Space-time internal algebra describing the hadronic mass spectrum

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A Lie algebra containing both the Poincaré and SU(6) algebras as subalgebras is abstracted from a Clifford algebra generated by seven elements. In a state in which the internal-symmetry quantum numbers are definite, the mass has a continuous spectrum peaked about certain values, and may be discrete in a few special states. An exact formula for the average squared mass is obtained which contains the Gell-Mann–Okubo expression in a natural way, and which also includes terms that correctly split the mass within isospin multiplets.

I. INTRODUCTION

The classification of particles as multiplets associated with the basis vectors of irreducible representations of an internal-symmetry algebra S of operators over a Hilbert space is a well established concept at present. The best known and most successful of these algebras are the SU(3) and SU(6) Lie algebras. Attempts to imbed S in an algebra A that includes space-time observables, so as to yield a discrete mass spectrum for the elements of a multiplet, have been thwarted by "no go" theorems, such as one proved by O' Raifeartaigh[1] that leads to no mass splitting if A is a Lie algebra containing the Poincaré algebra P as a subalgebra.

In this paper we shall formulate a Lie algebra A that contains P and SU(6) as subalgebras, but in which the mass operator has a continuous spectrum peaked at certain values, and may have discrete values in a few special states. This is in accordance with the observed mass spectrum, and is consistent with the results of the "no go" theorems.

The motivation for the particular algebra used here stems from the important role played by Clifford algebras[2] in physics. The Pauli spin algebra is abstracted from the Clifford algebra C_2 generated by two elements, the homogeneous Lorentz algebra is abstracted from C_3, and the Dirac algebra generated by the four γ_μ matrices is identical with C_4. The Clifford algebra C_5 was investigated by Basri and DeMeyer,[3] and was shown to contain P as a subalgebra. Since it is not possible to abstract both P and SU(3) from C_5, it was decided by the authors to investigate C_7 for the following reasons: (1) It is possible to abstract from C_7 both P and SU(6) as subalgebras. (2) C_7 has a nontrivial center consisting of the identity and another element [see (2.2) and (2.4)]. The latter element offers the possibility of interpretation as an operator such as CPT, whose eigenvalues differentiate between particles and antiparticles (the center of C_6 is trivial). (3) A Hilbert space may be constructed over C_7, which contains the basis for octonion Hilbert spaces.[4] Some applications of such spaces have recently been suggested by Günaydin and Gürsey.[5]

The algebra A is formulated and discussed in Sec. II. Then a maximal set of commuting operators is found in Sec. III, whose eigenvalues are used to specify the states. As a first test, A is used to derive an exact mass formula in Sec. IV, which gives the correct splitting of masses within SU(3) multiplets, both with respect to the hypercharge and isospin, and within each isospin muliplet.

II. THE ALGEBRA

We shall abstract the Lie algebra A we require from the commutation relations among the elements of the Clifford algebra C_7 generated by e_a ($a = 0, 1, \ldots, 7$), where e_0 is the identity and

$$e_a{}^2 = g_{aa} e_0, \quad g_{aa} = \pm 1 \tag{2.1a}$$

$$e_a e_b + e_b e_a = 0 \text{ for } a \neq b \ . \tag{2.1b}$$

C_7 is split by the central projection operators (these commute with all the elements of C_7)

$$p^{(\pm)} = \tfrac{1}{2}(e_0 \pm e_1 e_2 \cdots e_7) \ . \tag{2.2}$$

We shall choose the g_{aa} so that $e_1 e_2 \cdots e_7$ is self-adjoint in the finite-dimensional representation, where

$$e_a^\dagger = g_{aa} e_a \ . \tag{2.3}$$

Consider the even-closed Lie subalgebra generated by the even elements

$$\mathcal{E}=\{e_a e_b,\ e_a e_b e_c e_d,\ e_a e_b e_c e_d e_f e_g\}$$

containing 63 generators. This Lie algebra is isomorphic to that abstracted from C_6 and is the basic spinor module over the Cartan-Lie algebra B_6. Moreover, the two Lie subalgebras $\mathcal{E}^{(\pm)}=P^{(\pm)}\mathcal{E}$ are isomorphic to that generated by $\mathcal{E}$. Since

$$C_7=\text{center }(e_0, e_1\circ\cdots e_7)\oplus \mathcal{E}^{(+)}\oplus\mathcal{E}^{(-)}\ , \qquad (2.4)$$

it is sufficient to consider the Lie algebra $A^{(+)}$ which is abstracted from $\mathcal{E}^{(+)}$ alone. In what follows, we shall suppress the superscripts ($\pm$), with the understanding that we are working in the subspace characterized by $P^{(+)}$.

