# High-energy behavior in Abelian gauge theory, application to $\gamma^*$ decay, and high-energy estimates for form factors

Edward B. Manoukian

*Dublin Institute for Advanced Studies, Dublin 4, Ireland*

(Received 15 August 1974; revised manuscript received 6 January 1975)

Previously, we have made a study of the high-energy behavior of quantum electrodynamics in the presence of neutral-meson theory with pseudoscalar-pseudoscalar coupling proceeding loopwise. We have seen, by an elementary summation procedure, that the Adler-Baker-Johnson eigenvalue condition for the fine-structure constant $\alpha$ remains stable (unaltered) in the presence of the strong dynamics and that the photon self-energy part in the multi-fermion-loop contribution is asymptotically finite at $x=\alpha$ independently of the strong-coupling value. We have also learned that the strong dynamics damps out at high energies and no $\phi^4$ counterterm was required. Conversely, in the present work, we study this dynamics at high energies when the Callan-Symanzik function occurring in *pure* electrodynamics vanishes at $\alpha$ at the outset (with an infinite-order zero) by adding the $-\frac{1}{4}\lambda_0\phi^4$ term to our dynamics. We argue that the effective $\gamma_5$ and $\phi^4$ couplings in this Abelian gauge theory may become very small at high energies and that then they finally vanish even faster than the presently estimated *non*-Abelian gauge-theory ones. As an application, the ratio $R(Q^2)$ of the cross section associated with the reaction $\gamma^*(Q^2)\to$ anything to the corresponding one with the strong couplings set equal to zero is studied when the momentum squared of the virtual photon $Q^2\to\infty$. $R(Q^2)$ is shown to have a linear increase with $Q^2$ (broken possibly by logarithmic terms) at truly asymptotic energies, and, contrary to common belief, the leading contribution to it comes from the strong interaction. Some of the statements made in this work are also in pure electrodynamics. For example, the explicit asymptotic form of the photon spectral function in finite quantum electrodynamics *at* the eigenvalue $\alpha$ is given here for the first time. Finally, explicit high-energy estimates are obtained for elastic form factors in quantum electrodynamics in isolation and in the present correlated dynamics. At very large momenta for the external photon, the former are bounded by power-law behavior. Some interesting aspects of this Abelian gauge theory are also mentioned.

## I. INTRODUCTION

We continue with our investigation of the high-energy behavior[1] of quantum electrodynamics in the presence of neutral-meson theory with pseudoscalar-pseudoscalar coupling. In Ref. 1 we have learned, by an elementary summation procedure, that the Adler-Baker-Johnson (ABJ) eigenvalue condition[2,3] (defined in the single-fermion-loop contribution to the renormalized photon self-energy part) for the renormalized fine-structure constant[2] $\alpha$, $F^{[1]}(x)|_{x=\alpha}=0$, remains stable (unaltered) in the presence of the strong dynamics. The latter means that a possible zero of $F^{[1]}(x)$ does not "move" in the presence of the strong interaction. This, as we have mentioned in Ref. 1, leads to the beautiful idea that the value of $\alpha$ may be possibly determined within pure electrodynamics[2] in isolation from the rest of the world. We have then extended our study formally to the multi-fermion-loop contribution to the photon self-energy part and inferred that the latter is asymptotically finite at $x=\alpha$ independently of the strong-coupling value. The point $x=\alpha$ is the assumed (infinite order) zero[2] of the single-fermion-loop electromagnetic-current-correlation functions in mass-zero pure electrodynamics. We have also learned that the strong dynamics damps out at high energies and no $\phi^4$ counterterm is required. For completeness and for the convenience of the reader a very brief account of this work will be given in Sec. II.

