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What is the asymptotic limit of $\sigma(e^++e^-\to\text{hadrons})/\sigma(e^++e^-\to\mu^++\mu^-)$, if any?*

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More extensive discussion is given on the previously derived relation, $R\cong 16\pi^2/f_\rho{}^2 = 5.7\pm0.9$, where R is the asymptotic limit of the ratio $R(q^2)=\sigma(e^++e^-\to\text{hadrons})/\sigma(e^++e^-\to\mu^++\mu^-)$ and f_ρ is the γ-ρ coupling constant. It contains (1) a detailed derivation of the relation, (2) a rough estimate of possible corrections due to the extrapolation and higher vector-meson resonances, (3) the plausibility of the conjecture of Freund and Nandi, (4) some other ways of estimating R, and (5) the compatibility of the relation and a few other similar relations. The ratio $R(q^2)$, whose increase at $q^2=9$–$25\ \text{GeV}^2$ has recently been observed in the CEA and SLAC (SPEAR) e^+-e^- colliding-beam experiments, is expected not to increase much at higher energies.

I. INTRODUCTION

In 1966, Bjorken[1] predicted that the ratio $R(q^2)$ of the total cross section for $e^++e^-\to\gamma^*\to$ hadrons to that for $e^++e^-\to\gamma^*\to\mu^++\mu^-$ approaches a constant as q^2, the mass squared of the virtual photon γ^*, increases. Undoubtedly, the constant R, if it exists, is one of the most fundamental quantities in particle physics. It is given in the parton model[2] or in currently fashionable asymptotically free field theories[3] by[4]

$$R=\sum_i Q_i{}^2+\tfrac{1}{4}\sum_j Q_j{}^2, \qquad (1.1)$$

where Q_i and Q_j are the charges of spin-$\frac{1}{2}$ and -0 constituents of the electromagnetic current of hadrons, if there are such desirable (yet unseen) objects in nature. Since the surprisingly large total cross section for hadron production [e.g., $R(25\ \text{GeV}^2)=6\pm1.5$] was observed in the e^+-e^- colliding-beam experiments at the Cambridge Electron Accelerator (CEA),[5] many attempts[6] have been made to understand what is happening in hadron production by e^+-e^- colliding beams. More recently, the preliminary data reported from the Stanford Linear Accelerator Center (SPEAR)[7] have shown that $R(q^2)$ continues to rise as $(q^2)^{1/2}$ increases from 3 GeV to 4.8 GeV, which is consistent with the CEA data. Thus it is now widely accepted that the predicted scaling of the total cross section for $e^++e^-\to\gamma^*\to$ hadrons, i.e., $R(q^2)\cong$ constant for large q^2, if it exists, has not yet been reached at q^2 less than 25 GeV^2. What is the asymptotic value of R, if any? Where will the scaling set in, if anywhere? Is there such scaling at all? These are most challenging questions at the present stage of e^+-e^- colliding-beam physics. This paper is entirely devoted to the first question.

Obviously, the question could be answered immediately according to Eq. (1.1) if a particular model for hadron constituents were chosen *a priori*. The original Gell-Mann–Zweig model with a triplet of fractionally charged quarks, for example, predicts $R=(\frac{2}{3})^2+2(-\frac{1}{3})^2=\frac{2}{3}$. However, this would not give a final answer to the question since many different but equally attractive models such as the three-triplet fractionally charged quark model ($R=2$),[8] the three-quartet fractionally charged quark model ($R=3\frac{1}{3}$), the original Han-Nambu model with three triplets of integrally charged quarks ($R=4$),[9] the modified Han-Nambu model with three quartets of integrally charged quarks ($R=6$), etc., are proposed with different values of R.[6] Therefore, a definite prediction of R cannot be made without further experimental information or theoretical criterion. The PCAC (partially conserved axial-vector current) anomalous constant S is well known to serve as such a criterion. The low-energy theorem by Bell, Jackiw, and Adler[10] relates S to the $\pi^0\to2\gamma$ decay amplitude. The present data for the π^0 decay width ($\Gamma_{\pi^0\to2\gamma}=7.8\pm0.9$ eV) (Ref. 11) show that S is very close to $\frac{1}{2}$ ($S_{\text{exp}}=0.50\pm0.03$),[12] which seems to exclude the original Gell-Mann–Zweig quark model ($S=\frac{1}{6}$). However, it is also known that R is independent of S.[13] In fact, there exists more than one model with the same S but with different R. It should be noticed here that the $\pi^0\to2\gamma$ decay width teaches us even more: The classic application of vector-meson dominance by Gell-Mann, Sharp, and Wagner[14] is still consistent with the data, independent of the PCAC anomaly. It is very instructive to ask the following question on the hypothetical situation: How could we determine S if no information on the π^0 lifetime were available? We would probably estimate S by comparing two independent results, one by Bell and Jackiw and by Adler[10] and the other by Gell-Mann, Sharp, and Wagner.[14] Then, the result would be

$$S \cong 16\pi^2(f_\pi g_{\rho\omega\pi}/f_\rho f_\omega) = 0.61 \pm 0.09\,, \qquad (1.2)$$

