PHYSICAL REVIEW D VOLUME 11, NUMBER 3 1 FEBRUARY 1975

Real parts of the forward elastic $\pi^{\pm}p$, $K^{\pm}p$, $\bar{p}p$, and pp scattering amplitudes from 1 to 200 GeV/c*

R. E. Hendrick and B. Lautrup†
The Rockefeller University, New York, New York 10021
(Received 21 October 1974)

The recently measured total cross section data from the Fermi National Accelerator Laboratory are used to calculate the real parts of the forward scattering amplitude for $\pi^{\pm}p$, $K^{\pm}p$, $\bar{p}p$, and pp reactions from 1 to 200 GeV/c. The real parts and their uncertainties are calculated using dispersion relations which include subtraction constants, pole terms, and unphysical-cut contributions. A comparison to experiment shows the proton-proton real part measurements to be in excellent agreement with dispersion-relation results, while experiments show disagreement with the calculated π^-p and $K^{\pm}p$ real parts over at least a portion of the energy range where measurements exist.

I. INTRODUCTION

Recent experiments at the Fermi National Accelerator Laboratory[1] (Fermilab) have extended the total cross section measurements for $\pi^{\pm}p$, $K^{\pm}p$, and $\bar{p}p$ reactions up to a laboratory momentum of 200 GeV/c, and have accurately determined the pp cross section between 50 and 200 GeV/c. This extension of total cross section data permits a significant extension of the energy region over which the real part of the forward scattering amplitude may be accurately determined. The connection between the total cross section and the real part of the scattering amplitude is supplied by the optical theorem, analyticity, and crossing symmetry, usually written in the form of dispersion relations.[2] A number of detailed calculations of the real parts for the above reactions have been carried out at relatively low energies,[3-5] usually below 20 GeV/c. Recent calculations of the real parts at higher energies[6] have been based on assumptions concerning the high-energy total cross sections, and have ignored contributions to the real parts coming from the resonance region, pole terms, subtraction constants, and unphysical cuts.

In this paper we calculate the real parts of the $\pi^{\pm}p$, $K^{\pm}p$, $\bar{p}p$, and pp forward elastic amplitudes using the Fermilab total cross section data and recent determinations of the subtraction constants, coupling constants, and unphysical-cut contributions. The real parts and their uncertainties are calculated from 1 to 200 GeV/c, and the uncertainties are generally smaller than those coming from Coulomb interference measurements of the real parts.

A comparison of the dispersion-relation calculations and direct experimental measurements of the real parts is included. This serves as a check on the consistency of the total cross section measurements and the extrapolation of differential cross section measurements to the forward direction. Such a comparison also serves as a check on the validity of the dispersion relations themselves,[7] and to a certain extent on the assumption of spin independence of the forward and near-forward scattering amplitudes.[8] In proton-proton scattering, where accurate experimental determinations of the real part exist over the full energy range, experiment and dispersion-relation results are in excellent agreement (see Fig. 1). However, the experimental π^-p, K^+p, and K^-p real parts are in some disagreement with the dispersion-relation calculations. Further, in the $\bar{p}p$ and π^+p reactions, real-part measurements are scarce. The situation points to the need for more complete measurements of the $\pi^{\pm}p$, $K^{\pm}p$, and $\bar{p}p$ forward elastic real parts.

A quantity of considerable interest is the real part of the forward elastic antisymmetric πp amplitude, measured experimentally in the reaction $\pi^-p \to \pi^0 n$. Recent Fermilab measurements of this forward real part show some disagreement with dispersion-relation calculations.

II. NOTATION AND DISPERSION RELATIONS

Throughout this paper, amplitudes are normalized to dimensionless units[9]; the optical theorem has the form

$$\mathrm{Im}F_{ip}(E) = 2M(E^2 - M_i^{\,2})^{1/2}\sigma_{ip}(E), \tag{1}$$

where M is the proton mass, M_i is the mass of the incident particle, and $i = K^{\pm}$, $\pi^{\pm}$, $\bar{p}$, or p. [$\hbar = c = 1$, so 1 mb $= 2.568$ GeV^{-2}, 1 F $= 5.068$ GeV^{-1}.] The symmetric and antisymmetric amplitudes are defined by

$$F_S(E)=\tfrac{1}{2}(F_{i^-p}+F_{i^+p}),$$
$$F_A(E)=\tfrac{1}{2}(F_{i^-p}-F_{i^+p}), \qquad (2)$$

with corresponding relations for the symmetric and antisymmetric total cross sections.

