PHYSICAL REVIEW D VOLUME 11, NUMBER 3 1 FEBRUARY 1975

Single-particle contributions to certain classes of algebraic sum rules*

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(Received 24 June 1974)

Phenomenological and theoretical aspects of single-particle contributions to sum rules derived from commutation relations are considered. A derivation of sum rules arising from an equal-time axial-charge algebra evaluated between arbitrary single-particle states is given. A phenomenological analysis of these sum rules is carried out. An analogous derivation of sum rules associated with the sigma operator is shown to be invalid. An amended form for the sum rules is derived. Finally, we comment on relations obtained by taking vacuum-vacuum or vacuum-single-particle matrix elements of certain commutators.

I. INTRODUCTION

We present in this paper a study of phenomenological and theoretical aspects of single-particle contributions to sum rules arising from matrix elements of various commutator algebras. In all cases, we restrict the invariant momentum transfer, q^2, across these matrix elements to be either zero or small ($|q^2| \lesssim 0.5$ GeV^2).

The motivation for undertaking this study arose from a calculation by Golowich and Holstein[1] in which a model for vector and axial-vector current excitation of the pion into arbitrary-spin single-particle states was formulated and solved. This model parameterized the momentum transfer dependence, for small q^2, of matrix elements of the operators $V_a^\mu, A_a^\mu, \partial_\mu A_a^\mu$ ($a = 1, 2, 3$) by means of ρ, A_1, π poles, as well as allowing for higher-mass contributions by means of constants. Current algebra, the "partially conserved axial-vector current" hypothesis (PCAC), and the Bjorken-Johnson-Low (BJL) theorem were used to constrain the parameters of the model. A rigorous consequence of the conditions just enumerated is that excitation of the pion to single-particle states with spin $J \geq 4$ is forbidden. This result led to a study of sum rules associated with commutators of time components of currents at zero or small q^2 taken between single-pion states at infinite momentum. In particular, a Fourier transform of the charge-density algebra

$$[A_+^0(0, \vec{x}), A_-^0(0)] = 2\delta^3(\vec{x})V_3^0(0) \tag{1}$$

(in this paper, $J_\pm \equiv J_1 \pm iJ_2$ for any isospin-carrying operator J_a), its first derivative with respect to q^2, and the first derivative with respect to q^2 of the Fourier transform of

$$[V_+^0(0, \vec{x}), V_-^0(0)] = 2\delta^3(\vec{x})V_3^0(0) \tag{2}$$

were all evaluated at $q^2 = 0$. The sum rules were evaluated in resonance approximation. With experimentally determined decay widths as input, it was found that resonances of spin $J \lesssim 4$ were successful in nearly saturating the sum rules.[1] On this basis, it was conjectured in Ref. 1 that damping of small-q^2 current-induced transitions for which the difference in spin exceeds some moderate value (perhaps $\Delta J \sim 4$) might be a general hadronic phenomenon.

Further thought has tempered, to some extent, our enthusiasm for this outlook. It is possible that the underlying reason for the success of the saturated sum rules lies in the subtractive nature of the commutators in Eqs. (1) and (2). That is, the terms which arise from the two different orderings of the operators in these particular commutators contribute to the sum rule with opposite relative signs. Thus, terms associated with large mass contributions may have little effect on the commutator due to cancellation, although each could be individually large.

Without deeper theoretical understanding, it is not easy to judge the relative importance of these two mechanisms. Conceivably, either can be true to a greater or lesser degree. At any rate, we have been stimulated to examine a phenomenological aspect of this subject, a numerical evaluation of resonance saturated sum rules associated with the equal-time charge algebra

$$[F_+^5(0), F_-^5(0)] = 2F_3(0) \tag{3}$$

taken between arbitrary, diagonal single-particle states. This analysis is given in Sec. II. Our primary aim is simply to ascertain how the numbers come out in light of existing experimental data. As a matter of principle, it is important to keep subjecting relations like Eq. (3) to new experimental tests, even though previous studies[2] lead us to accept its validity. The cleanest signal of something wrong with Eq. (3) would be oversaturation, in which contributions to the left-hand side exceed the bound given by the right-hand side. We also wish to exhibit the problems one encounters in practice while attempting to evaluate the charge-algebra sum rules: What are the phenomenological limitations to these sum rules? Finally, we hope to stimulate experimental work in the difficult subject of higher meson and baryon resonances. The sum rules can provide, in individual cases, a quantitative measure of the extent to which further couplings to a given hadron are to be anticipated in order that the sum rule be saturated.

A natural extension of the work [based on Eq. (3)] just described is to apply the same methods to the equal-time commutator

$$i[F_+^5(0), \partial_\mu A_-^\mu(0)] = 2\sigma(0), \qquad (4)$$

where σ (the "σ operator") is assumed for simplicity to carry zero isospin. As will be described in Sec. III, this turns out to be impossible. The sum rules which result are not valid because the mathematical procedures used in deriving them are not legitimate. A method suggested by Jaffe and coworkers[3] for eliminating this difficulty is discussed, and an amended class of sum rules is written down. The emphasis in this section is almost entirely theoretical.

Thus far, we have discussed the contribution of single-particle intermediate states to certain commutation relations evaluated between single-particle states. In the interest of thoroughness, we devote Sec. IV to a brief survey of the status of "simpler" commutator matrix elements, in which one or both of the external states is the vacuum state.

The paper concludes in Sec. V with a summary of our results and a discussion of their significance.

II. AXIAL-CHARGE COMMUTATOR

Our goal, to perform a *phenomenological* analysis involving the commutation relation of Eq. (3) taken between *arbitrary* diagonal single-particle states, actually dictates that the single-particle intermediate states play a central role. Otherwise we would end up with formulas generally having no realistically obtainable experimental content.

We shall begin with derivations containing enough detail to establish our notation as well as to make the paper self-contained for the reader.[4] Suppose the commutation relation in Eq. (3) is sandwiched between initial and final states $|\alpha(\vec{p},r)\rangle\rangle, \langle\alpha(\vec{p}\,',r)|$ respectively, where r is a helicity label. In the numerical work to be discussed later, we shall consistently choose α to be the state of highest weight in its isospin multiplet.[5] For definiteness, we shall assume that it carries charge +1 in the following derivation. Let us insert an intermediate state consisting of some particle γ, not belonging to the same isotopic multiplet as α, and also sum over the helicity r of particle α. We find

$$\sum_{r,r'}\int \frac{d^3q}{(2\pi)^3 N_\gamma}\{\langle\alpha(\vec{p}\,',r)|F_+^5(0)|\gamma^0(\vec{q},r')\rangle\rangle\langle\gamma^0(\vec{q},r')|F_-^5(0)|\alpha(\vec{p},r)\rangle\rangle - \langle\alpha(\vec{p}\,',r)|F_-^5(0)|\gamma^{++}(\vec{q},r')\rangle\rangle\langle\gamma^{++}(\vec{q},r')|F_+^5(0)|\alpha(\vec{p},r)\rangle\rangle\}+\cdots$$
$$=2(2\pi)^3T_3(\alpha)(2J_\alpha+1)N_\alpha\delta^3(\vec{p}\,'-\vec{p}), \qquad (5)$$

where the normalization of single-particle states is given by

$$\langle\alpha(\vec{p}\,',r')|\alpha(\vec{p},r)\rangle\rangle = (2\pi)^3N_\alpha\delta_{rr'}\delta^3(\vec{p}\,'-\vec{p}). \qquad (6)$$

The normalization factor N_α need not be specified any further in this section because it will cancel out of our equations. Next, express each axial charge in terms of its charge density and use translation invariance to carry out the spatial integrals. As a result, all states α and γ have the same momentum $\vec{p}$. Thus, if we employ

$$\langle\alpha(\vec{p},r)|\partial_\mu A_+^\mu(0)|\gamma^0(\vec{p},r')\rangle\rangle = i(p_\alpha^0-p_\gamma^0)\langle\alpha(\vec{p},r)|A_+^0(0)|\gamma^0(\vec{p},r')\rangle\rangle \qquad (7)$$

along with the PCAC relation

$$\langle\alpha(\vec{p},r)|\partial_\mu A_+^\mu(0)|\gamma^0(\vec{p},r')\rangle\rangle = \frac{m_\pi^{\ 2}F_\pi\langle\alpha(\vec{p},r)|J_+^\pi(0)|\gamma^0(\vec{p},r')\rangle\rangle}{m_\pi^{\ 2}-(p_\alpha-p_\gamma)^2}, \qquad (8)$$

where J_+^π is the pion current[6] and $F_\pi \cong 94$ MeV, we obtain

$$\sum_{r,r'}\frac{(m_\pi^{\ 2}F_\pi)^2}{[m_\pi^{\ 2}-(p_\alpha-p_\gamma)^2]^2}\ \frac{|\langle\alpha(\vec{p},r)|J_+^\pi(0)|\gamma^0(\vec{p},r')\rangle\rangle|^2-|\langle\alpha(\vec{p},r)|J_-^\pi(0)|\gamma^{++}(\vec{p},r')\rangle\rangle|^2}{2N_\alpha N_\gamma T_3(\alpha)(2J_\alpha+1)(p_\alpha^0-p_\gamma^0)^2}+\cdots=1. \qquad (9)$$