where $g_{\rho\omega\pi}$, f_ρ, and f_ω are the ρ-ω-π, γ-ρ, and γ-ω coupling constants.[15] This exercise illuminates the way in which the present author, in previous papers,[16] has derived the approximate relation between R and f_ρ given by[15]

$$R \cong 16\pi^2/f_\rho{}^2 = 5.7 \pm 0.9 \qquad (1.3)$$

from the "enforced marriage" of the PCDC (partially conserved dilation current) anomaly[17] and vector-meson dominance. In Sec. II we shall present a more detailed derivation of this relation in order to exhibit all the assumptions and approximations involved. Possible corrections to the relation will be estimated in Sec. III. In Sec. IV, a physical interpretation and an underlying principle of the relation will be discussed, especially in connection with the conjecture, recently made by Freund and Nandi,[18] that the strengths of all interactions are inversely proportional to the number of fundamental fermions. In Sec. V we consider various other ways of predicting R and the mutual relationship between the relation (1.3) and some other similar relations.

II. DERIVATION OF THE RELATION

Wilson,[19] in 1969, demonstrated how the PCAC anomalous constant S, which appears in the Adler-Bell-Jackiw low-energy theorem for the π^0 decay amplitude, is completely determined by the short-distance behavior of the operator product of currents. Following him, Crewther[20] and, independently, Chanowitz and Ellis[21] have determined the coefficient of the PCDC anomalous term[17] existing in the trace identity for the product of the "improved" stress-energy tensor $\theta_{\mu\nu}$ (Ref. 22) and two electromagnetic currents J_μ of hadrons. We first review their results briefly. Let us start with the definitions of $\Delta_{\mu\nu}(q_1, q_2)$, $\Pi_{\mu\nu}(q)$, $\Delta(q^2)$, and $\Pi(q^2)$:

$$\Delta_{\mu\nu}(q_1, q_2) = \int dx\,dy\, e^{i(q_1\cdot x - Q\cdot y)} \times \langle 0 | T(J_\mu(x) J_\nu(0) \theta_\lambda^\lambda(y)) | 0\rangle\,, \qquad (2.1)$$

where $Q = q_1 - q_2$,

$$\Pi_{\mu\nu}(q) = i\int dx\, e^{iq\cdot x}\langle 0 | T(J_\mu(x)J_\nu(0))|0\rangle\,, \qquad (2.2)$$

$$\Delta_{\mu\nu}(q, q) = -(g_{\mu\nu}q^2 - q_\mu q_\nu)\Delta(q^2)\,, \qquad (2.3)$$

and

$$\Pi_{\mu\nu}(q) = -(g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2)\,. \qquad (2.4)$$

Assuming that the scale dimension of the electromagnetic current is canonical,[19,23] Chanowitz and Ellis[21] have derived the *anomalous* PCDC (or trace) Ward identity

$$\Delta(q^2) = -2q^2 \frac{\partial}{\partial q^2}\Pi(q^2) - \frac{R}{6\pi^2}\,. \qquad (2.5)$$

The strict definition of R is given by the Wilson expansion[19]

$$T(J_\mu(x)J_\nu(0)) \cong (R/12\pi^4)(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)(x^2 - i\epsilon)^{-2} + \cdots \quad \text{for } x \cong 0 \qquad (2.6)$$

or by[24]

$$R = \lim_{q^2\to\infty} R(q^2) \text{ and } R(q^2) = 12\pi\,\mathrm{Im}\Pi(q^2)\,. \qquad (2.7)$$

From Eq. (2.5) Crewther[20] and Chanowitz and Ellis[21] have derived the PCDC low-energy theorem[20,21]

$$\Delta(0) = -R/6\pi^2\,, \qquad (2.8)$$

which corresponds to the Bell-Jackiw-Adler low-energy theorem for the $\pi^0 \to 2\gamma$ decay derived from the anomalous PCAC Ward identity.[10]

It is now clear that an independent estimate of $\Delta(0)$ leads to the desired prediction of R. Let us suppose, for a moment, that there is only one vector meson, ρ (of mass m_ρ), dominating the isovector part of the electromagnetic current of hadrons. The function $\Delta_{\mu\nu}(q, q)$ has a double pole at $q^2 = m_\rho{}^2$:

$$\epsilon_1^\mu \epsilon_2^\nu \Delta_{\mu\nu}(q, q) \cong -\frac{1}{f_\rho{}^2}\left(\frac{m_\rho{}^2}{-q^2 + m_\rho{}^2}\right)^2 \times \langle \rho(\epsilon_1, q) | \theta_\lambda^\lambda(0) | \rho(\epsilon_2, q)\rangle \quad \text{for } q^2 \cong m_\rho{}^2\,, \qquad (2.9)$$

where ϵ_μ is the polarization vector of photons or ρ mesons and f_ρ is the familiar γ-ρ coupling constant defined by

$$\langle 0 | J_\mu(0) | \rho(\epsilon, q)\rangle = (m_\rho{}^2/f_\rho)\epsilon_\mu\,. \qquad (2.10)$$

