Gravitational field equations and the possibility of time variation of all masses*

S. Malin
Department of Physics and Astronomy, Colgate University, Hamilton, New York 13346
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Recent developments in spontaneously broken gauge theories as well as in group analysis of cosmological models indicate that rest masses may change on the cosmological scale. Such an effect contradicts Einstein's equations of general relativity. The possibility of replacing Einstein's equations by the equations $R_{\mu\nu} = -4\pi\kappa T_{\mu\nu}$ is explored. Cosmological models are calculated and time variation of all masses is derived for all isotropic, spatially homogeneous models. The result is $\dot{m}/m = -3H$, where H is Hubble's constant. It is shown that the equations $R_{\mu\nu} = -4\pi\kappa T_{\mu\nu}$ are not precluded by presently available observational and experimental data.

I. INTRODUCTION

The purpose of this note is to explore the possibility that the gravitational field equations in nonempty space are given by[1]

$$R_{\mu\nu} = -4\pi\kappa T_{\mu\nu} \qquad (1.1)$$

instead of the usual equation of general relativity

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi\kappa T_{\mu\nu}\,. \qquad (1.2)$$

To be sure, Eq. (1.1) is now new. It is often suggested in textbooks (without the calculation of the constant $4\pi\kappa$) as the first candidate for the gravitational field equations, and immediately rejected because the left-hand side does not have an identically vanishing four-divergence.[2] If, however, energy and rest mass are not conserved quantities on the cosmological scale, then the postulate that (1.1) are the correct gravitational field equations is worth exploring: They satisfy the basic postulates of the theory of general relativity (the equivalence principle, the principle of covariance) and are as simple and elegant as Eq. (1.2).

The possibility that energy and/or rest mass are not conserved quantities was recently raised in two entirely different contexts:

(1) Considerations based on Salam's and Weinberg's spontaneously broken gauge theories of weak and electromagnetic interactions[3]: As far as energy is concerned, Kirzhnits and Linde as well as Weinberg have demonstrated that the energy of the vacuum depends on the temperature of the medium and therefore varies on the cosmological scale.[4,5] Concerning rest masses: In spontaneously broken gauge theories various rest masses may vanish to zeroth order, and can then be calculated as higher-order effects,[6] which, in turn, may change with the temperature of the medium and therefore change with time.

(2) A group-theoretical analysis of isotropic, spatially homogeneous cosmological models[7] strongly suggests that the rest masses of all physical systems decrease as the universe expands.

The main consequences of Eq. (1.1) can be summed up as follows:

(1) For cosmological models, the variety of allowed models is similar to that of Einstein's equations (1.2); the detailed behavior of the metric tensor as a function of cosmic time is different. Explicit results are given in Sec. II.

(2) The masses of all physical systems decrease as the universe expands. The rate of change depends on Hubble's constant H only and is given by the formula

$$\dot{m}/m = -3H\,, \qquad (1.3)$$

where m is the mass and a dot denotes differentiation with respect to cosmic time. This result is certainly in line with Mach's principle: If the inertial mass of a system is due to the influence of all the other masses in the universe, one would expect the inertial mass to decrease as all the other masses get further and further away.

(3) The presently available observational and experimental data are not sensitive enough to decide between Eqs. (1.1) and (1.2), but measurements which are accurate enough for such a decision are expected to be available within a few years (see Sec. IV).

II. COSMOLOGICAL MODELS

As pointed out by Robertson and Noonan,[8] "the observational evidence allows us to assume the existence of a congruence of fundamental world lines which fills the universe." If the universe is also assumed to be isotropic and spatially homogeneous, then it is possible to choose a canonical coordinate system (t, x_1, x_2, x_3) such that the metric tensor in the coordinate system is of the form[9]

$$ds^2 = dt^2 - S^2(t)\,\frac{dx_1^2 + dx_2^2 + dx_3^2}{(1+\frac{1}{4}kr^2)^2}\,, \tag{2.1}$$

where

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2} \tag{2.2}$$

and on the fundamental world lines $ds^2 = dt^2$. The coordinate t in the canonical frame of reference is called "cosmic time."