Upon taking the limit $|\vec{p}|\to\infty$, we may relate the above squared matrix elements to decay widths if $|m_\gamma - m_\alpha| > m_\pi$. For definiteness, we temporarily take $m_\gamma > m_\alpha + m_\pi$. The association of the above matrix elements with physical decay widths is not exact. The latter are proportional to squared matrix elements having momentum transfer $q^2 = m_\pi^2$, whereas the former have $q^2 = 0$ in the limit $|\vec{p}| \to \infty$. We shall assume that the physical decay widths can be used without appreciable error.[7] This is the main point at which we employ the PCAC hypothesis. We can then express Eq. (9) in the form

$$\frac{8\pi\eta}{T_3(\alpha)} \frac{2J_\gamma + 1}{2J_\alpha + 1} \frac{F_\pi^2 m_\gamma^2}{k(m_\gamma^2 - m_\alpha^2)^2} \times [\Gamma(\gamma^0 \to \alpha\pi^-) - \Gamma(\gamma^{++} \to \alpha\pi^+)] + \cdots = 1, \quad (10)$$

where $\eta = 2$ if the particle α is a pion, $\eta = 1$ otherwise, and k is the decay momentum evaluated in the parent rest frame. There is a question as to whether k should be evaluated with the pion mass taken as zero or physical. We have chosen to use the former, thus implying

$$k = \frac{(m_\gamma^2 - m_\alpha^2)}{2m_\gamma}. \quad (11)$$

A relation analogous to Eq. (10) can be written down for the case $m_\alpha > m_\gamma + m_\pi$, where now $\eta = 2$ if γ is a pion. Summing over all contributions of single-particle states γ, we finally obtain

$$16\pi F_\pi^2 \left\{ \sum_\gamma^{(1)} \frac{2J_\gamma + 1}{2J_\alpha + 1} \frac{\eta}{T_3(\alpha)} \frac{\Gamma(\gamma^0 \to \alpha\pi^-) - \Gamma(\gamma^{++} \to \alpha\pi^+)}{m_\gamma^3(1 - m_\alpha^2/m_\gamma^2)^3} + \sum_\gamma^{(2)} \frac{\eta}{T_3(\alpha)} \frac{\Gamma(\alpha \to \gamma^0\pi^+) - \Gamma(\alpha \to \gamma^{++}\pi^-)}{m_\alpha^3(1 - m_\gamma^2/m_\alpha^2)^3} \right\} + \cdots = 1. \quad (12)$$

The superscripts on the summation symbols refer to (1) $m_\gamma > m_\alpha + m_\pi$ and (2) $m_\alpha > m_\gamma + m_\pi$. Contributions not explicitly included in Eq. (12) must have mass within a band $m_\alpha \pm m_\pi$. There are only a finite number of these.

Equation (12) will form the basis of our phenomenological analysis. Its content can be clarified somewhat by writing it in terms of spin-averaged total cross sections $\bar{\sigma}_\pm$ pertaining to the center-of-mass scattering of charged ($\pm$) pions off the particle α. Starting from a relation like Eq. (9) and taking account of the initial-state flux factor and final-state phase space, it is not difficult to express the sum rule as

$$\frac{2F_\pi^2}{\pi} \int_{s_0}^{\infty} \frac{ds}{s - m_\alpha^2} [\bar{\sigma}_-(s) - \bar{\sigma}_+(s)] + \text{discrete terms} = 2T_3(\alpha), \quad (13)$$

where s is the invariant energy and $s_0 = (m_\alpha + m_\pi)^2$. The "discrete terms" in Eq. (13) correspond to contributions with mass less than $m_\alpha + m_\pi$. We can recover Eq. (12) from Eq. (13) by using the narrow-resonance Breit-Wigner formula

$$\bar{\sigma}_\pm = \frac{4\pi}{k^2} \frac{2J_\gamma + 1}{2J_\alpha + 1} \frac{\pi\Gamma(\pi^\pm\alpha)}{2} \delta(m_\gamma - W), \quad (14)$$

where $W = s^{1/2}$ and k is given in Eq. (11).

Before commencing our numerical study of Eq. (12), we wish to point out two features of Eqs. (12) and (13). First is the convergent nature of the sum rules for large-mass contributions, as evidenced especially in Eq. (13) via the Pomeranchuk theorem. This is one of the mechanisms mentioned in the Introduction which tend to make this class of sum rules approximable in terms of single-particle contributions. Its origin lies in the antisymmetric behavior of Eq. (3) under ($+ \leftrightarrow -$), a property not universally shared by all commutators as we shall see in Sec. III. A second noteworthy feature of the sum rules, more easily apparent in Eq. (12), is the existence of contributions not expressible in terms of decay widths or cross sections. These terms are known only in special cases—more often, we lack even a reasonable theoretical estimate of them. It is this, along with the fact that these terms become more numerous (albeit finite in number) as the mass of the state chosen for α is increased, which constitutes the major limitation in confronting the sum rules (12) with experimental data. As candidates for the external states α, we shall consider first the baryons, then the mesons. Unless otherwise specified, the data are taken from Ref. 8.[9] In order of their appearance, the baryons to be surveyed here are $N^+(938)$, $\Delta^{++}(1233)$, $N^{*+}(1470)$, $\Sigma^+(1189)$, $Y_1^{*+}(1384)$, and $\Xi^0(1315)$.

$\underline{N^+(938).}$ This is naturally the case for which, of all the hadrons, the most data are available. Numerical results are exhibited in Table I. A summary of contributions is given by

$$\underset{\text{nucleon}}{g_A^2} + \underset{T=1/2}{0.544} - \underset{\Delta(1233)}{0.975} - \underset{T=3/2}{0.175} + \cdots = 1, \quad (15)$$

where the $T = \frac{1}{2}, \frac{3}{2}$ contributions group together all resonances of a given isospin. There are two ways in which Eq. (15) might naturally be interpreted: (i) Simply insert the existing experimental value for g_A, thereby testing how well Eq. (15) is saturated. With $g_A = 1.25$, we obtain $0.96 + \cdots = 1$. (ii) Use Eq. (15) to compute g_A. This is a traditional way of using algebras, often with poor success because the number of intermediate states taken into account is truncated too severely. From Eq. (15), we find $g_A^2 = 1.606$ or $g_A \cong 1.27$. In this case, nature has supplied us with enough data to give a reasonably good estimate of g_A.

$\Delta^{++}(1233)$. The only intermediate-state contribution not estimable in terms of a decay width is that of $\Delta(1233)$ itself. The parameter which characterizes this contribution is a form factor $F_A(q^2)$,

$$\langle \Delta^{++}(\vec{p}', r') | A_+^\mu(0) | \Delta^+(\vec{p}, r) \rangle$$
$$= i\bar{u}_\sigma(\vec{p}', r')\gamma_5[\gamma^\mu g^{\sigma\rho} F_A(q^2) + \cdots] u_\rho(\vec{p}, r), \quad (16)$$

evaluated at $q^2 = 0$. Analogous to g_A of the nucleon axial-vector matrix element, let us define $f_A \equiv F_A(0)$ for the $\Delta(1233)$ axial-vector matrix element. Referring to the values given in Table II, we have

$$\underset{\Delta}{\tfrac{5}{27} f_A^2} + \underset{\text{nucleon}}{0.244} + \underset{T=1/2}{0.149} + \underset{T=3/2}{0.03} + \cdots = 1. \quad (17)$$

Unfortunately, it is not realistic to expect an experimental determination of f_A. Rather than anticipate Eq. (17) will provide a good value for f_A provided that we assume the above numbers already saturate the sum rule, we prefer instead to adopt the more conservative stance of testing the degree of saturation in (17) by obtaining some estimate of f_A. This is done by first expressing f_A in terms of the coupling constant $g_{\pi\Delta\Delta}$ via a Goldberger-Treiman relation and then using $SU(6)_W$ to relate $g_{\pi\Delta\Delta}$ to the known quantity $g_{\pi NN}$.