Since ρ is an eigenstate of the Hamiltonian of hadrons $H = \int d^3x\, \theta_{00}(x)$,

$$\langle \rho(\epsilon_1, q) | \theta_{\kappa\lambda}(0) | \rho(\epsilon_2, q)\rangle = 2q_\kappa q_\lambda \epsilon_1^\mu \epsilon_2^\nu \times \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho{}^2}\right). \qquad (2.11)$$

Collecting Eqs. (2.3), (2.9), and (2.11), we find

$$\Delta(q^2) \cong -\frac{2}{f_\rho{}^2}\left(\frac{m_\rho{}^2}{-q^2 + m_\rho{}^2}\right)^2 \text{ for } q^2 \cong m_\rho{}^2\,. \qquad (2.12)$$

Extrapolating this result from $q^2 = m_\rho{}^2$ to $q^2 = 0$ yields

$$\Delta(0) \cong -(2/f_\rho{}^2) + (\text{isoscalar term}). \tag{2.13}$$

If the electromagnetic current of hadrons is a pure SU(3) octet,[25] the isoscalar contribution is $\frac{1}{3}$ times the isovector (ρ) contribution. Therefore,

$$\Delta(0) \cong -\tfrac{4}{3}(2/f_\rho{}^2). \tag{2.14}$$

Combining Eqs. (2.8) and (2.14), we finally obtain the desired relation (1.3).

It is important to notice that we have never applied vector-meson dominance directly to $\Pi_{\mu\nu}(q)$. If one were to do this, one would reach the contradictory result $R=0$ unless one adopts some new concepts such as an infinite series of vector-meson resonances proposed by Bramón, Etim, and Greco[26] or the "new duality" proposed by Sakurai.[27] If the scale dimension of θ_λ^λ is less than four,[19] $\Delta(q^2)$ is expected to vanish in the limit $q^2 \to \infty$ according to Weinberg's theorem in perturbation theory.[28,29] The decrease of $\Delta(q^2)$ for large q^2 is consistent not only with the presence of the anomalous term $-(R/6\pi^2)$ in the identity (2.5),[30] but with the phenomenological estimate (2.12) from vector-meson dominance. What we can learn from this lesson is that we may apply vector-meson dominance to "soft" operator products such as $T(J_\mu(x)J_\nu(0)\theta_\lambda^\lambda(y))$ consistently with scaling of "hard" operator products such as $T(J_\mu(x)J_\nu(0))$.

III. CORRECTIONS TO THE RELATION

Although little is known on higher vector-meson resonances at present, it seems worthwhile to discuss how the relation (1.3) will be affected by them. There are at least three independent experiments[31] reported for the evidence of another isovector vector meson ρ' whose mass is about 1600 MeV. The γ-ρ' coupling constant is estimated to be $f_{\rho'}{}^2/4\pi = 17\pm 5$, so that the ratio $f_\rho{}^2/f_{\rho'}{}^2$ is much smaller than unity:

$$f_\rho{}^2/f_{\rho'}{}^2 \cong \tfrac{1}{8}. \tag{3.1}$$

A naive estimate of the ρ' contribution seems to add a correction of about ten percent to the relation:

$$R \cong 16\pi^2(f_\rho{}^{-2} + f_{\rho'}{}^{-2}). \tag{3.2}$$

The problem of adding ρ', however, is not that simple because of possible interference between ρ and ρ'. We must also take account of the extraordinarily long distance for the extrapolation of $\Delta(q^2)$ from $q^2 = m_{\rho'}{}^2$ to $q^2=0$, which may totally suppress the contribution of such high mass states as ρ'. In our rather phenomenological approach of vector-meson dominance to scaling, it is difficult to evaluate these effects unambiguously. We consider the following three extreme cases:

(1) The strongest suppression of higher vector-meson resonances. Owing to the long distance for the extrapolation, the contribution of ρ', for example, would be much more suppressed than it is already by a factor of $f_\rho{}^2/f_{\rho'}{}^2$. In this case, the relation (1.3) would not be much changed.

(2) No interference and no suppression. The ρ' contribution would simply add to R:

$$R \cong 16\pi^2(f_\rho{}^{-2} + f_{\rho'}{}^{-2}) = 6.4 \pm 1.1. \tag{3.3}$$

(3) The maximal interference and no suppression. The interference term between ρ and ρ' would contribute to $\Delta_{\mu\nu}(q,q)$ as much as the diagonal (ρ-ρ and ρ'-ρ') terms. In this case,

$$R \cong 16\pi^2(f_\rho{}^{-1} + f_{\rho'}{}^{-1})^2 = \begin{cases} 10.8 \pm 2.4 & \text{for } f_\rho f_{\rho'} > 0 \\ 2.2 \pm 1.3 & \text{for } f_\rho f_{\rho'} < 0. \end{cases} \tag{3.4}$$

In the extremity (3), the effect of ρ' is most strongly enhanced, changing the value of R by as much as sixty percent. In any case, the corrections due to the extrapolation and to the higher-mass states can be substantial. The necessity of such corrections is also illustrated in solving the following puzzle pointed out by Orfanidis and Rittenberg.[32] If the isovector part of $\Delta(q^2)$, as a function of q^2, is exactly given by the ρ-dominance form (2.12) for arbitrary values of q^2, Eq. (2.5) can be taken as a differential equation of the form

$$\frac{2}{f_\rho{}^2}\left(\frac{m_\rho{}^2}{-s+m_\rho{}^2}\right)^2 = 2s\frac{d}{ds}\Pi^{(1)}(s) + \frac{R^{(1)}}{6\pi^2}. \tag{3.5}$$