In the present work, conversely to the above, we study this dynamics at high energies when the full Callan-Symanzik function[2,4] occurring in pure electrodynamics vanishes at $\alpha$ at the *outset* with an infinite-order zero[2] (see also Sec. II) by adding the $-\frac{1}{4}\lambda_0\phi^4$ term to our dynamics. (Needless to say, a loopwise summation procedure as in Ref. 1 is not meaningful in the usual perturbative theoretical sense for the present case.) In the present work we shall not discuss other possibilities than this case just mentioned. We argue that the effective $\gamma_5$ and $\phi^4$ couplings may become very small at very high energies and that then they finally vanish even faster than the presently estimated *non*-Abelian gauge theory ones.[5] The method of our study is that we rely on electrodynamics in a nontrivial manner, i.e., electrodynamics summed to all orders in $\alpha$. We then make an expansion in powers of the $\gamma_5$ and $\phi^4$ couplings *at* high energies. Such an expansion may indeed make sense if the corresponding effective couplings become small at high energies as we shall argue. The reason why electrodynamics is treated in a nontrivial manner is because of the well-known fact that the effective electromagnetic coupling may never vanish no matter how high we go in energies. The main reasons for this may be summarized as the restriction due to positivity $0 \leq Z_3 \leq 1$ (where $Z_3$ is the electromagnetic charge renormalization constant squared), and the restriction due to the Ward identity $Z_1 = Z_2$ (where $Z_1^{-1}$ and $\sqrt{Z_2}$ are the electromagnetic-vertex and charge-field renormalization constants, respectively). Luckily, there is no argument of this sort, for example, for the $\gamma_5$ coupling; no corresponding Ward identity exists for this latter case, as has been discussed in Ref. 1, and its damping property may be then self-consistently explored. No estimation, however, is given in this work of the magnitude of the renormalized $\gamma_5$ and $\phi^4$ couplings. As we shall discuss in Sec. III, this is an exceedingly complicated problem.

As an application of the work, we study the reaction $\gamma^*(Q^2) \to$ anything for $Q^2 \to \infty$. This virtual photon may be considered to be attached to some external charge distribution (external current) or "ideally" as the single photon contributing to $e^+e^-$ annihilation with the former decaying into anything. We shall emphasize that our result is presumably to be taken as a truly asymptotic one. We work out the ratio $R(Q^2)$ as $Q^2 \to \infty$ of the cross section associated with this reaction to the corresponding one with the strong couplings set equal to zero (and perhaps naively) as an asymptotic extrapolation of the object of experimental interest in this reaction. $R(Q^2)$ is shown to increase linearly with $Q^2$ (broken possibly by multiplicative logarithmic powers). Contrary to common belief, the leading contribution to $R(Q^2)$ comes from the strong interaction. The asymptotic form of the photon spectral function in finite quantum electrodynamics *at* the eigenvalue $\alpha$ is also explicitly given here for the first time. Finally we obtain explicit high-energy estimates for elastic form factors in quantum electrodynamics in isolation and in the present correlated dynamics. At very large momenta for the external photon, the former are bounded by a power-law behavior. We also mention some interesting aspects of this Abelian gauge theory.

In Sec. II we give a very brief account of the work in Ref. 1, emphasizing, however, only some of the key points. In Sec. III the high-energy behavior of the dynamics is discussed. The asymptotic form of the photon spectral functions are derived in Sec. IV. Sections V A and V B deal with the application of the work to virtual-photon decay and the derivation of the high-energy estimates for the form factors, respectively. A brief discussion on the work is given in Sec. VI with additional comments, and various aspects of this Abelian gauge theory are pointed out.

## II. VERY BRIEF DISCUSSION OF THE STABILITY OF THE EIGENVALUE CONDITION

For completeness and for the convenience of the reader a very brief account of the work in Ref. 1 will be given here, emphasizing, however, only some of the key points involved in the work. Our method of summation for the study of the stability of the eigenvalue condition defined within the single-fermion-loop contribution $\Pi_c^{[1]}$ to the photon self-energy part is the following. We first sum up all the virtual-photon-line corrections to $\Pi_c^{[1]}$ while holding the virtual-pion "variables" fixed. We then carry out the virtual-pion-line integrations as well. This amounts to making an expansion in powers of the $\gamma_5$ coupling ($g^2$) and treating $\alpha$ to all orders, and then summing up in the strong interaction as well. The motivation for this is two-fold. Firstly, this allows us to treat the explicit derivative with respect to $g^2$ appearing in the Callan-Symanzik equation[4] for $\Pi_c^{[1]}$ in an elementary fashion. Secondly, the damping of the effective $\gamma_5$ coupling is not ruled out—the corresponding electromagnetic one, however, is. This method of summation together with renormalization-group techniques[4] yield[1] (the last term comes from the strong corrections)