Forward dispersion relations follow from crossing symmetry, assumed polynomial boundedness, and analyticity of the scattering amplitude which has been proved from the axioms of field theory[10] for πp scattering; Kp and pp dispersion relations have not been proved from field theory, but have been shown true to all orders in perturbation theory, and we assume their validity. Experiments indicate that the symmetric dispersion relation requires a subtraction, while high-energy experiments are consistent with an unsubtracted antisymmetric dispersion relation. Writing dispersion relations in terms of the laboratory energy $E=(p_{\rm lab}^2+M_i^2)^{1/2}$ and subtracting the symmetric dispersion relation at $E=0$ gives

$$\mathrm{Re}F_S(E)=C+\sum_y\frac{R_yE_y}{E_y^2-E^2}+\frac{E^2}{\pi}\int_{\overline{M}}^{M_i}\frac{dE'\,\mathrm{Im}F_-(E')}{E'(E'^2-E^2)}+\frac{4ME^2}{\pi}\,\mathrm{P}\int_{M_i}^{\infty}\frac{dE'(E'^2-M_i^2)^{1/2}\sigma_S(E')}{E'(E'^2-E^2)}, \qquad (3)$$

$$\mathrm{Re}F_A(E)=E\sum_y\frac{R_y}{E_y^2-E^2}+\frac{E}{\pi}\int_{\overline{M}}^{M_i}\frac{dE'\,\mathrm{Im}F_-(E')}{E'^2-E^2}+\frac{4ME}{\pi}\,\mathrm{P}\int_{M_i}^{\infty}\frac{dE'(E'^2-M_i^2)^{1/2}\sigma_A(E')}{E'^2-E^2}, \qquad (4)$$

where

$$C=\mathrm{Re}F_S(0)-\sum_y(R_y/E_y). \qquad (5)$$

The summation $\sum_y$ is over all pole terms,

$$R_y=\frac{g_y^2}{2M}\left[(M_y-M)^2-M_i^2\right]. \qquad (6)$$

g_y^2 is the rationalized Watson-Lepore coupling constant; M_y is the mass of the intermediate particle; $E_y=(M_y^2-M^2-M_i^2)/2M$. P denotes a principal-value integral; $\overline{M}$ is the unphysical threshold, M_i the physical threshold, and $\mathrm{Im}F_-(E)$ the unphysical-cut discontinuity.

There are several difficulties in calculating the real parts of the forward scattering amplitudes using dispersion relations. First, the presence of a large unphysical cut, as in K^-p and $\bar{p}p$ scattering, complicates the determination of subtraction and coupling constants. The resultant uncertainties in subtraction and coupling constants, and in the total cross section near threshold, make it difficult to evaluate models for the unphysical-cut contributions. A discussion of the assumptions made concerning unphysical cuts and constants in calculating the real parts for each reaction is given in Sec. III.

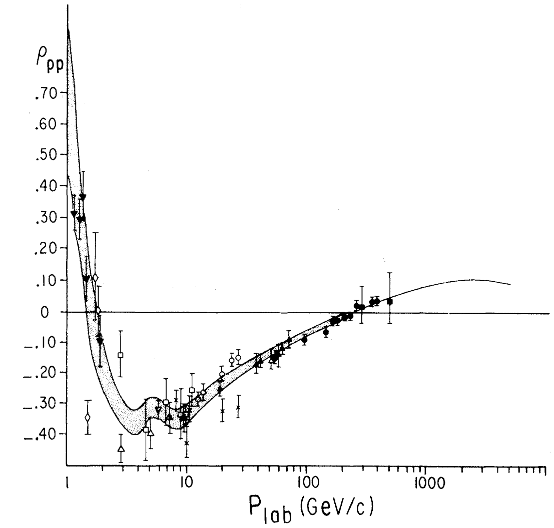

FIG. 1. Proton-proton ratio of forward real part to imaginary part. Experimental points are from the following references: ▼ A. A. Vorobyov *et al.*, Phys. Lett. 41B, 639 (1972). ◇ L. M. C. Dulton and H. B. Vander Raay, *ibid.* 26B, 11 (1968). △ A. R. Clyde, UCRL Report No. 16275, 1966 (unpublished). □ L. F. Kirillova *et al.*, Zh. Eksp. Teor. Fiz. 50, 76 (1966) [Sov. Phys.—JETP 23, 52 (1966)]. ○ K. J. Foley *et al.*, Phys. Rev. Lett. 19, 857 (1967). ∗ A. E. Taylor *et al.*, Phys. Lett. 14, 64 (1964). ▲ G. G. Beznogikh *et al.*, *ibid.* 39B, 411 (1972). × C. Bellettini *et al.*, *ibid.* 14, 164 (1965). ● V. Bartenev *et al.*, Phys. Rev. Lett. 31, 1367 (1973). ■ U. Amaldi *et al.*, Phys. Lett. 43B, 231 (1973).