This equation has the following solution:

$$\Pi^{(1)}(s) = -\frac{R^{(1)}}{12\pi^2}\ln s + \frac{1}{f_\rho{}^2}\left(\ln\frac{s}{s-m_\rho{}^2} - \frac{m_\rho{}^2}{s-m_\rho{}^2}\right). \tag{3.6}$$

By using the relation $R^{(1)} = 12\pi^2/f_\rho{}^2$ and the definition of $R(s)$ in (2.7), they have obtained the result

$$R^{(1)}(s) = (12\pi^2/f_\rho{}^2) \times [m_\rho{}^2\delta(s-m_\rho{}^2) + \theta(s-m_\rho{}^2)]. \tag{3.7}$$

The finite width of ρ, Γ_ρ, changes this result to a more realistic one:

$$R^{(1)}(s) = \frac{12\pi^2}{f_\rho{}^2}\left[\frac{1}{\pi}\frac{m_\rho{}^3\Gamma_\rho}{(s-m_\rho{}^2)^2 + m_\rho{}^2\Gamma_\rho{}^2} - \frac{1}{\pi}\frac{m_\rho{}^3\Gamma_\rho}{(s_0-m_\rho{}^2)^2 + m_\rho{}^2\Gamma_\rho{}^2} + \frac{1}{\pi}\tan^{-1}\left(\frac{s-m_\rho{}^2}{m_\rho\Gamma_\rho}\right) - \frac{1}{\pi}\tan^{-1}\left(\frac{s_0-m_\rho{}^2}{m_\rho\Gamma_\rho}\right)\right], \tag{3.8}$$

where s_0 is the threshold value of s for isovector hadronic states ($s_0=4m_\pi^2$). In the expression (3.8), they have found that the scaling $R^{(1)}(q^2) \to R \cong 12\pi^2/f_\rho^2$ sets in as soon as q^2 passes the peak of ρ [see the dashed curve (1) in Fig. 1]. Such behavior of $R^{(1)}(q^2)$ contradicts with the experimental data,[5,33] which show that, after passing through the ρ peak, $R(q^2)$ reaches a minimum at $q^2=1.0-1.4$ GeV^2 and then starts increasing towards the "Frascati plateau," where $R(q^2)=1-3$ for $q^2=2-9$ GeV^2. This puzzle can be solved, at least qualitatively, by either one of the following possibilities: (a) $\Delta(q^2)$ may differ substantially from the ρ-dominance form given in Eq. (2.12) when q^2 goes out of the ρ-peak region (i.e., for $|q^2-m_\rho^2|>m_\rho\Gamma_\rho$). In other words, if the extrapolation function $E(q^2)$ is defined by

$$\Delta^{(1)}(q^2)=-\frac{2}{f_\rho^2}\left(\frac{m_\rho^2}{-q^2+m_\rho^2}\right)^2 E(q^2), \quad \text{with } E(m_\rho^2)=1 \tag{3.9}$$

the approximation $E^{(1)}(q^2)\cong 1$ may not be good unless $q^2\cong m_\rho^2$. In this case, the relation (1.3) must be replaced by

$$R\cong 16\pi^2E(0)/f_\rho^2\,. \tag{3.10}$$

(b) The interference of ρ and ρ' may be almost maximal and the relative sign of f_ρ and $f_{\rho'}$ is negative. In this case,

$$\Delta^{(1)}(q^2)\cong -2\left(\frac{1}{f_\rho}\,\frac{m_\rho^2}{-q^2+m_\rho^2}+\frac{1}{f_{\rho'}}\,\frac{m_{\rho'}^2}{-q^2+m_{\rho'}^2}\right)^2. \tag{3.11}$$

The result for $R^{(1)}(s)$ is

$$\begin{aligned}R^{(1)}(s)=12\pi^2\Bigg[&\frac{m_\rho^2}{f_\rho^2}\delta(s-m_\rho^2)+\frac{1}{f_\rho^2}\left(1+\frac{2f_\rho}{f_{\rho'}}\,\frac{m_{\rho'}^2}{m_{\rho'}^2-m_\rho^2}\right)\theta(s-m_\rho^2)\\&+\frac{m_{\rho'}^2}{f_{\rho'}^2}\delta(s-m_{\rho'}^2)+\frac{1}{f_{\rho'}^2}\left(1-\frac{2f_{\rho'}}{f_\rho}\,\frac{m_\rho^2}{m_{\rho'}^2-m_\rho^2}\right)\theta(s-m_{\rho'}^2)\Bigg].\end{aligned} \tag{3.12}$$