The elements of $A^{(+)}$ are defined by

$$E^+_{ab}\equiv e_a P_0 e_b \equiv E^-_{ba}\ , \qquad (2.5a)$$

$$H_a\equiv P_a - e_0/8\ , \qquad (2.5b)$$

where the projection operators P_0 and P_a (see Ref. 4) are defined as follows:

$$P_0\equiv P_{123}P_{145}P_{264}P^{(+)} = P_0^\dagger\ , \qquad (2.6)$$

$$P_{123}\equiv \tfrac{1}{2}(e_0 - i_{123}e_1e_2e_3) = P^\dagger_{123}\ ,$$
$$P_{145}\equiv \tfrac{1}{2}(e_0 - i_{145}e_1e_4e_5) = P^\dagger_{145}\ , \qquad (2.7)$$
$$P_{264}\equiv \tfrac{1}{2}(e_0 - i_{264}e_2e_6e_4) = P^\dagger_{264}\ ,$$

$$i_{abc}=\begin{cases} i \text{ if } g_{aa}g_{bb}g_{cc}=+1\ ,\\ +1 \text{ if } g_{aa}g_{bb}g_{cc}=-1\ ,\end{cases} \qquad (2.8)$$

and

$$P_a\equiv g_{aa}e_aP_0e_a = g_{aa}E^+_{aa}\ . \qquad (2.9)$$

These definitions imply

$$[E^+_{ab}, E^+_{cd}] = g_{bc}E^+_{ad} - g_{ad}E^+_{cb}\ , \qquad (2.10a)$$

$$[E^+_{ab}, E^-_{ab}] = g_{aa}g_{bb}(H_a - H_b)\ , \qquad (2.10b)$$

$$[H_a, E^\pm_{ab}] = \pm E^\pm_{ab}\ , \qquad (2.10c)$$

$$\left[\sum_{a=0}^{7} H_a, E^\pm_{ab}\right] = 0\ , \qquad (2.10d)$$

$$[H_a, H_b]=0\ ; \qquad (2.10e)$$

$$\sum_{a=0}^{7} P_a = e_0,\ \sum_{a=0}^{7} H_a = 0\ ; \qquad (2.11)$$

$$(E^+_{ab})^\dagger = E^+_{ba}\, g_{aa}g_{bb}\ ,\ H^\dagger_a = H_a\ . \qquad (2.12)$$

In infinite-dimensional irreducible representations, $\sum_a H_a$ will be constant and may have different values in different representations.

We shall take the commutation relations (2.10) as the definition of the Lie algebra A. In what follows, we shall refer only to the operator-valued quantities defined in this way, which realize this algebra on unitary representations. In particular, we require that

$$(E^+_{ab})^\dagger = E^+_{ba} = E^-_{ab}\ ,\qquad H^\dagger_a = H_a\ . \qquad (2.13)$$

Then the self-adjoint operators

$$E^A_{ab}\equiv i(E^+_{ab} - E^-_{ab}),\ \ E^S_{ab}\equiv E^+_{ab}+E^-_{ab} \qquad (2.14)$$

satisfy the commutation relations

$$[E^A_{ab}, E^A_{cd}] = i[g_{bc}E^A_{ad} + g_{ad}E^A_{bc} - g_{ac}E^A_{bd} - g_{bd}E^A_{ac}]\ , \qquad (2.15a)$$

$$[E^S_{ab}, E^S_{cd}] = -i\{g_{bc}E^A_{ad} + g_{ad}E^A_{bc} + g_{ac}E^A_{bd} + g_{bd}E^A_{ac}]\ , \qquad (2.15b)$$

$$[E^A_{ab}, E^S_{cd}] = i[g_{bc}E^S_{ad} - g_{ad}E^S_{bc} - g_{ac}E^S_{bd} + g_{bd}E^S_{ac}]\ . \qquad (2.15c)$$

Taking

$$g_{11}=g_{22}=g_{33}=-g_{44}=-1\ , \qquad (2.16)$$

we find from (2.15a) that ($\kappa, \lambda = 1, 2, 3, 4$)

$$J_{\kappa\lambda}\equiv E^A_{\kappa\lambda} = i(E^+_{\kappa\lambda} - E^-_{\kappa\lambda}) \qquad (2.17)$$

generate the homogeneous Lorentz algebra L. Furthermore, if we choose

$$g_{55}=-g_{66} \qquad (2.18)$$

and write $H_0=H_8$, then the momenta

$$P_\kappa \equiv P'_\kappa f_1(H_7, H_8) + \{P'_\kappa, E^A_{56}\} f_2(H_7, H_8)\ , \qquad (2.19)$$

$$P'_\kappa \equiv E^A_{\kappa 5} + E^A_{\kappa 6}\ , \qquad (2.20)$$

along with $J_{\kappa\lambda}$, generate the Poincaré algebra P [the curly bracket in (2.19) denotes an anticommutator]. The elements H_7 and H_8 are the only ones that commute with P'_κ and $J_{\kappa\lambda}$, and E^A_{56} is the only other element of A that may be introduced into the expression for P_κ so that the Poincaré relations below are satisfied:

$$[P_\kappa, P_\lambda] = 0\ , \qquad (2.21)$$

$$[P_\kappa, J_{\kappa\lambda}] = i g_{\kappa\kappa} P_\lambda\ . \qquad (2.22)$$

The crucial property of E^A_{56} for this purpose is

$$[E^A_{56}, P'_\kappa] = iP'_\kappa\ . \qquad (2.23)$$

The relation (2.18) is necessary for both (2.23) and

$$[P'_\kappa, P'_\lambda] = 0\ . \qquad (2.24)$$

Note that P'_κ also satisfy

$$[P'_\kappa, J_{\kappa\lambda}] = i g_{\kappa\kappa} P'_\lambda \ , \tag{2.25}$$

and together with $J_{\kappa\lambda}$ generate a Poincaré algebra. However, we shall see in Sec. IV that the second term in (2.19) is essential to obtain correct electromagnetic mass splitting. The operator E^A_{56} will also be seen in Sec. III to play a crucial role in providing a nonvanishing lower bound to the dispersion of the mass in representations characterized by definite internal-symmetry quantum numbers.