TABLE I. Resonance contributions to axial-charge sum rule with proton external state. The first four columns list properties of each intermediate state and the final two columns give individual and cumulative contributions to the sum rule Eq. (12). See Eq. (15).

Mass (MeV)	Spin	Isospin	Partial width (MeV)	Individual	Cumulative
1470	0.5	0.5	140	0.125	0.125
1525	1.5	0.5	64	0.089	0.215
1550	0.5	0.5	37	0.023	0.238
1678	2.5	0.5	58	0.067	0.305
1685	2.5	0.5	86	0.097	0.402
1715	0.5	0.5	120	0.041	0.443
1755	0.5	0.5	40	0.012	0.455
1815	1.5	0.5	64	0.032	0.488
2130	3.5	0.5	74	0.035	0.522
2223	4.5	0.5	44	0.0214	0.544
1233	1.5	1.5	115	−0.975	−0.431
1655	0.5	1.5	46	−0.019	−0.451
1695	1.5	1.5	36	−0.026	−0.477
1880	2.5	1.5	48	−0.030	−0.507
1858	0.5	1.5	68	−0.015	−0.522
1955	3.5	1.5	99	−0.069	−0.591
2385	5.5	1.5	34	−0.015	−0.606

We find $f_A^2 \cong 2.1$, whereupon the sum rule (17) reads $0.82 + \cdots = 1$. See the Appendix for further discussion.

$N^{*+}(1470)$. This state is the lowest in mass of the essentially continuous spectrum of *highly* excited πN resonances. As such, it represents the first case where our ability to test the sum rule (12) becomes seriously hindered. There is a band of width $2m_\pi$ surrounding $N^*(1470)$ for which contributions to the sum rule cannot be estimated experimentally. This band contains the states $N^*(1520)$ and $N^*(1535)$ with spin-parity $J^P = \frac{3}{2}^-$ and $\frac{1}{2}^-$, respectively. Of course, there is also the contribution of $N^*(1470)$, expressed in terms of a parameter g_A^* entirely analogous to g_A. The sum rule reads

$$\underset{N^*(1470)}{g_A^{*2}} + \underset{\text{nucleon}}{0.126} - \underset{\Delta(1233)}{0.207}$$
$$+ N^*(1520) + N^*(1535) + \cdots = 1. \quad (18)$$

The possibility of theoretically estimating g_A^* is discussed in the Appendix. Naturally, for the more massive states α, the amount of data pertaining to resonances which decay into α plus a pion gets scarcer. This explains the paucity of numerical information in Eq. (18) relative to that in Tables I and II. However, this does not constitute a fundamental difficulty like the discrete contributions just discussed. If experiments in hadron spectroscopy continue, we can hope that transitions from one higher resonance to another can ultimately be unravelled. This is not easy but at least it is possible. At any rate, all we can infer from the numbers in Eq. (18) is that the sum of the discrete contributions equals 1.1, given the nucleon and $\Delta(1233)$ contributions.

TABLE II. Resonance contributions to axial-charge sum rule with $\Delta^+(1233)$ external state. The first four columns list properties of each intermediate state and the final two columns give individual and cumulative contributions to the sum rule Eq. (12). See Eq. (17).

Mass (MeV)	Spin	Isospin	Partial width (MeV)	Individual	Cumulative
939	0.5	0.5	115	0.244	0.244
1470	0.5	0.5	58	0.052	0.296
1525	1.5	0.5	35	0.035	0.331
1678	2.5	0.5	80	0.039	0.369
1685	2.5	0.5	38	0.018	0.387
1755	0.5	0.5	60	0.006	0.393
1655	0.5	1.5	57	0.008	0.402
1695	1.5	1.5	62	0.014	0.416
1858	0.5	1.5	27	0.001	0.418
1953	3.5	1.5	36	0.005	0.423

$\Sigma^+(1189)$. This state is of interest because it has the lowest mass for which the sum rule (12) is testable in a channel with nonzero strangeness and baryon number one. There are two discrete contributions, so that the sum rule reads

$$\tfrac{1}{2}(g_A^{(\Sigma^+\Sigma^0)})^2+\tfrac{1}{2}(g_A^{(\Sigma^+\Lambda)})^2+\text{resonances}=1\,. \tag{19}$$

We use SU(3), with an F/D parameter $\alpha\cong\frac{2}{3}$, to estimate the quantities $g_A^{(\Sigma^+\Sigma^0)}$ and $g_A^{(\Sigma^+\Lambda)}$. The resonance contributions are exhibited in Table III. Altogether, the numbers are

$$\underset{\Sigma}{0.174}+\underset{\Lambda}{0.231}+\underset{T=0}{0.142}+\underset{T=1}{0.055}+\cdots=1\,, \tag{20}$$

or in total, $0.60+\cdots=1$. There remains a fairly substantial contribution to be made from as yet unobserved decays of hypercharge-zero baryon resonances into $\Sigma\pi$. Note that the low-mass $\frac{1}{2}^+$ and $\frac{3}{2}^+$ contributions to the proton sum rule add to 0.588, whereas for the Σ^+ sum rule they give 0.442. However, the higher resonances for the intensively studied πN system sum to 0.369, whereas the more complicated system of hypercharge-zero higher resonances contributes only 0.16.

$Y_1^+(1385)$. The problem of unobservable contributions to the $Y_1^+(1385)$ sum rule is not serious. There is the contribution of $Y_1^0(1385)$ which we can estimate from $SU(6)_W$ just as we did for the $\Delta^+(1233)$ contribution to the $\Delta^{++}(1233)$ sum rule. In addition, the unitary singlet state $Y_0^*(1405)$ can contribute in principle. However, this transition proceeds only through SU(3) breaking effects. Thus, we can hope that its effect on the sum rule is minor. Our numerical analysis gives

TABLE III. Resonance contributions to axial-charge sum rule with $\Sigma^+(1189)$ external state. The first four columns list properties of each intermediate state and the final two columns give individual and cumulative contributions to the sum rule Eq. (12). See Eq. (20).

Mass (MeV)	Spin	Isospin	Partial width (MeV)	Individual	Cumulative
1405	0.5	0.0	40	0.094	0.094
1520	1.5	0.0	7	0.010	0.104
1670	0.5	0.0	11	0.003	0.107
1690	1.5	0.0	31	0.015	0.122
1815	2.5	0.0	9	0.004	0.125
1830	2.5	0.0	42	0.016	0.141
2100	3.5	0.0	5	0.001	0.142
1385	1.5	1.0	4	0.037	0.179
1670	1.5	1.0	20	0.016	0.195
1765	2.5	1.0	1	0.007	0.196
2030	3.5	1.0	5	0.002	0.197

$$\underset{Y_1(1385)}{0.383}+\underset{\Lambda}{0.120}+\underset{\Sigma}{0.019}+\underset{Y_0(1520)}{0.086}+\underset{Y_0(1815)}{0.008}+\underset{Y_1(1765)}{0.013}+\cdots=1\,, \tag{21}$$

or $0.62+\cdots=1$, which is a reasonably high amount of saturation considering the large mass of the external state.

$\Xi^0(1315)$. Despite the experimental effort which has been put into the hypercharge-1 baryon channel, distressingly little is known about the particle spectrum at this time. Our Ξ^0 sum rule reads

$$\underset{\Xi^-}{0.173}+\underset{\Xi^*(1530)}{0.196}+\cdots=1\,, \tag{22}$$

where we have used SU(3) to estimate the Ξ^- (discrete) contributions. Equation (22) sums to $0.37+\cdots=1$, so the dominant contributions to the Ξ^0 sum rule remain to be detected. We can only await a correct interpretation of the resonant behavior around energy 1820 MeV, which has for so long resisted efforts at classification. This concludes our survey of the baryon sum rules.

The main difference between the structure of the baryon and meson sum rules is that those intermediate states γ which lie in the same isotopic spin multiplet as the external state α are forbidden to contribute to the latter by G parity. We shall consider, in turn, sum rules for the following meson states: π^+, $\rho^+(765)$, $A_2^+(1310)$, $K^+(494)$, $K_V^+(892)$, $K_T^+(1421)$, where the subscripts V, A, and T denote vector (1^-), axial-vector (1^+), and tensor (2^+), respectively.