$$\alpha\Pi_c^{[1]}(Q^2) \underset{Q^2\to\infty}{\sim} \text{finite} + \alpha F^{[1]}(\alpha)\ln(Q^2/m^2) + O((m^2/Q^2)^{\beta_0(\alpha)/2}), \tag{1}$$

where $F^{[1]}(x)$ is the ABJ function.[6] The self-consistency for the finiteness of the self-mass of the fermion in pure electrodynamics (which we assume) requires[7] that $\beta_0(\alpha) > 0$. In the notation of Ref. 2, for example, $\beta_0(\alpha)$ is given by[1] $\beta_0(\alpha) = 2\delta(\alpha)$, where[7] $\delta(\alpha) = 3(\alpha/2\pi) + \frac{3}{4}(\alpha/2\pi)^2 + \cdots$ (with the self-consistency range for the latter[7] $0 < \delta(\alpha) < 2$). From (1) we immediately see that the ABJ condition for $\alpha$ remains unaltered, and at the eigenvalue $\alpha$, $\alpha\Pi_c^{[1]}$ is asymptotically finite. If we naively expand the second term in (1) in powers of $\beta_0(\alpha)$, we generate not only a single power of $\ln(Q^2/m^2)$ which "modifies" the ABJ function, but we also generate arbitrary powers of the former. So one should be very careful when making contact with perturbation theory results. As mentioned in Ref. 1 this sort of question originated our investigation. The dimensional parameter $m$ in (1) denotes the renormalized mass of the fermion. It is important to note that $\alpha\Pi_c^{[1]}$ contains one over-all closed fermion loop. Accordingly, it does not contain pion and photon self-energy parts and it does not contain pion-pion and photon-photon scattering graphs as well. To proceed to the multi-fermion-loop contribution to the photon self-energy part and discuss its asymptotic finiteness we have made the two very basic assumptions made by Adler[2] in pure electrodynamics:

(i) The theory may be correctly summed up by the loopwise summation manner.

(ii) The $2n$-point electromagnetic-current-correlation functions (in the single-fermion-loop contribution) for $m=0$ and $F^{[1]}(x)$ vanish simultaneously at $x=\alpha$ in pure electrodynamics.

The basic ingredients (facts) that have been shown[1] to be present in going from the single- to the multi-fermion-loop contribution to the photon self-energy part were[1] the following:

(a) A $\pi^0$-$\pi^0$ scattering graph (including the one with four external pion lines) in the single-fermion-loop contribution is finite when summed to *all* orders in $\alpha$ and vanishes very rapidly in the asymptotic region.

(b) The $\pi^0$-$\gamma$ scattering system also damps out quite fast at high energies.

(c) The strong corrections to $\gamma$-$\gamma$ scattering graphs damp out rapidly at high energies, and *at* $\alpha$ their *purely* electromagnetic contribution to the just-mentioned graphs in turn vanishes[2] rapidly asymptotically.

These are some key points in the study. We have mainly given a very brief summary of this work and discussed some basic steps involved in our study. For details as well as for other results we refer the reader to Ref. 1.

Finally, we wish to point out that from (i) and (ii) above it follows formally by an elementary induction proof, for example, that the infinite-order zero[2] of $F^{[1]}(\alpha)$ at $\alpha$ implies the same property for the full Callan-Symanzik function in pure electrodynamics (see Ref. 1). We also remind the reader that the vanishing of the ABJ function is *necessary* for the internal consistency for a completely finite quantum electrodynamics.[8]