The isotropic spatially homogeneous cosmological models are classified into the following three well-known types, according to the sign of k:

(i) $k=+1$ (spherical space). The hypersurfaces $t=$const have constant positive curvature.

(ii) $k=0$ (Euclidean space). The hypersurfaces $t=$const have zero curvature.

(iii) $k=-1$ (pseudospherical space). The hypersurfaces $t=$const have constant negative curvature.

For the purpose of calculating and discussing cosmological models the energy-momentum tensor will be assumed, as usual, to have the form

$$T^{\mu}{}_{\nu} = \begin{bmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.3}$$

where $\rho(t)$ is the matter density.[10] The function $S(t)$ is then obtained from the gravitational field equations, Eqs. (1.1) or (1.2), as the case may be.

The contracted Riemann tensor corresponding to the metric tensor (2.1) is given as follows:

$$R^0{}_0 = \frac{3\ddot{S}}{S}\,, \tag{2.4a}$$

$$R^i{}_i = \frac{2k}{S^2} + \frac{2\dot{S}^2}{S^2} + \frac{\ddot{S}}{S}\,, \quad i=1,2,3 \tag{2.4b}$$

where a dot denotes differentiation with respect to cosmic time. Substitution of Eq. (2.4) in Eq. (1.2) yields the usual equations for the function $S(t)$:

$$8\pi\kappa\rho = \frac{3\dot{S}^2}{S^2} + \frac{3k}{S^2}\,, \tag{2.5a}$$

$$2S\ddot{S} + \dot{S}^2 + k = 0\,. \tag{2.5b}$$

If, however, Eq. (1.1) is assumed, the following equations are obtained:

$$4\pi\kappa\rho = -3\ddot{S}/S\,, \tag{2.6a}$$

$$S\ddot{S} + 2\dot{S}^2 + 2k = 0\,. \tag{2.6b}$$

Equations (2.5) have the following well-known solutions:

(i) spherical space (in parametric representation):

$$\begin{aligned} t &= A(2u - \sin 2u)\,, \\ S &= A(1-\cos 2u)\,, \end{aligned} \tag{2.7}$$

where u is a parameter;

(ii) Euclidean space:

$$S(t) = Bt^{2/3}\,; \tag{2.8}$$

(iii) Pseudospherical space (in parametric representation):

$$\begin{aligned} t &= C(\sinh 2u - 2u)\,, \\ S &= C(\cosh 2u - 1)\,. \end{aligned} \tag{2.9}$$

A, B, and C are constants.

The solution of Eqs. (2.6) can be carried out up to a final quadrature as follows: If we denote $y=\dot{S}^2$, Eq. (2.6b) takes the form

$$S\,\frac{dy}{dS} + 4y + 4k = 0\,. \tag{2.10}$$

Substituting $z = \ln S$ we obtain

$$\frac{dy}{dz} + 4y + 4k = 0\,, \tag{2.11}$$

$$y = -k + DS^{-4}\,, \tag{2.12}$$

where D is a constant of integration. Recalling that $y = \dot{S}^2$ we finally obtain

$$t - t_0 = \int_{S_0}^{S} (DS^{-4} - k)^{-1/2} dS\,. \tag{2.13}$$

The following formulas will also be needed:

$$\frac{dS}{dt} = (DS^{-4} - k)^{1/2}\,, \tag{2.14}$$

$$\frac{d^2S}{dt^2} = -2DS^{-5}\,, \tag{2.15}$$

The general behavior of the cosmological models can be read off Eqs. (2.13)–(2.15) and compared with the standard general relativistic models:

(i) *Spherical space.* We have a pulsating universe, with $S(t)$ varying between 0 and $D^{+1/4}$. $S=0$ is a singularity [corresponding to the breakdown of the form (2.3) for the energy-momentum tensor as the density of matter gets high]; $S=D^{1/4}$ is an ordinary maximum.

(ii) *Euclidean space.* Since $k=0$ Eq. (2.13) can be integrated. It yields

$$S(t) = Kt^{1/3}\,, \tag{2.16}$$

as compared with the $t^{2/3}$ behavior of the Friedmann universe [Eq. (2.8)].

(iii) *Pseudospherical space.* At all time $dS/dt > 0$,

$d^2S/dt^2 < 0$; the universe keeps expanding at an ever decreasing rate.

