Asymptotic behavior of the cross section for $e^+e^- \to$ hadrons in asymptotically free field theories*

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Asymptotically free field theories make definite predictions for the behavior of the hadronic vacuum polarization tensor in the spacelike region. A theorem due to Meiman is used to combine these predictions with the analyticity of the vacuum polarization tensor to derive in a rigorous way the asymptotic behavior of the cross section for electron-positron annihilation into hadrons. Our theorems allow this cross section to oscillate indefinitely about a well-defined value. An explicit example is exhibited which demonstrates that such oscillations are not excluded and that the restrictions imposed on the annihilation cross section at finite energies are very weak. Even if oscillations of the leading term in the cross section which do not die out asymptotically are excluded as unphysical, very little can be said about the behavior of the nonleading terms without making very restrictive assumptions.

I. INTRODUCTION

Recently the electron-positron annihilation cross section has received much attention as a testing ground for asymptotically free field theories of the strong interactions. These theories, through the renormalization group, make definite predictions about the behavior of the renormalized hadronic vacuum polarization tensor

$$\pi_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu} q^2)\pi(q^2) \tag{1.1}$$

in the spacelike region. In particular, Appelquist and Georgi[1] and Zee[1] have shown that for asymptotically free field theories

$$T(s) \equiv \frac{d\pi(s)}{ds} \simeq \frac{1}{-s}\left[a + \frac{b}{\ln(-s/s_0)}\right], \quad -s \text{ large} \tag{1.2}$$

where a and b are exactly calculable in any particular theory [for example, in the SU(3)⊗SU(3)′ color triplet model $a=2$, $b=\frac{8}{9}$] and s_0 is an arbitrary momentum scale. With the precocious scaling of the deep-inelastic electroproduction structure functions as a guide, it is expected that the asymptotic behavior (1.2) will also set in for relatively small s, probably on the order of a few GeV^2.

The hadronic vacuum polarization tensor is related to the electron-positron annihilation cross section into hadrons through the Källén-Lehmann representation

$$\pi(s) = \frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'}\,\frac{\rho(s')}{s'-s+i\epsilon}, \tag{1.3}$$

where in the one-photon-annihilation approximation the spectral function is given by

$$\rho(s) = \frac{s}{32\pi\alpha^2}\sigma_{\text{had}}^{e^+e^-}(s). \tag{1.4}$$

We have written the spectral representation in once-subtracted form. If more subtractions were required, then (1.2) could not possibly be correct.

Adler[2] has pointed out that the implications of the annihilation reaction for the spacelike behavior of the vacuum polarization tensor can be most directly explored by studying the function $T(s)$, which, if one takes into account its definition and the Källén-Lehmann representation, can be written as

$$T(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds'\,\frac{\rho(s')}{(s'-s)^2}. \tag{1.5}$$

Now the observed annihilation cross section[3] into hadrons remains roughly constant from 5 to 25 GeV^2. Adler has shown that if this behavior continues up to 81 GeV^2 (this energy range will be available at SPEAR II) then the inequality

$$\frac{a+b\ln(|s|/s_0)}{|s|} \geq \int_{4m_\pi^2}^{s_{\max}} ds'\,\frac{\rho(s')}{(s'+|s|)^2}, \tag{1.6}$$

obtained by truncating (1.5) at $s_{\max} = 81\ \text{GeV}^2$ and comparing it with the prediction (1.2) for a precociously asymptotic color triplet model, would be violated. He has also shown that if $\sigma_{\text{had}}^{e^+e^-}(s)$ remains constant up to $s_{\max} = 900\ \text{GeV}^2$, then this procedure would rule out essentially all currently popular parton or asymptotically free models. On the other hand, if the annihilation cross section decreases sufficiently to satisfy (1.6), then we will learn nothing in this way.

The problem of obtaining directly the implications of the spacelike predictions (1.2) for the annihilation cross section is more difficult. Appelquist and Georgi[1] and Zee[1] simply calculated the discontinuity of the continuation to the timelike region of their asymptotic form for $\pi(s)$, e.g.,

$$\pi(s) = \int^{s} T(s')\,ds' \sim -a\ln(-s/s_0) - b\ln\ln(-s/s_0),$$

and found

$$\rho(s) \to \pi\left[a + \frac{b}{\ln(s/s_0)}\right]. \tag{1.7}$$

They further assumed that this behavior sets in for the same low values of s as was expected in the spacelike region. Of course, this procedure is not in general justified, and a more rigorous argument is desirable.

