Hamiltonian formulation of Wilson's lattice gauge theories

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Wilson's lattice gauge model is presented as a canonical Hamiltonian theory. The structure of the model is reduced to the interactions of an infinite collection of coupled rigid rotators. The gauge-invariant configuration space consists of a collection of strings with quarks at their ends. The strings are lines of non-Abelian electric flux. In the strong-coupling limit the dynamics is best described in terms of these strings. Quark confinement is a result of the inability to break a string without producing a pair.

I. INTRODUCTION

The quark model has systematized a very large amount of information concerning the hadron spectrum. However, free isolated quarks do not appear to exist. In order to confine quarks into baryons and mesons, one is then led to suppose that the field-theoretic coupling between quarks becomes strong at large distances. This explanation is, however, somewhat perplexing because the forces between quarks at small distances appear to be weak. Such behavior can in principle be found in renormalizable field theories in which effective coupling constants can change from one size scale to the next. Clearly, in order to understand the successes of the quarkless quark model we need a theory in which weak short-distance forces give rise to strong long-range forces. The only theory in which this behavior appears possible is one containing non-Abelian (Yang-Mills) gauge fields.

It is instructive to recall why this behavior does not occur in conventional formulations of Abelian vector-gluon theories (electrodynamics, for example). Consider a static free charge of magnitude e inserted into the vacuum of quantum electrodynamics. As is well known, the electrodynamic vacuum is an ordinary dielectric,[1] so the free charge creates a polarization charge of *opposite* sign. The polarization charge is distributed in the vicinity of the free charge. Therefore, the total charge contained within a sphere of radius r is $eZ(r)$, where $Z(r)$ is a fraction less than 1 which decreases as r increases. The factor $Z(r)$ causes the intensity of electromagnetic interactions to be dependent on the distance scales involved.[2] In fact, if we are only interested in long-wavelength phenomena in electrodynamics, we can ignore all the short-distance fluctuations of the theory and replace the bare electric charge e by the screened or renormalized charge. More precisely, long-wavelength phenomena are insensitive to a cutoff at length λ if the bare charge is replaced by $eZ(\lambda)$. Since $Z(\lambda)$ decreases as λ increases, this theory has just the reverse behavior of what we want.

In theories with Yang-Mills fields the interaction between a pair of static charges is also governed by an effective coupling constant $gZ(r)$. As in electrodynamics, a cutoff version of Yang-Mills theory must replace g by $gZ(\lambda)$. This time it is found, however, that $Z(\lambda)$ can be an increasing function of λ.[3] The implication is that the effective couplings between the low-momentum modes of the theory may become very strong although the shorter-distance behavior may not involve strong coupling.

In this paper we shall be interested in the large-distance properties of a non-Abelian theory assuming that the effective coupling $g(\lambda)$ is sufficiently large to use Wilson's strong-coupling methods.[4] An ultraviolet cutoff is introduced into the theory through a spatial lattice. This construction destroys most of the space-time symmetries of relativistic field theories. For this reason the theory discussed here is not a realistic Yang-Mills theory. However, following Wilson,[4] we are mainly interested in determining the special effects of exact gauge invariance in strongly coupled gauge theories. As a result of this study, we find that quarks can be confined in locally gauge-invariant theories. The confining mechanism is the appearance of one-dimensional electric flux tubes which must link separated quarks.[5] The appropriate description of the strongly coupled limit consists of a theory of interacting, propagating strings.

This paper is organized into eight sections. In Sec. II we describe the field theory of fermions on a spatial lattice. The theory has global but not local non-Abelian symmetry. In Sec. III we develop the principle of local gauge invariance. In the weak-coupling limit the theory reduces to a standard theory when the spatial cutoff is taken to zero. In Sec. IV we develop the canonical formalism for an SU(2) gauge theory. The fundamental gauge-field degree of freedom which links adjacent lattice points is mathematically equivalent to a rigid rotator. The theory of rigid rotators is a helpful guide in discussing the gauge theory. In Sec. V the physical space of states of the strong-coupling theory is constructed and is described in terms of stringlike excitations of a gauge-invariant vacuum. The strings are the non-Abelian analogs of electric flux lines. In Sec. VI we include the dynamics of the gauge field into the lattice model and obtain a Hamiltonian for a discrete theory of fermions and gauge fields. In the weak-coupling continuum limit the usual Yang-Mills theory with fermions is retrieved.[6] In Sec. VII we consider the dynamics of the stringlike excitations and calculate the energy in various configurations. In particular, we show that the energy of a well-separated quark pair increases linearly with the distance between them.[7] Then we develop a perturbation theory around the strong-coupling limit. It expresses the solution of the theory as a series expansion in inverse powers of the coupling constant. Physically, the higher-order effects cause quantum fluctuations in the string configurations. If these fluctuations grow too large, they could invalidate the quark binding mechanism. In the final section we discuss and summarize several of our conclusions.

II. FERMION FIELDS ON A LATTICE

We begin by formulating the Dirac equation on a spatial cubic lattice. An arbitrary point on the lattice is denoted by a triplet of integers $\vec{r} = (r_x, r_y, r_z)$. The unit lattice vectors are denoted $\hat{m}_x$, $\hat{m}_y$, $\hat{m}_z$, $\hat{m}_{-x}$, $\hat{m}_{-y}$, and $\hat{m}_{-z}$ pointing along the x, y, z, $-x$, $-y$, and $-z$ axes, respectively (Fig. 1). By definition $\hat{m}_{-x} = -\hat{m}_x$, etc. We use the convention that summations over the lattice vectors include the six directions. The spaces between neighboring lattice points *(links)* are denoted by a position and a lattice vector $(\vec{r}, \hat{m}_i)$. The same interval can be denoted $(\vec{r} + \hat{m}_i, -\hat{m}_i)$. It will prove convenient to label lattice sites as odd or even by the following prescription: A site r is called even (odd) if $(-1)^{r_x+r_y+r_z} \equiv (-1)^r$ is even (odd). On each lattice site we define a *two*-component spinor $\psi(r)$. A discrete Hamiltonian can easily be constructed such that it yields the conventional Dirac theory in the continuum limit. It reads

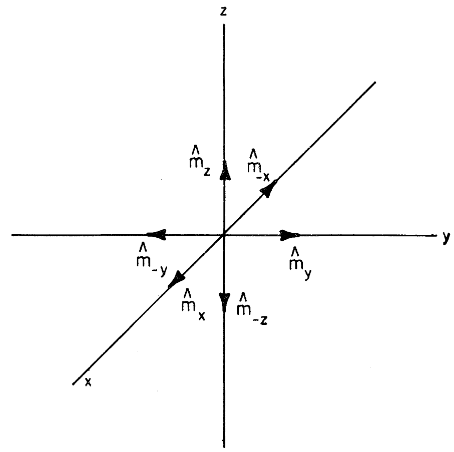

FIG. 1. Definition of unit lattice vectors.

$$H = a^{-1} \sum_{r,n} \psi^\dagger(r) \frac{\vec{\sigma}\cdot\vec{n}}{i} \psi(r+n) + m_0 \sum_r (-1)^r \psi^\dagger(r)\psi(r), \tag{2.1}$$

where a is the lattice spacing. If we postulate the canonical anticommutation relations

$$\{\psi_\alpha(r), \psi_\beta^\dagger(r')\} = \delta_{\alpha\beta}\,\delta_{r,r'}, \quad \text{etc.}, \tag{2.2}$$

the equation of motion of the spinor field becomes

$$i\dot{\psi}(r) = [\psi(r), H] = a^{-1} \sum_n \frac{\vec{\sigma}\cdot\vec{n}}{i} \psi(r+n) + m_0(-1)^r \psi(r). \tag{2.3a}$$

This equation may be rewritten

$$i\dot{\psi}(r) = \frac{1}{2a} \sum_n \frac{\vec{\sigma}\cdot\vec{n}}{i} [\psi(r+n) - \psi(r-n)] + m_0(-1)^r \psi(r). \tag{2.3b}$$

Consider the continuity properties of the solutions to this equation as $a \to 0$. The finite-energy solutions ($\dot{\psi}$ finite) require that

$$\psi(r+n) - \psi(r-n) \sim a \tag{2.4}$$

as $a \to 0$. However, $\psi(r) - \psi(r+m)$ is not constrained by the discrete Dirac equation. Thus, in order to define fields with finite derivatives, we must introduce two separate fields for even and odd lattice sites. In the continuum limit we represent these two fields as upper and lower components of a four-component Dirac spinor, the upper (lower) components being the fields on the even (odd) lattice points. The discrete Dirac equation may then be approximated,

$$i\dot{\psi}_{\text{upper}} = -i\vec{\sigma}\cdot\vec{\nabla}\psi_{\text{lower}} + m_0\psi_{\text{upper}}\,,$$
$$i\dot{\psi}_{\text{lower}} = -i\vec{\sigma}\cdot\vec{\nabla}\psi_{\text{upper}} - m_0\psi_{\text{lower}}\,, \tag{2.5}$$

which one can identify as the conventional continuum Dirac equation.