## III. HIGH-ENERGY BEHAVIOR

We wish to study the high-energy behavior of our dynamics with $\alpha$ as the infinite-order zero (see also Sec. II) of the Callan-Symanzik function [i.e., coefficient of $(\partial/\partial\alpha)$ in the Callan-Symanzik equations[4,2]] in pure electrodynamics. The interaction Lagrangian density under consideration is chosen to be

$$\mathcal{L}_I = g_0\bar{\Psi}\gamma_5\Psi\phi + e_0\bar{\Psi}\gamma_\mu\Psi A^\mu - \tfrac{1}{4}\lambda_0\phi^4,$$

where the symbols have their usual meaning. Let $\tilde{D}(Q^2)$ be the renormalized photon propagator with $\tilde{D}(Q^2)=d_c(Q^2)/Q^2$. The object $\alpha d_c(Q^2)$, for example, satisfies a Callan-Symanzik equation[4,2] for $Q^2\to\infty$ of the form

$$\left[m^2\frac{\partial}{\partial m^2}+\tfrac{1}{2}\alpha(\tilde{\chi}_0+\tilde{\chi}_1)\frac{\partial}{\partial\alpha}+\hat{\chi}_1\frac{\partial}{\partial g^2}+\hat{\chi}_2\frac{\partial}{\partial\lambda}\right]\alpha d_c(Q^2)=0, \tag{2}$$

where $m$—the scale parameter—is chosen to be the mass of the fermion, and the coefficients of the derivatives with respect to the couplings define the various Callan-Symanzik functions of the theory. $\tilde{\chi}_0(\alpha)$ is the corresponding one in pure electrodynamics [this is denoted by $\beta(\alpha)$ in Ref. 2; we have chosen our earlier notation in Ref. 1 for consistency]. We may also scale $Q^2\to\eta Q^2$ and define the parameter $\kappa\equiv\ln\eta$. The relevant effective couplings in the theory then formally satisfy the well-known form of renormalization-group equations[4,5,9]:

$$\frac{d}{d\kappa}\alpha(\kappa)=\frac{\alpha(\kappa)}{2}\left[\tilde{\chi}_0(\alpha(\kappa))+\tilde{\chi}_1(\alpha(\kappa),g^2(\kappa),\lambda(\kappa))\right], \tag{3}$$

$$\frac{d}{d\kappa}g^2(\kappa)=\hat{\chi}_1(\alpha(\kappa),g^2(\kappa),\lambda(\kappa)), \tag{4}$$

$$\frac{d}{d\kappa}\lambda(\kappa)=\hat{\chi}_2(\alpha(\kappa),g^2(\kappa),\lambda(\kappa)), \tag{5}$$

where $\alpha(\kappa)$, $g^2(\kappa)$, and $\lambda(\kappa)$ denote the effective electromagnetic, $\gamma_5$ (squared), and the $\phi^4$ couplings, respectively. The formal boundary conditions to the above equations are

$$\alpha(0)=\alpha,\quad g^2(0)=g^2,\ \text{and}\ \lambda(0)=\lambda, \tag{6}$$

respectively, with the latter parameters denoting renormalized quantities. Each of the above effective couplings satisfies an identical equation as in (2), i.e. (in long hand),

$$\left[-\frac{\partial}{\partial\kappa}+\tfrac{1}{2}\alpha(\tilde{\chi}_0+\tilde{\chi}_1)\frac{\partial}{\partial\alpha}+\hat{\chi}_1\frac{\partial}{\partial g^2}+\hat{\chi}_2\frac{\partial}{\partial\lambda}\right]\begin{Bmatrix}\alpha(\kappa)\\ g^2(\kappa)\\ \lambda(\kappa)\end{Bmatrix}=0. \tag{7}$$

As mentioned in the Introduction, our method of study will be to make an expansion in these equations in powers of the $\gamma_5$ and $\phi^4$ couplings. For simplicity, we study the equations for $g^2(\kappa)$ and $\lambda(\kappa)$ by omitting first the $\tilde{\chi}_1$ term in (7), then taking its action into consideration in a self-consistent manner, and in Eq. (7) setting $\hat{\chi}_{1,2}\to 0$ for $\alpha(\kappa)$ also. In this case the equations for $g^2(\kappa)$ and $\lambda(\kappa)$ "decouple" from the one for $\alpha(\kappa)$ and the expansion coefficients of $\hat{\chi}_{1,2}$ in powers of $g^2$ and $\lambda$ for the former effective couplings become *simply* parametrized by $\alpha$.[10] We shall argue that the effective couplings $g^2(\kappa)$ and $\lambda(\kappa)$ may become small for $\kappa\to\infty$. To do this we expand the right-hand sides of Eqs. (4) and (5) in powers of $g^2(\kappa)$ and $\lambda(\kappa)$,[10] e.g.,

$$\frac{d}{d\kappa}g^2(\kappa)=-\tfrac{1}{2}\beta_0(\alpha)g^2(\kappa)+\cdots \tag{8}$$