For the finite widths of ρ and ρ', we can simply replace

$$\begin{pmatrix}\delta(s-m^2)\\ \theta(s-m^2)\end{pmatrix}$$

by

$$\begin{pmatrix}\dfrac{1}{\pi}\,\dfrac{m\Gamma}{(s-m^2)^2+m^2\Gamma^2}-\dfrac{1}{\pi}\,\dfrac{m\Gamma}{(s_0-m^2)^2+m^2\Gamma^2}\\[2ex] \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{s-m^2}{m\Gamma}\right)-\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{s_0-m^2}{m\Gamma}\right)\end{pmatrix} \tag{3.13}$$

in Eq. (3.12). The expression (3.12) with (3.13) shows that $R^{(1)}(s)$ has a minimum between $q^2=m_\rho^2$ and $q^2=m_{\rho'}^2$ if

$$\left|1+\frac{2f_\rho}{f_{\rho'}}\,\frac{m_{\rho'}^2}{m_{\rho'}^2-m_\rho^2}\right|\ll 1\,. \tag{3.14}$$

It also shows that $R^{(1)}(q^2)$ will stay at the constant value given by (3.4) soon after q^2 passes $m_{\rho'}^2$. The condition (3.14) is surprisingly well satisfied experimentally:

$$1+\frac{2f_\rho}{f_{\rho'}}\,\frac{m_{\rho'}^2}{m_{\rho'}^2-m_\rho^2}=0.06\pm0.18 \quad \text{if } f_\rho f_{\rho'}<0\,. \tag{3.15}$$

It is rather amazing that Eq. (3.12) with (3.13), without any arbitrary unknown parameter, can explain all the observed features of $R(q^2)$ for $0<q^2<9$ GeV^2 [see the dashed curve (2) in Fig. 1]. However, this formula still needs some modification since it cannot explain the observed increase of $R(q^2)$ starting around at $q^2=9$ GeV^2. In this picture of the maximal interference without any suppression of higher-mass states, R can be modified further by unknown vector-meson resonances which are heavier than ρ'. Suppose that there exists a series of vector mesons V_n ($n=0,1,2,\ldots,$ and the masses m_n) which couple to a photon with the coupling constant f_n; a mathematical model for $\Delta(s)$ has the form

$$\Delta(s)=-2\sum\left(\sum\frac{1}{f_n}\,\frac{m_n^2}{-s+m_n^2}\right)^2, \tag{3.16}$$

where the summation inside the parenthesis runs over vector mesons in a set in which they interfere maximally with each other and the summation outside runs over such independent sets. In this model, the relation (1.3) is replaced by

$$R=12\pi^2\sum\left(\sum f_n^{-1}\right)^2. \tag{3.17}$$

Because of the appearance of unknown parameters f_n and m_n (and Γ_n), one can fit the data arbitrarily. In fact, one needs to assume only one unknown vector meson whose mass is larger than 4 GeV in order to make this model consistent with all the experimental data available for q^2 less than 25 GeV^2. This can be demonstrated by the following example

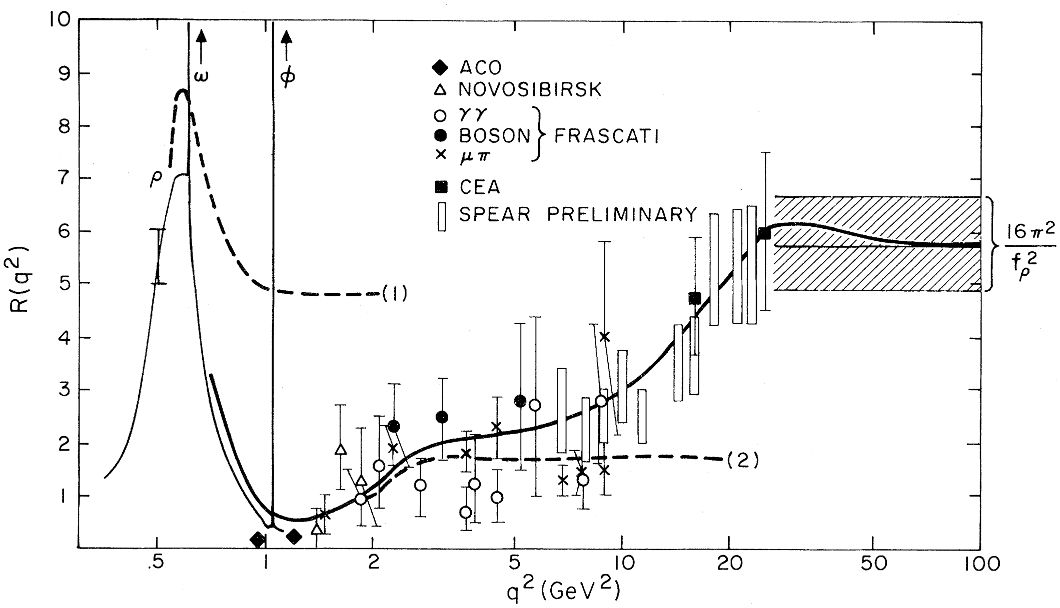

FIG. 1. Typical solutions of the differential equation (2.5) with the "strong" vector-meson dominance applied to $\Delta(q^2)$. The dashed curves (1) and (2) are given by Eqs. (3.8) and (3.12) with (3.13), respectively. The solid curve given by Eq. (3.18) is a naive mathematical example which fits all the available data, but which should not be taken seriously. The data are taken from Refs. 5, 7, and 33. The shaded area represents the range of R predicted by the relation (1.3).