For physical applications ψ will have both "color"[8] and ordinary SU(3) indices. The role of color is to provide a locally conserved quantum number whose vanishing in the finite-energy physical spectrum of the theory implies the absence of triality. The quark-confining mechanism then becomes one of color confinement. To carry this scheme out, the color degrees of freedom [not the ordinary SU(3)] will be coupled to colored Yang-Mills fields. For illustrative purposes we will ignore ordinary SU(3) and replace the color group by an SU(2) group.

The transformation of the fermion field under *global* gauge transformations reads

$$\hat{\psi}(r) = e^{i\vec{\tau}\cdot\vec{\omega}/2}\psi(r) \equiv V\psi(r)\,. \tag{2.6}$$

The Hamiltonian Eq. (2.1) is clearly invariant under such transformations. Since the global transformation rotates the fermion field identically over all points of space, this global invariance of the theory still permits color to be compared at separated points. This freedom will be lost when local gauge invariance is built into the theory.[6]

III. PRINCIPLE OF LOCAL GAUGE INVARIANCE

A local gauge transformation on the fermion field is written

$$\hat{\psi}(r) = e^{i\vec{\tau}\cdot\vec{\omega}(r)/2}\psi(r) \equiv V(r)\,\psi(r)\,, \tag{3.1}$$

where $\vec{\omega}(r)$ can now depend on the position r. In general the full gauge group consists of transformations which depend upon time as well as position. The canonical formalism is significantly more difficult when the full time-dependent gauge transformations are considered. We will therefore only discuss the invariance for spatially dependent gauge functions. This will then allow us to set the time component of the vector potential to zero when the gauge field enters the theory. There is, in fact, no loss of generality in this procedure.[9]

The Hamiltonian in Eq. (2.1) is not locally gauge-invariant since it involves the product of fermion fields at separated points. To compensate this lack of local invariance, we introduce a gauge field. This is done as follows.[4] On each link (r,m) we place a gauge field $\vec{B}(r,m)$ and a unitary transformation,

$$U_{1/2}(r,m) = \exp[i\tfrac{1}{2}\vec{\tau}\cdot\vec{B}(r,m)]\,. \tag{3.2}$$

The subscript $\frac{1}{2}$ on U denotes the fundamental representation of the SU(2) color group. We make the convention $B(r,m) = -B(r+m,\,-m)$. We note that the Fermi field is associated with the lattice points themselves, but the gauge fields are associated with *links* between points. This is so because the gauge field *transports* color information between lattice points. The two indices of the matrix

$$U_{1/2}(r,m)^i{}_j$$

are identified with the two ends of the link m. The upper (lower) index is associated with the beginning (end) of the link as depicted in Fig. 2. The gauge transformation acts on $U(r,m)$ according to,

$$U'(r,m) = V(r)\,U(r,m)\,V^{-1}(r+m)\,. \tag{3.3}$$

Since in general $V(r)$ and $V(r+m)$ are different, our gauge-invariant equations will require invariance under right multiplication and left multiplication separately. The matrices $U(r,m)$ can now be used to convert gauge-noninvariant products of spatially separated fields to gauge-invariant products. For example, an operator such as,

$$\psi^\dagger(r)\,\psi(r+m)$$

transforms under gauge transformations to

$$\psi^\dagger(r)\,V^{-1}(r)\,V(r+m)\,\psi(r+m)\,.$$

However, the operator

$$\psi^\dagger(r)\,U(r,m)\,\psi(r+m) \tag{3.4}$$

is gauge-invariant. The gauge transformations $V(r)$ and $V(r+m)$ acting on the ends of the link [indices of $U(r,m)$] undo the gauge transformations of the fermion fields. We can now apply this procedure to the Hamiltonian to render it gauge-invariant:

$$H = a^{-1}\sum_{r,m}\psi^\dagger(r)\frac{\vec{\sigma}\cdot\vec{m}}{i}\,U(r,m)\,\psi(r+m) + m_0\sum(-1)^r\psi^\dagger(r)\,\psi(r)\,. \tag{3.5}$$

Let us now consider the continuum limit ($a\to 0$) of this Hamiltonian. To do this we write

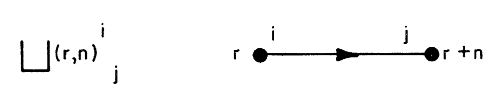

FIG. 2. The gauge field U is defined on the links. The two indices of U refer to the ends of the links.

$$H=a^3\sum \psi'^\dagger(r)\,\frac{\vec{\sigma}\cdot\vec{m}}{ia}\,e^{i\vec{\tau}\cdot\vec{B}(r,m)/2}(1-e^{-i\vec{\tau}\cdot\vec{B}(r,m)/2})\,\psi'(r+m)$$
$$+a^3\sum \psi'^\dagger(r)\,\frac{\vec{\sigma}\cdot\vec{m}}{ia}\,\psi'(r+m)+m_0a^3\sum(-1)^r\psi'^\dagger(r)\psi'(r)\,, \tag{3.6}$$

where $\psi'(r)=a^{-3/2}\psi(r)$. In order to take the continuum limit smoothly, it is essential to assume that the operator $1-\exp[-i\,\frac{1}{2}\vec{\tau}\cdot\vec{B}(r,m)]$ tends to zero as the lattice spacing goes to zero. That is, a small lattice spacing is only compatible with dynamics in which the magnitude of the operator $\vec{B}(r,m)$ is small, $\sim a$. Then the exponential in Eq. (3.6) can be expanded and the Hamiltonian becomes

$$H=a^3\sum_{r,m}\psi'^\dagger(r)\left(\frac{\vec{\sigma}\cdot\vec{m}}{ia}\right)[i\,\tfrac{1}{2}\vec{\tau}\cdot\vec{B}(r,m)]\psi'(r+m)$$
$$+a^3\sum\psi'^\dagger(r)\,\frac{\vec{\sigma}\cdot\vec{m}}{ia}\,\psi'(r+m)$$
$$+m_0a^3\sum(-1)^r\psi'^\dagger(r)\,\psi'(r)\,. \tag{3.7}$$

In the continuum limit the quantity

$$\psi'^\dagger(r)\,\frac{\vec{\sigma}\cdot\vec{m}}{ia}\,\psi'(r+m)$$

becomes the kinetic-energy term

$$\bar{\psi}'(r)\,\gamma_i\,\partial_i\,\psi'(r)\,.$$

Then the full Hamiltonian becomes

$$H=\int_r\Big[\bar{\psi}'(r)\,i\gamma_i\partial_i\,\psi'(r)-\bar{\psi}'(r)\,\gamma_i\,\frac{\vec{B}_i(r)}{a}\cdot\frac{\vec{\tau}}{2}\,\psi'(r)$$
$$+m_0\,\bar{\psi}'(r)\,\psi'(r)\Big]\,. \tag{3.8}$$

This is the usual Yang-Mills gauge theory with fermions if we identify

$$B_i(r)=agA_i(r)\,,$$

where $A_i(r)$ is the vector potential and g is the coupling constant. At this stage the gauge field does not enter the dynamics as a bona fide degree of freedom. This limitation will be remedied in Sec. VI.

IV. THE RIGID ROTATOR

In this section we shall consider the nature of the gauge-field degree of freedom on a single link. In ordinary scalar-particle field theory, the field degree of freedom at a point is an anharmonic oscillator. The derivative terms in the Hamiltonian couple adjacent oscillators. In Yang-Mills theory the local degree of freedom $U(r,m)$ is an element of a group. In our example the group is O(3). Since this is a non-Abelian compact group, the topology of the configuration space at a link is closed and nontrivial. This leads to complexities in the canonical formalism. Fortunately, the configuration space of a well-known mechanical system, the rigid rotator, is identical to these degrees of freedom.

We shall first review the kinematics of the quantum rigid rotator. A configuration of the rigid rotator is specified by a rotation from the space-fixed to a set of body-fixed axes.[10] The rotation may be represented in the form

$$U_j=\exp(i\,\vec{T}_j\cdot\vec{\Omega})\,,$$

where $T_{j\alpha}$ $(\alpha=1,2,3)$ are representation matrices of the generators of the rotation group for angular momentum j. In the spinor representation the elements of the matrix $\exp(i\,\vec{T}_{1/2}\cdot\vec{\Omega})$ are Cayley-Klein parameters. We introduce a notation for matrices in which lower (upper) components refer to space (body) axes. For example, if V_i are components of a vector in the space-fixed frame, then $(U_1)^l{}_iV_i=V^l$ are the corresponding body components.