π^+. This sum rule, also studied in Ref. 1, gives

$$\underset{\epsilon(700)}{0.292}+\underset{\epsilon'(1000)}{0.047}+\underset{f(1260)}{0.097}+\underset{\rho(765)}{0.47}+\underset{g(1680)}{0.043}+\cdots=1\,, \tag{23}$$

where $\pi\pi$ total widths $\Gamma(\epsilon\pi\pi)=300$ MeV, $\Gamma(\epsilon'\pi\pi)=50$ MeV have been employed.[10] The above contributions come close to saturating the sum rule, yielding $0.95+\cdots=1$. Thus the lowest-mass baryon and meson states have their resonant-dominated sum rules saturated to within five percent.

$\rho^+(765)$. The only contribution to the ρ^+ sum rule which is not directly measurable comes from the ω intermediate state. However, we may use Eq. (9) in conjunction with the Gell-Mann–Sharp–Wagner model[11] of the $\omega\to\pi\gamma$ transition to estimate it. We define a coupling constant $g_{\omega\rho\pi}$ for the process $\omega(\vec{k},r')\to\rho(\vec{q},r)+\pi(\vec{p})$, with a momentum-space interaction amplitude

$$g_{\omega\rho\pi}\epsilon_{\alpha\beta\mu\nu}k^\alpha\epsilon^\beta(k,r')q^\mu\epsilon^{\dagger\nu}(q,r)\,. \tag{24}$$

The ρ^+ sum rule can then be written as

$$\tfrac{1}{6}(F_\pi g_{\omega\rho\pi})^2+\underset{\pi}{0.078}+\underset{A_1(1100)}{0.189}+\underset{A_2(1310)}{0.042}+\cdots=1\,. \tag{25}$$

A recent determination[12] of $g_{\omega\rho\pi}$ is quoted as $g_{\omega\rho\pi}=14.4\ \mathrm{GeV}^{-1}$, so the degree of saturation of the ρ^+ sum rule is $0.62+\cdots=1$.

$\underline{A_2^+(1310)}$. This is the largest-mass nonstrange meson whose sum rule we shall analyze numerically. The extent to which contributions from states lying within energy m_π above or below the A_2 mass affect the sum rule is not clear because of our relative ignorance of the particle spectrum at these energies. An amusing example of the difficulty in estimating one of these contributions is provided by the axial-vector meson $B(1237)$. In principle, $A_2(1310)$ can decay into $B(1237)\pi$ because the finite widths of these resonances provides a certain amount of phase space.[13] The decay $A_2\rightarrow\omega\pi\pi$ has been observed,[14] and noting that $B\rightarrow\omega\pi$ is essentially the only decay mode of $B(1237)$, we can obtain an upper bound on the rate for $A_2\rightarrow B\pi$. This upper bound would imply a huge contribution from $B(1237)$ of 0.78 to the A_2 sum rule. However, it would also imply a dimensionless p-wave coupling constant $g^2(A_2B\pi)/4\pi\cong 70$, which in our opinion is too large to be believed. Therefore, we summarize the present situation as

$$\underset{\rho(765)}{0.024}+\underset{\eta(549)}{0.01}+B(1237)\text{ term}+\cdots=1 \tag{26}$$

for the A_2^+ sum rule, with the "$B(1237)$ term" $\lesssim 0.78$.

$\underline{K(494)}$. There are data at present for us to take the contribution of just two resonances into account,

$$\underset{K_V(892)}{0.375}+\underset{K_T(1421)}{0.084}+\cdots=1\,, \tag{27}$$

or $0.46+\cdots=1$. The analogous two states contribute 0.57 to the pion sum rule, so at this level the difference in convergence between the π and K sum rules is not large. However, the predominantly isoscalar meson $\epsilon(700)$ contributes significantly to the pion sum rule but not at all to that of the kaon. The kaon sum rule must make up the difference with the more massive states,[15] a situation which suggests a deep relation between the chiral algebra and the spectrum of hadron states.

$\underline{K_V(892)}$. The status of the $K_V(892)$ sum rule would be clarified if more information on the axial-vector kaons in the mass range 1200–1400 MeV were available. For our calculation, we have assumed that the axial-vector kaon with mass 1242 decays into $K_V(892)\pi$ with a width of 127 MeV.[8] We find

$$\underset{K(494)}{0.0168}+\underset{K_T(1421)}{0.0272}+\underset{K_A(1242)}{0.33}+\cdots=1\,, \tag{28}$$

or $0.48+\cdots=1$, a fair degree of saturation.

$\underline{K_T(1421)}$. Like its SU(3) partner $A_2^+(1310)$, the $K_T^+(1421)$ has but two well-determined contributions to its sum rule,

$$\underset{K(494)}{0.017}+\underset{K_V(892)}{0.027}+\cdots=1\,, \tag{29}$$

or $0.044+\cdots=1$. The 0^- and 1^- states are thus seen to contribute almost negligibly to the sum rules of the tensor mesons $A_2^+(1310)$ and $K_T^+(1421)$.

Further comments on the analysis just presented are reserved for the Conclusion. In Sec. III, we consider an algebra which at first sight appears amenable to a similar treatment.

III. σ OPERATOR

The σ operator has been defined in terms of an equal-time commutation relation in Eq. (4). Not much empirical knowledge exists regarding matrix elements of this operator. The nucleon matrix element is thought to be given, to a good approximation, by the combination of isospin-even pion-nucleon amplitudes $A^{(+)}+\nu B^{(+)}$ evaluated at $s=m_N^2$, $t=2m_\pi^2$. A variety of recent phenomenological efforts[16] points to a value $40\lesssim\langle N|\sigma|N\rangle\lesssim 70$ MeV although a definitive evaluation has yet to be performed. The pion matrix element can be estimated in terms of a low-energy theorem to be $\langle\pi|\sigma|\pi\rangle\cong m_\pi^2$. This evaluation is suspect because it involves extrapolation over a distance m_π^2 of a quantity itself of order m_π^2. However, it does provide an order-of-magnitude estimate.

We may use the methods of the previous section to derive a class of sum rules for the σ-operator matrix element taken between single-particle states. Upon doing so we find[17]

$$\phi_\alpha\langle\alpha(\vec{p},\lambda)|\sigma(0)|\alpha(\vec{p},\lambda)\rangle=-16\pi F_\pi^2\left[\sum{}^{(1)}\frac{2J_\gamma+1}{2J_\alpha+1}\,\frac{\Gamma(\gamma^0\rightarrow\alpha\pi^-)+\Gamma(\gamma^{++}\rightarrow\alpha\pi^+)}{m_\gamma(1-m_\alpha^2/m_\gamma^2)^2}-\sum{}^{(2)}\frac{\Gamma(\alpha\rightarrow\gamma^0\pi^+)+\Gamma(\alpha\rightarrow\gamma^{++}\pi^-)}{m_\alpha(1-m_\alpha^2/m_\gamma^2)^2}\right]+\cdots\,, \tag{30}$$

where the superscripts on the summation symbols have the same meaning as in Eq. (12). The constant

$$\phi_\alpha=\begin{cases}1 & \text{meson}\\ 2m_\alpha & \text{baryons}\end{cases} \tag{31}$$

occurs because meson and baryon matrix elements of the σ operator have different units. A formula relating the σ-operator matrix elements to cross sections, analogous to Eq. (13), can also be derived,

$$\phi_\alpha\langle\alpha(\vec{p},\lambda)|\sigma(0)|\alpha(\vec{p},\lambda)\rangle\rangle$$
$$=-\frac{2F_\pi^2}{\pi}\int ds[\bar{\sigma}_-(s)+\bar{\sigma}_+(s)]+\text{discrete terms}. \tag{32}$$

There are several features of Eqs. (30) and (32) that warrant immediate discussion. The widths and cross sections are seen to add, whereas in Eqs. (12) and (13) they contribute with opposite relative sign. This is not a mistake, but rather reflects the behavior of the commutation relation (4) under the interchange $(+\longleftrightarrow-)$. Of greater significance is that, given existing estimates (e.g., Regge) of the asymptotic behavior of hadronic cross sections, our formulas for the σ-operator matrix elements are seen to diverge. Even if these asymptotic estimates turn out to be wrong and the integrals in (32) actually converge, the situation is still bleak because the dominant contributions to (30) and (32) are of the wrong sign.[18] As an example, the resonances exhibited in Table I give $\langle N|\sigma(0)|N\rangle\cong-0.92m_N$ according to Eq. (30). In the same manner, we calculate $\langle\pi|\sigma(0)|\pi\rangle\cong-36m_\pi^2$ upon using the $\pi\pi$ resonances listed in Sec. III. Neither of these values can coexist with the estimates discussed at the beginning of this section.