The difficulty with trying to obtain more precise restrictions imposed by (1.2) on $\sigma_{\text{had}}^{e^+e^-}(s)$ for any finite region of s is that in general we cannot exclude the possibility that $\sigma_{\text{had}}^{e^+e^-}(s)$ has some oscillatory behavior which averages out in the spectral representation to produce the smooth behavior predicted in the spacelike region.[4] Indeed, we cannot even rule out the possibility that such unpleasant behavior persists asymptotically.

In this paper we consider the problem of determining what restrictions on the asymptotic behavior of the annihilation cross section follow from (1.2). We show in Sec. II that if $\lim_{s\to\infty}[\pi(s)/\ln s]$ exists or if, for large enough s, $d\rho(s)/ds$ has one sign [actually we require only that these properties be true when $\pi(s)$ is averaged over some suitable interval around s] then

$$\lim_{s\to\infty} \rho(s) = \pi a, \tag{1.8}$$

at least on some infinite sequence of points $\{s_n\}$, $s_n \to \infty$ as $n\to\infty$. In Sec. III we prove that if $d\rho(s)/ds$ has one sign for large enough s and in addition $\lim_{s\to\infty}\{[sT(s)-a]\ln s\}$ exists, then

$$\rho(s) \xrightarrow[s\to\infty]{} \pi\left[a + \frac{b}{\ln(s/s_0)}\right]. \tag{1.9}$$

This result also need be true only on some infinite sequence $\{s_n\}$. The assumptions required to obtain the nonleading behavior described by (1.9) are of course much stronger than those used to derive (1.8).

These results cannot be compared with data at any finite s, since our theorems need hold only in the limit $s\to\infty$. If one has other reasons for believing that the asymptotic behavior sets in for precociously small values of s and that oscillations can be excluded, then the spectral function must approach the form (1.9) for small s. However, in Sec. IV we present an explicit example which demonstrates that it is impossible to improve our theorems to rule out asymptotic oscillation of $\rho(s)$ about the value πa and, in addition, that the asymptotic-freedom prediction (1.2) for the spacelike region does not severely restrict the behavior of the cross section for finite s. Even if $\lim_{s\to\infty}\rho(s) = \pi a$, the approach to this limit need not take the simple logarithmic form (1.9) unless the rather strong assumptions used to obtain this behavior are fulfilled. Thus it is quite possible that measurements of the annihilation cross section for larger values of s may provide no evidence to confirm or contradict the prediction (1.2).

One of our principal tools in obtaining our results is a theorem of Meiman[5] which is a consequence of an inequality on harmonic measures. This theorem has been used by Khuri and Kinoshita[6] to examine the relation between the asymptotic behaviors of the total cross section and phase of the scattering amplitude for strong interactions. We refer the interested reader to their paper for a discussion of the origin, use, and limitations of Meiman's theorem.

II. LEADING ASYMPTOTIC BEHAVIOR

We examine here the restrictions imposed on the asymptotic behavior of the spectral function $\rho(s)$ by the leading spacelike behavior of the hadronic vacuum polarization tensor

$$\lim_{s\to-\infty} \frac{\pi(s)}{\ln(-s)} = -a, \tag{2.1}$$

which is predicted in asymptotically free field theories. We assume throughout that $\pi(s)$ is analytic and bounded by a polynomial in s in the once-cut plane and continuous[7] on the cut, and that $\rho(s)$ is positive-definite for $s \geq 4m_\pi^2$. Define the function $\tilde{\pi}(s)$ as the average of $\pi(s)$ over an interval of width Δs around s:

$$\tilde{\pi}(s) = \frac{1}{\Delta s}\int_{s-\Delta s/2}^{s+\Delta s/2} ds'\pi(s'). \tag{2.2}$$

This averaged function obviously possesses the same analyticity, continuity, and positivity properties as does $\pi(s)$. We will present two theorems which determine the asymptotic behavior of $\rho(s)$ from (2.1).