The relationship between body and space axes for the rigid rotator is the same as the relation between the indices on the two ends of a link in the Yang-Mills theory. The action of a rotation of space axes on U is given by left multiplication by the appropriate rotation matrix, V, say. Similarly, a rotation of the body axes relative to the body is given by right multiplication. The requirement of local gauge invariance in Yang-Mills theory translates into invariance under *separate* space and body rotations. This invariance requires that the rotator be spherical, since only the spherical rotator has invariance under rotations of the body axes.

The angular velocity vector of the rigid rotator is defined as the time derivative of $\vec{\Omega}$,

$$\vec{\omega}=\frac{d}{dt}\vec{\Omega}\,.$$

The angular momentum $\vec{J}$ (generator of space rotations) of the rotator is given by $I\vec{\omega}$, where I is the moment of inertia of the rigid body. The Hamiltonian reads

$$H=J^2/(2I)=\tfrac{1}{2}I\omega^2=\tfrac{1}{2}I\,\dot{\Omega}^2\,. \tag{4.1}$$

Since the moment of inertia tensor is diagonal, the Hamiltonian is invariant under individual body and space rotations as required by gauge invariance. In fact, if $\vec{\mathcal{J}}=U_1\vec{J}$ is the generator of *body* rotations, the Hamiltonian may be rewritten

$$H=\mathcal{J}^2/(2I)\,. \tag{4.2}$$

The fundamental canonical commutation relations are most easily given in terms of $\vec{J}$, $\vec{\mathcal{J}}$, and U. The commutator of J with U follows from the fact that $\vec{J}$ generates spatial (lower indices) rotations,

$$[J_i, (U_j)^a{}_b] = (T_{ji})_{bc}(U_j)^a{}_c \tag{4.3a}$$

or

$$[J_i, U_j] = T_{ji}\, U_j \quad (\text{no sum on } j)\,, \tag{4.3b}$$

where j labels the representation of the rotation group. Similarly, the commutator of the generator of body rotations with U reads

$$[\mathcal{J}^i, U_j] = U_j\, T_j{}^i \quad (\text{no sum on } j)\,. \tag{4.4}$$

In the Yang-Mills theory a *global* color rotation is given by rotating the degrees of freedom over all space equally, i.e., $V(r,m)$ is independent of r and m. Thus, each link transforms according to $U \to VUV^{-1}$. Therefore, the rigid rotator analog is a simultaneous and equal rotation of body and space axes. This transformation is generated by the difference of the body and space angular momenta.[11]

In the quantum theory of the spherical rotator, the eigenvectors are classified as simultaneous eigenvectors of the operators J^2, J_z, $\mathcal{J}^2$, and $\mathcal{J}_z$. They transform as multiplets under body and space rotations. Since $J^2 = \mathcal{J}^2$ as operators, we label states with quantum numbers J^2, J_z, and $\mathcal{J}_z$. The energy of a state is

$$E = j(j+1)/(2I) \tag{4.5}$$

and its multiplicity is $(2j+1)^2$. The space of states may be constructed from the singlet ground state $|0\rangle$ by using the matrix elements of U in the spin-$\frac{1}{2}$ representation as ladder operators. Alternatively one may apply $\exp(i\,\vec{T}_j\cdot\vec{\Omega})$ to the ground state. Consider first the quantities

$$(U_j)^i{}_l|0\rangle = [\exp(i\,\vec{T}_j\cdot\vec{\Omega})]^i{}_l|0\rangle\,. \tag{4.6}$$

Using Eqs. (4.1) and (4.3b) and the fact that the ground state satisfies

$$\vec{J}|0\rangle = \vec{\mathcal{J}}|0\rangle = 0\,,$$

we compute

$$\begin{aligned} H(U_j)^i{}_l|0\rangle &= \frac{1}{2I} J^2(U_j)^i{}_l|0\rangle \\ &= \frac{1}{2I}[J^2, (U_j)^i{}_l]|0\rangle \\ &= \frac{1}{2I} J_k[J_k, (U_j)^i{}_l]|0\rangle \\ &= \frac{1}{2I}[J_k, (T_k)_{lm}(U_j)^i{}_m]|0\rangle \\ &= \frac{1}{2I} T^2{}_{ln}(U_j)^i{}_n|0\rangle \\ &= \frac{1}{2I} j(j+1)(U_j)^i{}_l|0\rangle \end{aligned} \tag{4.7}$$

as claimed. In particular, U_j generates states of definite energy $j(j+1)/(2I)$.

To use spin-$\frac{1}{2}$ representation matrices as ladder operators, we note that the representations $\vec{T}_j$ can can be constructed from $\vec{T}_{1/2}$. For example, in the case $j=1$,

$$(U_1)^\alpha{}_\beta = \mathrm{tr}\, U_{1/2}{}^{-1}\sigma^\beta U_{1/2}\sigma^\alpha\,. \tag{4.8}$$

In summary, we can make the following set of correspondences between the Yang-Mills theory and the rigid rotator:

Simultaneous body and space rotation

$\to$ global color rotation,

separate body and space rotations

$\to$ local gauge transformation,

body index $\to$ final end of link,

space index $\to$ beginning end of link,

$\vec{\Omega} \to \vec{B}$,

$\vec{\omega} = \vec{J}/I \to \dfrac{d\vec{B}}{dt}$.

The body- (space-) fixed angular momenta correspond to the generators of gauge transformations which rotate one end of a link and do not affect the other end. We denote these operators in the Yang-Mills theory Q_- and Q_+, where the $-$ $(+)$ indicates the beginning (end) of a link. The total color Q carried by a link is the difference $Q_+ - Q_-$.[12] Since the operators $\mathcal{J}$ and J are related by $\mathcal{J}^\alpha = (U_1)^\alpha{}_\beta J_\beta$ for a rigid rotator, it follows by analogy that

$$Q_+^\alpha = (U_1)^\alpha{}_\beta(Q_-)_\beta \tag{4.9}$$

and

$$\begin{aligned} Q &= Q_+ - Q_- \\ &= [U(r,m) - 1]Q_- \\ &= [1 - U^{-1}(r,m)]Q_+\,. \end{aligned} \tag{4.10}$$

V. GAUGE-INVARIANT SPACE OF STATES

A. Pure Yang-Mills field

Evidently the space of states of the Yang-Mills field is the product of an infinite number of rigid-rotator spaces. However, not all of the states are physically relevant. The physical states are drawn from the space of gauge-invariant states. Let us first consider the generator of gauge transformations. An arbitrary gauge transformation can be built from individual gauge transformations at the points of the lattice. Therefore, it suffices to consider just a gauge transformation at the lattice site i. Six links emanate from the point and each is effected by the gauge transformation

$$[G^\alpha, U] = [Q_+^\alpha, U],$$

where G is the generator of gauge transformations at position r. Thus, the generator must be equal to the sum of the Q_+ over the six links,

$$G(r) = \sum_m Q_+(r,m)\,. \tag{5.1}$$

We have seen in Sec. IV that Q_+ is proportional to $\dot{B}$ (in analogy with the space-fixed angular momentum). Accordingly, the generator $G(r)$ may be written as

$$G(r) = \text{const} \times \sum_m \dot{B}(r,m)\,. \tag{5.2a}$$

The time derivative of the vector potential can be identified with the component of the non-Abelian electric field at position r in the direction m. The sum over electric fields emanating from a single site is the lattice analog of $\vec{\nabla}\cdot\vec{\mathrm{E}}$ at the lattice site r. Because the electric field itself varies from $Q_-(r,m)$ to $Q_+(r,m)$ along a link, there is an additional contribution to the lattice analog of $\vec{\nabla}\cdot\vec{\mathrm{E}}$ which is associated with the links. This additional source is just $Q_+(r,m) - Q_-(r,m)$, or the charge carried by the link. Thus the generator may be rewritten

$$G(r) = \vec{\nabla}\cdot\vec{\mathrm{E}}(r) - \tfrac{1}{2}\sum_m Q(r,m)\,. \tag{5.2b}$$

The gauge invariance of the physical sector is defined by $G(r)|\psi\rangle = 0$. Identifying $\frac{1}{2}\sum_m Q(r,m)$ as the local color density ρ_G, this constraint becomes the familiar condition $\vec{\nabla}\cdot\vec{\mathrm{E}} = \rho_G$.

