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SU(3) trace identities and the calculation of effective couplings in a baryon-loop model of weak radiative kaon decays

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A simple algorithm for the recursive reduction of the trace of a product of any number of the f and d matrices of SU(3) is given. SU(3) trace identities (through fourth order) are used to construct effective interaction Lagrangians in the baryon-loop model for PVV, $PV\gamma$, and weak (strangeness-changing) $PV\gamma$ couplings. At the same time it is shown that the effects of baryon-mass breaking on the PVV coupling are negligible in the loop model. The usefulness of this formal compaction is further illustrated in the calculation of the contributions of all vector-meson poles to $K_2^0 \to \pi^+\pi^-\gamma$ decay. It is found that the previous neglect of the contribution of K^* poles in this case is justified.

I. INTRODUCTION

Recently we explored in a succession of papers[1-6] some of the predictive consequences of a baryon-loop model for weak radiative kaon decays with SU(3) symmetry. In the baryon-loop approach to which we allude, one replaces the parity-conserving and parity-violating parts of the weak nonleptonic Hamiltonian density relevant for hyperon decay,

$$\mathcal{H}_G = \sqrt{2}\, G \cos\theta \sin\theta \tfrac{1}{2}\{J_\mu^{(1-i2)}, J^{\mu(4+i5)}\}, \tag{1}$$

by equivalent weak Hamiltonians,

$$\mathcal{H}_W^{(\text{p.c.})} = 2\sqrt{2}\, \bar{\psi}_j(-if_{6ji}F + d_{6ji}D)\psi_i\,, \tag{2}$$

$$\mathcal{H}_W^{(\text{p.v.})} = -c d_{6il}\bar{\psi}_j\left(-if_{ljk} + \frac{\delta}{\phi} d_{ljk}\right)\gamma^\mu \psi_k \partial_\mu \varphi_i\,, \tag{3}$$

expressed in terms of physical baryon fields. The parameters F, D, δ, ϕ, and c derive from Gronau's remarkable fit[7] of a semiphenomenological current-algebraic treatment of nonleptonic hyperon decays to experiment with

$$\begin{aligned} &F = 4.7\times 10^{-5}\ \text{MeV}\,,\\ &D/F = -0.85\,,\\ &\delta/\phi = -0.5\,,\\ &c = 3.2\times 10^{-9}\ \text{MeV}^{-1}\,. \end{aligned} \tag{4}$$

To complete the interaction Lagrangian for this model one next adjoins to the weak Lagrangian $\mathcal{L}_W$ determined by Eqs. (2) and (3) the strong and electromagnetic interactions

$$\begin{aligned}\mathcal{L}_{\text{strong}} = &-\sqrt{2}\, gf \operatorname{Tr}([\bar{B}i\gamma_5, B]M) + \sqrt{2}\, gd \operatorname{Tr}(\{\bar{B}i\gamma_5, B\}M)\\ &-\sqrt{2}\,\phi \operatorname{Tr}([\bar{B}\gamma_\mu, B]V^\mu) + \sqrt{2}\,\delta \operatorname{Tr}(\{\bar{B}\gamma_\mu, B\}V^\mu)\\ &- i(g_\rho/\sqrt{2}) \operatorname{Tr}([M, \partial_\mu M]V^\mu)\,,\end{aligned} \tag{5}$$

$$\mathcal{L}_{\text{em}} = -\tfrac{1}{2} e A^\mu \operatorname{Tr}\{([\bar{B}\gamma_\mu, B] + 2i[M, \partial_\mu M])Q\}\,, \tag{6}$$

where $B = \lambda_i \psi_i/\sqrt{2}$ is the traceless baryon matrix, $M = \lambda_i\varphi_i/\sqrt{2}$ is the traceless pseudoscalar-meson matrix, $V^\mu = \lambda_i \varphi_i^\mu/\sqrt{2}$ is the traceless vector-meson matrix, and $Q = \lambda_3 + \lambda_8/\sqrt{3}$. Following Gronau,[7] we take $d/f = 1.8$ $(d+f=1)$ with $g^2/4\pi = 14.6$, and use the Barger-Olsson values,[8] $g_\rho{}^2/4\pi \simeq 2.5$ and $2\phi = (1.25/1.03)g_\rho = 6.7$.