The general structure of the Lie algebra we have generated can be summarized in Fig. 1. The number 78 at the bottom vertex stands for the isospin subalgebra generated by

$$I_\pm = E^\pm_{78}, \qquad 2I_3 = [I_+, I_-] = H_7 - H_8 \ . \tag{2.26}$$

This choice for the isospin stems from the requirement that it commute with P'_κ.

Since the isospin algebra is a subalgebra of SU(3), we place the SU(3) algebra in the 678 triangle by taking

$$U_\pm = E^\pm_{67}, \quad V_\pm = -E^\pm_{68} \ , \tag{2.27}$$

$$2U_3 = [U_+, U_-] = H_7 - H_8 = \tfrac{3}{2} Y - I_3 \ , \qquad 2V_3 = [V_+, V_-] = H_6 - H_8 = \tfrac{3}{2} Y + I_3 \ , \tag{2.28}$$

where the hypercharge is given by

$$Y = \tfrac{1}{3}(2H_6 - H_7 - H_8) \ . \tag{2.29}$$

The fact that SU(3) is compact implies that

$$g_{66} = g_{77} = g_{88} \ . \tag{2.30}$$

One has the option of taking $g_{88} = +1$ or -1. To obtain the largest compact subalgebra of A, we choose the negative sign by defining

$$e_8 = i e_0, \quad g_{88} = -1 \ . \tag{2.31}$$

Then it follows from (2.16), (2.18), and (2.30) that

$$g_{11} = g_{22} = g_{33} = -g_{44} = -g_{55} = g_{66} = g_{77} = g_{88} = -1 \ . \tag{2.32}$$

This means that the 123678 lowermost triangle of five numbers on a side contains an SU(6) algebra.

The spin subalgebra lies in the 123 triangle on the upper left of the SU(6) triangle. The three spin components, according to (2.17), are

$$J_1 \equiv J_{23} = E^A_{23}, \quad J_2 \equiv E^A_{31}, \quad J_3 \equiv E^A_{12} \ . \tag{2.33}$$

In addition to these elements, the spin triangle also contains the three elements E^S_{jk} $(j, k = 1, 2, 3)$. These six elements, together with two linearly

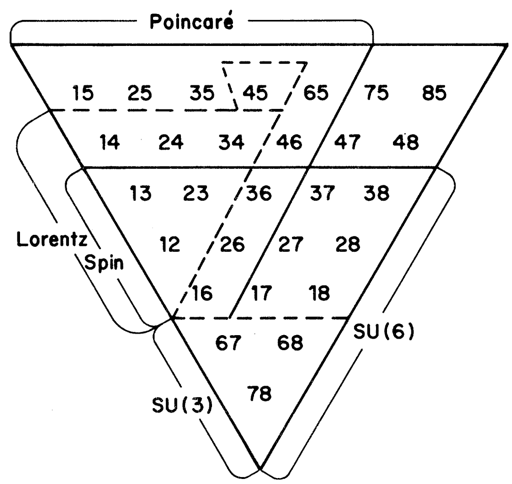

FIG. 1. Structure of the algebra A.

independent expressions of H_1, H_2, H_3, form a "spin-SU(3)" algebra. The magnetic moment components, being the product of electric charge with the spin, lie in the region of the SU(6) triangle outside the 123 and 678 triangles. Note, however, that all the elements of the SU(6) algebra are generators of A, and not constructed from Kronecker products of spin and SU(3) elements.

The spin triangle is included in the 1234 triangle containing all the elements $J_{\kappa\lambda}$ of the homogeneous Lorentz algebra. The latter triangle is in turn part of the 123456 upper left triangle of five numbers on a side, containing all the generators of the Poincaré algebra, except for H_7 and H_8 needed in (2.19).

Successively larger triangles, starting from the bottom vertex, contain SU(2), SU(3), SU(4), SU(5), SU(6), SU(6, 1), and SU(6, 2) algebras, with the last algebra being the whole of A. The intricate coupling of the space-time and internal observables is succinctly described by Fig. 1.

III. PHYSICAL STATES

In view of Fig. 1, a complete set of quantum numbers (QN) labeling the physical states may be obtained from the 20 QN describing the SU(6) states, the two additive QN $H_4 \pm H_5$, the six invariants of the SU(6, 1) algebra, and the seven invariants of A. This gives a total of 35 QN.

Note that the elements $E^\pm_{45}, H_4 - H_5$ form an SU(2) subalgebra that commutes with the SU(6) algebra. The QN

$$\vec{I}_{45}{}^2 \equiv \tfrac{1}{2}\{E^+_{45}, E^-_{45}\} + \tfrac{1}{4}(H_4 - H_5)^2 \tag{3.1}$$

is related to the energy P_4. It may be used instead of one of the invariants of SU(6, 1).