Second, there are regions in which the total cross sections are unknown. In the region just above the physical threshold, where direct total cross section measurements do not exist, we have used the effective-range approximations compiled by Barashenkov.[11] In the energy region beyond 200 GeV/c, we assume that the antisymmetric cross sections continue to decrease as inverse powers of energy.[12] We assume that at higher energies the symmetric cross sections do not decrease below their values at 200 GeV/c, while they may increase as fast as any smooth $\ln^2 E$ extrapolation of the existing data. In the low-energy regions, where total cross section measurements are dense, we assume a straight-line interpolation between data points. In the region between 50 and 200 GeV/c, where only a few measurements exist, cross sections have been interpolated according to the parametrizations in Ref. 12. In calculating the real parts, we have used the most accurate total cross section measurements, taken from the compilations of total cross sections.[13] Principal-value integrals have been evaluated as described in the Appendix.

III. CONSTANTS AND UNPHYSICAL CUTS

In this section we describe the pole terms, subtraction constants, and unphysical-cut contributions used to calculate the real parts for each reaction. The purpose of this work has not been to arrive at better determinations of these contributions, but to include the best known values of these parameters in calculating the real parts.

A. πp

The πp dispersion relation has one pole term coming from the neutron intermediate state in $\pi^- p$ scattering, with the single coupling constant usually written as

$$f^2 = \frac{g_{\pi^- pn}{}^2 M_\pi{}^2}{32\pi M^2}, \tag{7}$$

with $g_{\pi^- pn}{}^2$ the rationalized coupling constant appearing in Eq. (6). The process also has a small unphysical-cut contribution due to the reactions

$$\pi^- p \to \pi^0 n,$$
$$\pi^- p \to \gamma n.$$

The contributions of these unphysical cuts have been analyzed and found to be small, their major effect being to increase the uncertainties in the subtraction and coupling constants.[14] For our purposes, the best determination of the subtraction and coupling constants is that of Samaranayake and Woolcock.[14] Using a large number of $\pi^{\pm}p$ phase-shift analyses of the real parts below 2 GeV/c, they determine f^2 and C directly, obtaining

$$f^2 = 0.0763 \pm 0.0020,$$

$$C = \frac{8\pi M}{M_\pi}(-0.1057 \pm 0.0024),$$

where C is in dimensionless units.

B. Kp

The Kp dispersion relations contain two pole terms due to Λ^0 and Σ^0 intermediate states in the $K^- p$ reaction. Rather than attempt to determine both coupling constants, it is sufficient to determine the constant

$$R = \frac{1}{4\pi}\left[g_{K^- p\Lambda_0}{}^2 + g_{K^- p\Sigma_0}{}^2 \frac{(M_{\Sigma^0} - M)^2 - M_K{}^2}{(M_{\Lambda^0} - M)^2 - M_K{}^2}\right]$$

and approximate the two poles by a single pole located between the two. Even that is difficult, owing to substantial unphysical-cut contributions from the processes

$$K^- p \to \Lambda^0 \pi^0,$$
$$K^- p \to \Sigma^0 \pi^0,\ \ \Sigma^{\pm}\pi^{\mp},\ \text{etc.}$$

In our calculations, the unphysical cuts are approximated by the constant-scattering-length model,[15] which gives

$$\mathrm{Im}F_-(E) = 4\pi(M^2 + M_K{}^2 + 2ME)^{1/2} \times \left[\frac{b_0}{(1 + a_0 K)^2 + b_0{}^2 K^2} + \frac{b_1}{(1 + a_1 K)^2 + b_1{}^2 K^2}\right],$$

$$K^2 = \frac{M^2(M_K{}^2 - E^2)}{M^2 + M_K{}^2 + 2ME}.$$

The first term in brackets is cut off at the $\Sigma\pi$ threshold and the second at the $\Lambda\pi$ threshold, and a_0, a_1, b_0, b_1 are taken from the analysis of Kim.[16] The value of R used in the calculation was taken from the most reliable value given by Pilkuhn *et al.*[17] The value of C was chosen so that the calculated real parts agree with the real-part measurements below 1 GeV/c,[18] giving

$$R = 12.2 \pm 2.5,$$
$$C = -9.4 \pm 1.9.$$

A 20% uncertainty has been assigned to the determination of C and the unphysical-cut approximation.

C. pp

The pp dispersion relations have pole terms due to π^0 and η^0 exchange in the $\bar{p}p$ reaction, and a large unphysical-cut contribution due to the reactions

$$\bar{p}p \to 2\pi^0,\ \ \pi^{\pm}\pi^{\mp},\ \ 3\pi^0,\ \ \pi^+\pi^-\pi^0,\ \text{etc.}$$

below the elastic threshold. Very little work has been done evaluating the contribution of this unphysical cut. Following Söding,[3] we approximate the cut discontinuity by poles, located at energies corresponding to the ρ and ω, in addition to the π and η poles, with residues

$$R_\pi = \frac{8\pi}{M}(0.036)$$

$$R_\eta = \frac{8\pi}{M}(0.376)$$

$$R_\rho = \frac{8\pi}{M}(2.53)$$

$$R_\omega = \frac{8\pi}{M}(0.89)$$

The subtraction constant was chosen so that the pp real parts below 1 GeV/c correspond to the results of pp phase-shift analyses,[19]

$$C = -135 \pm 27.$$

A 20% uncertainty in the determination of C and in the unphysical-cut approximation is assumed.