(see the solid curve in Fig. 1):

$$\Delta^{(1)}(s) = -2\left[\left(\frac{1}{f_\rho}\,\frac{m_\rho^2}{-s+m_\rho^2}+\frac{1}{f_{\rho'}}\,\frac{m_{\rho'}^2}{-s+m_{\rho'}^2}\right)^2+\frac{1}{f_V^2}\left(\frac{m_V^2}{-s+m_V^2}\right)^2\right], \tag{3.18}$$

with $m_V \cong 5$ GeV, $\Gamma_V \cong 3$ GeV, $f_V^2 \cong f_\rho^2$, and $f_\rho f_{\rho'} < 0$.

In the model of maximal interference and no suppression, case (3) or (b), we thus have less confidence in the original relation (1.3). If this is the case, however, how can we understand many successful applications of ρ-, ω-, and ϕ-meson dominance such as the one shown in Eq. (1.2)? It is hard to imagine that all the agreements between the results of vector-meson dominance and the experimental data were just accidental. In the following sections, we shall simply assume that neither the extrapolation correction nor the higher-mass correction is important. In concluding this section, it should be pointed out that there is another (though not compelling) way to estimate a systematic error possibly involved in the relation (1.3). The magnitude of the error in the relation (1.2) can be found most practically by comparing the predicted value of S with the experimental value:

$$\frac{S_{\rm exp}}{S_{\rm VMD}} = \frac{0.50 \pm 0.03}{0.61 \pm 0.09} = 0.82 \pm 0.17\,. \tag{3.19}$$

Because of the similarity of the relations (1.2) and (1.3), it may be that the error in (1.3) can be corrected by

$$R_{\rm exp} = \frac{S_{\rm exp}}{S_{\rm VMD}} \times R_{\rm VMD} = 4.7 \pm 1.6\,. \tag{3.20}$$

IV. CONJECTURE OF FREUND AND NANDI

Is there any simple principle underlying the relation (1.3)? Freund and Nandi[18] have recently obtained the relation between the PCAC anomalous constant S and the γ-ρ coupling constant,

$$S \cong \frac{4\sqrt{2}\,\pi^2}{3 f_\rho^2}, \tag{4.1}$$

based on vector-meson dominance (1.2), SU(6)$_W$ symmetry, and the "elusive" Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation.[34] They have suggested that the typical strong coupling constant $f_{\rho\pi\pi}$, which equals f_ρ in vector-meson dominance, is thus inversely proportional to the number of quarks, N, as is shown in Eq. (4.1). Their principle is further illuminated in the relation (1.3) or, equivalently,

$$f_\rho^2 \cong \frac{16\pi^2}{R} = \frac{16\pi^2}{\langle Q^2\rangle N}, \tag{4.2}$$

where $\langle Q^2\rangle$ is the average value of the quark charge squared. Freund and Nandi have further conjectured that the strengths of all interactions including electromagnetic and gravitational ones are inversely proportional to the number of fundamental fermions. The plausibility of their conjecture has been demonstrated in a field-theoretical model (or in the N^{-1} expansion) where diagrams with a chain of fermion loops are assumed to be dominant. In such a model,[35] if a vector field V couples to a conserved current $J_\mu^{(V)}$, a renormalized coupling constant f is related to its unrenormalized one f_0 by

$$f^2 = \frac{f_0^2}{1+f_0^2\Pi^{(V)}(m_V^2)}, \tag{4.3}$$

where $\Pi^{(V)}(q^2)$, which is unrenormalized, is defined for $J_\mu^{(V)}$ in the same way as $\Pi(q^2)$ for the electromagnetic current J_μ in Eqs. (2.2) and (2.4). If $f_0^2\Pi^{(V)}(m_V^2) \gg 1$, this relation approximately yields[36]

$$f^2 \cong \frac{1}{\Pi^{(V)}(m_V^2)}. \tag{4.4}$$

Since $\Pi^{(V)}(m_V^2)$ is proportional to the number of independent fermions, Eq. (4.4) certainly supports their conjecture. It should be emphasized here that the relations (4.2) and (4.4) have remarkable similarity. In fact, the former becomes identical to the latter if

$$\Pi^{(\rho)}(m_\rho^2) \cong \frac{1}{\pi}\,\mathrm{Im}\Pi^{(1)}(\infty). \tag{4.5}$$

There are many other models whose results satisfy the Freund-Nandi principle. The best example is massless quantum electrodynamics in two dimensions[37,38] in which the dimensionless coupling constant e renormalized at $q^2=0$ is given exactly by

$$e^2 = \frac{\pi}{N}. \tag{4.6}$$

Similar N^{-1} dependence of renormalized coupling constants can also be found in the analysis of various field-theoretical models in less than four dimensions.[39] Although all the N^{-1} dependence of these coupling constants is subject to the validity of the N^{-1} expansion (or approximation) except for that in two-dimensional massless quantum electrodynamics (where no approximation is necessary), the conjecture of Freund and Nandi seems to be very plausible, at least for hadron dynamics.