The gauge-invariant space of states may be constructed by starting with the gauge-invariant state $|0\rangle_G$, which is defined as the product over lattice sites of the individual gauge-field ground states. The full space of states is given by acting with any product of components of the $U_{1/2}(r,m)$,

$$\prod_{r,m\in\{s\}} U_{1/2}(r,m)^i{}_l|0\rangle_G\,, \tag{5.3}$$

where the product goes over all r and m belonging to some set $\{s\}$. The set $\{s\}$ may include any link any number of times. In general Eq. (5.3) describes a gauge-invariant state only if the color indices at each point are contracted to form a local singlet. Indices associated with different lattice sites may not be contracted since they do not transform identically under *local* gauge transformations. For example, the state,

$$U_{1/2}(r,m)^i{}_l|0\rangle$$

is not gauge-invariant since it has uncontracted indices. The state

$$U_{1/2}(1)^i{}_j U_{1/2}(2)^j{}_k U_{1/2}(3)^k{}_l U_{1/2}(4)^l{}_i|0\rangle\,,$$

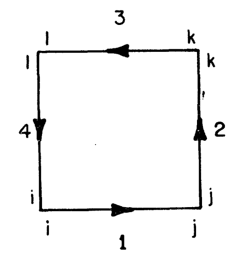

FIG. 3. Graphical representation for the gauge-invariant operator $\mathrm{tr}U_{1/2}(1)U_{1/2}(2)U_{1/2}(3)U_{1/2}(4)$.

where 1, 2, 3, and 4 refer to the links shown in Fig. 3, is gauge-invariant. However, if the contraction of indices did not involve the same lattice site, a similar object such as

$$U_{1/2}(1)^i{}_j U_{1/2}(3)^j{}_k U_{1/2}(2)^k{}_l U_{1/2}(4)^l{}_i|0\rangle$$

would not be gauge-invariant.

A simple pictorial representation of the construction of gauge-invariant states can be given. We begin with an oriented closed path of links Γ. For each link on Γ we associate an operator

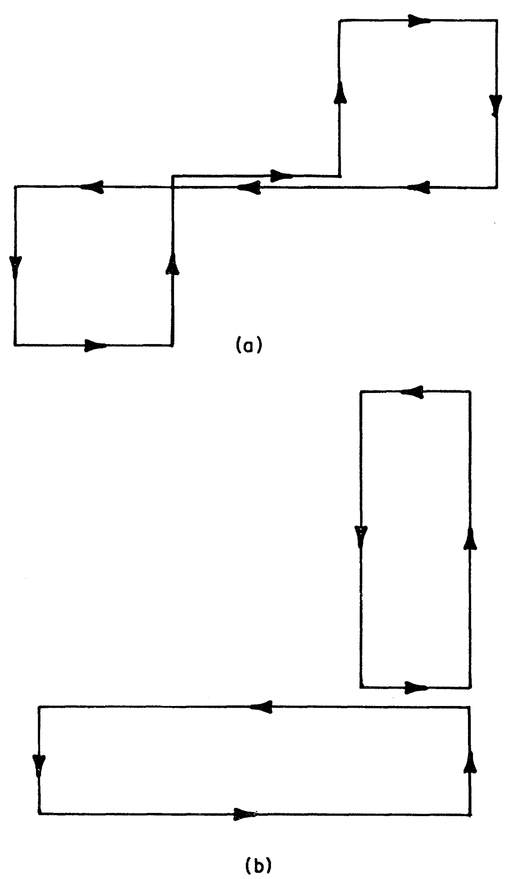

FIG. 4. The operators (a) $U_{1/2}(\Gamma)$ and (b) $U_{1/2}(\Gamma_1)U_{1/2}(\Gamma_2)$.

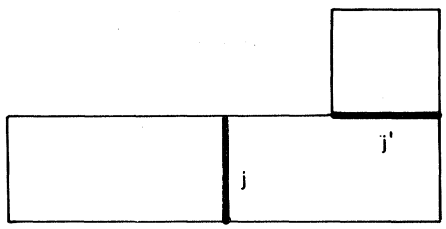

FIG. 5. Replacing overlapping "spin"-$\frac{1}{2}$ flux lines by lines of "spin" j.

$U_{1/2}(r,m)$. We construct the matrix trace,

$$U(\Gamma) = \mathrm{tr}U_{1/2}(r,n)\,U_{1/2}(r+n,m)\times U_{1/2}(r+n+m,\,l)\cdots U_{1/2}(r-s,\,-s)\,,$$

where the factors occur in the order indicated by the path Γ. Obviously $U(\Gamma)$ is gauge-invariant. One can now apply any product of $U(\Gamma)$'s to the ground state $|0\rangle_G$ to produce the gauge-invariant subspace. Individual links may be covered more than once either by an individual Γ as in Fig. 4(a) or by two or more Γ's as in Fig. 4(b). Thus an alternative to the field description is provided by a configuration space in which a set of closed flux lines or strings are specified.

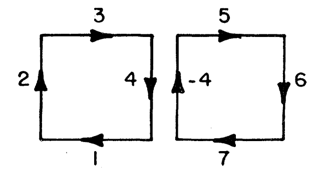

FIG. 6. Product of boxes with an overlapping link.

Another way to characterize the gauge-invariant states is to consider products of the $U_j(r,m)$ where j is not necessarily $\frac{1}{2}$. Again all indices must be contracted as before, but no link is covered more than once. The pictorial representation associated with this construction involves closed paths which can branch as in Fig. 5. Each segment is labeled by a value of j and the vertices involve the proper Clebsch-Gordan coefficients. We will construct an example of the equivalence of the two procedures. Consider the paths shown in Fig. 6. We associate with them the operator

$$\mathrm{tr}U_{1/2}(1)\,U_{1/2}(2)\,U_{1/2}(3)\,U_{1/2}(4)\,\mathrm{tr}U_{1/2}(-4)\,U_{1/2}(5)\,U_{1/2}(6)\,U_{1/2}(7) = \mathrm{tr}U_{1/2}(1)\,U_{1/2}(2)\,U_{1/2}(3)\,U_{1/2}(4)\times \mathrm{tr}U_{1/2}{}^{-1}(4)\,U_{1/2}(5)\,U_{1/2}(6)\,U_{1/2}(7)\,. \qquad (5.4)$$

Using the identity

$$(U_{1/2})^i{}_j(U_{1/2}{}^{-1})^k{}_l = \tfrac{1}{2}\delta^{ik}\,\delta_{jl} + \tfrac{1}{4}(\mathrm{tr}U^{-1}\tau_\beta U\tau_\alpha)(\tau_\beta)_{ik}\,(\tau_\alpha)_{jl}\,, \qquad (5.5)$$

the operator in Eq. (5.4) can be written as

$$\tfrac{1}{2}\mathrm{tr}U_{1/2}(1)\,U_{1/2}(2)\,U_{1/2}(3)\,U_{1/2}(5)\,U_{1/2}(6)\,U_{1/2}(7) + \tfrac{1}{4}[\mathrm{tr}U_{1/2}(1)\,U_{1/2}(2)\,U_{1/2}(3)\;\tau_\alpha U_{1/2}(5)\,U_{1/2}(6)\,U_{1/2}(7)\tau_\beta]\,U_1(4)^\alpha{}_\beta\,. \qquad (5.6)$$

The equality of Eqs. (5.4) and (5.6) is illustrated in Figs. 6 and 7.

The reader should realize through these examples that the fact that every index must be matched at each site is the non-Abelian equivalent of the continuity of electric flux lines in an Abelian theory.[13]

B. Yang-Mills theory with fermions

Let us next consider the gauge-invariant states which can be formed when we include the fermion field ψ. We can construct a gauge-invariant state by considering the lowest eigenstate of the gauge-invariant charge-conjugation-invariant operator

$$\sum_r (-1)^r\,\psi^\dagger(r)\,\psi(r)|0\rangle_F\,. \qquad (5.7)$$

The state $|0\rangle_F$ is a product of fermion vacua over all the lattice sites. The product state $|0\rangle = |0\rangle_F|0\rangle_G$ is gauge-invariant.

Now, in addition to the operators $U(\Gamma)$ formed from closed paths, we can form gauge-invariant operators from paths with ends (Fig. 8). For ex-

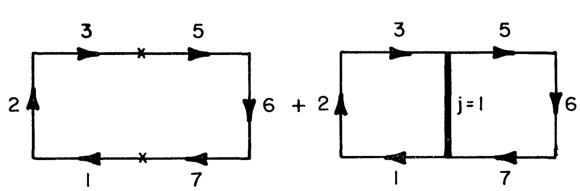

FIG. 7. Replacing the flux in link 4 of Fig. 6 by $j=0$ and $j=1$ flux lines.

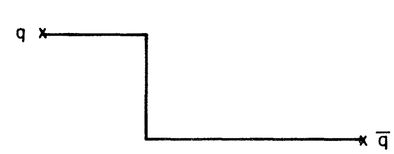

FIG. 8. A $q\bar{q}$ state with its accompanying flux line.

ample, consider a path Γ beginning at r and ending at site s. One can form the gauge-invariant operators

$$U(\Gamma,\Sigma)=\psi^{\dagger}(r)\,\Sigma\, U_{1/2}(r,n)\,U_{1/2}(r+n,m)\cdots \times U_{1/2}(s-l,\,l)\,\psi(s)\,, \tag{5.8}$$

where Σ is any 2×2 spin matrix.