IV. RESULTS AND CONCLUSIONS

In each reaction, the subtraction and coupling constants are determined using low-energy measurements of the real parts, usually below 1 GeV/c. Thus, agreement of the calculated real parts with some fit to the real-part data below 1 GeV/c is automatically guaranteed. Above 200 GeV/c, symmetric and antisymmetric total cross sections have not been measured. Any statement about the real parts beyond this energy takes the form of a prediction, based on assumptions about the behavior of the total cross sections beyond 200 GeV/c.

Accordingly, we present the calculated real parts, their uncertainties, and a comparison to experimental measurements of the real parts from 1 to 200 GeV/c (Figs. 1–6). The shaded region in each figure represents the calculated range of values for real parts based on uncertainties in the total cross section measurements, unphysical-cut approximations, and subtraction- and coupling-constant determinations. The solid line beyond 200 GeV/c is a prediction for the real parts based on parametrizations of existing cross sections data[12] which have been extrapolated beyond 200 GeV/c.

Accurate measurements of the real parts over the entire energy range exist only for proton-proton scattering. For this reaction, agreement between calculated real parts and experiment is good over the entire energy range (Fig. 1). There is no evidence that the proton-proton dispersion relation or the assumption of spin independence is invalid.[20]

For πp and Kp scattering, the situation is not so clear. Real-part measurements exist mainly at low energies. Several of the higher-energy real-part measurements for $\pi^- p$ and $K^\pm p$ scattering are in some disagreement with the dispersion-relation results (Figs. 4–6). The situation points to the need for more detailed experimental results on the $\pi^\pm p$, $K^\pm p$, and $\bar{p}p$ real parts above a few GeV/c.

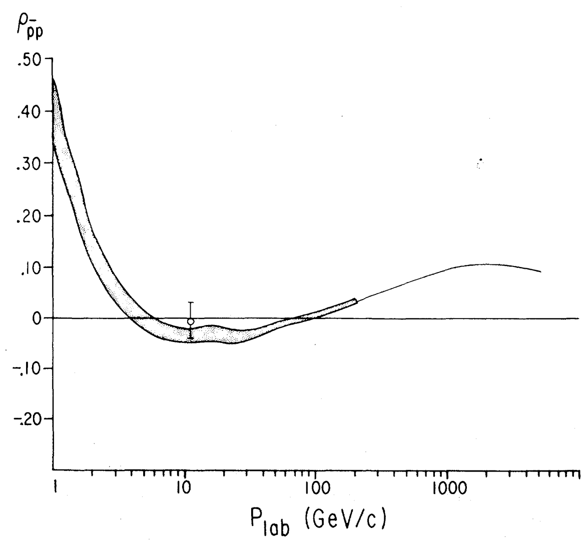

FIG. 2. $\bar{p}p$ ratio of forward real part to imaginary part. Experimental point labeled ○ is from K. J. Foley *et al.*, Phys. Rev. Lett. 19, 857 (1967).

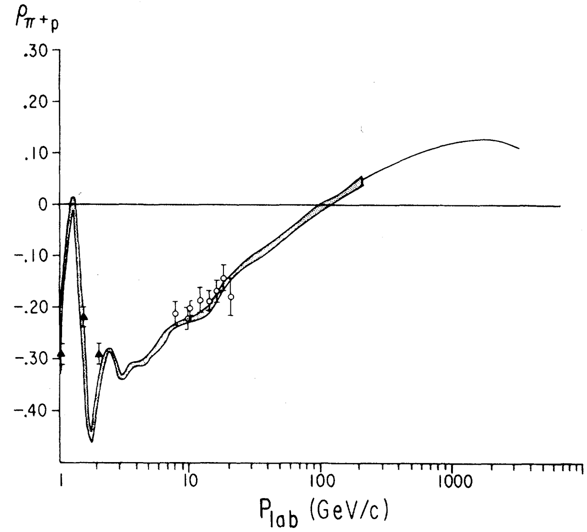

FIG. 3. π^+p ratio of forward real part to imaginary part. Experimental points are from the following references: ▲ P. Baillon *et al.*, Phys. Lett. 50B, 387 (1974). ○ K. J. Foley *et al.*, Phys. Rev. 181, 1755 (1969).