[note $\hat{\chi}_1 \propto O(g^2)$], where the parameter $\beta_0(\alpha)$ has been discussed in the previous section and is certainly very welcome here since it introduces a negative contribution to (8). Other corrections on the right-hand side of (8) are of the order $g^4(\kappa)$, $g^2(\kappa)\lambda^2(\kappa)$ (and so on), and are expected to vanish faster than the $g^2(\kappa)$ term retained in (8) if $g^2(\kappa)$ and $\lambda(\kappa)$ do indeed vanish as $\kappa\to\infty$, as we shall argue. Equation (8) may be readily integrated to yield

$$g^2(\kappa)=g^2\exp[-\tfrac{1}{2}\beta_0(\alpha)\kappa]. \tag{9}$$

Similarly, we expand the right-hand side of (5) to obtain[10]

$$\frac{d}{d\kappa}\lambda(\kappa)=-\eta_0(\alpha)g^4(\kappa)+\eta_2\lambda^2(\kappa)+\cdots, \tag{10}$$

where $\eta_2$ is easily extracted to be $\eta_2=9/8\pi^2$ and is independent of $\alpha$. The positive sign of $\eta_2$ is directly related to the negative sign of the $\phi^4$ coupling chosen in our Lagrangian.[11] Other corrections on the right-hand side of (10) are of the order $g^2(\kappa)\lambda^2(\kappa), g^4(\kappa)\lambda(\kappa), \ldots$ . From Eq. (10) we learn that $\lambda(\kappa)$ cannot vanish slower than $g^2(\kappa)$. Similarly, we see that it cannot vanish like $g^2(\kappa)$ (as is usually expected) since $dg^2(\kappa)/d\kappa \propto g^2(\kappa)$ [and not $\propto g^4(\kappa)$ (Ref. 12)]. In perturbation theory, $\eta_0(\alpha)$ may be computed, and has the form

$$\eta_0(\alpha)=\frac{1}{4\pi^2}[1+O(\alpha)], \tag{11}$$

and, at least in the perturbative calculation sense in $\alpha$, it is strictly positive [the requirement of the knowledge of the property of the full $\eta_0(\alpha)$, i.e., to all orders in $\alpha$, may be somewhat relaxed for the internal consistency of our solutions as we shall discuss shortly]. Equation (10) then may be integrated to give