The 20 QN of SU(6) may be chosen as follows: From the isospin SU(2) algebra we have I_3 and $\vec{I}^2$, from the SU(3) algebra we have Y and the two invariants of SU(3), and from the U(3) containing the SU(3) we have the additive QN

$$B=-\tfrac{1}{3}(H_6+H_7+H_8)\ , \tag{3.2}$$

which commutes with all the elements of SU(3). By means of (2.26), (2.28), and (3.1) we obtain

$$H_6=Y-B\ , \tag{3.3}$$

$$H_7=-B-\tfrac{1}{2}Y+I_3,\ \ H_8=-B-\tfrac{1}{2}Y-I_3\ . \tag{3.4}$$

Thus we can interpret B as the baryon number and H_6 as the strangeness.

So far, we have the 6 QN

$$I_3,\ \vec{I}^2,\ Y,\ \text{two invariants of SU(3), and } B\ . \tag{3.5}$$

In addition, we have six analogous QN from the "spin U(3)," namely

$$J_3,\ \vec{J}^2,\ H_3,\ \text{two invariants of spin SU(3),}$$
$$\text{and } H_1+H_2+H_3\ . \tag{3.6}$$

These 12 QN plus the three invariants of SU(4) and five invariants of SU(6) complete the 20 QN of SU(6).

Instead of H_4+H_5, one may take the QN associated with U(6), i.e.,

$$C\equiv H_1+H_2+H_3+H_6+H_7+H_8$$
$$=(H_1+H_2+H_3)-3B\ . \tag{3.7}$$

The QN $(H_1+H_2+H_3)$ is related to J, and in view of the relation

$$(-1)^{B+L}=(-1)^{2J}\ , \tag{3.8}$$

the QN C might be related to the lepton number L.

Although the spin listed in (3.6) can be used to label particle states, the Poincaré-invariant spin

$$S^2=W_\kappa W^\kappa/m^2,\ \ W^\kappa=\tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu}J_{\lambda\mu}P_\nu \tag{3.9}$$

cannot, since it does not commute with H_6. The same is true of the squared mass operator

$$m^2=P_\kappa P^\kappa=g^{\kappa\lambda}P_\kappa P_\lambda\ . \tag{3.10}$$

Thus S^2 and m^2 have dispersion in states of definite H_6. The extent of the dispersion of m^2 is estimated below.

If $\Delta X\equiv(\langle X^2\rangle-\langle X\rangle^2)^{1/2}$ for some $X\in A$, then the uncertainty relation between m^2 and X is

$$\Delta m^2\Delta X\geq\tfrac{1}{2}|\langle i[m^2,X]\rangle|\ . \tag{3.11}$$

The commutator on the right-hand side can be written as a polynomial in terms of $E^{\pm}_{ab}$ and H_a. Since $E^{\pm}_{ab}$ act as ladder operators on the eigenvalues of H_a labeling the states, the right-hand side of (3.11) vanishes unless there are equal numbers of raising and lowering operations. This implies that the only X that leads to a nonvanishing right-hand side of (3.11) is E^A_{56}. Note that X cannot be H_a because $\Delta H_a\simeq 0$ (choosing suitable wave packets for the case of continuous spectra of H_a).

Making use of (3.9) and (2.23), we find

$$\Delta m^2\Delta E^A_{56}\geq\langle m^2\rangle\ . \tag{3.12}$$

For zero-rest-mass states, $\langle m^2\rangle=0$ and Δm^2 has a zero lower bound. For the electron and proton we expect ΔE^A_{56} to be exceptionally large, leading to a practically vanishing Δm^2. In general, $\Delta m^2\neq 0$, and we do not have an exactly discrete isolated mass spectrum.

IV. MASS FORMULA

The best known mass formula for hadrons is the Gell-Mann–Okubo formula[6]

$$m=a-bY+c[I(I+1)-\tfrac{1}{4}Y^2]\ . \tag{4.1}$$

This formula was derived on the assumption that the mass operator contains an SU(3) irreducible tensor that transforms like Y, and it is treated as a first-order perturbation to the mass of an SU(3) multiplet.

Similarly, by assuming that the term in the mass responsible for the splitting within an isospin multiplet is an irreducible tensor that transforms like the squared electric charge Q^2 ($Q=\tfrac{1}{2}Y+I_3$), one may derive a formula for this splitting in agreement with observed facts.[7,8] Various relations for electromagnetic mass splitting have been derived. The best known are the Coleman-Glashow[9] relation

$$m(\Xi^-)-m(\Xi^0)+m(n)-m(p)=m(\Sigma^-)-m(\Sigma^+)\ , \tag{4.2}$$

and the Rosen and Oakes relations[10]

$$m(\Delta^-)-m(\Delta^{++})=3[m(\Delta^0)-m(\Delta^+)]\ , \tag{4.3}$$

$$m(\Delta^0)-m(\Delta^+)=m(Y^0)-m(Y^+)\ , \tag{4.4}$$

$$m(\Delta^-)-m(\Delta^0)=m(Y^-)-m(Y^0) \tag{4.5}$$

$$=m(\Xi^{*-})-m(\Xi^{*0})\ . \tag{4.6}$$

In this section a mass formula will be derived directly from the expression for the mass

$$m^2=g_{\kappa\kappa}[P'^2_\kappa f_1^2+\{P'_\kappa,\{P'_\kappa,E^A_{56}\}\}f_1f_2+\{P'_\kappa,E^A_{56}\}^2f_2^2] \tag{4.7}$$

obtained from (3.10) and (2.19). This formula will lead naturally to (4.1) as well as the mass splitting within isospin multiplets.