Unfortunately, calculation in this promising model is hampered by the necessity for evaluating traces of products of Gell-Mann's f and d tensors (loops in unitary spin space). Since we are not aware of any simple approach to the expansion of SU(3) traces ("loops") in SU(3) tensors ("trees"), in spite of the considerable literature[9] dealing with the algebra of SU(3) and with the properties of the f and d tensors,[9] our brief presentation of a recursive solution to this problem in the next section may also be useful in other contexts. In Sec. III SU(3) trace identities through fourth order are employed to summarize succinctly the effective PVV (with an estimate of the effects of the usually neglected baryon mass-breaking), $PV\gamma$, and weak $PV\gamma$ couplings which emerge in the baryon-loop model. These last effective couplings are used to calculate the (small) contribution of the strange vector-meson poles to $K_2^0 \to \pi^+\pi^-\gamma$ decay which is seen to be justifiably omitted in the earlier[2] loop-model calculation of this process.

II. RECURSIVE APPROACH TO SU(3) TRACE IDENTITIES

The problem is to express $\operatorname{Tr}(f_i d_j d_k \cdots)$, the trace of a product of n f and d matrices (a trace of nth order), in terms of traces of $(n-1)$th order or lower. The f and d matrices, with

$$(f_k)_{lm} \equiv -if_{klm}\,, \tag{7a}$$

$$(d_k)_{lm} \equiv d_{klm}\,, \tag{7b}$$

obey the commutation relations

$$[f_j, f_k] = if_{jkl}f_l\,, \tag{8a}$$

$$[f_j, d_k] = if_{jkl}d_l\,, \tag{8b}$$

$$[d_j, d_k]_{\alpha\beta} = if_{jkl}(f_l)_{\alpha\beta} - \tfrac{2}{3}(\delta_{\alpha j}\delta_{k\beta} - \delta_{\alpha k}\delta_{j\beta})\,, \tag{8c}$$

and by virtue of the independent relation,[10]

$$(d_jd_k)_{\alpha\beta} - (f_jf_k)_{\alpha\beta} = -2d_{jkl}(d_l)_{\alpha\beta} - \tfrac{1}{3}(\delta_{jk}\delta_{\alpha\beta} - \delta_{k\beta}\delta_{j\alpha}) + \delta_{j\beta}\delta_{k\alpha}\,, \tag{9}$$

the *anticommutation relations*,

$$\{f_j, f_k\}_{\alpha\beta} = 3d_{jkl}(d_l)_{\alpha\beta} + \delta_{jk}\delta_{\alpha\beta} - (\delta_{\alpha j}\delta_{k\beta} + \delta_{\alpha k}\delta_{j\beta})\,, \tag{10a}$$

$$\{d_j, d_k\}_{\alpha\beta} = -d_{jkl}(d_l)_{\alpha\beta} + \tfrac{1}{3}(\delta_{\alpha j}\delta_{k\beta} + \delta_{\alpha k}\delta_{j\beta} + \delta_{jk}\delta_{\alpha\beta})\,. \tag{10b}$$

One has

$$\mathrm{Tr} f_r = \mathrm{Tr} d_r = 0\,, \tag{11}$$

and from the trace of the anticommutation relations

$$\mathrm{Tr}(f_jf_k) = 3\delta_{jk}\,, \tag{12a}$$

$$\mathrm{Tr}(d_jd_k) = \tfrac{5}{3}\delta_{jk}\,. \tag{12b}$$

Since we may assign to any such trace a *parity* determined by its sign under transposition, i.e.,

$$\mathrm{Tr}(f_jd_kd_l\cdots) = \mathrm{Tr}[(f_jd_kd_l\cdots)^T] = \mathrm{Tr}(\cdots d_l^T d_k^T f_j^T) = (-1)^{N_f}\,\mathrm{Tr}(\cdots d_ld_kf_j)\,, \tag{13}$$

where N_f = number of f-type matrices in the product $f_jd_kd_l\cdots$, we may write

$$\mathrm{Tr}(f_jd_kd_l\cdots) = \tfrac{1}{2}[\mathrm{Tr}(f_jd_kd_l\cdots) + (-1)^{N_f}\,\mathrm{Tr}(\cdots d_ld_kf_j)]\,. \tag{14}$$