It turns out that one can employ a rather different derivation for the $q^2=0$ sum rule (32), the result of which exhibits remnants of its original form, while at the same time patching up its weak spots. Much of the work is already done in Ref. 3, so in the following we shall just outline the necessary steps.

First, we rederive Eq. (32) as a $q^2=0$ sum rule involving structure functions for neutrino scattering.[19] Consider the spin-averaged quantity

$$W^{\mu\nu}(q,p)=\frac{1}{4\pi}\int d^4x\,e^{iq\cdot x}\times\langle\alpha(\vec{p})|[J_+^\mu(x),J_-^\nu(0)]|\alpha(\vec{p})\rangle\rangle, \tag{33}$$

where J_+^μ is the $\Delta Y=0$ weak current which raises the hadronic charge. The σ-operator matrix element is obtained by contracting Eq. (33) with q_ν and setting $\vec{q}=0$,

$$\phi_\alpha\langle\alpha(\vec{p})|\sigma(0)|\alpha(\vec{p})\rangle\rangle=-\int_{-\infty}^{\infty}dq^0q^0W^{00}(q^0,p). \tag{34}$$

If we define a set of kinematic-singularity-free structure functions[20]

$$W^{\mu\nu}=-g^{\mu\nu}W_1+\frac{p^\mu p^\nu}{m_\alpha^2}W_2-i\frac{\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta}{2m_\alpha^2}W_3+\frac{q^\mu q^\nu}{m_\alpha^2}W_4+\frac{(p^\mu q^\nu+p^\nu q^\mu)}{2m_\alpha^2}W_5, \tag{35}$$

where $W_i=W_i(q^2,\nu)$, $\nu=q\cdot p$, and use the crossing property

$$W_2^\nu(q^2,\nu)=-W_2^{\bar{\nu}}(q^2,-\nu), \tag{36}$$

we obtain

$$\phi_\alpha\langle\alpha(\vec{p})|\sigma(0)|\alpha(\vec{p})\rangle\rangle=-\frac{1}{m_\alpha^2}\int_0^\infty dx\,x[W_2^\nu(0,x)+W_2^{\bar{\nu}}(0,x)]. \tag{37}$$

But if the lepton mass is neglected, we can use the $q^2=0$ relation

$$W_2^{\bar{\nu},\nu}(0,\nu)=\frac{2m_\alpha^2F_\pi^2}{\pi kW}\sigma_\pm(W) \tag{38}$$

[where $W=s^{1/2}$, $k=(s-m_\alpha^2)/2s$] in conjunction with Eq. (37) to regain Eq. (32). In other words, the steps leading to Eq. (37) are not valid.

The method suggested in Ref. 3 to deal with these difficulties is to consider the spin-averaged amplitude

$$T^{\mu\nu}(q^2,\nu)=i\int d^4x\,e^{iq\cdot x}\theta(x^0)\times\langle\alpha(p)|[J^\mu(x),J^\nu(0)]|\alpha(p)\rangle\rangle+\text{seagull terms}, \tag{39}$$

where

$$\operatorname{Im}T^{\mu\nu}(q^2,\nu)=2\pi W^{\mu\nu}(q^2,\nu). \tag{40}$$

The seagull terms are polynomials in q which might be needed in order for $T^{\mu\nu}$ to be a legitimate second-rank tensor. The decomposition used in Ref. 3 for $W^{\mu\nu}$ is the same as our Eq. (35) except for the presence of

$$W_4'=W_4-\frac{m_\alpha^2}{q^2}W_1-\frac{\nu^2}{q^4}W_2,\qquad W_5'=W_5+\frac{2\nu}{q^2}W_2 \tag{41}$$

in place of our $W_{4,5}$. The above decomposition, although not kinematic singularity free for $q^2\to0$, has the advantage that only $W'_{4,5}$ contribute to $q_\mu W^{\mu\nu}$. The amplitude $T^{\mu\nu}$ of Eq. (39) has a similar decomposition in terms of amplitudes T_i', $i=1,\ldots,5$. The key point is that appropriately subtracted dispersion relations can be written for each of the $T_i'(q^2,\nu)$. Thus, the high-energy $(\nu\to\infty)$ behavior is properly accounted for. In particular, the dispersion relation for $T_4'(q^2,\nu)$ is seen to contain a subtraction constant $T_4'(q^2,0)$. Information regarding the σ operator is obtained by taking the BJL limit of $q_\nu T^{\mu\nu}(q^2, \nu)$. It is found that

$$\phi_\alpha\langle\alpha(p)|\sigma(0)|\alpha(p)\rangle = \frac{1}{2m_\alpha{}^2} \lim_{q^2\to-\infty} q^4 \bar{T}'_4(q^2, 0) , \tag{42}$$

where the quantity $\bar{T}'_4(q^2, 0)$ is that part of $T'_4(q^2, 0)$ which varies as q^{-4} in the limit $q^2 \to \infty$. At this point, the situation for expressing $\langle\alpha|\sigma|\alpha\rangle$ in terms of measurable quantities admittedly looks hopeless.

One way out of this impasse is to conjecture the existence of $J=0$ fixed poles[21] in the $T'_{4,5}$ amplitudes. The fixed poles can be extracted from $T'_{4,5}$ by subtracting off from these amplitudes all Regge contributions in the range $1 \geq \alpha(0) > 0$. One finds for their residues in the limit $q^2 \to -\infty$,

$$\begin{aligned} C_4(q^2) &\to T'_4(q^2, 0) - \frac{8m_\alpha{}^4}{q^4} \int_0^\infty (\tilde{F}_4^\nu + \tilde{F}_4^{\bar{\nu}}) \frac{dx}{x^3} , \\ C_5(q^2) &\to \frac{4m_\alpha{}^2}{q^2} \int_0^\infty (\tilde{F}_5^\nu + \tilde{F}_5^{\bar{\nu}}) \frac{dx}{x^2} , \end{aligned} \tag{43}$$

where $F_{4,5}$ are the scaling limits of $\nu^2 W_{4,5}/m_\alpha{}^4$, and $\tilde{F}_{4,5} = F_{4,5} - F_{4,5}^{(R)}$, where $F_{4,5}^{(R)}$ are the Regge fits which include all singularities with $1 \geq \alpha(0) > 0$. Notice that Eq. (43) gives information on the subtraction constant $T'_4(q^2, 0)$. The $C_{4,5}(q^2)$ can be related to fixed-pole residues $P_i(q^2)$ $(i=1, \ldots, 5)$ of the kinematic-singularity-free amplitudes by an equation identical in form to (41). With the assumptions that the $P_i(q^2)$ are polynomials in q^2 and that $\nu^2 T'_{4,5}/m_\alpha{}^4$ scale, it follows from Eqs. (42), (43) that

$$\lim_{q^2\to-\infty} q^4 \bar{T}'_4(q^2, 0) = -m_\alpha{}^4 P_2(0) + 8m_\alpha{}^4 \int_0^\infty (\tilde{F}_4^\nu + \tilde{F}_4^{\bar{\nu}}) \frac{dx}{x^3} . \tag{44}$$

But by its very definition

$$P_2(0) = \frac{2}{m_\alpha{}^4} \int_0^\infty dx\, x[\tilde{W}_2^\nu(0, x) + \tilde{W}_2^{\bar{\nu}}(0, x)] . \tag{45}$$

Finally, we have

$$\begin{aligned} \phi_\alpha\langle\alpha(p)|\sigma(0)\alpha(p)\rangle &= 4m_\alpha{}^2 \int_0^\infty dx\, x(\tilde{F}_4^\nu + \tilde{F}_4^{\bar{\nu}}) \\ &\quad - \frac{1}{m_\alpha{}^2} \int_0^\infty dx\, x[\tilde{W}_2^\nu(0, x) + \tilde{W}_2^{\bar{\nu}}(0, x)] . \end{aligned} \tag{46}$$

Upon comparison with the original formula Eq. (37) for the σ-operator matrix element, Eq. (46) is seen to solve the divergence problem as well as include the necessary positive contribution. However, the triumph is rather hollow because even for the nucleon, the structure function F_4 will be extremely difficult to measure. That is, the formula (46), while sound in principle, is not likely to be of any use in practice.

