Comments and Addenda

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Comment on a recent calculation of the anomalous magnetic moment of the electron to order α

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We evaluate analytically the results given in integral form by Lai *et al.* for the anomalous magnetic moment of the electron to order α. We point out that for small cutoff energies ($\omega_c \lesssim \frac{1}{2} m_e$) the result is negative. A short critique of the nonrelativistic approximation is included.

Recently an expression for a_e, the anomalous magnetic moment of the electron, was derived using the equations of motion for the spin precession.[1] The result was expressed in terms of two integrals [Eqs. (9a) and (9b) of Ref. 1]:

$$C = \frac{4e^2\omega_0}{3\pi} \int_0^{\omega_c} \frac{(m^2+\omega^2)^{1/2}+m}{2(m^2+\omega^2)^{1/2}} \frac{\omega\, d\omega}{(\omega - \omega^2/2m - 2m)^2} , \tag{1}$$

$$S = \frac{ie^2\omega_0^3}{3m^2} - 8\,\frac{e^2\omega_0}{3\pi} \int_0^{\omega_c} \frac{\omega^3 d\omega}{2(m^2+\omega^2)^{1/2}[(m^2+\omega^2)^{1/2}+m][\omega+\omega^2/2m]^2} . \tag{2}$$

In these equations m is the electron mass, $\omega_0 = eB_0/mc$ is the Larmor frequency in a uniform magnetic field $\vec{B}_0$, and ω_c is a cutoff supplied by the authors of Ref. 1 "as a reminder of the low-energy character of our calculation." In terms of C and S,

$$a_e = \lim_{\omega_0 \to 0} \mathrm{Re}\left[\frac{C+S}{\omega_0}\right] , \tag{3}$$

and it is pointed out in Ref. 1 that for $\omega_c \simeq 2m$ the exact value[2] of a_e to order α is obtained:

$$a_e^{\text{2nd order}} = \frac{\alpha}{2\pi} . \tag{4}$$

Intrigued by this result, we tried to evaluate (1) and (2) analytically. After a long calculation, the following expressions are obtained:

$$\begin{aligned} C = \frac{2e^2\omega_0}{3\pi}\Bigg\{ & \frac{2(y-4)}{3X} + \frac{2}{3\sqrt{3}} \tan^{-1}\frac{y-1}{\sqrt{3}} + \frac{2}{39X}(4+5y)(1+y^2)^{1/2} \\ & + \frac{14}{(39)^{3/2}}\left[(a-2\sqrt{3}\,b)\tfrac{1}{2}\ln\frac{\gamma^2+\beta^2}{X^2} - (b+2\sqrt{3}\,a)\cos^{-1}\frac{\gamma}{(\gamma^2+\beta^2)^{1/2}}\right] \\ & + \frac{8}{13} + \frac{\pi}{9\sqrt{3}} - \frac{14}{(39)^{3/2}}\left[(a-2\sqrt{3}\,b)\tfrac{1}{2}\ln\frac{1+2b+\sqrt{13}}{4} - (b+2\sqrt{3}\,a)\cos^{-1}\frac{-(1+b+\sqrt{3}\,a)}{2(1+2b+\sqrt{13})^{1/2}}\right]\Bigg\} , \end{aligned} \tag{5}$$

$$S=\frac{ie^2\omega_0^3}{3m^2}-\frac{16e^2\omega_0}{3\pi}\left\{\frac{4-3z}{5X'}+\tfrac{1}{4}\ln\frac{z^2}{-X'}+\frac{9}{20\sqrt{5}}\ln\frac{\sqrt{5}+3-z}{\sqrt{5}-3+z}-\frac{1}{10}-\frac{9}{10\sqrt{5}}\ln\frac{1+\sqrt{5}}{2}\right\}. \tag{6}$$

The following symbols are used in (5) and (6):

$$\begin{aligned} y&=\omega_c/m\,,\\ z&=1-y+(1+y^2)^{1/2}\,,\\ X&=y^2-2y+4\,,\\ X'&=z^2-6z+4\,,\\ \gamma&=[b(y-1)-\sqrt{3}\,a](1+y^2)^{1/2}+y^2-1-3y\,,\\ \beta&=[a(y-1)+\sqrt{3}\,b](1+y^2)^{1/2}+\sqrt{3}\,(1+y^2)\,, \end{aligned} \tag{7}$$

while the constants a and b are given by

$$a=\left[\frac{\sqrt{13}+1}{2}\right]^{1/2},\quad b=\left[\frac{\sqrt{13}-1}{2}\right]^{1/2}. \tag{8}$$

The anomalous magnetic moment of the electron is obtained by putting together Eqs. (3), (5), and (6). The numerical values agree with Fig. 3 of Ref. 3, from which, however, it is not clear that for $\omega_c \lesssim \frac{1}{2}m$ the result is very small and *negative*. The exact value (4) is reached at $\omega_c \simeq 2m$, and for $\omega_c \to \infty$ about twice as large a number is obtained.

From Eqs. (5) and (6), we see that the value $\alpha/2\pi$ at $\omega_c \simeq 2m$ must not be taken seriously. In fact, $a_e(\omega_c/m)$ is changing most rapidly in the neighborhood of this point. A more severe problem, however, is the following.

By using translational invariance *after* a nonrelativistic approximation is applied to the momentum operator, one has effectively introduced another condition $\omega \ll m$ which is not justified (transitions involve energy changes of $2m$). This gives rise to the $\omega^2/2m$ terms in (1) and (2). After such an approximation is made in these integrands, one should hardly allow oneself to take the integration limit ω_c to $2m$ to get $\alpha/2\pi$ for a_e. It would perhaps be more consistent to take a very low cutoff, in which case, as observed above, a negative result is obtained.

I wish to thank Professor H. Grotch for discussions on this problem.

[1] S. B. Lai, P. L. Knight, and J. H. Eberly, Phys. Rev. Lett. 32, 494 (1974).

[2] J. Schwinger, Phys. Rev. 73, 416L (1948).

[3] S. B. Lai, P. L. Knight, and J. H. Eberly, Rochester Univ. report (unpublished).