On comparing these results with Eqs. (2.7)–(2.9) we see an over-all similarity along with some significant differences.

III. MASS VARIATION

In the case of the usual cosmological models it follows from Eqs. (2.5) that

$$\rho(t)S^3(t) = \text{const}, \tag{3.1}$$

which is an expression of the law of conservation of mass.

When Eqs. (2.6) are considered, ρS^3 is a changing quantity, on the cosmological scale. It can be understood to imply that all masses in the universe change at the same rate. This rate of change is derived from Eqs. (2.6) and (2.14): Since ρS^3 is proportional to S^{-3}, we obtain

$$m(t) = m_0[S(t)]^{-3}, \tag{3.2}$$

where $m(t)$ is the mass of a particle (or any physical system) at cosmic time t and m_0 is a constant. Differentiating Eq. (3.2) and dividing by m we obtain

$$\dot{m}/m = -3\dot{S}/S = -3H, \tag{3.3}$$

where H is Hubble's constant. Equation (3.3) is valid for all isotropic, spatially homogeneous cosmological models, with energy-momentum tensor given by Eq. (2.3). For recent values of the Hubble constant, such as[11]

$$H_0 = 55 \text{ km sec}^{-1}\,\text{Mpc}^{-1} = (17.8\times 10^9 \text{ years})^{-1} \tag{3.4}$$

the rate of change of mass is of the order[12]

$$|\dot{m}/m| \sim 10^{-10} \text{ per year}. \tag{3.5}$$

Mass variations of similar magnitude were previously obtained using an entirely different method—a group analysis of cosmological models. As will be shown in a forthcoming paper, the group method[7] gives identical results, namely Eq. (3.3), for all the cosmological models of Eq. (2.6).

Equation (3.3), which shows that all masses decrease as the universe expands, is in line with Mach's idea that the inertial mass of a particle is due to the influence of all other matter on it. This influence can be expected to decrease as the particles get further and further away from each other.

IV. OBSERVATIONAL AND EXPERIMENTAL EVIDENCE

Equation (1.1) can, in principle, be put to the observational and experimental test via three approaches:

(i) observation of its consequences as far as particle trajectories are concerned; (ii) cosmological consequences; (iii) observstions of time variation of rest masses.

(i) Particle trajectories are involved in the first observational confirmations of the theory of general relativity—the perihelic motion of Mercury and the deflection of light rays passing near the sun. Unfortunately, these are tests of Einstein's equations *in empty space*, for which case Eqs. (1.1) and (1.2) are the same. The only difference, as far as the Schwarzschild solution is concerned, is in the possible time variation of the mass which generates the field (see Sec. III). However, the effects of these variations on the perihelic motion are 6–8 order of magnitude too small for detection.

(ii) The results of Sec. II show significant variations between cosmological models based on Eq. (1.1), as compared with Eq. (1.2). There are two factors, however, which make it impractical to use cosmological models to test Eq. (1.1): (1) the great uncertainty in the numerical values of cosmological quantities, such as Hubble's constant or the mean density of matter in the universe;[13] (2) the fact that both Eqs. (1.1) and (1.2) allow for a whole range of cosmological models [in fact, there are models based on Eq. (1.1), as well as models based on Eq. (1.2), which agree with presently available cosmological data].

(iii) perhaps the major difference between physical consequences of Eqs. (1.1) and (1.2) concerns the possibility of time variation of all rest masses. According to Eq. (1.2) energy and momentum are strictly conserved. According to Eq. (1.1), however, all rest masses vary according to Eqs. (3.3) and (3.5). This result of Eq. (1.1) has consequences which are close to the threshold of observational verification. An analysis of the experimental and observational evidence for such a mass variation was presented in a previous article.[7] The analysis was largely based on an article by Dyson concerning evidence for time variation of fundamental constants.[14] It leads to the conclusion that the wealth of available data from beta-decay experiments, planetary orbit observations, interplanetary ranging experiments, as well as solar and stellar evolution considerations do not prelude such a time variation of all masses. Furthermore, some of these experiments are expected to be accurate enough to detect such an effect, if it exists, within the next few years. See Ref. 7 for details.