Theorem I. If $\lim_{s\to-\infty}[\pi(s)/\ln(-s)] = -a$ and $\lim_{s\to\infty+i\epsilon}[\pi(s)/\ln s]$ exists, then either

$$\lim_{s\to\infty} \rho(s) = \pi a, \tag{2.3}$$

or there exists a sequence of points $\{s_n\}$, $s_n\to\infty$ as $n\to\infty$, on which (2.3) holds. If $\lim_{s\to\infty+i\epsilon}[\pi(s)/\ln(s)]$ does not exist, but we can choose a Δs large enough that $\lim_{s\to\infty+i\epsilon}[\tilde{\pi}(s)/\ln s]$ does exist, then the result (2.3) must again be true.

Proof. We define $E = \sqrt{s}$ and construct the function

$$g(E) = \frac{-\pi(E^2)}{2[\ln E - i\pi/2]}. \tag{2.4}$$

Here $g(E)$ is analytic in the upper half E plane outside of the unit semicircle, continuous on the real axis, and satisfies

$$g(-E+i\epsilon)=g^*(E+i\epsilon). \tag{2.5}$$

The asymptotic freedom prediction for g is obviously

$$\lim_{E\to i\infty} g(E)=a. \tag{2.6}$$

The Phragmén-Lindelöf theorem[8] requires that the value a is contained in the manifold of limit points of $g(E)$ for $E\to\pm\infty$. For the moment we assume that $\lim_{E\to\infty} g(E)$ exists, in which case

$$\lim_{|E|\to\infty} g(E)=a, \tag{2.7}$$

uniformly in $0\le \arg E\le\pi$.

For large real E

$$\mathrm{Im}\, g(E)\cong -\left[\mathrm{Im}\pi(E)+\frac{\pi}{2\ln E}\mathrm{Re}\pi(E)\right]\Big/2\ln E, \tag{2.8a}$$

$$\mathrm{Re}\, g(E)\cong -\left[\mathrm{Re}\pi(E)-\frac{\pi}{2\ln E}\mathrm{Im}\pi(E)\right]\Big/2\ln E. \tag{2.8b}$$

There are now three cases to consider:

(a) $\mathrm{Im}\pi(E)>\dfrac{-\pi\,\mathrm{Re}\pi(E)}{2\ln E}$ for $E>E_0$ large and $\lim_{E\to\infty}\dfrac{\mathrm{Im}\pi(E)\ln E}{\mathrm{Re}\pi(E)}\neq\dfrac{-\pi}{2}$,

(b) $0<\mathrm{Im}\pi(E)<-\dfrac{\pi\mathrm{Re}\pi(E)}{2\ln E}$ for $E>E_0$ and $\lim_{E\to\infty}\dfrac{\mathrm{Im}\pi(E)\ln E}{\mathrm{Re}\pi(E)}\neq\dfrac{-\pi}{2}$,

(c) $\lim_{E\to\infty}\dfrac{\mathrm{Im}\pi(E)}{\mathrm{Re}\pi(E)}\ln E=\dfrac{-\pi}{2}$ or $\dfrac{\mathrm{Im}\pi(E)}{\mathrm{Re}\pi(E)}\ln E=\dfrac{-\pi}{2}$ on some infinite sequence of points $\{E_n\}$, $E_n\to\infty$ as $n\to\infty$.

Note that this covers the possibility where $[\mathrm{Im}\pi(E)/\mathrm{Re}\pi(E)]\ln E$ oscillates about the value $-\pi/2$.

In case (c) we have the desired result since then

$$\lim_{E\to\infty} g(E)=a=-\lim_{E\to\infty}\frac{\mathrm{Re}\pi(E)}{2\ln E}=\lim_{E\to\infty}\frac{\mathrm{Im}\pi(E)}{\pi}, \tag{2.9}$$

at least on some sequence $\{E_n\}$.