We begin by analyzing the SU(3) irreducible tensor properties of the terms of P_K given in (2.19). According to (2.20) and (2.14),

$$-iP'_K = E^+_{K5} - E^-_{K5} + E^+_{K6} - E^-_{K6} \ . \qquad (4.8)$$

$E^{\pm}_{K5}$ commutes with all the generators of SU(3) and is thus an SU(3) singlet. Moreover, E^+_{K6} commutes with $I_\pm$, U_-, V_-, and satisfies the relations

$$[[E^+_{K6}, U_+], U_+] = 0,$$
$$[[E^+_{K6}, V_+], V_+] = 0 \ ,$$
$$[E^+_{K6}, Y] = -\tfrac{2}{3} E^+_{K6} \ .$$

Thus E^+_{K6} transforms (~) like the triplet $T_3(-\frac{2}{3},0,0)$, where $T_n(Y, I, I_3)$ is an irreducible tensor of the SU(3) multiplet $\underline{n}$. Similarly, $E^-_{K6} \sim T_{\bar{3}}(\frac{2}{3}, 0, 0)$.

Summarizing,

$$P'_K \sim T_1(000) + T_3(-\tfrac{2}{3}, 0, 0) + T_{\bar{3}}(\tfrac{2}{3}, 0, 0) \ . \qquad (4.9)$$

In the same way, we conclude that

$$E^A_{56} \equiv i(E^+_{56} - E^-_{56}) \sim T_3(-\tfrac{2}{3}, 0, 0) + T_{\bar{3}}(\tfrac{2}{3}, 0, 0) \ . \qquad (4.10)$$

Using the known Clebsch-Gordan expansion of SU(3) irreducible representations,[11] one obtains

$$\{P'_K, E^A_{56}\} \sim (\underline{1} + \underline{3} + \underline{\bar{3}}) \times (\underline{3} + \underline{\bar{3}}) = \underline{1} + \underline{3} + \underline{\bar{3}} + \underline{6} + \underline{\bar{6}} + \underline{8} \ . \qquad (4.11)$$

Consequently,

$$P'^2_K \sim \underline{1} + \underline{3} + \underline{\bar{3}} + \underline{6} + \underline{\bar{6}} + \underline{8} \ , \qquad (4.12a)$$

$$\{P'_K, \{P', E^A_{56}\}\} \sim \underline{1} + \underline{3} + \underline{\bar{3}} + \underline{6} + \underline{\bar{6}} + \underline{8} + \underline{10} + \underline{\overline{10}} + \underline{15} + \underline{\overline{15}} \ , \qquad (4.12b)$$

$$\{P'_K, E^A_{56}\}^2 \sim \text{multiplets on right-hand side of (4.12b)} + \underline{27} \ . \qquad (4.12c)$$

All the terms T on the right-hand side of (4.12) are isosinglets and satisfy

$$[T, Y] = aT \ .$$

For all terms, except $\underline{1}$, $\underline{8}$, and $\underline{27}$, $a \neq 0$ and

$$0 = \langle YII_3 | [T, Y] | YII_3 \rangle = a \langle YII_3 | T | YII_3 \rangle$$

implies $\langle T \rangle = 0$. Therefore,

$$\langle P'^2_K \rangle = \langle \{E^+_{K5}, E^-_{K5}\} \rangle + \langle \{E^+_{K6}, E^-_{K6}\} \rangle \sim T_1(000) + T_8(000) \ , \qquad (4.13a)$$

$$\langle \{P'_K, \{P'_K, E^A_{56}\}\} = i \langle \{E^+_{K5}, \{E^-_{K6}, E^+_{56}\}\} - \{E^+_{K6}, \{E^-_{K5}, E^-_{56}\}\} - \{E^-_{K5}, \{E^+_{K6}, E^-_{56}\}\} + \{E^-_{K6}, \{E^+_{K5}, E^+_{56}\}\} \rangle$$
$$\sim T_1(000) + T_8(000) \ , \qquad (4.13b)$$

$$\langle \{P'_K, E^A_{56}\}^2 \rangle = 2 \langle \{E^+_{K5}, E^+_{56}\}\{E^-_{K5}, E^-_{56}\} + \{E^+_{K5}, E^-_{56}\}\{E^-_{K5}, E^+_{56}\} + \{E^+_{K6}, E^+_{56}\}\{E^-_{K6}, E^-_{56}\} + \{E^+_{K6}, E^-_{56}\}\{E^-_{K6}, E^+_{56}\} \rangle$$
$$\sim T_1(000) + T_8(000) + T_{27}(000) \ . \qquad (4.13c)$$