Thus we have the algorithm that *odd-parity traces are reduced one order by the pairwise rearrangement of* $\mathrm{Tr}(\cdots d_ld_kf_j)$ *into* $\mathrm{Tr}(f_jd_kd_l\cdots)$ *via commutation relations,* while *even-parity traces are reduced one order by the pairwise rearrangement of* $\mathrm{Tr}(\cdots d_ld_kf_j)$ *into* $\mathrm{Tr}(f_jd_kd_l\cdots)$ *via commutation relations together with an odd number of anticommutations.*

By way of example, one finds for the odd-parity fifth-order trace

$$\begin{aligned}\mathrm{Tr}(f_jf_kf_lf_mf_n) &= \tfrac{1}{2}[\mathrm{Tr}(f_jf_kf_lf_mf_n) - \mathrm{Tr}(f_nf_mf_lf_kf_j)]\\ &= -\tfrac{1}{2}i[f_{kjr}\,\mathrm{Tr}(f_nf_mf_lf_r) + f_{mlr}\,\mathrm{Tr}(f_nf_rf_jf_k)\\ &\quad + f_{nlr}\,\mathrm{Tr}(f_rf_mf_jf_k)\\ &\quad + f_{nmr}\,\mathrm{Tr}(f_lf_rf_jf_k)]\,;\end{aligned} \tag{15}$$

the complete reduction of this trace is then carried out by means of the fourth- and third-order relations,

$$\begin{aligned}\mathrm{Tr}(f_jf_kf_lf_m) &= \tfrac{1}{2}[\mathrm{Tr}(f_jf_kf_lf_m) + \mathrm{Tr}(f_mf_lf_kf_j)]\\ &= \tfrac{1}{2}if_{kjr}\,\mathrm{Tr}(f_mf_lf_r) + \tfrac{1}{2}\mathrm{Tr}(\{f_m, f_l\}f_jf_k)\\ &= \tfrac{1}{2}if_{kjr}\,\mathrm{Tr}(f_mf_lf_r) + \tfrac{3}{2}d_{mlr}\,\mathrm{Tr}(d_rf_jf_k)\\ &\quad + \tfrac{1}{2}(3\delta_{ml}\delta_{jk} + f_{jlr}f_{krm} + f_{klr}f_{jrm})\,,\end{aligned} \tag{16}$$

and

$$\mathrm{Tr}(f_mf_lf_r) = \tfrac{1}{2}[\mathrm{Tr}(f_mf_lf_r) - \mathrm{Tr}(f_rf_lf_m)] = -\tfrac{1}{2}if_{lms}\,\mathrm{Tr}(f_rf_s) = -\tfrac{3}{2}if_{lmr}\,, \tag{17}$$

$$\mathrm{Tr}(d_rf_jf_k) = \tfrac{1}{2}[\mathrm{Tr}(d_rf_jf_k) + \mathrm{Tr}(f_kf_jd_r)] = \tfrac{1}{2}\mathrm{Tr}(\{f_k, f_j\}d_r) = \tfrac{3}{2}d_{kjr}\,. \tag{18}$$