The physical significance of the lines between occupied sites is interesting. They represent lines of electric flux. To see this observe that the operator $Q_+(r,m)$ is proportional to the electric field at site r and points in the direction m. At all links where there is no $U_{1/2}(r,m)$, $Q_+(r,m)$ gives zero. On the links through which a single Γ_j has passed, $Q_+{}^2(r,m)$ gives $j(j+1)$. In this sense one can think of these lines as containing electric flux of magnitude $[j(j+1)]^{1/2}$.

Now that the fermions have been added to the theory, the generator of gauge transformations at point r must include the additional operator $\psi^{\dagger}(r)\,\frac{1}{2}\vec{\tau}\,\psi(r)$, which generates color rotations of ψ. The full gauge-invariance condition on the space of states becomes

$$\left[\sum_m Q_+^{\alpha}(r,m)-\psi^{\dagger}(r)\,\tfrac{1}{2}\,\tau^{\alpha}\,\psi(r)\right]\big|\big\rangle=0\,. \tag{5.9}$$

This is analogous to the condition

$$\vec{\nabla}\cdot\vec{\mathrm{E}}=\rho_G+\rho_F\,,$$

where ρ_G and ρ_F are the color densities of the gauge and Fermi fields.

VI. THE GAUGE-FIELD HAMILTONIAN

We must add a pure gauge-field term to the Hamiltonian describing fermions in order to give the field B some nontrivial dynamics. Since we are requiring local gauge invariance, the Hamiltonian must be built from gauge-invariant operators. Thus the Hamiltonian may contain objects like $U(\Gamma)$. However, since all components of the U's commute with one another, such terms will not be enough to produce nontrivial dynamics.

In addition to the U's, gauge-invariant operators can be built from the $Q_{\pm}(r,m)$. In particular $Q_+{}^2$ $(=Q_-{}^2)$ is the analog of the total angular momentum J^2 of the rigid rotator. Since J^2 commutes with both space and body rotations of the rigid rotator, $Q_+{}^2$ commutes with left and right gauge transformations. $Q_+{}^2$ is therefore gauge-invariant. Furthermore, since $Q_+{}^2$ does not commute with U, its appearance in the Hamiltonian will generate nontrivial dynamics. Accordingly, we include in the Hamiltonian a term

$$\sum_{r,m} Q_+{}^2(r,m)/(2I)\,, \tag{6.1}$$

where I is a constant. This expression is, of course, analogous to the energy of an assembly of uncoupled rotators.

Clearly, from the rotator analogy $(J=I\omega)$ we may deduce the equation of motion

$$\dot{B}=-i[B,H]=Q_+/I\,. \tag{6.2}$$

This equation states that the operator $Q_+(r,m)$ is proportional to the electric flux emanating from the lattice point r in the direction m. This fact was noted previously in Secs. IV and V.

In order that the pure Yang-Mills theory be nontrivial we must introduce terms which couple different links. To do this we make use of the operators $U(\Gamma)$. There is a great deal of arbitrariness in choosing the additional term(s). Following Wilson[4] we pick the simplest object which reproduces continuum Yang-Mills theory when $a\to 0$. Accordingly, let us consider the continuum limit of $U(\Gamma)$, where Γ is shown in Fig. 9. As $a\to 0$ we require as in Sec. III that the field B tend to zero $\sim a$. Thus we write the expansion,

$$U_{1/2}\cong 1+iag\vec{\mathrm{A}}\cdot\tfrac{1}{2}\vec{\tau}-\tfrac{1}{2}a^2g^2\vec{\mathrm{A}}\cdot\tfrac{1}{2}\vec{\tau}\vec{\mathrm{A}}\cdot\tfrac{1}{2}\vec{\tau}+\cdots\,. \tag{6.3}$$

We assume that in the limit $a\to 0$ the field A becomes sufficiently smooth so that

$$A(r,n)-A(r+m,n)\cong a\vec{\nabla}A_n(r)\cdot\hat{m}\,.$$

Now we can expand $U_{1/2}(\Gamma)$ in powers of a. After some algebra we obtain

$$\mathrm{tr}U(\Gamma)=-\tfrac{3}{2}a^4g^4F^2\,, \tag{6.4}$$

where

$$F^{\alpha}_{mn}=\partial_m A^{\alpha}_n-\partial_n A^{\alpha}_m+g\epsilon^{\alpha\beta\gamma}A^{\beta}_nA^{\gamma}_m\,. \tag{6.5}$$

This quantity in Eq. (6.4) is familiar from the conventional continuum form of the Yang-Mills Hamiltonian.

Now we can collect together the various pieces of the Hamiltonian defining the discrete theory:

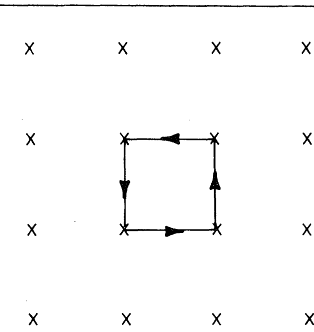

FIG. 9. The term U (box) in the gauge-field Hamiltonian.

$$H=\frac{a}{2g^2}\sum_{r,m}\dot{B}^2(r,m)+\frac{4}{ag^2}\sum \mathrm{tr}U_{1/2}(r,n)\,U_{1/2}(r+n,m)\,U_{1/2}(r+n+m,\,-n)\,U_{1/2}(r+m,\,-m)$$
$$+a^{-1}\sum \psi^\dagger(r)\,\frac{\vec{\sigma}\cdot\vec{\mathrm{n}}}{i}\,U(r,n)\,\psi(r+n)+m_0\sum(-1)^r\psi^\dagger(r)\,\psi(r)\,. \tag{6.6}$$

The coefficients of the various terms are determined by requiring that H must have the usual continuum ($a\to 0$) limit.

Comparing the first term of Eq. (6.6) with the energy of a rigid rotator and recalling the correspondence $\dot{B}\to\omega$, we note the correspondence

$$I\to a/g^2\,.$$

It then follows from Eq. (6.1) that

$$J\to Q_+(r,n)=a\dot{B}(r+n,\,-n)/g^2\,.$$

Finally, recall that the color carried by a link is

$$Q=[U_1(r,n)-1]\,Q_-\,, \tag{4.10$'$}$$

which becomes in the continuum limit

$$Q^i\cong(iagA\circ T_1)^i{}_j\,\frac{a^2}{g}\,\dot{A}_j\,,$$
$$Q^i/a^3\cong(\vec{\mathrm{A}}\times\vec{\mathrm{E}})_i\,. \tag{6.7}$$

This is the familiar expression for the color carried by the gauge field in the continuum Yang-Mills theory.

VII. ENERGY CONSIDERATIONS AND PERTURBATION THEORY

The qualitative features of a solution to the Hamiltonian in Eq. (6.6) depend upon which term dominates. For large g, the first term

$$H_0=\frac{g^2}{2a}\sum_{r,n}Q_+{}^2(r,n)$$
$$=\frac{a}{2g^2}\sum\dot{B}^2(r,n) \tag{7.1}$$

dominates and the remaining terms may be treated as perturbations. Since the first term does not couple adjacent lattice sites, its eigenvectors are product eigenvectors of the individual rigid rotators. The gauge-invariant eigenvectors are simply the states defined by products of $U(\Gamma)$ and $U(\Gamma,\Sigma)$ applied to $|0\rangle=|0\rangle_G|0\rangle_F$. Evidently in this strong-coupling approximation there are no fluctuations in the stringlike flux-line states. The energy of a state is most easily described in a representation utilizing the $U_j(r,m)$ operators introduced earlier. A typical state is pictured in Fig. 5. The energy of that state receives a contribution from each link equal to $j(j+1)/(2I)$.

This picture of the strongly coupled Yang-Mills theory in terms of a collection of stringlike flux lines is the central result of our analysis. It should be compared with the phenomenological use of stringlike degrees of freedom which has been widely used in describing hadrons.[14]

An important element of the Yang-Mills theory is that the electric flux is quantized. Since electric flux ($Q_\pm$) satisfies the commutation relations of the non-Abelian generators of gauge transformations, it cannot be indefinitely subdivided. This means that a unit of electric flux of magnitude $Q_\pm{}^2=\frac{1}{2}(\frac{1}{2}+1)$ cannot split. This is to be contrasted with a conventional Abelian gauge theory in which both charge and flux can be arbitrarily subdivided.[15]

Now consider some examples. The ground state is the state $|0\rangle$ with no excited flux lines. The next-lowest-energy gauge-invariant states involve a single excited link with a fermion-antifermion pair at its ends. All such states have energy $\frac{1}{2}(\frac{1}{2}+1)/(2I)$ in the strong-coupling limit in which only the term in Eq. (7.1) is considered in H. In the pure Yang-Mills theory the lowest-energy gauge-invariant excitation involves four links as in Fig. 9. It has energy $4\times\frac{1}{2}(\frac{1}{2}+1)/(2I)$.