It only remains to eliminate (a) and (b). To do this we employ Meiman's theorem. Consider the region of the upper half E plane bounded by two semicircles centered at the origin and with radii E_0 and E, both large and such that $E\gg E_0$. This region is mapped onto some region of the g plane. Let Γ_1 and Γ_2 be the images of the segments of the upper edge of the real E axis from E_0 to E and from $-E_0$ to $-E$, respectively. Because of the symmetry (2.5) the curves Γ_1 and Γ_2 are symmetrically located with respect to the real g axis. Furthermore, Γ_1 and Γ_2 do not intersect because the positivity of $\rho(s)$ implies that on the positive real axis $\mathrm{Im}\, g$ is strictly negative in case (a) and strictly positive in case (b). In both cases the curves Γ_1 and Γ_2 lie in the region (see Fig. 1) defined by

$$\left|\frac{\mathrm{Im}\, g(E)}{\mathrm{Re} g(E)}\right|\ge\tan\theta_E\ge\frac{\mathrm{const}}{\ln|E|}. \tag{2.10}$$

Let $g=u+iv$ and define u_0 to be the farthest (from the origin) intersection with the positive real g axis of the image of the semicircle in the E plane of radius E_0. Similarly, define u_E to be the nearest intersection with the positive real g axis of the image of the semicircle of radius E. Let $r(u)$ be the shortest distance between the point $(u,0)$ and the curve Γ_1 (or equivalently Γ_2). Finally define

$$u_{\min}=\mathrm{Min}(u_0,u_E),$$
$$u_{\max}=\mathrm{Max}(u_0,u_E). \tag{2.11}$$

Meiman's theorem states that[6]

$$\int_{u_{\min}}^{u_{\max}}\frac{du}{r(u)}\ge\tfrac{1}{4}\ln(E/E_0). \tag{2.12}$$

We know from (2.10) that

$$r(u)\ge u\sin\theta_E\ge u\frac{\mathrm{const}}{\ln E}. \tag{2.13}$$

Inserting (2.13) into (2.12) we obtain the inequality

$$\ln\frac{u_{\max}}{u_{\min}}\ge\mathrm{const}\frac{\ln(E/E_0)}{\ln E}\simeq\mathrm{const}. \tag{2.14}$$

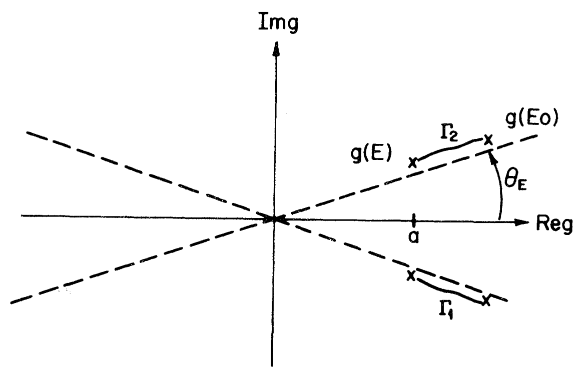

FIG. 1. The g plane for large E in case (a). For case (b) Γ_2 and Γ_1 are interchanged.

Since $g(E)\to a$ uniformly in $0\leq \arg E\leq\pi$, by choosing E_0 and E large enough we can make u_0 and u_E both arbitrarily close to a. The left-hand side of (2.14) can then be made arbitrarily small, which is a contradiction since the constant on the right-hand side is fixed. This rules out cases (a) and (b) and so (2.9) must be true.

If $\lim_{s\to\infty}[\pi(s)/\ln s]$ exists only on the average in the sense that $\lim_{s\to\infty}[\bar{\pi}(s)/\ln s]$ exists, then by replacing $\pi(E)$ by $\bar{\pi}(E)$ in Eqs. (2.4) through (2.14) we find that

$$\lim_{s\to\infty}\bar{\rho}(s)=\pi a\,, \tag{2.15}$$

at least on some infinite sequence of points $\{s_n\}$, $s_n\to\infty$ as $n\to\infty$. Since $\rho(s)$ is a continuous function of s, this completes the proof of the theorem.

We would like to remove the condition that $\lim_{s\to\infty}[\pi(s)/\ln s]$ exists. We are able to do this by replacing it with a different condition, namely that for large enough s the spectral function $\rho(s)$ is an increasing or decreasing function of s, at least when averaged over a suitable interval around s.