$$\lambda(\kappa)=\lambda\exp[-\beta_0(\alpha)\kappa], \tag{12}$$

with $\lambda\sim g^4\eta_0(\alpha)/\beta_0(\alpha)>0$. Clearly, this latter solution goes beyond perturbation theory and is based on electrodynamics in a nontrivial manner. This in turn signals the important fact that when the theory is summed to all orders in $\alpha$, a typical $\pi^0$-$\pi^0$ scattering graph (with four external pion lines) of order $g^4$ is already finite and the so-called $\phi^4$ term does not have the usually expected compensating role.[1] Equation (12) also suggests that the $\lambda$ coupling be treated as a $g^4$ effect. In precise terms, this suggests scaling the couplings $g^2\to\xi g^2$ and $\lambda\to\xi^2\lambda$ and treating $\xi$ as an expansion parameter, and finally setting $\xi=1$.[13] Before we proceed we would like to make some important remarks. Although the solutions (9) and (12) vanish rapidly as $\kappa\to\infty$, it is not clear how high in $\kappa$ one has to go so that these are the leading ones for $\kappa$ large. Clearly, this will depend on how large $\beta_0(\alpha)$ is. We cannot also give an estimate of the allowed magnitude for the renormalized coupling $g^2$, for example. This presumably requires detailed knowledge of the other expansion coefficients in (8) (as functions of $\alpha$) and is a problem which goes beyond perturbation theory. We shall make no attempt, however, to discuss these points as they are actually beyond our reach. Accordingly, we do not necessarily mean that these renormalized couplings may be allowed to be arbitrarily large. On the other hand, if, for example, $g^2$ is very small, say of the order $\alpha$, then this problem is almost understood since one may treat the above-mentioned coefficients by their lowest nontrivial expressions in $\alpha$, and the main ones [$\eta_0(\alpha)$ and $\beta_0(\alpha)$] we already know. Now we come back to Eq. (7) for $g^2(\kappa)$ and $\lambda(\kappa)$ and take the $\tilde{\chi}_1(\partial/\partial\alpha)$ term into account. We note that $\tilde{\chi}_1$ is of the order $\xi$. Accordingly, our solutions are indeed correct to the leading order in $\xi$. We make an expansion in powers of $\xi$ in Eq. (7) for $g^2(\kappa)$ and $\lambda(\kappa)$ and readily infer (in a standard manner[1]) self-consistently that these effective couplings have the following most general forms[10]:

$$g^2(\kappa)\sim O(\exp(-\tfrac{1}{2}\beta_0\kappa)\times\text{powers of }\kappa)$$

and

$$\lambda(\kappa)\sim O(\exp(-\beta_0\kappa)\times\text{powers of }\kappa)$$

for $\kappa\to\infty$. The powers of $\kappa$ multiplying the *already* damping factor $\exp(-\tfrac{1}{2}\beta_0\kappa)$ come from the $\tilde{\chi}_1(\partial/\partial\alpha)$ term when expanding in powers of $\xi$. Similarly, we make an expansion for $\alpha(\kappa)$, $\alpha(\kappa)=\alpha+g^2f_1(\kappa)+\cdots$, and substitute the latter in Eq. (7) to obtain the following at the eigenvalue $\alpha$:[14]

$$\alpha(\kappa)\sim\text{finite}+O(\exp(-\tfrac{1}{2}\beta_0\kappa)\times\text{powers of }\kappa)$$

Similarly, by making an expansion in powers of $\xi$, it is easily seen[1] that any basic Green's function, when the external momenta (nonexceptional and spacelike) are taken to infinity simultaneously, is of the form

$$G(\kappa)\underset{\kappa\to\infty}{\simeq}G_0(\kappa)[C_0+O(\exp(-\tfrac{1}{4}\beta_0N\kappa)\times\text{powers of }\kappa)]$$
$$\times\exp[-\gamma_0(\alpha)\kappa][\text{powers of }(\partial/\partial\alpha)]$$
$$\times\exp[\gamma_0(\alpha)\kappa], \tag{13}$$

where $G_0(\kappa)$ denotes the corresponding Green's function in pure electrodynamics (if there is one) and $\gamma_0(\alpha)$ denotes the purely electromagnetic contribution (if there is one) of the sum of the anomalous dimensions of the fields making up $G_0(\kappa)$. The constant $C_0$ is either finite or zero. $N$ inside the bracket is a positive integer and is a "measure" of the minimum number of pion lines contributing to $G(\kappa)$ for $\xi \neq 0$. Equation (13) is a compact notation. The powers of $(\partial/\partial\alpha)$ come from the $\tilde{\chi}_1(\partial/\partial\alpha)$ term and it is understood that the identity $(\partial/\partial\alpha)^0$ contributes to them. For the renormalized photon propagator, for example, $\gamma_0 \equiv 0$, and the object $d_c(\kappa)$ [from (13)] is given by

$$d_c(\kappa) \underset{\kappa\to\infty}{\simeq} \text{finite} + O(\exp[-\tfrac{1}{2}\beta_0\kappa]\times\text{powers of } \kappa), \quad (14)$$

and a similar expression results for the corresponding object for the renormalized pion propagator. For a $\pi^0$-$\pi^0$ scattering graph with four external lines, $\gamma_0 \equiv 0$, $G_0 \to 1$, $C_0 \equiv 0$, $N=4$, i.e.,

$$\Gamma_{\pi^0-\pi^0}(\kappa) \underset{\kappa\to\infty}{\simeq} O(\exp[-\beta_0\kappa]\times\text{powers of } \kappa). \quad (15)$$