The states involving fermions can have nontrivial SU(3) quantum numbers and may be identified as mesons. The Yang-Mills gauge-invariant excitations are SU(3) singlets. Excited states of these objects can be identified with states in which more than the minimum number of links

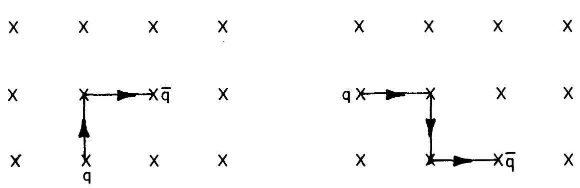

FIG. 10. Two excited states of a $q\bar{q}$ system.

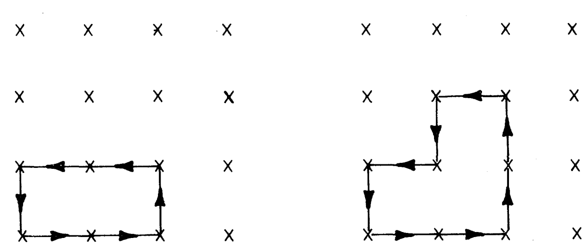

FIG. 11. Two excited states of a gauge-invariant excitation.

are involved. For example, in Fig. 10 we show two possible $q\bar{q}$ states, and in Fig. 11 we show two possible Yang-Mills gauge-invariant excitations. The open-ended $q\bar{q}$ flux lines are similar to the original "dual string"[14] used phenomenologically to describe ordinary mesons. The excitations along trajectories are analogous to excited $q\bar{q}$ states occupying more than one link. The Yang-Mills gauge-invariant excitation also has a counterpart in the dual model. It is closed SU(3)-singlet dual Pomeron string.[16]

Let us consider the force law between widely separated quarks in the strongly coupled limit. The potential energy is defined as the lowest energy compatible with the presence of a quark at site r and an antiquark at site s. For simplicity choose the sites r and s to lie in a given row as depicted in Fig. 12. The minimum-energy gauge-invariant state is obviously given by exciting the shortest path of links connecting the $q\bar{q}$ pair. The energy associated with the configuration shown in Fig. 12 is

$$(L/a)\,\tfrac{1}{2}(\tfrac{1}{2}+1)/(2I)\,, \tag{7.2}$$

where L is the separation distance between the quarks. Since the potential energy increases linearly with distance, the force between the quarks is independent of their separation. Such a force is clearly sufficient to confine quarks. As discussed in Refs. 5 and 7, this force law between unscreened charges is identical to the classical force laws in one-dimensional gauge theories.[17] It is also clear that the energy of confinement is stored on the line (flux tube) between the quarks (Fig. 12).

The force law between $j=\frac{1}{2}$ (color triplet) objects contrasts sharply with the force law between hypothetical $j=1$ (color octet) static objects. Physically the reason is that the low-energy state of a distance pair of $j=1$ objects can be constructed by screening the color of the static objects by the color gauge field. To see this explicitly, define the field $\phi_\alpha(r)$, which creates the $j=1$ particle at site r. The minimum-energy gauge-invariant state compatible with a single such object is

$$\phi_\alpha(r)\,\mathrm{tr}\,U_{1/2}(r,n)\,U_{1/2}(r+n,m) \times U_{1/2}(r+n+m,-n)\,U_{1/2}(r+m,-m)\tau^\alpha|0\rangle\,.$$

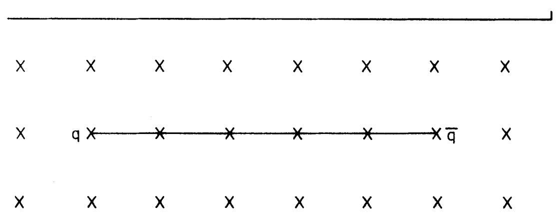

FIG. 12. Minimum-energy configuration for a separated $q\bar{q}$ system in the strong-coupling limit.

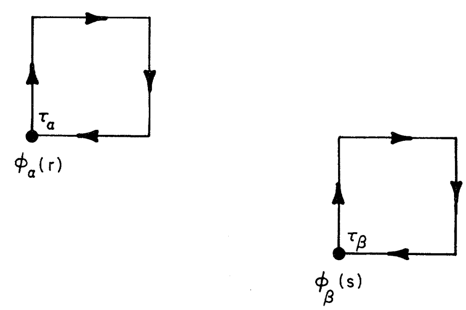

FIG. 13. Minimum-energy configuration for a separated pair of color-1 objects.

This state clearly has finite energy. The state of minimum energy of two such particles at large distance is obtained by independently applying such operators at distant sites (Fig. 13). The resultant energy is independent of the distance between the objects, so the force is entirely screened. The resulting well-separated objects are each colorless.

In summary, the strong-coupling limit is characterized by nonfluctuating flux-line configurations. This means that the fluctuations of the vector potentials A are as large as possible. This corresponds to the fact that the relative orientation of the body- and space-fixed axes in the rigid rotator's ground state is maximally uncertain.

Now we shall consider perturbations around the strong-coupling limit. The effects of the second term in H,

$$\begin{aligned} V &= \sum_{r,n} V(r,n) \\ &= \frac{4}{ag^2}\sum \mathrm{tr}\,U_{1/2}(r,n)U_{1/2}(r+n,m) \\ &\qquad \times U_{1/2}(r+n+m,-n)U_{1/2}(r+m,-m) \\ &\quad + \mathrm{H.c.} \end{aligned} \tag{7.3}$$

correct the eigenstates of H_0 to $O(g^{-1})$ and correct the energy of these states to $O(g^{-2})$. In fact the term V plays three roles. The first is to diminish the fluctuations of the magnetic field.[18] The second is to create fluctuations of the string configurations. And the third is to propagate excitations through the lattice.

The first role is very simple. This follows because in the continuum limit V becomes the square of the magnetic field.

The second two roles can be studied in perturbation theory. For example, consider the corrections to the vacuum state of pure Yang-Mills theory. The first-order correction to a state Ψ_0 is

$$\Psi_1 = \frac{1}{E_0-H_0}\,V\Psi_0\,. \tag{7.4}$$

Therefore, the correction to the vacuum state is

$$\Psi_1 = \frac{8I}{3ag^2}\Big(\sum_{r,n} \mathrm{tr}\, U_{1/2}(r,n)U_{1/2}(r+n,m)U_{1/2}(r+n+m,-n)U_{1/2}(r+m,-m) + \mathrm{H.c.}\Big)\,|0\rangle \,. \tag{7.5}$$

Thus, the corrected state consists of a superposition of states in which a gauge-invariant excitation occurs anywhere on the lattice. Clearly higher-order perturbation theory will create many gauge-invariant excitations as well as more complicated excitations of a single box. Thus the vacuum becomes a fluctuating sea of closed flux loops. The first-order correction to the vacuum energy is given by

$$\Delta E(\text{vacuum}) = \left\langle 0 \left| V \frac{-1}{H_0} V \right| 0 \right\rangle , \tag{7.6}$$

and an explicit calculation shows that

$$\Delta E(\text{vacuum})/\text{vol.} = -32/a^4 g^6 \,. \tag{7.7}$$

Next let us consider the corrections to the energy and state of a separated $q\bar{q}$ pair in lowest-order strong-coupling perturbation theory. The correction to the state is

$$\Delta\Psi(q\bar{q}) = \frac{1}{E_0 - H_0} V\psi^\dagger(r)U_{1/2}(r,n)U_{1/2}(r+n,m)\cdots \times U_{1/2}(s-l,l)\psi(s)\,|0\rangle \tag{7.8}$$

and is best analyzed graphically. First there are terms in which the boxes of V do not overlap with the original string connecting the $q\bar{q}$ pair. They are shown in Fig. 14 and may be regarded as corrections to the vacuum. Next there are corrections in which one of the lines in V overlaps with the original string. There are two possibilities as indicated in Figs. 15(a) and 15(b). Namely, the overlapping flux lines may be parallel or antiparallel. The two diagrams of Fig. 15 may be rewritten as in Fig. 16 in terms of $j=0$ and $j=1$ flux lines. Clearly these corrections cause both the position and structure of the string to fluctuate.