Theorem II. If for $s\geq s_0$, s_0 large, $d\rho(s)/ds$ exists, is bounded, and has one sign, and if the Källén-Lehmann representation for $\pi(s)$ requires at most one subtraction, then the asymptotic-freedom prediction (2.1) implies that

$$\lim_{s\to\infty}\rho(s)=\pi a\,. \tag{2.16}$$

If $\rho(s)$ is not monotonically increasing or decreasing, but $\bar{\rho}(s)\equiv \mathrm{Im}\bar{\pi}(s)$ is, then

$$\lim_{s\to\infty}\bar{\rho}(s)=\pi a\,. \tag{2.17}$$

Proof. The proof is very simple. For $s<0$ we have

$$(-s)T(s)=\frac{|s|}{\pi}\int_{4m_\pi^2}^{\infty}ds'\,\frac{\rho(s')}{(s'+|s|)^2}\,. \tag{2.18}$$

Integrating by parts for $s'>s_0$, taking $s\to-\infty$, and dropping terms which vanish in that limit we have

$$\pi a=\rho(s_0)+\lim_{s\to\infty}\int_{s_0}^{\infty}ds'\,\frac{d\rho/ds'}{(1+s'/s)}\,. \tag{2.19}$$

Consider first the case of $d\rho/ds\geq 0$ for $s\geq s_0$. Cutting off the integral at some $s_1\gg s_0$ we get the inequality

$$\pi a\geq\rho(s_0)+\int_{s_0}^{s_1}ds'\,\frac{d\rho}{ds'}\,. \tag{2.20}$$

Now we can choose s_1 as large as we please, so

$$\pi a\geq\rho(s_0)+\int_{s_0}^{\infty}ds'\,\frac{d\rho}{ds'}\,. \tag{2.21}$$

On the other hand,

$$\begin{aligned}\pi a&=\rho(s_0)+\lim_{s\to\infty}\int_{s_0}^{\infty}ds'\,\frac{d\rho}{ds'}\,\frac{s}{s'+s}\\&=\rho(s_0)+\int_{s_0}^{\infty}ds'\,\frac{d\rho}{ds'}-\lim_{s\to\infty}\int_{s_0}^{\infty}ds'\,\frac{d\rho}{ds'}\,\frac{s'}{s'+s}\\&\leq\rho(s_0)+\int_{s_0}^{\infty}ds'\,\frac{d\rho}{ds'}\,.\end{aligned} \tag{2.22}$$

Therefore, the equality sign must hold in (2.21) and (2.22). Since $d\rho(s)/ds$ exists and is bounded for $s\geq s_0$ we can perform the integration and obtain the desired result (2.16). The proof of (2.16) for the case $d\rho/ds\leq 0$, $s\geq s_0$, is the same as for $d\rho/ds\geq 0$ except that the inequalities (2.21) and (2.22) are reversed.

If $\rho(s)$ is not a monotonic function of s but $\bar{\rho}(s)$ is, then $\rho(s)$ should be replaced by $\bar{\rho}(s)$ in Eqs. (2.18) through (2.22). Then we must have (2.17) and the theorem is proved.

III. NONLEADING ASYMPTOTIC BEHAVIOR

Here we consider the following question: If

$$\lim_{s\to\infty}\rho(s)=\pi a\,, \tag{3.1}$$

then how is this limit approached? As we shall see, much stronger (physically) assumptions are required to treat this problem, and in the general case we cannot determine the behavior of the nonleading terms.

To begin we define the function

$$\hat{T}(s)=[-sT(s)-a]\,. \tag{3.2}$$

Like $T(s)$, our new function $\hat{T}(s)$ is analytic in the cut plane. Its discontinuity is $-s(d\rho/ds)$, which we will assume to exist, and furthermore we will assume that $T(s)$ is continuous[7] on the real axis. For spacelike s, the asymptotic-freedom prediction for $\hat{T}(s)$ is

$$\lim_{s\to-\infty}\hat{T}(s)\ln\left(\frac{-s}{s_0}\right)=b\,. \tag{3.3}$$

We see that the properties of $\hat{T}(s)$ are very similar to those of $\pi(s)$, the main difference being that $\mathrm{Im}\hat{T}(s)$ is not positive-definite and $\hat{T}(s)$ decreases logarithmically, whereas $\pi(s)$ increases logarithmically for $s\to-\infty$. If we assume that $d\rho/ds$, and hence $\mathrm{Im}\hat{T}(s)$, has one sign for $s\geq s_0$, s_0 large, and that $\lim_{s\to\infty}[\hat{T}(s)\ln(s)]$ exists, then the function

$$\begin{aligned}\hat{g}(E)&=2(\ln E-i\pi/2)\hat{T}(E^2)\\&=2(\ln E-i\pi/2)[-E^2T(E)-a]\end{aligned} \tag{3.4}$$

satisfies the same conditions as did $g(E)$ in (2.4). Applying Meiman's theorem in precisely the same fashion as we did to prove theorem I we obtain the following theorem.