Clearly, $T_1(000)$ is just a constant. Moreover (see Ref. 7, pp. 576–578),

$$T_8(000) = a_8 Y - b_8[I(I+1) - \tfrac{1}{4} Y^2 - C_2] \ , \qquad (4.14a)$$

$$T_{27}(000) = a_{27}[I(I+1) + \tfrac{9}{4} Y^2 - \tfrac{9}{4} C_2] \ , \qquad (4.14b)$$

where C_2 is the eigenvalue of the Casimir operator of SU(3). The functions f_1 and f_2 will be taken to be the simplest functions possible, consistent with known facts. Since meson isospin multiplets contain both particles and antiparticles, f_1 and f_2 must be even functions of H_7 and H_8. Moreover, no function of H_7 and H_8 can distinguish between K and Ξ multiplets because of identical value of H_7 and H_8 for these two multiplets. Thus E^A_{56} is essential for obtaining the correct splitting within isospin multiplets. Accordingly, we take

$$f_1 = 1, \ f_2 = (aH_7 + bH_8)^2 \ . \qquad (4.15)$$

From (3.4) we have

$$-(aH_7 + bH_8) = (a+b)(B + \tfrac{1}{2} Y) - (a-b) I_3 = (a+b)(B + \tfrac{1}{2} Y - kI_3) \ , \qquad (4.16)$$

where

$$k \equiv (a-b)/(a+b) \ . \qquad (4.17)$$

Since the octet baryons in an isospin multiplet have larger mass for more negative I_3, we expect $k > 0$.

The mass formula obtained from (4.7) and (4.13) to to (4.17) is

$$\langle m^2 \rangle = m_0^2 + m_1^2 \ , \qquad (4.18a)$$

$$m_0^2 = a_0 - b_0 Y + c_0[I(I+1) - \tfrac{1}{4} Y^2] \ , \qquad (4.18b)$$

$$m_1^2 = \{a_1 - b_1 Y + c_1[I(I+1) - \tfrac{1}{4} Y^2]\}(B + \tfrac{1}{2} Y - kI_3)^2 + \{a_2 - b_2 Y + c_2[I(I+1) - \tfrac{1}{4} Y^2] + d[I(I+1) + \tfrac{9}{4} Y^2]\}$$
$$\times (B + \tfrac{1}{2} Y - kI_3)^4 \ , \qquad (4.18c)$$

where the constant $(a+b)$ in (4.16) has been absorbed in the coefficients. Aside from k, all the coefficients in (4.18) are irreducible matrix elements that in principle can be calculated from the properties of the algebra, and are expected to have different values in different irreducible representations of SU(3). The parameter k has the same value in all representations. The expression for $m_0{}^2$ is entirely due to $P_K'^2$ and is identical with the Gell-Mann–Okubo formula. The term $m_1{}^2$ accounts for the mass splitting within isospin multiplets. The coefficient d in (4.18c) gives the contribution of the $\underline{27}$ multiplet [see (4.14b)].

That $m_0{}^2$ satisfies the Gell-Mann–Okubo formula follows directly from the fact that P_K' commutes with the isospin component I_3. This can be seen as follows: The structure of P_K' within the framework of a theory generated by the commutation relations of a Clifford algebra necessarily has the form (2.20) with two indices outside of the space-time indices 1, 2, 3, 4. One of these indices is associated with the compact internal-symmetry group. The requirement that P_K' commute with the isospin subgroup implies that this index must be strange, and hence that $P_K'^2$ satisfies the Gell-Mann–Okubo formula [see (4.8), (4.9), and (4.12a)].

For the baryon octet,

$$\begin{aligned} m_0{}^2(N) &= a_0 - b_0 + \tfrac{1}{2}c_0 \,, \\ m_0{}^2(\Xi) &= a_0 + b_0 + \tfrac{1}{2}c_0 \,, \\ m_0{}^2(\Sigma) &= a_0 + 2c_0 \,, \\ m_0{}^2(\Lambda) &= a_0 \,; \end{aligned} \tag{4.19}$$

$$\begin{aligned} m_1{}^2(p) &= a_N(1-k/3)^2 + b_N(1-k/3)^4 \\ &= 0.8803 - m_0{}^2(N) \,, \\ m_1{}^2(n) &= a_N(1+k/3)^2 + b_N(1+k/3)^4 \\ &= 0.8828 - m_0{}^2(N) \,, \\ m_1{}^2(\Xi^0) &= a_\Xi(1-k)^2 + b_\Xi(1-k)^4 \\ &= 1.729 - m_0{}^2(\Xi) \,, \\ m_1{}^2(\Xi^-) &= a_\Xi(1+k)^2 + b_\Xi(1+k)^4 \\ &= 1.746 - m_0{}^2(\Xi) \,, \\ m_1{}^2(\Sigma^+) &= a_\Sigma(1-k)^2 + b_\Sigma(1-k)^4 \\ &= 1.415 - m_0{}^2(\Sigma) \,, \\ m_1{}^2(\Sigma^0) &= a_\Sigma + b_\Sigma \\ &= 1.422 - m_0{}^2(\Sigma) \,, \\ m_1{}^2(\Sigma^-) &= a_\Sigma(1+k)^2 + b_\Sigma(1+k)^4 \\ &= 1.434 - m_0{}^2(\Sigma) \,, \\ m_1{}^2(\Lambda) &= a_1 + a_2 \\ &= 1.245 - m_0{}^2(\Lambda) \,, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} &a_N = (\tfrac{3}{2})^2(a_1 - b_1 + \tfrac{1}{2}c_1), \quad b_N = (\tfrac{3}{2})^4(a_2 - b_2 + \tfrac{1}{2}c_2 + 3d) \,, \\ &a_\Xi = 2^{-2}(a_1 + b_1 + \tfrac{1}{2}c_1), \quad b_\Xi = 2^{-4}(a_2 + b_2 + \tfrac{1}{2}c_2 + 3d) \,, \\ &a_\Sigma = a_1 + 2c_1, \quad b_\Sigma = a_2 + 2c_2 + 2d \,. \end{aligned} \tag{4.21}$$