On the other hand, the (independent) even-parity fifth-order trace, $\mathrm{Tr}(d_jf_kf_lf_mf_n)$, has as a possible initial reduction

$$\begin{aligned}\mathrm{Tr}(d_jf_kf_lf_mf_n) &= \tfrac{1}{2}[\mathrm{Tr}(d_jf_kf_lf_mf_n) + \mathrm{Tr}(d_jf_nf_mf_lf_k)]\\ &= \tfrac{1}{2}\mathrm{Tr}(d_j[f_n, f_m]f_lf_k) + \tfrac{1}{2}\mathrm{Tr}(d_jf_m[f_n, f_l]f_k) + \tfrac{1}{2}\mathrm{Tr}(d_jf_mf_l[f_n, f_k]) + \tfrac{1}{2}\mathrm{Tr}(d_j[f_m, f_l]f_kf_n)\\ &\quad + \tfrac{1}{2}\mathrm{Tr}(d_jf_l[f_m, f_k]f_n) + \tfrac{1}{2}\mathrm{Tr}(d_j\{f_l, f_k\}f_mf_n)\,.\end{aligned} \tag{19}$$

III. SOME EFFECTIVE COUPLINGS IN THE BARYON-LOOP MODEL

SU(3) trace identities find their natural application in the construction of effective interaction Lagrangians for PVV, $PV\gamma$, and weak $PV\gamma$ couplings which are *derived* couplings in the loop model. We treat these briefly in turn. (This technique may also be applied to the pentagon graphs of Ref. 2; in this case one would require the systematic reduction of the relevant fifth-order traces.)

A. *PVV* couplings

From the straightforward consideration of loop graphs [see Fig. 1(a)], one finds

$$\mathcal{L}_{PVV} = -\frac{g}{2\pi^2 m}[d(3\phi^2 - \delta^2) + 6f\delta\phi] \times d_{abc}\epsilon_{\mu\nu\rho\sigma}\partial^\mu\varphi_a^\nu\partial^\rho\varphi_b^\sigma\varphi_c\,, \tag{20}$$

with the numerical result

$$\frac{g}{2\pi^2 m}[d(3\phi^2-\delta^2)+6f\delta\phi]=\frac{10.7}{m}. \qquad (21)$$

It is interesting to compare this result with (a) the author's earlier phenomenological determination of PVV coupling,[11] which yields $7.2/m$ for the same quantity, and with (b) the analogous coupling in the octet-broken SU(3) treatment of Brown, Munczek, and Singer,[12] where one finds for

$$2h=\frac{2}{(1+\epsilon_1)}\left(\frac{m}{m_\pi}\right)(0.4\pi)^{1/2}\frac{1}{m} \qquad (22)$$

the bounds

$$\frac{7.3}{m}\leqslant 2h\leqslant\frac{8.7}{m}\quad(1.18\geqslant\epsilon_1\geqslant 0.85) \qquad (23)$$

for an "average" baryon mass (m) of 1 GeV.

The first-order effects on $\mathcal{L}_{PVV}$ [Eq. (20) above] of baryon mass-breaking are easily accommodated in this formalism. The replacement

$$\mathcal{L}_{\text{mass}}=-\bar{\psi}_i m\delta_{ij}\psi_j\rightarrow-\bar{\psi}_i M_{ij}\psi_j\,, \qquad (24)$$

where

$$M_{ij}=m_0\delta_{ij}+\delta m_F(-if_{8ij})+\delta m_D d_{8ij}\,, \qquad (25)$$

generates the mass-breaking interaction Lagrangian,

$$\mathcal{L}_{\text{mass-breaking}}=-\bar{\psi}_i[\delta m_F(-if_{8ij})+\delta m_D d_{8ij}]\psi_j\,, \qquad (26)$$

with the F- and D-mixing parameters given by fitting M_{ij} to the observed octet-baryon mass spectrum:

$$\begin{aligned}
&M_\Sigma-M_\Lambda=\frac{2}{\sqrt{3}}\delta m_D=77\text{ MeV},\\
&M_\Xi-M_P=-\sqrt{3}\,\delta m_F=94\text{ MeV},\\
&m_0=\tfrac{1}{2}(M_\Sigma+M_\Lambda)=1155\text{ MeV},\\
&\delta m_D=67\text{ MeV},\\
&\delta m_F=-55\text{ MeV}.
\end{aligned} \qquad (27)$$