IV. VACUUM MATRIX ELEMENTS

Thus far, we have studied diagonal single-particle matrix elements of the axial-charge and σ-operator commutation relations. The sum rules thereby generated are expressible in terms of pion cross sections and structure functions pertaining to neutrino-induced processes. We have shown that at least two of the axial-charge sum rules are almost entirely saturated by single-particle intermediate states and that several others give promise of behaving accordingly as more data become available. In this section, we allow one or both of the external states to be the vacuum.[22] Again, we focus on the contributions of the single-particle intermediate states and also clarify the physical content of the algebra sum rules.

First, we treat the vacuum-vacuum matrix elements. Suppose we have an algebra in which some charge operator Q is commuted at equal times with local operator $A(0)$ to produce local operator $B(0)$,

$$i[Q(0), A(0)] = B(0) . \tag{47}$$

Sandwiching this commutator between vacuum states and inserting single-particle intermediate states yields

$$\langle 0|B(0)|0\rangle = -\sum_\gamma \langle 0|\partial_\mu J^\mu(0)|\gamma\rangle\langle\gamma|A(0)|0\rangle , \tag{48}$$

where $\partial_\mu J^\mu$ is the divergence of the current associated with charge Q. The sum rule (48) can also be derived as a low-energy theorem associated with the propagator

$$\Delta(q^2) = i \int d^4x\, e^{iq\cdot x}\langle 0|T\partial_\mu J^\mu(x)A(0)|0\rangle . \tag{49}$$

Therefore, this type of sum rule relates the vacuum expectation value of a local operator to the zero-energy value of a related propagator.

Let us briefly explore the consequences of single-particle dominance in a model where the chiral nonsymmetric part of the energy density is

$$\theta'_{00} = u_0 + cu_8 \tag{50}$$

and the trace of the energy-momentum tensor is

$$\theta = (4-d)(u_0 + cu_8) , \tag{51}$$

where $u_{0,8}$ transform as 0^+ isoscalar members of $(3, 3^*) + (3^*, 3)$ with dimension d. Letting operators $\partial_\mu J^\mu$ and A of Eq. (49) become $\partial_\mu A_a^\mu$, $\partial_\mu A_b^\mu$, first with $a, b = 1, 2, 3$, then with $a, b = 4, 5, 6, 7$, we find

$$m_\pi{}^2F_\pi{}^2 = -\tfrac{1}{3}(\sqrt{2}+c)(\sqrt{2}\,\langle 0|u_0|0\rangle + \langle 0|u_8|0\rangle) \tag{52}$$

and

$$m_K{}^2F_K{}^2 = -\frac{2\sqrt{2}-c}{2\sqrt{3}} \times\left(\frac{\sqrt{2}}{\sqrt{3}}\langle 0|u_0|0\rangle - \frac{1}{2\sqrt{3}}\langle 0|u_8|0\rangle\right) . \tag{53}$$

If the vacuum is taken to be approximately SU(3)-invariant, $\langle 0|u_8|0\rangle \cong 0$, then the approximate numerical relation

$$c = -2\sqrt{2}\left[\frac{13(F_K/F_\pi)^2 - 1}{26(F_K/F_\pi)^2+1}\right] \tag{54}$$

is obtained.[23] Next replace $\partial_\mu J^\mu$ of Eq. (49) by the operator θ and for A substitute first θ, then σ. We therefore find

$$\left(\frac{F_\epsilon}{F_\pi}\right)^2 = \frac{3}{\sqrt{2}}\,\frac{d(4-d)}{\sqrt{2}+c}\left(\frac{m_\pi}{m_\epsilon}\right)^2 \tag{55}$$

and then

$$\frac{F_\pi}{F_\epsilon} = g_\epsilon , \tag{56}$$

where F_ϵ, g_ϵ are defined by

$$\langle 0|\theta(0)|\epsilon(p)\rangle = m_\epsilon{}^2F_\epsilon \tag{57}$$

and

$$\langle 0|\sigma(0)|\epsilon(p)\rangle = m_\pi{}^2F_\pi g_\epsilon . \tag{58}$$

We have obtained Eqs. (52)–(58) by approximating zero-energy two-point functions in terms of $0^-(\pi, K)$ and then $0^+(\epsilon)$ intermediate states. The over-all picture given by Eqs. (54) and (55)–(58) is that c is near $-\sqrt{2}$ and that F_π and F_ϵ are of the same order of magnitude.

There is nothing outlandish about the results just derived. They are in qualitative accord with estimates of c and F_π/F_ϵ arising from at least nominally different approaches.[24] In fact, given the structure of the vacuum-vacuum matrix elements, dominance of the π, K, ϵ states in the relations (52)–(56) can be given an aura of respectibility by appealing to the "nearby singularity" argument of analytic function theory. However, in our opinion, justification for these single-particle truncations is not so clear. Unfortunately, because of the difficulty in detecting low-spin hadrons with high mass and then revealing their properties, it is not likely that more than a few intermediate states can be explicitly taken into account in the vacuum-vacuum sum rules (48). Thus, calculable corrections to these relations are not expected to be forthcoming. This is in marked contrast to the sum rules of Sec. II. Moreover, the "nearby singularity" justification mentioned above is probably specious. In a recent paper,[25] Baluni and Broadhurst have used rigorous theoretical bounds on K_{l3} form factors along with reliable experimental data to show that the dimension of the $(3,3^*)+(3^*,3)$ operators more than likely exceeds the value two. Since the high-q^2 behavior of two-point functions goes as $(q^2)^{d-2}$, this means that spectral representations of the $(3,3^*)+(3^*,3)$ propagators must be at least once subtracted. Thus, the singular high-energy behavior is capable of upsetting zero-energy estimates by introducing an unknown subtraction constant into the calculation. The only means of evasion from this dilemma is to view propagator pole and cut contributions perturbatively in the context of chiral and scale symmetry breaking. It is then argued that any effect arising from the cut is of second order in symmetry breaking and hence negligible relative to the pole. However, this argument is certainly not compelling for the kaon channel and is even less so for the ϵ channel.

It is instructive to consider commutation relations of various of the $(3,3^*)+(3^*,3)$ operators taken between a vacuum and a single-particle state. The problems associated with truncating the number of contributing intermediate states remain but without the "successes" of the vacuum-vacuum matrix elements. It will suffice to give some examples. Employing the notation of Eq. (47), the general form of the "vacuum–single-particle" sum rule becomes

$$-\langle 0|B(0)|\alpha(\vec{p})\rangle = \sum_\gamma \frac{\langle 0|A(0)|\gamma(\vec{p})\rangle\langle\gamma(\vec{p})|\partial_\mu J^\mu(0)|\alpha(\vec{p})\rangle}{2p_\gamma^0(p_\gamma^0 - p_\alpha^0)} + \sum_\beta \frac{\langle 0|\partial_\mu J^\mu(0)|\beta(0)\rangle\langle\beta(0)|A(0)|\alpha(\vec{p})\rangle}{2m_\beta{}^2} . \tag{59}$$

In the following we shall choose the charge operator to be F_a^5, $a=1,2,3$, and we shall take the limit $|\vec{p}|\to\infty$. If hadronic form factors of local operators vanish for infinite momentum transfer q^2 $q^2\to\infty$, then a truncated form of the second term in Eq. (59) will not contribute. The physical content of the sum rule at this level of approximation is seen to involve relations between $q^2=0$ form factors and various constants associated with vacuum–single-particle matrix elements of local operators,

$$-\langle 0|B(0)|\alpha(\vec{p})\rangle = \sum_\gamma \frac{\langle 0|A(0)|\gamma(\vec{p})\rangle\langle\gamma(\vec{p})|\partial_\mu J^\mu(0)|\alpha(\vec{p})\rangle}{m_\gamma{}^2 - m_\alpha{}^2} . \tag{60}$$

In our examples, $\partial_\mu J^\mu$ will be the axial-vector divergence, so we can estimate the $q^2=0$ form factors by means of Goldberg-Treiman formulas. Substituting $\partial_\mu A_b^\mu$ ($b=1,2,3$) for A, σ for B, and ϵ for α in Eq. (60), we obtain

$$g_\epsilon = \frac{F_\pi g_{\epsilon\pi\pi}}{m_\epsilon{}^2 - m_\pi{}^2} + \cdots , \tag{61a}$$

whereas the replacements σ for A, $-\partial_\mu A_a^\mu$ for B, and π for α yield

$$1 = \frac{g_\epsilon F_\pi g_{\epsilon\pi\pi}}{m_\epsilon{}^2 - m_\pi{}^2} + \cdots . \tag{61b}$$

From the estimate $\Gamma(\epsilon\pi\pi) \cong 300$ MeV, we obtain $F_\pi g_{\epsilon\pi\pi}/m_\epsilon{}^2 \cong 0.5$. Thus Eq. (61a) implies $g_\epsilon \cong 0.5$, whereas (61b) gives $g_\epsilon \cong 2.0$. This disagreement can probably be blamed on the deficiency of the truncation approximation—not enough of the intermediate states have been taken into account. A more striking failure of the truncation approximation emerges upon making the replacements θ for A, $-(4-d)\partial_\mu A_a^\mu$ for B, and ϵ for α. Then we find from Eq. (60) that

$$4-d = \left(\frac{m_\epsilon}{m_\pi}\right)^2 \frac{F_\epsilon g_{\epsilon\pi\pi}}{m_\epsilon{}^2 - m_\pi{}^2} + \cdots . \tag{61c}$$

The left-hand side is an order of magnitude or so smaller than the right-hand side of this "equation." A result like this increases one's appreciation for the success of hard-meson off-shell methods.[26]

We have derived Eqs. (59)–(61c) by considering scalar operators and states. Analogous relations can be obtained for operators (like $V^\mu, A^\mu, \theta^{\mu\nu}$) and states (like ρ, A_1, f) which carry spin.[27] Similar negative results are found.