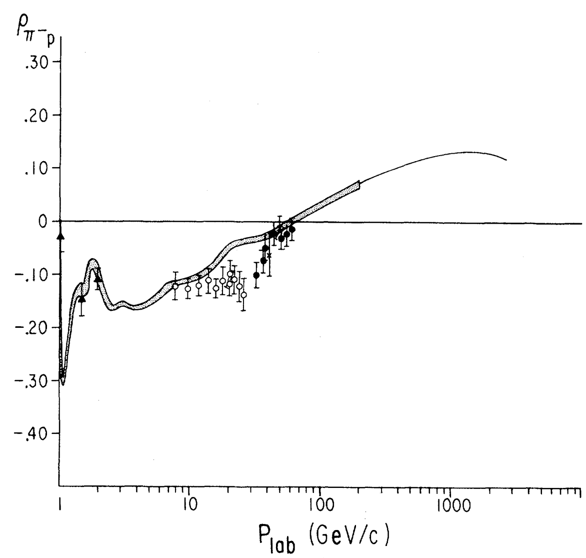

FIG. 4. π^-p ratio of forward real part to imaginary part. Experimental points are from the following references: ▲ P. Baillon *et al.*, Phys. Lett. 50B, 387 (1974). ○ K. J. Foley *et al.*, Phys. Rev. 181, 1755 (1969). ●, × V. D. Apokin *et al.*, papers contributed to the Second International Conference on Elementary Particles, Aix-en-Provence, 1973 (unpublished).

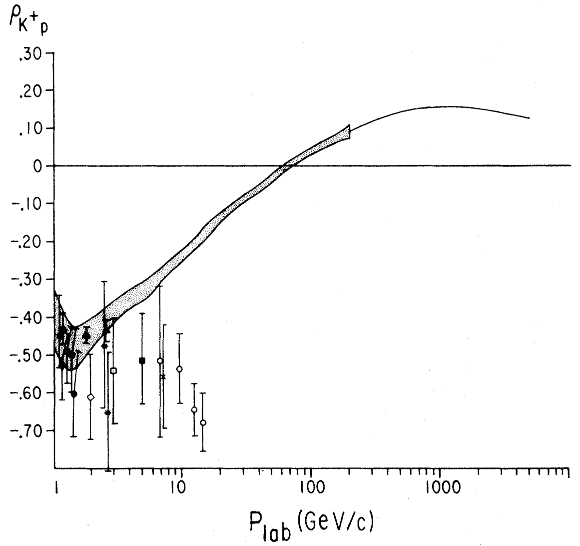

FIG. 5. K^+p ratio of forward real part to imaginary part. Experimental points are from the following references: ● G. Giacomelli *et al.*, Nucl. Phys. B20, 301 (1970). ▲ P. Baillon *et al.*, Phys. Lett. 50B, 377 (1974). ◆ G. Goldhaber, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, Univ. of Miami, 1967*, edited by A. Perlmutter and B. Kurşunoğlu (Freeman, San Francisco, 1967), p. 190. ◇ W. Chinowsky *et al.*, Phys. Rev. 139, B1411 (1965). * J. A. Danysz *et al.*, Nucl. Phys. B14, 161 (1969). □ J. Debaisieux *et al.*, Nuovo Cimento 43A, 142 (1966). ■ T. H. J. Bellm *et al.*, Nuovo Cimento Lett. 3, 389 (1970). ○ K. J. Foley *et al.*, Phys. Rev. Lett. 11, 503 (1963). × C. Y. Chien *et al.*, Phys. Lett. 28B, 615 (1969).

The calculated antisymmetric πp real part is compared to experiment in Fig. 7. The recent Caltech-LBL data[21] on $\pi^- p \to \pi^0 n$ are in at least some disagreement with the dispersion-relation results. It should be noted, however, that the behavior of the antisymmetric πp cross section above 200 GeV/c contributes to the calculated real part, and that a sufficient change in the slope of the antisymmetric total cross section above 200 GeV/c may remove the discrepancy.[12] However, if the antisymmetric πp cross section continues to agree roughly with the parametrization of Ref. 12 up to 300–400 GeV/c, then a serious discrepancy remains.

At sufficiently high energies, the effect of the unphysical cuts, subtraction constants, and pole terms will be negligible. One way to determine this energy is to calculate the real parts ignoring these contributions, and to determine the energy at which the results fall within the uncertainty of the real parts calculated including all contributions, that is, fall within the shaded region in the