The effective Lagrangian of first order in the breaking parameters $\delta m_F, \delta m_D$, constructed from the loop graphs of Fig. 1(b), is[13]

$$\begin{aligned}
\mathcal{L}^{(1)}_{PVV}=&\frac{2g}{3\pi^2 m_0^{\,2}}\epsilon_{\mu\nu\rho\sigma}\partial^\mu\varphi_b^\nu\partial^\rho\varphi_a^\sigma\varphi_c\\
&\times\{(d_{8bn}d_{nac}+d_{8an}d_{nbc})[\tfrac{7}{6}d\delta m_D(2\phi^2-\delta^2)+\tfrac{1}{6}f\delta m_F(9\phi^2+14\delta^2)+\tfrac{8}{3}(d\delta m_F+f\delta m_D)\delta\phi]\\
&+d_{8cn}d_{nab}[\tfrac{1}{6}d\delta m_D(2\phi^2-7\delta^2)+\tfrac{1}{6}f\delta m_F(9\phi^2+2\delta^2)+\tfrac{14}{3}(d\delta m_F+f\delta m_D)\delta\phi]\\
&+\delta_{ab}\delta_{8c}[\tfrac{1}{36}d\delta m_D(29\phi^2+31\delta^2)+\tfrac{1}{36}f\delta m_F(63\phi^2+29\delta^2)-\tfrac{31}{18}(d\delta m_F+f\delta m_D)\delta\phi]\\
&+(\delta_{8b}\delta_{ac}+\delta_{8a}\delta_{bc})[\tfrac{1}{36}d\delta m_D(-31\phi^2+31\delta^2)+\tfrac{1}{36}f\delta m_F(63\phi^2-31\delta^2)-\tfrac{1}{18}(d\delta m_F+f\delta m_D)\delta\phi]\}.
\end{aligned} \qquad (28)$$

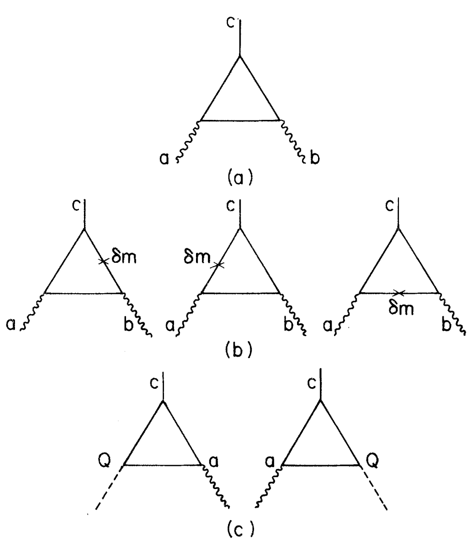

FIG. 1. Baryon-loop graphs for (a) PVV coupling, (b) first-order baryon-mass-breaking correction to PVV coupling via $\mathcal{L}_I=-\bar{\psi}_i\,\delta m_{ij}\psi_j$, (c) $PV\gamma$ coupling. c is the unitary-spin label of the pseudoscalar; b and a are the unitary-spin labels of the vector mesons.

Note that because of the presence of terms proportional to $(\delta_{8b}\delta_{ac}+\delta_{8a}\delta_{bc})$, the structure of $\mathcal{L}^{(1)}_{PVV}$ is *more general* than the "most general form" of Ref. 12. However, since the numerically signifi-

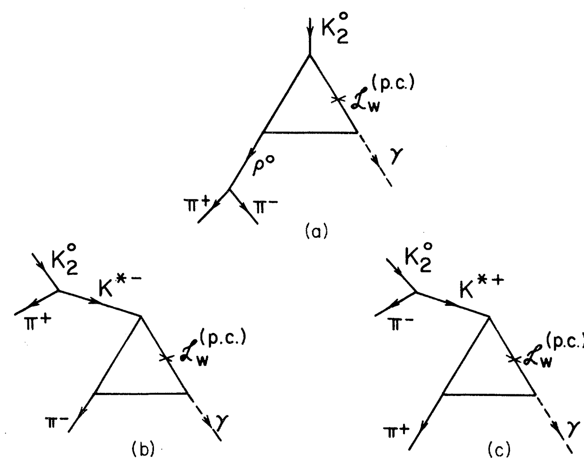

FIG. 2. Contributions of the vector-meson poles to $K_2^0\rightarrow\pi^+\pi^-\gamma$ decay. Graphs of types (b) and (c) were justifiably neglected in Ref. 2.

