{P}_{\rm classical} = -Q\vec{R}_{\rm c.m.} = -(Q/M)\int \vec{r} T^{tt}\, dV \tag{4.15}$$

is associated with the system and this classical dipole precisely corresponds to the fields produced.

For $l=1$ and even parity we have then a simple picture of the origin of the electromagnetic radiation. The black hole is not stationary for an even-parity dipole perturbation but rather it moves about the center of mass of the black hole plus source system. The radiation emitted, at least in the nonrelativistic approximation, is just the classically expected radiation from the resulting time-dependent dipole. Relativistic corrections, which can be calculated from Eq. (4.7a) with the techniques of Sec. IV B, also take into account the distortion of the Coulomb field of the black hole. These effects on the dipole are smaller by a factor of order ωM.

APPENDIX A: DIPOLE GRAVITATIONAL PERTURBATIONS

A simple argument suffices to prove that, in our formalism, even-parity dipole gravitational perturbations of a Schwarzschild black hole can be removed by a restricted gauge transformation of the type discussed in Sec. II C: It is know that there are no physically real (non-gauge-removable) even-parity dipole perturbations,[19] hence some gauge transformation will remove the perturbation to give the unperturbed Schwarzschild geometry. Since the unperturbed geometry, in our formalism, is described by the special system of Eqs. (2.2) and (2.3) and since all physically equivalent special systems are related by restricted transformations, the gauge transformation which removes the perturbation must be a restricted one.

In the remainder of this appendix we consider odd-parity dipole perturbations. These are known to correspond physically to a small addition of angular momentum to the gravitational fields, so that the geometry is gauge-transformable to a stationary form.[21] We show here that in the special system an appropriate gauge exists in which Ψ_{-1} is stationary and ignorable in radiation calculations.

To facilitate the proof we use here a retarded-time coordinate $u \equiv t - r^*$ so that

$$D = \partial_r\,, \quad \Delta = \partial_u - \tfrac{1}{2} f \partial_r\,. \tag{A1}$$

Since we are considering only odd parity and $l=1$ we immediately have several simplifications: (i) All spin-weight ± 2 quantities vanish, thus $\Psi_{-2} = \Psi_{+2} = \lambda = \sigma = 0$. (ii) We are dealing only with the imaginary (odd-parity) parts of all despun quantities so the perturbations of real variables must vanish, e.g., $U_B = \rho_B = 0$. With these simplifications we have from [B5i], [B5j], and [B4g] that

$$r\hat{\Psi}_1 = -(\hat{\tau} + \hat{\omega}/r)\,, \tag{A2a}$$

$$\hat{\Psi}_{0B} = (\hat{\tau} - \hat{\mu})\,, \tag{A2b}$$

$$\hat{\omega} = -r\hat{\mu}\,, \tag{A2c}$$

so that

$$\hat{\Psi}_1 = -\hat{\Psi}_{0B}\,. \tag{A3}$$

The Bianchi identities [B6a] and [B6f] then give us

$$D r^4 \hat{\Psi}_1 = 0\,, \tag{A4a}$$

$$f^{-1}\Delta r^2 f \hat{\Psi}_1 + r\hat{\Psi}_{0B} - 3Mr^{-1}(\hat{\tau} + \hat{\omega}/r) = r^{-2}\Delta r^4 \hat{\Psi}_1 = 0 \tag{A4b}$$

and therefore $\hat{\Psi}_1 r^4 = A =$ a pure imaginary "constant" (i.e., a function of θ, ϕ only). Two of the remaining Bianchi identities [B6g] and [B6c] or

$$\Delta r^3 \hat{\Psi}_{0B} = r^2 \hat{\Psi}_{-1} + 3M\hat{\mu}\,, \tag{A5a}$$

$$D r^2 \hat{\Psi}_{-1} = -r\hat{\Psi}_{0B} - 3M\hat{\omega}/r^2 \tag{A5b}$$

can now be combined with (A2c) to give

$$D r^2 \hat{\Psi}_{-1} + r\hat{\Psi}_{-1} = r^{-1}\Delta r^3 \hat{\Psi}_{0B} - r\hat{\Psi}_{0B} \tag{A6a}$$