The change in energy of the $q\bar{q}$ configuration can be computed by standard perturbation theory. After subtracting off terms contributing to the vacuum energy, we find that the correction is a sum over the original links of the flux line. The contribution of each link is $\sim\text{const}/g^6$. Therefore the new potential energy is

$$\frac{L}{a}\left[\frac{\frac{1}{2}(\frac{1}{2}+1)}{(2I)} - \frac{\text{const}}{g^6}\right] . \tag{7.9}$$

It is important to note that the energy is still proportional to the distance between the fermions. The fermions are still confined.

When higher orders in the perturbation are considered, the energy continues to grow linearly with the length of the string for large interquark separation. However, deviations occur for short strings. To see why this is so, consider an intermediate state in which two adjacent boxes are involved as in Fig. 17. The contribution from such diagrams will again be summed along the length of the flux line and therefore lead to a linear force law for large distances. However, when the original flux line is less than two lattice sites long, this term becomes inoperative.

We now turn to a perturbation theory description of the propagation of gauge-invariant excitations through the lattice. To illustrate the effects of V, consider the propagation of the symmetric gauge-invariant excitation.[19] Recall that any localized box is an eigenvector of H_0 with eigenvalue $4\times\frac{1}{2}(\frac{1}{2}+1)/(2I)$. So, in the strong-coupling limit the momentum eigenstates (which are linear superpositions with weighting factors e^{ikr} of localized boxes at position r) exhibit no momentum dependence in their energy spectrum. Now consider the effects of V acting upon a localized gauge-invaraint excitation. The interesting terms occur when one of the sides of the gauge-invariant excitation coincide with a link in V. This gives rise to the states shown in Fig. 18. Allowing V to act twice, it may act on the original site of the gauge-invariant excitation and annihilate it, leaving a displaced gauge-invariant excitation. In this way

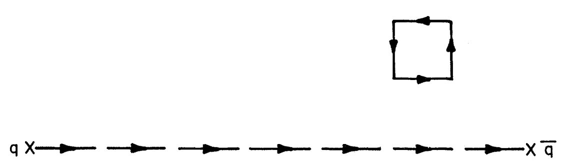

FIG. 14. A disconnected contribution to V acting on a $q\bar{q}$ state.

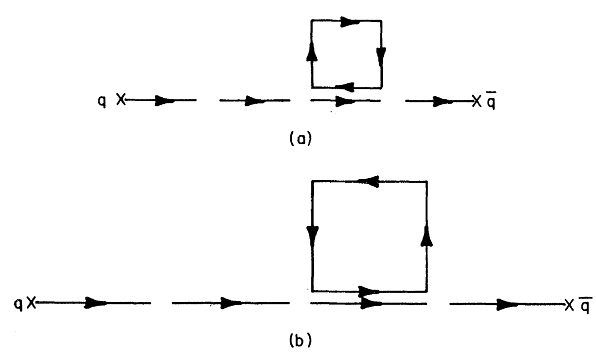

FIG. 15. Connected contributions to V acting on a $q\bar{q}$ state.

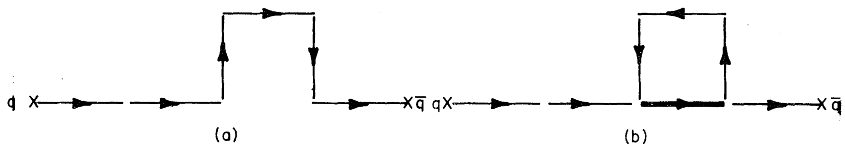

FIG. 16. Replacing the doubly excited link in Fig. 15 by an unexcited link or a $j=1$ link.

the perturbation can cause the gauge-invariant excitation to propagate through the lattice.

In order to study the propagation properties, we introduce the following notation. We denote $|r, m\rangle$ as a gauge-invariant excitation at r with polarization m. To define this terminology consider the three box states at r: $|r, x\rangle$, $|r, y\rangle$, and $|r, z\rangle$. The state $|r, x\rangle$ is bounded by the links (r, m_y), (r, m_z), $(r+m_y, m_z)$, and $(r+m_z, m_y)$ as depicted in Fig. 19.

Now consider a momentum eigenstate

$$|k, n\rangle = \sum_r e^{ik\cdot r}|r, n\rangle \ . \qquad (7.10)$$

We will calculate the lowest-order nonvanishing correction to the energy of the state $|k, m\rangle$ and thereby exhibit a nontrivial dispersion law for these excitations. The first-order matrix element of the perturbation vanishes identically,

$$\langle k, l\,|V|\,k, l\rangle = 0\,.$$

To see this, note that if the box created by $V(r, m)$ has no line in common with $\langle r', l'|$ or $|r, l\rangle$, then $\langle r', l'|V(r, m)|r, l\rangle$ is zero. If it has one or more line in common with the states $|r, l\rangle$ or $|r', l'\rangle$, then one has two possible situations. Either $V(r, m)$ has one line in common with $|r, l\rangle$ or $|r', l'\rangle$, in which case a rectangle having six links is made (such a rectangle cannot project back onto the final state) or if $V(r, m)$ completely overlaps with $|r, l\rangle$ or $|r', l'\rangle$, then each side of the box is excited to a state of color $j=1$ or 0, and does not project back onto the final state which is a gauge-invariant excitation with $j=\frac{1}{2}$ sides.

The next order of perturbation theory is described with the matrix elements

$$\left\langle r', l'\left|V\frac{1}{E-H_0}\,V\right|r, l\right\rangle \ .$$

We must compute the eigenvalues of this matrix. The possible nonvanishing matrix elements are as follows. If $|r, l\rangle = |r', l'\rangle$, V may create a nonadjacent box, a box with one line in common or a completely overlapping box with $|r, l\rangle$. The nonoverlapping box contributes

$$\left\langle 0\left|V\frac{1}{H_0}\,V\right|0\right\rangle ,$$

which we identify as the correction to the vacuum energy per box to this order. Since only the difference of the excitation energy with the vacuum is physically significant, we must subtract off this vacuum energy. This leaves over

$$\sum_{\substack{\text{boxes having no lines}\\ \text{in common with } |r,l\rangle}} = \sum_{\text{all boxes}} - \sum_{\substack{\text{boxes with a line}\\ \text{in common}}} \ .$$

So, subtracting out the vacuum energy gives us -13 times the energy of a vacuum box. The counting factor 13 comes from the fact that there are 12 boxes with one line in common with a given box and one box which completely overlaps with that box. Another contribution to the matrix element has an intermediate state with an adjacent box (Figs. 6 and 7). Finally, there is the case in which V creates a box completely overlapping $|r, l\rangle$. Call the magnitude of this contribution A. The other matrix elements which can exist involve $|r, l\rangle$ and $|r', l'\rangle$ with a single line in common. There are four possible contributions to this matrix element which are illustrated in Fig. 18 and

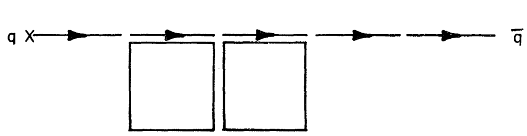

FIG. 17. A higher-order correction to a separated $q\bar{q}$ state.

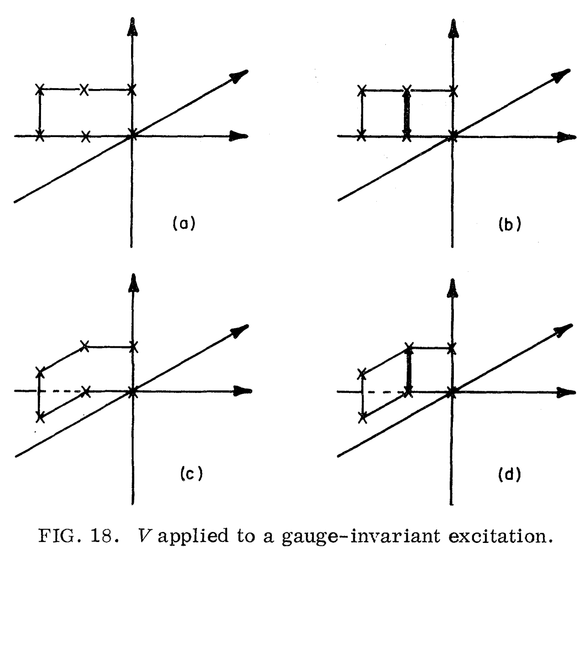

FIG. 18. V applied to a gauge-invariant excitation.

TABLE I. Matrix elements of $V(E_0-H)^{-1}V$ corresponding to the propagation of a box according to the processes of Fig. 18. The values in the table should be multiplied by a factor $-B$. Here $k_{x,y,z}$ are the three components of spatial momentum of the box. The components k_i satisfy the constraint $-\pi a^{-1}<k_i<\pi a^{-1}$.