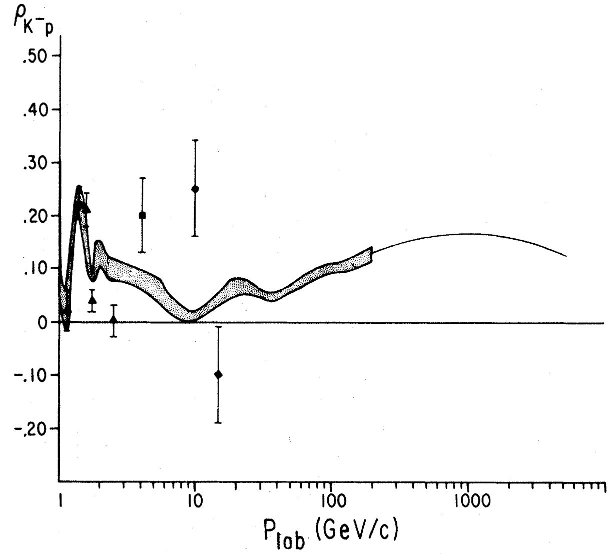

FIG. 6. K^-p ratio of forward real part to imaginary part. Experimental points are from the following references: ▲ P. Baillon *et al.*, Phys. Lett. 50B, 377 (1974). ■ T. H. J. Bellm *et al.*, Phys. Lett. 33B, 438 (1970). ● J. R. Campbell *et al.*, Nucl. Phys. B64, 1 (1973). ◆ Amsterdam-Nijmegen-Paris Collaboration, paper submitted to the Second International Conference on Elementary Particles, Aix-en-Provence, 1973 (unpublished).

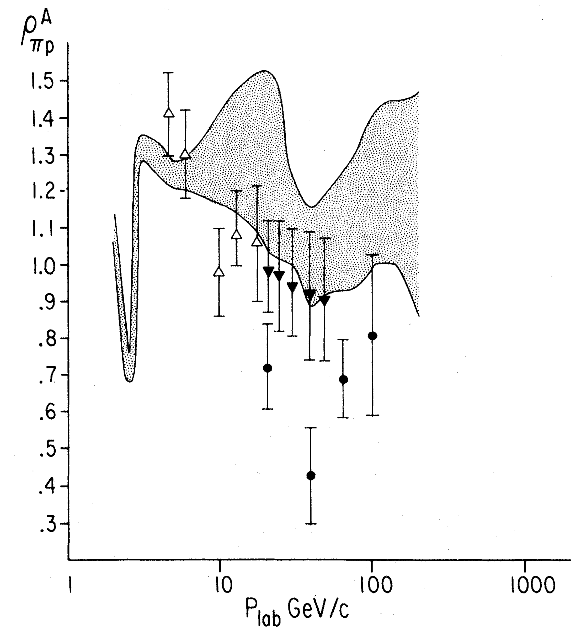

FIG. 7. Ratio of forward antisymmetric πp real part to imaginary part. Experimental points are taken from the $\pi^- p \to \pi^0 n$ measurements listed in the following references: △ A. V. Stirling *et al.*, Phys. Rev. Lett. 14, 736 (1965); P. Sonderegger *et al.*, *ibid.* 20, 75 (1966). ▼ V. N. Bolotov *et al.*, Phys. Lett. 38B, 120 (1971); Nucl. Phys. B73, 365 (1974). ● Barnes *et al.*, Ref. 21.

figures. This energy was found to correspond to p_{lab} of 45, 50, and 150 GeV/c for πp, Kp, and pp real parts, respectively.

The same method can be used to determine the energy at which the derivative analyticity relations[22] give a precise representation of the real parts. At high energies the contributions mentioned above and the detailed effects of the resonance region will wear off, and the total cross sections appear to become sufficiently smooth to make derivative analyticity predictions accurate. The energies at which their predictions agree with the real parts calculated from dispersion relations, within the stated errors, is 50, 30, and above 200 GeV/c for πp, Kp, and pp, respectively.

ACKNOWLEDGMENTS

We would like to thank Dr. R. L. Cool, Dr. V. Rittenberg, and Dr. N. N. Khuri for their advice and encouragement in this work.

APPENDIX

In this appendix we outline the method used to evaluate principal-value integrals. Consider the integral

$$I = \int_a^b dx\, y(x)F(x)\,, \tag{A1}$$

where $F(x)$ is a given function and $y(x)$ is known experimentally at discrete values of x in the range (a,b). For each value x_i $(i=1,2,\ldots,N)$ there are corresponding values y_i and z_i, where z_i is the uncertainty in the measured y_i value.

In order to evaluate the integral it is necessary to introduce assumptions about the behavior of $y(x)$ between the data points. However, any estimate, $y_{\text{est}}(x)$, should be linear in the values y_i, that is, of the form

$$y_{\text{est}}(x) = \sum_{i=1}^N y_i G_i(x)\,, \tag{A2}$$

where $G_i(x)$ are known functions. The choice of $G_i(x)$ depends on our theoretical assumptions about the smoothness of $y(x)$ and is a source of systematic error which is difficult to estimate, particularly when one extrapolates far outside the range of the data points.