cant terms of $\mathfrak{L}^{(1)}_{PVV}$, those proportional to $d_{8bn}d_{nac} + d_{8an}d_{nbc}$ and $\delta_{8b}\delta_{ac} + \delta_{8a}\delta_{bc}$, have coefficients only 0.05 and −0.05, respectively, of the coupling characterizing the zeroth $\mathfrak{L}_{PVV}$ Lagrangian, baryon mass-breaking plays a negligible role in the loop model.

B. $PV\gamma$ couplings

Retaining only the coupling of charged baryons with the electromagnetic field, we find in the same manner as before [see Fig. 1(c).] the effective Lagrangian

$$\mathfrak{L}_{PV\gamma} = -\frac{3ge}{4\pi^2 m}(d\phi + f\delta)\left(d_{ac3} + \frac{1}{\sqrt{3}}d_{ac8}\right)\epsilon_{\mu\nu\rho\sigma}\partial^{\mu}\varphi^{\nu}_{a}F^{\rho\sigma}\varphi_c . \tag{29}$$

C. Weak $PV\gamma$ couplings and the vector-meson pole contributions to $K^0_2 \to \pi^+\pi^-\gamma$ decay

Proceeding as in subsections (A) and (B), one can construct the effective weak $PV\gamma$ interaction Lagrangian

$$\mathfrak{L}^{(\text{weak})}_{PV\gamma} = \frac{\sqrt{2}ge}{3\pi^2 m^2}\Gamma(a,c)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\partial^{\rho}\varphi^{\sigma}_{a}\varphi_c , \tag{30}$$

with

$$\begin{aligned}\Gamma(a,c) = {} & D\delta f\left[\tfrac{7}{3}(d_{can}d_{n63} + d_{3an}d_{n6c}) + \tfrac{1}{3}d_{a6n}d_{n3c} + \tfrac{29}{36}\delta_{6a}\delta_{3c} - \tfrac{31}{36}\delta_{3a}\delta_{c6}\right] \\ & + D\phi d\left[\tfrac{7}{3}(d_{can}d_{n63} + d_{3cn}d_{n6a}) + \tfrac{1}{3}d_{c6n}d_{n3a} + \tfrac{29}{36}\delta_{c6}\delta_{3a} - \tfrac{31}{36}\delta_{3c}\delta_{a6}\right] \\ & + F\delta d\left[\tfrac{7}{3}(d_{6cn}d_{na3} + d_{6an}d_{nc3}) + \tfrac{1}{3}d_{can}d_{n63} - \tfrac{31}{36}(\delta_{3a}\delta_{c6} + \delta_{3c}\delta_{a6})\right] \\ & + F\phi f\left[\tfrac{3}{2}(d_{6cn}d_{na3} + d_{6an}d_{nc3} + d_{can}d_{n63}) + \tfrac{7}{4}(\delta_{3a}\delta_{c6} + \delta_{3c}\delta_{a6})\right] + \frac{1}{\sqrt{3}}(3 \to 8) .\end{aligned} \tag{31}$$