	$\vert k, m_z\rangle$	$\vert k, m_y\rangle$	$\vert k, m_x\rangle$
$\vert k, m_z\rangle$	$2(\cos k_x+\cos k_y)$	$1+e^{ikz}+e^{-iky}+e^{ik(z-y)}$	$1+e^{ikz}+e^{-ikx}+e^{ik(z-x)}$
$\vert k, m_y\rangle$		$2(\cos k_z+\cos k_x)$	$1+e^{iky}+e^{-ikx}+e^{ik(y-x)}$
$\vert k, m_x\rangle$			$2(\cos k_z+\cos k_y)$

have been discussed above. These terms contribute a coefficient $-B$ to the matrix element.

Now we can collect together the matrix elements of $V(E_0-H)^{-1}V$ in the states $|k, l\rangle$ and $|k', l'\rangle$. They are tabulated in Table I. The energy eigenvalues are given by $4\times\frac{1}{2}(\frac{1}{2}+1)/(2I)+A$ plus the eigenvalues of the 3×3 matrix of Table I. For small momentum in the z direction the eigenvalues and eigenvectors read

$$\tfrac{3}{2}I+A+Bk_z^2 \qquad (0,1,-1) \quad \text{spin 2}$$

$$\tfrac{3}{2}I+A+\tfrac{2}{3}Bk_z^2 \qquad (1,a,a)$$

$$\tfrac{3}{2}I+A-6B+\tfrac{1}{3}Bk_z^2 \qquad (1,a',a') \quad \text{spin 0}\ .$$

At rest the three eigenvectors are $(0, 1, -1)$, which is a spin-2 object, $(2, 1, -1)$, and $(1, 1, 1)$, which is a rotational singlet. In general it is not possible to classify these states according to the rotation group because the lattice has only cubic symmetry. However, if we consider a rotation by 90° about the z axis, we see that the states $(1, 1, 1)$ and $(2, -1, -1)$ are invariant. The state $(0, -1, 1)$ changes sign under this rotation and is therefore classified as spin 2.

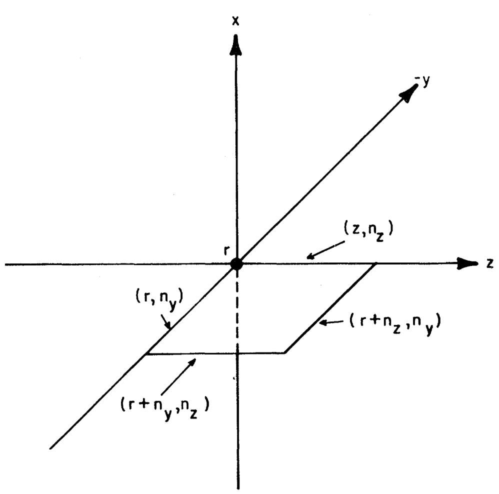

FIG. 19. Notation for the polarization of a gauge-invariant excitation.

Let us now return to the problem of quark confinement when the fermion piece of the Hamiltonian Eq. (6.6) is accounted for. The state with a single fluctuating flux line connecting two quarks is no longer an energy eigenstate. The fermion term considered as a perturbation describes processes in which a $q\bar{q}$ pair and its flux line is created. If that flux line overlaps with a link of the original flux line connecting the initial quarks, it may leave that link in an unexcited state (Fig. 20). Thus these processes allow the original string to break, i.e., they screen the long-range interquark forces. However, they do not allow free quarks to escape since each segment in Fig. 20 must be colorless. This situation is clearly very closely analogous to the phenomenon of vacuum polarization and screening in one-dimensional quantum electrodynamics which also confines quarks[17] and eliminates long-range forces.

Nevertheless, quark confinement can fail in the four-dimensional lattice theory if the very high-order terms in the perturbation-series expansion become important. If the terms in the Hamiltonian which cause fluctuations in the flux line become dominant, then electric flux will fail to be collimated along a line between quarks. Then the long-range force which would permanently bind quarks may disappear.

VIII. CONCLUSIONS AND DISCUSSION

The main result of this paper is that strongly coupled Yang-Mills theory on a lattice describes interacting propagating stringlike disturbances. The strings are elementary quantized lines of electric flux which can only end on charges.

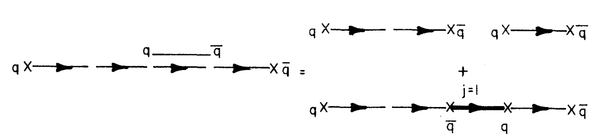

FIG. 20. The possible breaking of a flux line by $q\bar{q}$ production.

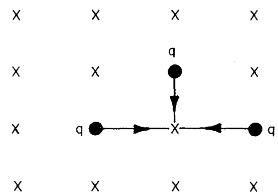

FIG. 21. A typical baryon configuration in a SU(3)-color theory.

For physical applications the gauge group should be SU(3) (color) rather than SU(2) (color). The main new feature of this generalization is that three quark indices can be contracted at a point with the antisymmetric coupling ϵ_{ijk}. This allows baryons to be formed from three flux lines joined at the center. It would be interesting to study these objects (see Fig. 21.)

We have not discussed in this article a number of difficult theoretical questions which remain unsolved. The most important is to show that renormalization effects really do lead to a strongly coupled theory at large distances. This will presumably require a renormalization-group approach to the theory in which it is first formulated on a lattice with a small spacing and a small coupling constant. The degrees of freedom may be "thinned out" according to Wilson's method[20] until an effective description with large lattice spacing is found. We hope such an analysis will justify our use of strong-coupling methods. In addition, one must hope that the effective description will define a theory which is not sensitive to the initial lattice of small spacing (except through certain coupling and wave-function renormalization constants). Then covariance will be restored and the theory will be (potentially) realistic. If this can not be done, our approach will be invalid.

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[1] The earliest discussion of the phenomenon in a field-theoretic framework appears in V. F. Weisskopf, Phys. Rev. 56, 72 (1939).

[2] M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954); K. Symanzik, Commun. Math. Phys. 23, 49 (1971); C. G. Callan, Phys. Rev. D 5, 3202 (1972).

[3] G. 't Hooft, in Proceedings of the Marseilles Conference on Gauge Theories, 1972 (unpublished); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973).

[4] Kenneth G. Wilson, Phys. Rev. D 10, 2445 (1974).

[5] J. Kogut and Leonard Susskind, Phys. Rev. D 9, 3501 (1974).

[6] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[7] Linear force laws have been suggested by E. P. Tryon [Phys. Rev. Lett. 28, 1605 (1972)] and more recently by K. Kaufman [private communication from R. P. Feynman to one of us (L.S.)]. The theories discussed in Ref. 5 also have this behavior.

[8] For a recent discussion of color quark models, see H. Fritzsch and M. Gell-Mann, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972*, edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 2, p. 135.

[9] To prove this one can derive the equations of motion and explicitly verify that they can be written in a manifestly gauge-invariant form.

[10] The rigid rotator is discussed in H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1958).

[11] The reader may be confused by the simultaneous use of two linear vector spaces. Each component of the $(2j+1)^2$-dimensional matrices U_j and the $(2j+1)$-dimensional vector J are operators in the Hilbert space of states of the quantum system. The matrices T_j are c numbers in this space. The commutation relations refer to this space and not to the $(2j+1)$-dimensional vector space on which T_j acts.

[12] It is the *difference* because the commutation relations of $\mathcal{J}$ differ in sign from those of J. See, for example, L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1958).

[13] The notion of continuous flux lines was invented and used by Faraday.

[14] The string model of hadrons was originally suggested by L. Susskind, Phys. Rev. Lett. 23, 545 (1969). See Y. Nambu, in *Symmetries and Quark Models*, edited by R. Chand (Gordon and Breach, New York, 1970); H. B. Nielsen, Nordita report, 1969 (unpublished).

[15] However, Wilson's Abelian gauge theory formulated on a lattice has quantized flux since the gauge field B is again treated as an angular variable. See Ref. 4 for details.

[16] G. Frye and L. Susskind, Phys. Lett. 31B, 589 (1970).

[17] A. Casher, J. Kogut, and Leonard Susskind, Phys. Rev. Lett. 31, 792 (1973).

[18] The term "magnetic field" refers to $F_{ij}^{\alpha} = \partial_i A_j^{\alpha} - \partial_j A_i^{\alpha} + g\epsilon^{\alpha\beta\gamma} A_i^{\beta} A_j^{\gamma}$, $i, j = 1, 2, 3$.

[19] The term "symmetric gauge-invariant excitation" indicates the operator $\frac{1}{2}(\mathrm{tr}U_1U_2U_3U_4 + \mathrm{tr}U_4U_3U_2U_1)$, where the links 1, 2, 3 and 4 bound a box.

[20] For a discussion of and references to the Kadanoff-Wilson block-spin analysis see the review article, K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).