For gauge-dependent objects, we can always find a suitable gauge[1] so that all the expansion coefficients of $\gamma_0(\alpha)$ are identically equal to zero (see the first paper in Ref. 1 for the explicit gauges). In this case the powers of $(\partial/\partial\alpha)$ will not contribute since all the expansion coefficients of $\gamma_0(\alpha)$ may be consistently chosen to be identically equal to zero by appropriate choice of a gauge. The asymptotic finiteness of the theory may be directly discussed from (13) (see also Ref. 1).

Although we do obtain very interesting damping for the $\gamma_5$ and $\phi^4$ couplings at high energies, we shall make no attempt to discuss convergence problems and shall not dwell further on the powers of logs multiplying the *already* damping factor $\exp(-\frac{1}{2}\beta_0\kappa)$. Here it should be noted that our definition of the summation procedure by expanding in $\xi$ and taking $\kappa\to\infty$ (term by term) is not in general equivalent to a reversed way of summation, e.g., through the replacement $\alpha(\kappa)\to\alpha(\infty)$ everywhere first, since $\tilde{\chi}_0$ does not in general vanish at $\alpha(\infty)$. (This reminds us of the well-known situation of the inequivalence (*vis-à-vis*) of the loopwise[2] summation procedure and the so-called vacuum-polarization-insertion-wise[3] summation procedure in pure electrodynamics with the correction [see Eq. (30) in Ref. 2] $h(Q^2)$ to $\alpha d_c(Q^2)-\alpha_0$ replacing $\alpha(\kappa)-\alpha(\infty)$ (in here)). Summarizing then, our method of study is based directly on the Callan-Symanzik equations together by making an expansion in powers of $\xi$ with $\kappa\to\infty$ (and not otherwise). This has been applied to the effective couplings *as well as* to the Green's functions. Needless to say, our solutions are true only for $\alpha\neq 0$—the latter as the infinite-order zero of $\tilde{\chi}_0(\alpha)$.

For applications, we shall omit, from now on, the powers of $\kappa$ multiplying the already damping object $\exp(-\frac{1}{2}\beta_0\kappa)$ in Eq. (14) for the renormalized photon propagator with no striving for rigor.

## IV. PHOTON SPECTRAL FUNCTION

In this section we derive the asymptotic form of the photon spectral function. We first consider pure electrodynamics. The Callan-Symanzik equation for $\alpha d_c(Q^2)$, for all $Q^2$, is given by (see, e.g., Ref. 2) in our notation (see Sec. II)

$$\left[m\frac{\partial}{\partial m}+\alpha\tilde{\chi}_0(\alpha)\frac{\partial}{\partial\alpha}\right][\alpha d_c(Q^2)]^{-1}=[1+\delta(\alpha)]\tilde{\Gamma}_{\gamma\gamma S}(Q^2), \quad (16)$$

where ($\Pi$ is the unrenormalized photon self-energy part)

$$\tilde{\Gamma}_{\gamma\gamma S}(Q^2)=m_0(\partial/\partial m_0)\Pi(Q^2),$$
$$d_c^{-1}(Q^2)=1+\alpha[\Pi(Q^2)-\Pi(0)], \quad (17)$$
$$\frac{m}{m_0}\frac{d}{dm}m_0=1+\delta(\alpha),$$

and $2\delta(\alpha)=\beta_0(\alpha)$. The parameter $m_0$ denotes the unrenormalized mass of the fermion. The total differentiation in the last expression in (17) with respect to $m$ is taken with the unrenormalized fine-structure constant $\alpha_0$ and the ultraviolet cutoff introduced in the unrenormalized theory fixed.[2] All the quantities in Eq. (16) denote renormalized objects. The cutoff independence of $\tilde{\Gamma}_{\gamma\gamma S}$ is easily established.[2,15] For $Q^2\to\infty$, the right-hand side of (16) vanishes like $m/Q$ from Weinberg's theorem.[16] More precisely, $\tilde{\Gamma}_{\gamma\gamma S}$ is an even function of $m$ (as may be checked in perturbation theory) and hence it vanishes like $m^2/Q^2$ for $Q^2\to\infty$. Accordingly, *at* the eigenvalue $\alpha$, the leading contribution to $\alpha d_c(Q^2)$ is given by[2,10]