V. CONCLUSION

When the possibility of replacing Eq. (1.2) by Eq. (1.1) as the gravitational field equations is considered, it is important to bear in mind that Eq.

(1.1) is in full accord with the basic principles of the theory of general relativity. Einstein's view on Eq. (1.2) is expressed in the following quotation:[15]

> The second member on the left side is added because of formal reasons; for the left side is written in such a way that its divergence disappears identically in the sense of the absolute differential calculus. The right side is a formal condensation of all things whose comprehension in the sense of a field-theory is still problematic. Not for a moment, of course, did I doubt that this formulation was merely a makeshift in order to give the general principle of relativity a preliminary closed expression. For it was essentially not anything *more* than a theory of the gravitational field, which was somewhat artificially isolated from a total field of as yet unknown structure.
>
> If anything in the theory as sketched—apart from the demand of the invariance of the equations under the group of the continuous co-ordinate-transformations—can possibly make the claim to final significance, then it is the theory of the limiting case of the pure gravitational field and its relation to the metric structure of space.

Notes added in proof

When the effects of mass variation on spectral lines is taken into account, it turns out that Eqs. (1.3), (3.3) should be replaced by $\dot{m}/m = +\frac{3}{2}H$ and observed red shifts correspond to a contracting universe. Details will be given in a forthcoming paper.

Equations (1.1) imply, in general, $T^{\mu\nu}{}_{;\nu} \neq 0$ and therefore imply that the matter action does not transform like a scalar under coordinate transformations (see Ref. 10, Chap. 12).

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[1]$R_{\mu\nu}$ is the Ricci tensor, R is the Riemann scaler, κ is the gravitational constant, and $T_{\mu\nu}$ is the energy-momentum tensor. The constant $4\pi\kappa$ in Eq. (1.1) was derived by the requirement that Newton's law of gravitation is obtained as a limiting case of Eq. (1.1), in complete analogy with the derivation of the constant $8\pi\kappa$ in Eq. (1.2).

[2]See, e.g., R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965), p. 276.

[3]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367. For review articles on spontaneously broken gauge theories see K. T. Mahanthappa, University of Texas at Austin report, 1973 (unpublished); E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

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[5]The consequences of spontaneously broken gauge theories for the gravitational field equations were recently discussed by A. D. Linde, Zh. Eksp. Teor. Fiz. Pisma'Red, 19, 320 (1974) [JETP Lett. 19, 183 (1974)], J. Dreitlein, Phys. Rev. Lett. 33, 1243 (1974); M. Veltman, University of Utrecht, Netherlands, report, 1974 (unpublished). In particular, Linde has pointed out that within the framework of the usual equations of general relativity the temperature dependence of the energy of the vacuum leads to an addition of a time-dependent cosmological constant to Eq. (1.2).

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[8]H. P. Robertson and T. W. Noonan, *Relativity and Cosmology* (Saunders, Philadelphia, Pa., 1968), Chaps. 14–18.

[9]H. P. Robertson, Astrophys. J. 82, 284 (1935).

[10]For a discussion of the validity of Eq. (2.3) see S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Chap. 15.

[11]A. Sandage, Hale Observatories report, 1972 (unpublished), quoted in Ref. 13.

[12]In fact, we have a bootstrap situation here: Changing mass will affect luminosities and thus "the cosmological ladder." The result is a decrease in the value of H. This effect is not easy to estimate (see Refs. 7 and 13), but is certainly not big enough to change the order of magnitude of $\dot{m}/m$ [Eq. (3.5)].

[13]For a recent critical review see J. R. Gott, III, J. E. Gunn, D. N. Schramm, and B. M. Tinsley, California Institute of Technology Report No. OAP-354, 1974 (unpublished).

[14]F. J. Dyson, in *Aspects of Quantum Theory, in Honour of P. A. M. Dirac's 70th Birthday*, edited by A. Salam and E. P. Wigner (Cambridge Univ. Press, Cambridge, England, 1972).

[15]A. Einstein, autobiographical notes in *Albert Einstein: Philosopher-Scientist*, edited by P. A. Schilpp (Harper, New York, 1951), Vol. 1, p. 75.