Theorem III. If, for large real s, $d\rho/ds$ has one sign, and if $\lim_{s\to\infty}\{[-sT(s)-a]\ln s\}$ exists, then the asymptotic-freedom prediction (3.3) implies that

$$\lim_{s\to\infty}\left[\frac{d\rho(s)}{ds}\, s\ln^2 s\right]=-\pi b\,, \tag{3.5}$$

at least on some sequence of points $\{s_n\}$, $s_n\to\infty$ as $n\to\infty$. If the above conditions are satisfied only on the average, in the sense that with

$$\tilde{T}(s)\equiv\frac{d\tilde{\pi}(s)}{ds}=\frac{\pi(s+\Delta s/2)-\pi(s-\Delta s/2)}{\Delta s}$$

and suitably chosen Δs

$$\frac{d\tilde{\rho}}{ds}=\frac{\rho(s+\Delta s/2)-\rho(s-\Delta s/2)}{\Delta s}$$

has one sign for large enough real s and $\lim_{s\to\infty}\{[-s\tilde{T}(s)-a]\ln s\}$ exists, then we find instead of (3.5)

$$\lim_{s\to\infty}\left(\frac{d\tilde{\rho}}{ds}\, s\ln^2 s\right)=\lim_{s\to\infty}\left[\left(\frac{\rho(s+\Delta s/2)-\rho(s-\Delta s/2)}{\Delta s}\right)s\ln^2 s\right]=-\pi b\,. \tag{3.6}$$

It follows from this theorem that on some sequence of points the spectral function $\rho(s)$ is decreasing like $\pi b/\ln s$ to a constant value. If $\lim_{s\to\infty}\rho(s)$ does not exist [e.g., $\rho(s)$ oscillates around the value πa], then $\rho(s)$ need not approach the value πa on this sequence. But if (3.1) holds and in addition $\lim_{s\to\infty}[(d\rho/ds)s\ln^2 s]$ exist, then combining (3.1) and (3.5) we find that for large enough s

$$\rho(s)\approx\pi\left[a+\frac{b}{\ln(s/s_0)}\right]. \tag{3.7}$$

We want to emphasize at this point that from a physical point of view the assumption that $\lim_{s\to\infty}\{[-sT(s)-a]\ln s\}$ exists is much stronger than the assumption, used in proving theorem I, that $\lim_{s\to\infty}[\pi(s)/\ln s]$ exists. The reason of course is that $\pi(s)$ is the quantity with physical meaning, and even if $\pi(s)$ itself has "nice" (e.g., nonoscillatory) behavior in the limit of large real s, there is no reason to believe that every function (in this case the function which describes its nonleading asymptotic behavior) constructed from it will also be so well behaved.

IV. DISCUSSION

We would like to make a few remarks about our theorems. First we must point out that from a mathematical point of view the theorems do not cover all cases since they do not apply if $\pi(s)/\ln s$ and $\rho(s)$ oscillate with unbounded amplitude or period as $s\to\infty$. However, it is not clear that any meaningful conclusion could be reached in such a pathological situation, and while this behavior cannot be ruled out mathematically it is extremely unphysical. Therefore, we will not consider this possibility further.

Next we ask whether theorems I and II could be strengthened to exclude the possibility that $\rho(s)$ oscillates indefinitely about the value πa as $s\to\infty$. That such an improvement is impossible, without more information, is demonstrated by the following simple example: Suppose we have a function $\pi_0(s)$ which is analytic in the cut plane, is continuous on the real axis, has positive absorptive part $\rho_0(s)$, and satisfies

$$T_0(s)=\frac{d\pi_0(s)}{ds}\simeq\left[a+\frac{b}{\ln(-s/s_0)}\right]\frac{1}{-s}\,,\quad s\leqslant-|s_1| \tag{4.1a}$$

$$\rho_0(s)\simeq a+\frac{b}{\ln(s/s_0)}\,,\quad s\geqslant|s_2|\,. \tag{4.1b}$$

Let

$$\pi_1(s)=c\exp[i\alpha(s-M^2)^{1/2}]\,,\quad \alpha>0 \text{ and } M^2\geqslant 4m_\pi^{\;2} \tag{4.2a}$$