V. CONCLUSION

The underlying theme of our study has been to survey the contributions of single-particle intermediate states to sum rules generated by various commutation relations.

For the class of axial-charge sum rules catalogued in Sec. II, considered as a whole, single-particle contributions afford the only realistic phenomenological test. Aside from using theoretical estimates for certain nonmeasurable terms, our analysis was phenomenologically oriented. In particular, we made no effort to classify intermediate states according to algebraic representations. We found almost complete saturation for two sets of external states (π, N), and varying degrees of saturation of all the rest: 82% for $\Delta(1233)$, roughly 60% for Σ, $Y_1(1385)$, and ρ, somewhat under 50% for K and $K_V(892)$, under 40% for $\Xi(1315)$, and not much information for the remaining cases examined. Notice from Eq. (13) that if the asymptotic equality of particle and antiparticle cross sections were not valid, we might expect to see some effect of nonconvergence in our sum rules. However, in none of the cases was oversaturation detected. We are optimistic that with further experimental effort in hadron spectroscopy enough information can be gathered to allow almost complete saturation of the $\Delta(1233)$, Σ, $Y_1(1385)$, ρ, K, and $K_V(892)$ sum rules, and substantially more information gathered regarding the $\Xi(1315)$ sum rule. In principle, there is nothing to prevent this. However, we do not envisage there being substantial phenomenological applicability of the sum rules associated with external states α of higher mass. Since contributions for which the mass of the external state exceeds that of the intermediate state do not appear to be large, decay widths where α appears in a final state will be needed. These are hard to measure. Moreover, the number of nonmeasurable contributions will increase. Whether our theories of hadrons will improve enough to allow calculation of these is a matter of conjecture. A relevant example is discussed in the Appendix.

The work of Sec. III essentially speaks for itself and warrants little discussion here. While it is commendable to see that the $q^2=0$ σ-operator sum rule can be written in such a way that its original defects are eliminated, the resulting phenomenological distortion is such that the sum rule loses almost all its attractiveness. In particular, the low-energy pion-nucleon system will remain the best area in which to attempt determination of the nucleon matrix element of the σ operator. However, we would like to point out here one possible extension of our analysis. It is observed in Ref. 3 that in a model where hadrons contain quarks interacting with gluons, $P_2(0)$ [defined in Eq. (45)] is proportional to the quark-gluon coupling constant. Ordinarily, one might expect this quantity to be large. But, in models of hadrons such as the "MIT bag,"[28,29] it is possible for the quark-gluon coupling to be small, thus allowing $P_2(0)\cong 0$. If so, it follows that[3]

$$F_4(x) = \frac{m_P{}^2}{4x^3 m_\alpha{}^2} F_2(x) , \tag{62}$$

where m_P is the quark mass and m_α is the mass of hadron α. Equation (46) then becomes

$$\phi_\alpha \langle \alpha(\vec{p}) | \sigma(0) | \alpha(\vec{p}) \rangle = m_P{}^2 \int_0^\infty dx \, \frac{\tilde{F}_2^\nu(x) + \tilde{F}_2^{\bar{\nu}}(x)}{x^2} . \tag{63}$$

This relation affords an estimate of the bare quark mass in terms of the σ matrix element and observable neutrino structure functions. Although its theoretical foundations are open to criticism, the content of Eq. (63) is sufficient to warrant further study.

Despite the rather extensive employment by previous workers of the vacuum-vacuum and vacuum–single-particle sum rules of the general type in Sec. IV, it is our conclusion that these systems are far from being under theoretical control. At the very least, the probable need for subtraction constants[25] in the $(3, 3^*)+(3^*, 3)$ propagators is an ominous signal that the usual truncation procedures adopted might be inadequate. In addition, there is the recent $e\bar{e}$ annihilation data, which, for example, shows that the ρ contribution to Weinberg's first sum rule is only $\frac{1}{30}$ that of the higher-mass continuum[27] in the vector current propagator probed so far by the experiments. It remains to be seen whether other calculations based on single-particle dominance of the vacuum-vacuum and vacuum–single-particle matrix elements will fail so resoundingly.

ACKNOWLEDGMENTS

We wish to acknowledge use of the Cornell University computing facilities as well as the warm hospitality of the Cornell Physics Department during the time when part of the work described in this paper was done. We also wish to thank Mr. V. Kapila for assistance in various stages of the calculations.

APPENDIX

In Sec. II, we have carried out a phenomenological analysis of the axial-charge algebra, Eq. (3), taken between arbitrary hadronic single-particle states. As the mass of the external state becomes larger, we encounter increasingly the difficulty that matrix elements occurring in our analysis are unmeasurable because of kinematic reasons. To overcome this problem, we must refer to dynamical models to provide estimates for the unknown matrix elements. In the following, we shall employ the "MIT bag model"[28] as an illustration of how one could proceed.

Consider the following case. The axial-charge algebra taken between Δ^{++} states has an unknown contribution from the Δ^{+} intermediate state. In particular, the unknown quantity is a parameter, f_A, defined in Eq. (16) and again below. We employ a Goldberger-Treiman relation in Sec. II to relate f_A to a coupling constant $g_{\pi\Delta\Delta}$ which is itself expressed in terms of $g_{\pi NN}$ by means of $SU(6)_W$ symmetry. An alternative method for obtaining an estimate for f_A is to use a variant of the MIT bag model.[29] Recall the definition given in Eq. (16),

$$\langle \Delta^{++}(\vec{p}\,', \lambda')\,|A_+^\mu(0)|\,\Delta^+(\vec{p}, \lambda)\rangle = \bar{u}_{++}^{\sigma}(\vec{p}\,', \lambda')\,[F_A(t)\,g_{\sigma\rho}\gamma^\mu\gamma_5 + \cdots]\,u_+^\rho(\vec{p}, \lambda)\ .$$

Performing an isospin rotation and, for convenience, evaluating the above in the static limit, we find for $\mu = i$ $(i = 1, 2, 3)$

$$f_A \vec{\chi}^\dagger(\lambda')\cdot\sigma_i\,\vec{\chi}(\lambda) = \frac{2}{\sqrt{3}}\langle \Delta^{++}(\lambda')\,|A_3^i(0)|\,\Delta^{++}(\lambda)\rangle\ , \tag{A1}$$

where $f_A \equiv F_A(0)$ and the Δ^{++} wave function with spin component λ is given by the vector-spinor $\vec{\chi}(\lambda)$. These spin wave functions obey

$$\sum_{\lambda=-3/2}^{3/2} (\chi_i(\lambda)\,\chi_j^\dagger(\lambda))_{ab} = (\delta_{ij} - \tfrac{1}{3}\sigma_i\,\sigma_j)_{ab}\ . \tag{A2}$$

If the axial-vector current is carried by quarks, and baryons are entities (called bags) in which quarks are permanently bound, we have

$$f_A = \frac{1}{\sqrt{3}}\left\langle \Delta^{++}(\lambda' = \tfrac{3}{2})\left|\int_{\text{bag}} d^3x\,\psi^\dagger(x)\tau_3\sigma_3\psi(x)\right|\Delta^{++}(\lambda = \tfrac{3}{2})\right\rangle\ . \tag{A3}$$