or

$$D r^3 \hat{\Psi}_{-1} = \Delta \hat{\Psi}_{0B} r^3 - r^2 \hat{\Psi}_{0B} = (\tfrac{1}{2} + M/r) A/r^2\,, \tag{A6b}$$

and hence

$$\hat{\Psi}_{-1} = -\tfrac{1}{2}(1 + M/r) A/r^4 + F(u)/r^3\,, \tag{A7}$$

where $F(u)$ is an arbitrary function. Finally, with the special gauge transformation of Eq. (2.25b) the time-varying part of Ψ_{-1} can be eliminated by choosing $\hat{a} = -F(u)/3M$.

APPENDIX B: WAVE EQUATIONS FOR A FLAT BACKGROUND

In this appendix the derivations of wave equations (2.21) and (2.22) are outlined. The starting point is Eq. (2.7), rewritten as

$$W(r\hat{\Phi}_{-1}) = 2Q(\hat{\nu} - \tfrac{1}{2} L\hat{\mu} + \tfrac{1}{2}\hat{\omega}^*/r) \tag{B1}$$

by use of [B4h]. [Here $L \equiv l(l+1)$ and $\mathfrak{L}$ will denote $(l-1)(l+2)$.] The terms on the right-hand side can be manipulated, using (NP4.2j) and (NP4.2m) (which are [B5h], [B5k], plus source terms), to give

$$\hat{\nu} - \tfrac{1}{2} L\hat{\mu} + \tfrac{1}{2}\hat{\omega}^*/r = -\mathfrak{L}^{-1}(2r^{-1}\Delta r^2 \hat{\lambda} - \mathfrak{L}\hat{\lambda} + \mathfrak{L} r\hat{\Phi}_{21} - \tfrac{1}{2}\mathfrak{L}\hat{\tau}^* - \mathfrak{L} r\hat{\Psi}_{-1} + 2r\hat{\Psi}_{-2})\,. \tag{B2}$$

A wave equation for $\hat{\lambda}$ is now formed by eliminating $\hat{\sigma}$ in (NP4.2g) and (NP4.2m) (or [B5e] and

[B5m] plus source terms):

$$W(\hat{\lambda}) + 2\Delta r\hat{\lambda} + \hat{\lambda} - \tfrac{1}{2}\mathcal{L}\hat{\tau}^* = 2r\Delta r\hat{\Phi}_{20} + L\hat{\lambda}\,. \qquad \text{(B3)}$$

When this equation is combined with Eq. (B1) the result is

$$W(r\hat{\Phi}_{-1} - 2Q\mathcal{L}^{-1}\hat{\lambda}) = -2Q\mathcal{L}^{-1}(2r\Delta r\hat{\Phi}_{20} + \mathcal{L}r\hat{\Phi}_{21} - \mathcal{L}r\hat{\Psi}_{-1} + 2r\Psi_{-2})\,. \qquad \text{(B4)}$$

The goal is to have only source terms remaining on the right-hand side. To achieve this, the last two terms on the right-hand side in Eq. (B4) are reexpressed, first by use of Eqs. (2.18a) and (2.18b), with $M=0$:

$$\begin{aligned}-\mathcal{L}r\hat{\Psi}_{-1} + 2r\hat{\Psi}_{-2} = {} & r^2D\hat{\Psi}_{-1} - \mathcal{L}r\hat{\Psi}_{-1} - W(r\hat{\Psi}_{-1}) \\ & - \mathcal{L}r\hat{\Phi}_{21} + 2r^2Dr^{-1}\Delta r^2\hat{\Phi}_{21} \\ & + LrDr\hat{\Phi}_{22} - 2r\Delta r\hat{\Phi}_{20} - Lr\hat{\Phi}_{22}\,.\end{aligned} \qquad \text{(B5)}$$