Illustrative of the usefulness of this formal compaction is the calculation of the contribution of vector-meson poles to $K^0_2 \to \pi^+\pi^-\gamma$ decay.[2] One has (see Fig. 2)

$$\begin{aligned}&\left\langle \pi^+(p_+)\pi^-(p_-)\gamma(q)\left| i^2\int\int d^4x_1 d^4x_2 T(\mathfrak{L}_{VPP}(x_1)\mathfrak{L}^{(\text{weak})}_{PV\gamma}(x_2))\right| K^0_2(k)\right\rangle \\ &\qquad \simeq -i(2\pi)^4\delta^4(k-(p_+ + p_- + q))A^{(\rho+K^{*+}+K^{*-})}\epsilon(p_+p_-k\epsilon(q,\lambda))(16k_0p^+_0p^-_0q_0)^{-1/2} ,\end{aligned} \tag{32}$$

with

$$\begin{aligned}A^{(\rho+K^{*+}+K^{*-})} = \frac{\sqrt{2}egg_\rho\phi}{\pi^2 m^2}\Bigg\{&\left[3(fF+dD) - \frac{\delta}{\phi}(fD+dF)\right]\frac{1}{(p_+ + p_-)^2 - m_\rho{}^2} \\ &+ \frac{2}{9}\left[Dd + \frac{\delta}{\phi}(fD - 2dF)\right]\left[\frac{1}{(p_+ - k)^2 - m_{K^*}{}^2} + \frac{1}{(p_- - k)^2 - m_{K^*}{}^2}\right]\Bigg\} .\end{aligned} \tag{33}$$

Since one finds the coefficient of the contribution to $A^{(\rho+K^{*+}+K^{*-})}$ from the K^* poles only 0.08 of that from the ρ^0, our previous omission[2] of these contributions is seen to be justified.

[1] R. Rockmore and T. F. Wong, Phys. Rev. Lett. 28, 1736 (1972).

[2] R. Rockmore and T. F. Wong, Phys. Rev. D 7, 3425 (1973).

[3] R. Rockmore, J. Smith, and T. F. Wong, Phys. Rev. D 8, 3224 (1973).

[4] R. Rockmore, Phys. Rev. D 8, 3226 (1973).

[5] A. N. Kamal and R. Rockmore, Phys. Rev. D 9, 752 (1974).

[6] R. Rockmore and A. N. Kamal, Phys. Rev. D 10, 2091 (1974).

[7] M. Gronau, Phys. Rev. Lett. 28, 188 (1972); Phys. Rev. D 5, 118 (1972).

[8] V. Barger and M. Olsson, Phys. Rev. 146, 1080 (1966).

[9] See, for example, the detailed studies of A. J. Macfarlane, A. Sudbery, and P. H. Weisz, Commun. Math. Phys. 11, 77 (1968); and P. Dittner, *ibid.* 22, 238 (1971), for references to the abundant earlier literature.

[10] This relation follows, for example, from the expansion of the interesting identity

$$\begin{aligned}\ln\det(1 + g_\alpha\lambda_\alpha) &= \ln(1 - g^2 + \tfrac{2}{3}d_{\alpha\beta\gamma}g_\alpha g_\beta g_\gamma) \\ &= \operatorname{Tr}\ln(1 + g_\alpha\lambda_\alpha).\end{aligned}$$

[11] R. Rockmore, Phys. Rev. D 1, 226 (1970). In this work we used the effective PVV coupling given by

$$\mathcal{H}_{PVV} = (4\lambda/C_i)d_{ijk}\epsilon_{\mu\nu\alpha\beta}\varphi_i\partial^{\mu}\phi^{\nu}_j\partial^{\alpha}\varphi^{\beta}_k ,$$

with $4\lambda = 0.348$.

[12] L. M. Brown, H. Munczek, and P. Singer, Phys. Rev. Lett. 21, 707 (1968).

[13] In view of relations (8) and (9), one needs only *one* of the even-parity trace identities of fourth order, say,

$$\begin{aligned}\operatorname{Tr}(f_if_jf_kf_l) = {} & d_{ijn}d_{nkl} + d_{jkn}d_{nli} - \tfrac{1}{2}d_{ikn}d_{njl} \\ & + \tfrac{11}{12}(\delta_{ij}\delta_{kl} + \delta_{jk}\delta_{li}) - \tfrac{1}{12}\delta_{ik}\delta_{jl}.\end{aligned}$$