The numbers on the right-hand side of (4.20) are the values of m^2 in GeV^2.[12]

The only mass differences within the same isospin multiplet are the following four:

$$\begin{aligned} D_N \equiv m^2(n) - m^2(p) &= \tfrac{4}{3}[a_N + 2b_N(1+k^2/9)]k \\ &= 2.43 \times 10^{-3} \,, \end{aligned} \tag{4.22a}$$

$$\begin{aligned} D_\Xi \equiv m^2(\Xi^-) - m^2(\Xi^0) &= 4[a_\Xi + 2b_\Xi(1+k^2)]k \\ &= 16.89 \times 10^{-3} \,, \end{aligned} \tag{4.22b}$$

$$\begin{aligned} D_\Sigma \equiv m^2(\Sigma^-) - m^2(\Sigma^+) &= 4[a + 2b(1+k^2)]k \\ &= 18.95 \times 10^{-3} \,, \end{aligned} \tag{4.22c}$$

$$\begin{aligned} D_\Sigma' &\equiv \tfrac{1}{2}[m^2(\Sigma^-) + m^2(\Sigma^+)] - m^2(\Sigma^0) \\ &= [a_\Sigma + b_\Sigma(6+k^2)]k^2 = 2.12 \times 10^{-3} \,. \end{aligned} \tag{4.22d}$$

The last is a second difference, whereas the others are first differences. Of those, the relative smallness of D_N may be attributed to the occurrence of $k/3$ in $m_1{}^2(N)$ instead of k in the other expressions for $m_1{}^2$.

Since $m_1{}^2 \ll m_0{}^2$, we conclude that the second term in the expression (2.19) for P_K must be much smaller than the first. This implies that the first, second, and third terms in expression (4.7) for m^2 must be successively smaller. Thus we expect the second terms in (4.18c), (4.20), and (4.22) to be considerably smaller than the first terms.

If we neglect the second terms in (4.22), as well as the terms in b_1 and c_1 [see (4.21)], then we immediately obtain $D_N + D_\Xi = D_\Sigma$, which is the Coleman-Glashow formula (4.2). We shall see below that although b_1 and c_1 are not negligible compared to a_1, their total contribution will be.

Another conclusion that can be drawn from (4.22), after dropping the second terms, is the relation

$$\tfrac{1}{2}[m^2(\Sigma^-) + m^2(\Sigma^+)] - m^2(\Sigma^0) = \tfrac{1}{4}k[m^2(\Sigma^-) - m^2(\Sigma^+)] \,. \tag{4.23}$$

An analogous relation holds generally for any baryon ($B=1$) isotriplet. In particular, for the decuplet we have

$$\tfrac{1}{2}[m^2(Y^-) + m^2(Y^+)] - m^2(Y^0) = \tfrac{1}{4}k[m^2(Y^-) - m^2(Y^+)] \,. \tag{4.24}$$

For the mesons, $B=0$, and we get $0=0$, instead of a relation analogous to (4.23).

In the following we shall carry out a fit to the actual baryon data to evaluate the universal constant k, and to get an estimate of the coefficients in (4.20). We have eight equations for 11 unknowns. However, according to the comments made above about the second terms in (4.20) and (4.22), we shall drop b_N, b_Ξ, and b_Σ and solve for the eight quantities k, a_0, b_0, c_0, a_1, b_1, c_1, and a_2.

From (4.22) we obtain

$$k=0.447,\quad k^2=0.200\;; \tag{4.25}$$

$$\begin{aligned}&10^3 a_N=4.07,\quad 10^3 a_\Xi=9.42,\quad 10^3 a_\Sigma=10.59\,,\\ &10^3 a_1=22.8,\quad 10^3 b_1=17.9,\quad 10^3 c_1=-6.10\;;\end{aligned} \tag{4.26}$$

$$\begin{aligned}&10^3 m_1{}^2(\Sigma^0)=10.59,\quad 10^3[m_1{}^2(p)+m_1{}^2(n)]=8.31\,,\\ &10^3[m_1{}^2(\Xi^0)+m_1{}^2(\Xi^-)]=26.0\,.\end{aligned} \tag{4.27}$$

Then by means of (4.27), (4.18a), and (4.19) we get

$$\begin{aligned}&m_0{}^2(N)=0.8774,\quad m_0{}^2(\Xi)=1.725,\quad m_0{}^2(\Sigma)=1.411\,,\\ &a_0=1.265,\quad b_0=0.422,\quad c_0=0.0732\,.\end{aligned} \tag{4.28a}$$