$$\pi_2(s)=\frac{s}{\pi}\sum_{i=1}^{n}\int_{x_i}^{y_i}\frac{ds'}{s'}\,\frac{\rho_2(s')}{s'-s}\,,\quad 4m_\pi^{\;2}\leqslant x_i<y_i<x_{i+1} \tag{4.2b}$$

$$\pi(s)=\pi_0(s)+\pi_1(s)+\pi_2(s)\,. \tag{4.2c}$$

Then for $\pi a\geqslant c\geqslant-\pi a$ and $b\geqslant 0$ we can always choose a weight function $\rho_2(s)$, nonzero only in the set of n (finite) intervals $\{x_i, y_i\}$, such that the function $\pi(s)$ satisfies the required analyticity, continuity, and positivity properties and obeys the asymptotic-freedom prediction (1.2). However, with this choice of $\pi(s)$ the spectral function $\rho(s)$ oscillates about the value πa with amplitude c and period α/π [in the variable $E_m=(s-M^2)^{1/2}$] for $s\to\infty$.

In addition to providing a counter example to the improvement of theorems I and II, functions of the type (4.2) demonstrate that the restrictions imposed on the behavior of the annihilation cross section at finite s are very weak. To be more concrete consider the $SU(3)\otimes SU(3)'$ color triplet model in which $a=2$ and $b=\frac{8}{9}$. Suppose that in (4.1) we choose $s_0 = 2$ GeV^2, $s_1 = 6$ GeV^2, and $s_2 = 10$ GeV^2. Then with $c = 2\pi$, $\alpha = \pi/5$ GeV, and $M^2 = 10$ GeV^2 the absorptive part of $\pi_0(s) + \pi_1(s)$ is positive and we do not even need $\pi_2(s)$ (unless of course we want to fit some data). In the spacelike region $\pi_1(s)$ is exponentially decreasing and at $s = -6$ GeV^2, $T_1(s) \equiv d\pi_1(s)/ds$ is already less than 10% of $T_0(s)$. But between 10 and 35 GeV^2, $\rho(s)$ reaches a maximum which is nearly twice as high as $\rho_0(s)$. Clearly, by choosing more complicated functions of the type (4.2a) and adjusting $\rho_2(s)$ in (4.2b) we can accommodate an enormous range of behaviors for $\rho(s)$ while still maintaining the spacelike behavior (1.2).

The situation with respect to the nonleading asymptotic behavior of $\rho(s)$ is still worse. Even if oscillations in the leading term of the type just described are ruled out on physical grounds, very little can be said about the nonleading terms. The proof of theorem III requires very strong assumptions and even when these assumptions are true, the theorem is not very restrictive.

To conclude we suggest that the cross section for electron-positron annihilation into hadrons may not after all provide a good testing ground for the asymptotic-freedom prediction (1.2). As Adler has pointed out, if the measured cross section remains constant then eventually all reasonable asymptotically free models will be ruled out. On the other hand, if $\rho(s)$ begins to settle down to a behavior of the type (1.2) then this will help to select among the theories which have been proposed. But the annihilation cross section has at its disposal a vast range of possibilities which would neither support nor contradict the asymptotic-freedom prediction for the spacelike region.

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[4]R. Shrock and F. Wilczek have obtained sum rules for $\sigma_{\text{had}}^{e^+e^-}(s)$ over finite regions of s by using an approximate inversion of the spectral representation and effectively assuming that oscillations in the cross section average out over a suitably small range of s. Using these sum rules they find that the data favor the existence of a charmed quark. Their work is summarized by F. Wilczek, in *Particles and Fields*—1974, proceedings of the 1974 Williamsburg meeting of the Division of Particles and Fields of the American Physical Society, edited by Carl E. Carlson (A.I.P., New York, 1975), p. 596. We thank Dr. Wilczek for informing us of their results.

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[6]N. N. Khuri and T. Kinoshita, Phys. Rev. 137, B720 (1965).

[7]In the case when $\pi(s)$ is a distribution on the cut this will only be true for some averaged $\pi(s)$ like the one defined in (2.2).

[8]See, for example, E. C. Titchmarsh, *The Theory of Functions* (Oxford Univ. Press, London, 1931), pp. 176–180.