V. WHAT IS R?

Is there any way to estimate R without using vector-meson dominance? Based on the PCAC anomaly and ϵ-meson dominance for the normal part of the trace of the stress-energy tensor θ_λ^λ, Crewther[20] and, independently, Chanowitz and Ellis[21] have predicted the coupling constant $g_{\epsilon\gamma\gamma}$ defined by $\mathcal{L}_{\epsilon\gamma\gamma}^{\mathrm{eff}} = -\frac{1}{2}e^2 g_{\epsilon\gamma\gamma}F_{\mu\nu}F^{\mu\nu}$ to be

$$g_{\epsilon\gamma\gamma} \cong R/12\pi^2 f_\epsilon, \tag{5.1}$$

where f_ϵ is defined by $\langle 0|\theta_\lambda^\lambda(0)|\epsilon\rangle = m_\epsilon^2 f_\epsilon$. The $\epsilon\pi\pi$ coupling constant was predicted earlier by Crewther[40] and Ellis[41] to be

$$g_{\epsilon\pi\pi} \cong m_\epsilon^2/f_\epsilon. \tag{5.2}$$

These coupling constants are related to the decay widths of ϵ:

$$\begin{aligned}\Gamma_{\epsilon\to\gamma\gamma} &= \pi\alpha^2 g_{\epsilon\gamma\gamma}^2 m_\epsilon^3\\ &\cong (\alpha^2/144\pi^3)(m_\epsilon/f_\epsilon)^2 m_\epsilon R^2\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\Gamma_\epsilon &\cong \Gamma_{\epsilon\to\pi^+\pi^-} + \Gamma_{\epsilon\to\pi^0\pi^0}\\ &= (3g_{\epsilon\pi\pi}^2/32\pi m_\epsilon)(1-4m_\pi^2/m_\epsilon^2)^{1/2}\\ &\cong (3/32\pi)(m_\epsilon/f_\epsilon)^2 m_\epsilon(1-4m_\pi^2/m_\epsilon^2)^{1/2}.\end{aligned} \tag{5.4}$$

Therefore,

$$R \cong (3\pi/\alpha)(3\Gamma_{\epsilon\to\gamma\gamma}/2\Gamma_\epsilon)^{1/2}(1-4m_\pi^2/m_\epsilon^2)^{1/4}. \tag{5.5}$$

The first experimental study of the reaction $\gamma+\gamma\to\pi^++\pi^-$ has recently been carried out by Orito, Ferrer, Paoluzi, and Santonico[42] in the region of $\epsilon(700)$ through the two-photon process $e^++e\to e^++e^-+\pi^++\pi^-$.[43] Based on the two observed events, they have concluded that

$$\Gamma_{\epsilon\to\gamma\gamma} = (9.6^{+13.3}_{-8.0})\ \mathrm{keV}. \tag{5.6}$$

From (5.5) and (5.6), they have finally estimated R to be[42]

$$R = 5.8^{+3.2}_{-3.5} \text{ for } m_\epsilon = 660 \text{ MeV and } \Gamma_\epsilon = 640 \text{ MeV}. \tag{5.7}$$

There is still another way of estimating R. Sarkar[44] and many others[45] have estimated the product of the coupling constants $g_{\epsilon\gamma\gamma}$ and $g_{\epsilon\pi\pi}$ from finite-energy sum rules for the amplitude of forward pion Compton scattering. Although they have obtained different results depending on how they saturate the sum rules, all of their predictions lie in the range of

$$1.4 \lesssim g_{\epsilon\gamma\gamma}g_{\epsilon\pi\pi} \lesssim 2.0. \tag{5.8}$$

On the other hand, collecting Eqs. (5.1), (5.2), and (5.4) leads to

$$R \cong (9\pi/8)(g_{\epsilon\gamma\gamma}g_{\epsilon\pi\pi})(m_\epsilon/\Gamma_\epsilon)(1-4m_\pi^2/m_\epsilon^2)^{1/2}. \tag{5.9}$$

Combining (5.8) and (5.9), we obtain

$$R = 6.5 \pm 1.2 \quad \text{for } m_\epsilon = 700 \text{ MeV and } \Gamma_\epsilon = 600 \text{ MeV}. \tag{5.10}$$

It is remarkable that all of these estimates, (1.3), (5.7), and (5.10), agree with each other within their errors. Therefore, insofar as the PCDC anomaly on which these estimates are based exists, the limit R is expected to lie between 4 and 8. If this is the case, the expected value for R is in strong contradiction with all the predictions in the known fractionally charged quark models (where $R = \frac{2}{3}$, 2, or $3\frac{1}{3}$).

Let us next discuss the compatibility of the relation (1.3) and some other relations, all of which have been derived more or less from combinations of vector-meson dominance and the algebra of currents. Such relations are the Freund-Nandi relation (4.1),[18] the KSRF relation[34]

$$f_\pi^2 \cong m_\rho^2/2f_\rho^2, \tag{5.11}$$

and the Suura-Young relation[46]

$$f_\pi^2 \cong m_\rho^2/8\pi^2. \tag{5.12}$$

The combination of Eqs. (1.3) and (4.1) leads to[47]

$$R \cong 6\sqrt{2}\, S$$
$$= 3\sqrt{2} \cong 4 \times 1.06 \quad \text{for } S = \tfrac{1}{2}, \tag{5.13}$$

while that of Eqs. (1.3), (5.11), and (5.12) leads to

$$R \cong 4. \tag{5.14}$$

Strangely, the particular value of R is chosen preferably by these combinations in spite of the fact that no particular quark model has been assumed in deriving these relations. It should also be noticed that, when Eq. (5.13) and Crewther's relation[20]

$$4S = KR \tag{5.15}$$

are combined, the constant K would be completely fixed:

$$K \cong \tfrac{1}{3}\sqrt{2} \cong \tfrac{1}{2} \times 0.94. \tag{5.16}$$

This can be tested by determining K with Bjorken's sum rule[48] for polarized deep-inelastic electroproduction.