Finally, from

$$m_0{}^2(\Lambda)=a_0=1.265=m^2(\Lambda)-m_1{}^2(\Lambda) \tag{4.28b}$$

and (4.26) it follows that

$$m_1{}^2(\Lambda)=a_1+a_2=-0.020,\quad a_2=-0.0428\,. \tag{4.29}$$

For the meson octet we have

$$m_1{}^2(\pi^0)=m_1{}^2(\eta)=0\,, \tag{4.30}$$

$$m_1{}^2(\pi^\pm)=(a_1'+2c_1')k^2+(a_2'+2c_2')k^4\,, \tag{4.31}$$

$$m_1{}^2(K^\pm)=a_K(1-k)^2+b_K(1-k)^4\,, \tag{4.32}$$

$$m_1{}^2(K^0)=a_K(1+k)^2+b_K(1+k)^4\,, \tag{4.33}$$

where $[b_1'=0$ because $m^2(K^+)=m^2(K^-)]$

$$a_K=a_1'+\tfrac{1}{2}c_1',\quad b_K=a_2'+\tfrac{1}{2}c_2'+3d'\,. \tag{4.34}$$

Equations (4.30) can be solved for a_0' and c_0' $[b_0'=0$ because $m^2(\pi^+)=m^2(\pi^-)]$, and (4.33) for a_k and b_k. It is interesting to note here the occurrence of the zero values (4.30) in contrast to the nonzero values of $m_1{}^2(\Sigma^0)$ and $m_1{}^2(\Lambda)$. This is due to the value of $B+\frac{1}{2}Y$ in (4.18c).

Finally, for the baryon decuplet we have

$$\begin{aligned}&m_1{}^2(\Delta^{++})=a_\Delta(1-k)^2+b_\Delta(1-k)^4\,,\\ &m_1{}^2(\Delta^{+})=a_\Delta(1-k/3)^2+b_\Delta(1-k/3)^4\,,\\ &m_1{}^2(\Delta^{0})=a_\Delta(1+k/3)^2+b_\Delta(1+k/3)^4\,,\\ &m_1{}^2(\Delta^{-})=a_\Delta(1+k)^2+b_\Delta(1+k)^4\,;\end{aligned} \tag{4.35}$$

$$\begin{aligned}&m_1{}^2(Y^{+})=a_Y(1-k)^2+b_Y(1-k)^4\,,\\ &m_1{}^2(Y^{0})=a_Y+b_Y\,,\\ &m_1{}^2(Y^{-})=a_Y(1+k)^2+b_Y(1+k)^4\,;\end{aligned} \tag{4.36}$$

$$\begin{aligned}&m_1{}^2(\Xi^{*0})=a_{\Xi^*}(1-k)^2+b_{\Xi^*}(1-k)^4\,,\\ &m_1{}^2(\Xi^{*-})=a_{\Xi^*}(1+k)^2+b_{\Xi^*}(+k)^4\,;\end{aligned} \tag{4.37}$$

$$m_1{}^2(\Omega^-)=0\,. \tag{4.38}$$

Here again, the zero value in (4.38) is due to $B+\frac{1}{2}Y=0$ for Ω^-. If the second terms in (4.35) are neglected, then the relation (4.3) follows at once, without further approximations.

If b_Δ is not neglected, we obtain a relation between the mass differences

$$\begin{aligned}&D_1\equiv m^2(\Delta^-)-m^2(\Delta^{++})\,,\\ &D_2\equiv m^2(\Delta^0)-m^2(\Delta^{+})\,,\\ &D_3\equiv m^2(\Delta^-)-m^2(\Delta^{0})\end{aligned} \tag{4.39}$$

by solving two equations for a_Δ and b_Δ and substituting the results into the third equation. In this way we find

$$\begin{aligned}\tfrac{8}{3}k^2D_3=&(1+\tfrac{2}{3}k)[3(1+k^2)D_2-(1+k^2/9)D_1]\\ &-(1+2k+13k^2/9+10k^3/27)(3D_2-D_1)\,.\end{aligned} \tag{4.40}$$

The form $(a_\Delta x^2+b_\Delta x^4)$ for the Δ multiplet may exhibit a nonmonotonic variation with x, such as that predicted by Eliezer and Singer,[8] if the signs of a_Δ and b_Δ are opposite to each other.

V. CONCLUDING REMARKS

In this paper we have constructed an algebra in which the Poincaré algebra is coupled nontrivially to the internal-symmetry SU(6) and SU(3) algebras. The algebra is characterized by continuous mass spectra. However, we have obtained mass formulas for $\langle m^2\rangle$ that are consistent with the known data, including splitting within isospin multiplets. In each particle state, Δm^2 is generally not zero. It may, however, be zero in some cases, such as zero rest mass, and be very small in cases where E_{56}^A is very large (E_{56}^A is unbounded).

Although there are a large number of reduced matrix elements which are chosen to fit the data, these can in principle be evaluated when the representations of the Lie algebra are worked out. The present fit to experimental data with the universal parameter k provides a qualitative understanding of the structure of the mass spectrum.

The interesting results obtained here provide a first test of the theory. There are many other interesting problems to be worked out by the theory. One is the possibility of including leptons in the algebraic scheme. Another is the calculation of decay rates and cross sections. Concerning this, it should be noted that in the framework of this algebra, the S operator is a functional of tensor operators.

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