Using the estimate $y_{\text{est}}(x)$ we find an estimate of the *integral*

$$I_{\text{est}} = \int_a^b dx\, y_{\text{est}}(x)F(x) = \sum_{i=1}^N c_i y_i\,, \tag{A3}$$

where

$$c_i = \int_a^b dx\, G_i(x)F(x) \tag{A4}$$

is a set of explicitly known coefficients. The statistical uncertainty in the estimated integral due to the uncorrelated uncertainties z_i in the y_i is

$$\Delta I_{\text{est}} = \left(\sum_{i=1}^N c_i^2 z_i^2\right)^{1/2}. \tag{A5}$$

Thus, if the y_i values fluctuate as Gaussians having widths z_i then the integral will fluctuate as a Gaussian with the width ΔI_{est} around the value I_{est}. Again we emphasize that ΔI_{est} does not include the systematic uncertainty due to the interpolation chosen.

In the following we take $x_i = a$ and $x_N = b$. This may always be obtained by adding "artificial" data points consistent with the extrapolation assumptions. We furthermore assume that the data points are sufficiently dense in the range (a,b) to justify the use of a linear interpolation formula

$$y_{\text{est}}(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}}\, y_i + \frac{x - x_i}{x_{i+1} - x_i}\, y_{i+1}\,,$$
$$x_i \le x \le x_{i+1}\,, \quad i = 1,2,\ldots,N-1\,. \tag{A6}$$

Then the coefficients c_i become

$$\begin{aligned}
c_1 &= \int_{x_1}^{x_2} \frac{x - x_2}{x_1 - x_2} F(x)dx\,,\\
c_i &= \int_{x_i}^{x_{i+1}} \frac{x - x_{i+1}}{x_i - x_{i+1}} F(x)dx\\
&\quad + \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{x_i - x_{i-1}} F(x)dx, \quad i = 2,\ldots,N-1\\
c_N &= \int_{x_{N-1}}^{x_N} \frac{x - x_{N-1}}{x_N - x_{N-1}} F(x)dx\,.
\end{aligned} \tag{A7}$$

For the special case of a principal-value integral

$$I(t) = \mathrm{P}\int_a^b dx\, \frac{y(x)}{x - t}\,,$$

i.e.,

$$F(x) = \mathrm{P}\left(\frac{1}{x - t}\right), \tag{A8}$$

we find when $t \ne x_i$, $i = 1,2,\ldots N$,

$$\begin{aligned}
c_1 &= \frac{t - x_2}{x_1 - x_2} \ln\left|\frac{x_2 - t}{x_1 - t}\right| - 1\,,\\
c_i &= \frac{t - x_{i+1}}{x_i - x_{i+1}} \ln\left|\frac{x_{i+1} - t}{x_i - t}\right| + \frac{t - x_{i+1}}{x_i - x_{i+1}} \ln\left|\frac{x_i - t}{x_{i-1} - t}\right|,\\
c_N &= \frac{t - x_{N-1}}{x_N - x_{N-1}} \ln\left|\frac{x_N - t}{x_{N-1} - t}\right| + 1\,.
\end{aligned} \tag{A9}$$

If we take $t = x_j$ $(3 < j < N-2)$, then we have

$$c_{j-1} = \frac{x_j - x_{j-2}}{x_{j-1} - x_{j-2}} \ln \left| \frac{x_{j-1} - x_j}{x_{j-2} - x_j} \right| ,$$
$$c_j = \ln \left| \frac{x_{j+1} - x_j}{x_{j-1} - x_j} \right| , \qquad \text{(A10)}$$
$$c_{j+1} = \frac{x_j - x_{j+2}}{x_{j+1} - x_{j+2}} \ln \left| \frac{x_{j+2} - x_j}{x_{j+1} - x_j} \right| ,$$

while the other coefficients are unmodified.

In the case of a more complicated integral of the form

$$I(t) = \mathrm{P} \int_a^b \frac{y(x) f(x) dx}{x - t} , \qquad \text{(A11)}$$

where $y(x)$ is experimentally measured, while $f(x)$ is not a function of t and is smoothly varying in the range of integration, one may interpolate linearly between the values of $y_i f(x_i)$ and proceed as above.

---

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-2232B.

†Present Address: The Niels Böhr Institute, Copenhagen Denmark.

[1]A. S. Carroll *et al.*, Phys. Rev. Lett. 33, 928 (1974).

[2]M. Gell-Mann, M. L. Goldberger, and W. Thirring, Phys. Rev. 95, 1612 (1954); M. L. Goldberger, *ibid.* 99, 979 (1955); M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. (N.Y.) 2, 226 (1957); N. N. Bogoliubov, report at the International Congress on Theoretical Physics, Seattle, 1956 (unpublished); N. N. Bogoliubov, B. V. Medvedev, and M. K. Polivanov, *Problems in the Theory of Dispersion Relations* (Fizmatgiz, Moscow, 1958).