The particular bag model[29] we employ here involves the quantum modes of massless Dirac fields, which carry quark quantum numbers, enclosed within a static spherical boundary. The scale of the model is fixed by normalizing its only free parameter to the $N(938)$-$\Delta(1233)$ average mass. The following kind of spectrum for strangeness-zero baryons is found: degenerate N, Δ at 1180 MeV, degenerate spin $\frac{1}{2}$ and $\frac{3}{2}$ negative-parity states at 1421 MeV, degenerate spin $\frac{1}{2}$ and $\frac{3}{2}$ positive-parity states at 1626 MeV and a similar set at 1657 MeV, etc. The mass of each state is determined from an eigenfrequency $\omega_{n\kappa}$, where the index n labels the mode and κ distinguishes the parity. For example, some of the modes which a quark may occupy are $1S_{1/2}$ $(\omega_{1,-1} = 2.04)$, $1P_{1/2}$ $(\omega_{1,1} = 3.84)$, $2S_{1/2}$ $(\omega_{2,-1} = 5.4)$, etc. Further details of the model are given in Ref. 29. Upon evaluating Eq. (A3) in the bag model, we find

$$f_A = \sqrt{3}\,h(\omega_{1,-1})\ , \tag{A4}$$

where

$$h(\omega_{n,\kappa}) = 1 - \frac{2\omega_{n,\kappa} + \kappa}{3(\omega_{n,\kappa} + \kappa)}\ . \tag{A5}$$

The $\sqrt{3}$ in Eq. (A4) is the value of f_A in the most naive quark model, uncorrected by any kind of dynamical considerations.[30] Numerically, we find $f_A = 1.16$ from (A4). A similar phenomenon occurs for the nucleon axial-vector coupling in this bag model; the naive quark value, $g_A = \frac{5}{3}$, is reduced to $g_A = 1.09$. Below, we summarize the situation as regards the axial-charge sum rule evaluated between Δ^{++} states.

	f_A	Status of sum rule, Eq. (17)
naive quark	$\sqrt{3}$	$0.98+\cdots=1$
bag model	1.16	$0.66+\cdots=1$
Goldberger-Treiman relation	1.43	$0.80+\cdots=1$

The naive quark estimate for f_A is seen to give rise to almost total saturation of the charge-algebra sum rule, in light of existing data. In view of the rather limited information pertaining to resonance decay of baryons into $\Delta\pi$ modes, and the fact that each contribution to the Δ^{++} sum rule is positive, total saturation of the sum rule would have to be considered with some suspicion. On this basis, we feel that both the bag model and Goldberger-Treiman estimates of the unmeasured parameter f_A are superior to the naive quark estimate. Unfortunately, it is not possible to judge the relative merit of the former two estimates solely on the basis of the Δ^{++} charge-algebra sum rule without further data.

In principle, it is possible to analyze even-higher-mass external states in the same way. The next baryon external state would be $N^*(1470)$. The $N^*(1470)$ sum rule has a contribution coming from the $N^*(1470)$ intermediate state itself. This contribution depends on the value of an axial-vector coupling g_A^*, defined analogously to the nucleon axial-vector coupling g_A. Upon calculating g_A^* in the bag model, we find

$$g_A^*=\tfrac{10}{18}[2h(\omega_{1,-1})+h(\omega_{2,-1})]\ , \tag{A6a}$$

where we assign $N^*(1470)$ to the $\frac{1}{2}^+$ state occurring in the bag model with mass 1626 MeV ($1S_{1/2}{}^2\, 2S_{1/2}$ configuration). Numerically, we find $g_A^*=0.96$. However, in the bag model, there is a $\frac{1}{2}^+$ state with nearly the same mass (1657 MeV, $1S_{1/2}\, 2P_{1/2}{}^2$ configuration) for which

$$g_A^*=\tfrac{10}{18}[2h(\omega_{1,1})+h(\omega_{1,-1})]\ , \tag{A6b}$$

or numerically $g_A^*=0.66$. The relation between either of these $\frac{1}{2}^+$ states and the one seen in Nature at 1470 MeV is far from clear. Evidently, a considerable degree of mixing can take place, the calculation of which must await a more realistic model. In other words, to pursue the consequences of this particular bag model any further in our phenomenological analysis seems to us of little more than academic interest. We await further progress in our theoretical understanding of excited baryon states.

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*Work supported by the U. S. Atomic Energy Commission.

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[1]E. Golowich and B. R. Holstein, Phys. Rev. D 8, 1486 (1973).

[2]S. L. Adler and R. F. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968).

[3]R. L. Jaffe and C. H. Llewellyn Smith, Phys. Rev. D 7, 2506 (1973); D. J. Broadhurst, J. F. Gunion, and R. L. Jaffe, Ann. Phys. (N.Y.) 81, 88 (1973); J. Ellis and R. L. Jaffe, Stanford Linear Accelerator Center Report No. SLAC-PUB-1353, 1973 (unpublished).

[4]The over-all approach we follow in this section is standard and is described in Ref. 2. The only stipulation we wish to add is that unstable states are treated in the narrow-resonance approximation throughout.

[5]However, α cannot have isospin zero as this reduces the content of Eq. (3) to a triviality. The right-hand side vanishes because $T_3(\alpha)=0$ and the left-hand side is zero term by term due to cancellation between $F_+^5F_-^5$ and $F_-^5F_+^5$ with isospin-one intermediate states.

[6]Beware that $J_+^\pi=J_1^\pi+iJ_2^\pi$, so that an inverse factor of $\sqrt{2}$ must ultimately be appended to J_+^π for comparison with physical processes.

[7]An additional assumption in this approach is that the limit $|\vec{p}|\to\infty$ can be taken term by term.

[8]Particle Data Group, Rev. Mod. Phys. 45, S1 (1973).

[9]In those cases where ranges for masses and/or widths are given, we use the average value.

[10]We take the $\epsilon'(1000)$ width from D. M. Binnie *et al.*, Phys. Rev. Lett. 31, 1534 (1973), and S. D. Protopopescu *et al.*, Phys. Rev. D 7, 1279 (1973).

[11]M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Lett. 8, 261 (1962).

[12]F. J. Gilman and M. Kugler, Phys. Rev. Lett. 30, 518 (1973).

[13]The problem of calculating a transition rate, given this kinematical situation, is discussed by E. Golowich, Phys. Rev. D 10, 3861 (1974).

[14]For example, see U. Karshon *et al.*, Phys. Rev. Lett. 32, 852 (1974); J. Diaz *et al.*, *ibid.* 32, 260 (1974).

[15]We do not anticipate a very large contribution from the lowest-mass 0^+ kaon, whose effect can be estimated from SU(3) in terms of the $\epsilon'(1000)$ contribution to the pion sum rule. Admittedly, mixing effects involving the ϵ' could strain this argument somewhat.

[16]For example, see Y.-C. Liu and J. A. M. Vermaseren, Phys. Rev. D 8, 1602 (1973), and references cited therein.

[17]At this point we specify our normalization of states by choosing $N_\alpha=2p_\alpha^0$ (mesons) and $N_\alpha=p_\alpha^0/m_\alpha$ (baryons).

[18]A further criticism of Eqs. (30) and (32) is that their derivation involves an extrapolation over a range $m_\pi{}^2$ and so cannot be used with absolute confidence to provide an accurate estimate of $\langle\alpha|\sigma(0)|\alpha\rangle$. However, our interest in these relations is not so much pheno-

menological as theoretical. The real issue is to determine what modifications are necessary to obtain meaningful sum rules.

[19]Hereafter, we concentrate on the continuum contribution to $\langle\alpha|\sigma|\alpha\rangle$, completely ignoring the discrete terms arising from contributions below the $\pi\alpha$ physical threshold.

[20]We assume time-reversal invariance.

[21]A $J=0$ fixed pole shows up as a contribution to Re T_1' in the limit $\nu\rightarrow\infty$ with energy dependence the same as that of an $\alpha=0$ Regge pole.

[22]Unfortunately the charge algebra reduced to a triviality in this case.

[23]We realize that this formula is already familiar to many readers. We are simply using it as an example in the course of our discussion.

[24]For example, see M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968), and P. A. Carruthers, Phys. Rev. D 3, 959 (1971), respectively.

[25]V. Baluni and D. J. Broadhurst, CERN Report No. 1871, 1974 (unpublished).

[26]For example, see J. Ellis, P. H. Weisz, and B. Zumino, Phys. Lett. 34B, 91 (1971).

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[28]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974).

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[30]Of course, even though $SU(6)_W$ is used to relate the coupling constants $g_{\pi\Delta\Delta}$ and $g_{\pi NN}$, there is no reason that the Goldberger-Treiman and naive quark estimates should be identical.