In conclusion, from a simple observation that the relation (1.3) is already in good agreement with the CEA and SPEAR data at the highest energies available at present (see Fig. 1), we expect that the ratio $R(q^2)$ will not increase much at higher energies and that the predicted scaling in the total cross section for $e^+ + e^- \to$ hadrons will be seen in future SLAC and DESY colliding-beam experiments.

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$$\partial_\lambda A_\pi^\lambda = f_\pi m_\pi^2 \phi_\pi + (e^2 S/16\pi^2)\epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta},$$

which defines both S and the pion decay constant f_π ($\cong$ 93 MeV).

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[13] $S = \sum_i Q_i^2 (I_3)_i$, where $(I_3)_i$ is the third isospin component of quarks (Ref. 10). Therefore, R and S are related each other in the sense that both of them are

proportional to the number of independent multiplets of quarks. They, however, still vary independently when the charge and isospin structures of individual quark multiplets change. If the electromagnetic current is a pure SU(3) octet, R and S should obey Crewther's relation $4S=KR$, where K is the constant defined as a coefficient of the isovector axial-vector current in the equal-time commutation relation of space components of the electromagnetic currents.

$$[J_i(x),\ J_j(0)]\delta(x_0)=i\tfrac{2}{3}K\delta(x)\epsilon_{ijk}A_k^{(1)}(0)+\dots\ .$$

This point will be discussed later in Sec. V.

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$$\Gamma_{V\to l^+l^-}=(e^4/12\pi f_V^2)m_V(1+2m_l^2/m_V^2)\times(1-4m_l^2/m_V^2)^{1/2}\ \text{for}$$

for the decay width of the vector meson V into a lepton pair l^+l^- and

$$\Gamma_{\omega\to\pi^0\gamma}=(e^2g_{\rho\omega\pi}{}^2m_\omega^2/6\pi f_\rho^2)m_\omega(1-m_\pi^2/m_\omega^2)^3$$

for that of ω into π^0 and γ, and the present particle data (Ref. 11),

$$\Gamma_{\rho\to e^+e^-}=(146\pm10)\times(0.0043\pm0.0005)\times10^{-2}\ \text{MeV},$$

$$\Gamma_{\omega\to e^+e^-}=(9.8\pm0.5)\times(0.0076\pm0.0017)\times10^{-2}\ \text{MeV},$$

$$\Gamma_{\phi\to e^+e^-}=(4.2\pm0.2)\times(0.032\pm0.003)\times10^{-2}\ \text{MeV},$$

and

$$\Gamma_{\omega\to\pi^0\gamma}=(9.8\pm0.5)\times(9.1\pm0.5)\times10^{-2}\ \text{MeV}.$$

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$$-(g_{\mu\nu}q^2-q_\mu q_\nu)\mathrm{Im}\Pi(q^2)=\tfrac{1}{2}\int dx\,e^{iqx}\langle 0|J_\mu(x)J_\nu(0)|0\rangle=\pi\sum_n(2\pi)^3\delta(q-p_n)\langle 0|J_\mu(0)|n\rangle\langle n|J_\nu(0)|0\rangle\,.$$

The total cross sections for $e^++e^-\to$ hadrons and for $e^++e^-\to\mu^++\mu^-$ are given by $\sigma(e^++e^-\to\text{hadrons})=16\pi^2\alpha^2\mathrm{Im}\Pi(q^2)/q^2$ and $\sigma(e^++e^-\to\mu^++\mu^-)\cong4\pi\alpha^2/3q^2$, respectively. Therefore, $R(q^2)=\sigma(e^++e^-\to\text{hadrons})/\sigma(e^++e^-\to\mu^++\mu^-)=12\pi\mathrm{Im}\Pi(q^2)$, which agrees with the formula (2.7).

[25]This assumes the usual Gell-Mann–Nishijima rule $Q=I_3+\frac{1}{2}Y$. If this is not the case, the isoscalar contribution is larger than $\frac{1}{3}$ times the isovector one. Then, the relation (1.3) becomes an inequality: $R\geq16\pi^2/f_\rho^2$. In a model of the Han-Nambu type with $\mathrm{Tr}Q^2=\frac{8}{3}\mathrm{Tr}(I_3)^2$, the relation (1.3) is replaced by $R=32\pi^2/f_\rho^2=11.4\pm1.8$.

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$$-(g_{\mu\nu}q^2-q_\mu q_\nu)\Pi^{(\rho,1)}(q^2)=i\int dx\,e^{iqx}\langle 0|T(J_\mu^{(\rho)}(x)J_\nu(0))|0\rangle\,.$$

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