[3]Calculations of the $\bar{p}p$ and pp real parts: V. S. Barashenkov, Forschr. Phys. 10, 204 (1962); P. Söding, Phys. Lett. 8, 285 (1964); G. Bialkowski and S. Pokorski, Nuovo Cimento 57A, 219 (1968); P. Pascual and F. Y. Yndurain, *ibid.* 61, 225 (1969); O. V. Dumbrais, Yad. Fiz. 13, 1096 (1971) [Sov. J. Nucl. Phys. 13, 626 (1971)]; P. Kroll, Nuovo Cimento Lett. 7, 745 (1973).

[4]Calculation of $\pi^\pm p$ real parts: J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963); G. Höhler, G. Ebel, and J. Giesecke, Z. Phys. 180, 430 (1964); B. Lautrup, P. M. Nielsen, and P. Olesen, Phys. Rev. 140, B984 (1965); J. K. Foley *et al.*, *ibid.* 181, 1775 (1969); G. Höhler, F. Steiner, and R. Strauss, Z. Phys. 233, 430 (1970); G. Höhler and H. P. Jakob, Nuovo Cimento Lett., 6, 389 (1973); Z. Phys. 268, 75 (1974).

[5]Calculations of the Kp real parts: N. Zovko, Z. Phys. 192, 346 (1966); 196, 16 (1966); M. Lusignoli, M. Restignoli, G. Violini, and G. A. Snow, Nuovo Cimento 45, 792 (1966); N. M. Queen, Nucl. Phys. B1, 207 (1967); A. D. Martin and R. Perrin, *ibid.* B20, 287 (1970); Y. A. Chao and E. Peitarinen, Phys. Rev. Lett. 26, 1060 (1971).

[6]W. Bartel and A. N. Diddens, CERN NP Internal Report No. 73-4 (unpublished); C. Bourrely and J. Fischer, Nucl. Phys. B61, 513 (1973); C. Bourrely, J. Fischer, and Z. Sekera, *ibid.* B67, 452 (1973); J. B. Bronzan, G. L. Kane, and U. P. Sukhatme, Phys. Lett. 49B, 272 (1974); G. K. Eichmann and J. Dronkers, *ibid.* 52B, 428 (1974); D. P. Sidhu and U. P. Sukhatme, Phys. Rev. D (to be published). The paper of Sidhu and Sukhatme using the derivative analyticity approximation includes the new Fermilab data.

[7]S. J. Lindenbaum, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1967,* edited by A. Perlmutter and B. Kurşunoğlu (Freeman, San Francisco, 1967), p. 122; Foley *et al.*, Ref. 4; T. N. Pham and T. N. Truong, Phys. Rev. D 8, 3980 (1973).

[8]C. Bourrely, J. Soffer, and D. Wray, Nucl. Phys. B77, 386 (1974).

[9]See Lautrup *et al.*, Ref. 4, or R. J. Eden, *High Energy Collisions of Elementary Particles* (Cambridge Univ. Press, New York, 1967) p. 117.

[10]A. Martin, Nuovo Cimento 42, 930 (1966); 44, 1219 (1966).

[11]V. S. Barashenkov, *Interaction Cross Sections of Elementary Particles* (Moscow, 1966) [English translation: Israel Program for Scientific Translation, Jerusalem, 1968].

[12]Parametrizations of the total cross sections are given by R. E. Hendrick, P. Langacker, B. Lautrup, S. J. Orfanidis, and V. Rittenberg, this issue, Phys. Rev. D 11, 536 (1975).

[13]See E. Bracci *et al.*, *CERN/HERA Cross Section Compilations* (CERN, Geneva).

[14]V. K. Samaranayake and W. S. Woolcock, Nucl. Phys. B48, 205 (1972); see also H. Pilkuhn *et al.*, *ibid.* B65, 460 (1973).

[15]B. R. Martin, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, New York, 1970), Vol. 55, p. 100; R. Perrin and W. S. Woolcock, Nucl. Phys. B4, 671 (1968); B12, 26 (1969).

[16]J. K. Kim, Phys. Rev. Lett. 14, 29 (1965).

[17]H. Pilkuhn *et al.*, Ref. 14; E. Pietarinen and C. P. Knudsen, Nucl. Phys. B57, 637 (1973).

[18]B. Conforto *et al.*, Nucl. Phys. B8, 265 (1968).

[19]J. D. Dowell *et al.*, Phys. Lett. 12, 252 (1964).

[20]Following R. Oehme, Phys. Rev. 100, 1503 (1965), and Lindenbaum, Ref. 7, this extends the fundamental length (the length at which microcausality could fail and still be undetectable by experiment) down to $l_0 \sim 1/200$ GeV $= 10^{-16}$ cm, and perhaps even somewhat lower.

[21]A. V. Barnes *et al.*, paper submitted to the XVII International Conference on High Energy Physics, Imperial College, London, 1974 (unpublished).

[22]See Bronzan *et al.* and Sidhu and Sukhatme, Ref. 6.