Next, Eqs. (2.9a) and (2.9b) are used to rewrite the first two terms on the right-hand side of (B5):

$$r^2D\hat{\Psi}_{-1} - \mathcal{L}r\Psi_{-1} = \tfrac{1}{4}W(r\hat{\Psi}_{0B}) - \tfrac{1}{2}\mathcal{L}S_{\rm I} - \tfrac{1}{2}Lr^2Dr^{-2}S_{\rm II}\,. \qquad \text{(B6)}$$

Finally, Eqs. (B4), (B5), and (B6) are combined to give a wave equation with only stress-energy terms on the right-hand side:

$$\begin{aligned}W[r\hat{\Phi}_{-1} - 2Q\mathcal{L}^{-1}(\hat{\lambda} + r\hat{\Psi}_{-1} - \tfrac{1}{4}Lr\hat{\Psi}_{0B})] &= \mathcal{S} \\ &= -2Q\mathcal{L}^{-1}(2r^2Dr^{-1}\Delta r^2\hat{\Phi}_{21} + Lr^2D\hat{\Phi}_{22} - \tfrac{1}{2}\mathcal{L}S_{\rm I} - \tfrac{1}{2}Lr^2Dr^{-2}S_{\rm II})\,.\end{aligned} \qquad \text{(B7)}$$

The equations of motion, Eqs. (2.15), can be used to effect two simplifications of the source terms: (i) expressing all second derivatives in terms of the W operator, and (ii) separating the remaining terms as to parity. Both conveniences are realized when the source is put in the form

$$\begin{aligned}&\mathcal{S} = S_1 + S_2 + S_3\,,\\ &S_1 = W\{2Q\mathcal{L}^{-1}[-r\hat{\Phi}_{21} + \tfrac{1}{4}l(l+1)r(\hat{\Phi}_{11} + \hat{\Lambda})]\}\,,\\ &S_2 = -(\tfrac{1}{3}Q)(2r^2D\hat{\Phi}_{[12]} + 4r^2\Delta\hat{\Phi}_{[10]} + r\hat{\Phi}_{[02]} - 3r\hat{\Phi}_{[10]})\,,\\ &S_3 = -Q[r\hat{\Phi}_{(02)} + r\hat{\Phi}_{(01)} - 4r\hat{\Phi}_{(12)} + \tfrac{1}{2}l(l+1)r(\hat{\Phi}_{11} + 3\hat{\Lambda})] + Q\mathcal{L}^{-1}[-4r^2D\hat{\Phi}_{(12)} + l(l+1)r^2D(-\hat{\Phi}_{22} + \tfrac{1}{2}\hat{\Phi}_{11} + \tfrac{3}{2}\hat{\Lambda})]\,,\\ &\hat{\Phi}_{[ij]} \equiv \tfrac{1}{2}(\hat{\Phi}_{ij} - \hat{\Phi}_{ji}), \quad \hat{\Phi}_{(ij)} \equiv \tfrac{1}{2}(\hat{\Phi}_{ij} + \hat{\Phi}_{ji})\,.\end{aligned} \qquad \text{(B8)}$$

Since real and imaginary parts of despun quantities correspond to even and odd parity, and since $\Phi_{ij} = \Phi_{ji}^*$, it follows that S_2 contains purely odd-parity terms and S_3 purely even-parity terms.

For purely odd-parity perturbations Eq. (B7) can be put in a more useful form. The odd-parity parts of (NP4.2m), (NP4.2q), and (NP6.10g) (or equivalently the imaginary parts of [B5k], [B5n], and [B4g] plus sources) give

$$-\hat{\lambda} - r\hat{\Psi}_{-1} + \tfrac{1}{4}Lr\hat{\Psi}_{0B} = \tfrac{1}{4}\mathcal{L}(\hat{\mu} + \hat{\tau}) - r\hat{\Phi}_{21} \qquad \text{(B9)}$$

so that for pure odd-parity perturbations Eq. (B7) becomes Eq. (2.22).

Since Eq. (B7) is clearly inapplicable to $l=1$ modes, the above simple derivation of Eq. (2.22) is valid only for $l>1$. It is, however, possible to start from Eq. (2.7), assuming odd parity at the outset, and to derive Eq. (B9) without requiring $l>1$ at any step. Equation (B9) is then in fact valid for magnetic dipoles.

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