$$\alpha d_c(Q^2)=q(\alpha),\quad Q^2\to\infty \quad (18)$$

where[9]

$$q(\alpha)=\alpha-(5/9\pi)\alpha^2+\cdots.$$

[The relation (18) is true with $\alpha$, the renormalized fine-structure constant as the zero (of infinite order) of $\tilde{\chi}_0(\alpha)$.] To find the next-to-the-leading, energy-dependent corrections to the right-hand side of (18) (at $\alpha$) we have to consider the behavior of the inhomogeneous part in (16) for $Q^2\to\infty$.[17] To do this, we consider in turn the Callan-Symanzik equation for $\tilde{\Gamma}_{\gamma\gamma S}$, which is easily derived to be

$$\left\{m \frac{\partial}{\partial m} + \alpha\tilde{\chi}_0(\alpha) \frac{\partial}{\partial \alpha} - [1+\delta(\alpha)]\right\} \tilde{\Gamma}_{\gamma\gamma S} = [1+\delta(\alpha)]\tilde{\Gamma}_{\gamma\gamma SS}\,, \quad (19)$$

where

$$\tilde{\Gamma}_{\gamma\gamma SS} = m_0{}^2 \frac{\partial}{\partial m_0{}^2}\,\Pi\,. \quad (20)$$

The cutoff independence of $\tilde{\Gamma}_{\gamma\gamma SS}$ is easily seen.[2,15] Weinberg's theorem[16] states that $\tilde{\Gamma}_{\gamma\gamma SS}(Q^2)$ vanishes like $m^2/Q^2$. The solution to the homogeneous part of (19) (at $\alpha$) is

$$\tilde{\Gamma}_{\gamma\gamma S}(Q^2) \sim C(m^2/Q^2)^{[1+\delta(\alpha)]/2}\,, \quad (21)$$

and hence[18] $C \equiv 0$ [note $\delta(\alpha) \propto \alpha$] since $\tilde{\Gamma}_{\gamma\gamma S}$ is even in $m$ (as may be checked in perturbation theory with gauge invariance invoked). Finally we write the Callan-Symanzik equation for $\tilde{\Gamma}_{\gamma\gamma SS}$ [to study the particular solution of (19)], which is given by

$$\left\{m \frac{\partial}{\partial m} + \alpha\tilde{\chi}_0(\alpha) \frac{\partial}{\partial \alpha} - 2[1+\delta(\alpha)]\right\} \tilde{\Gamma}_{\gamma\gamma SS}(Q^2) \sim 0\,, \quad Q^2 \to \infty \quad (22)$$

where the right-hand side vanishes like $(m^2/Q^2)^2$ in perturbation theory.[16] From Eqs. (16)–(22) we obtain, at $\alpha$,[17] the solution

$$\alpha d_c(Q^2) \underset{Q^2\to\infty}{\sim} q(\alpha) + \alpha q_0(\alpha)(m^2/Q^2)^{1+\beta_0(\alpha)/2} \quad (23)$$

[Note the factor of 2 multiplying $1+\delta(\alpha)$ on the left-hand side of (22)]. The very rapid energy-dependent correction in (23) for $Q^2 \to \infty$ is certainly very interesting.[17] We would like to emphasize rather strongly that an expression like (23) is presumably to be taken as a *truly asymptotic* one. Accordingly, a completely different behavior at "relatively lower" energies is not ruled out. This may happen, for example, if smaller energy-dependent corrections add up to a different expression than the leading one at lower energies. With this we emphasize that the expressions to be obtained below then are, presumably, also to be taken as truly asymptotic ones.

We now turn to the photon spectral function. The object $d_c(Q^2)$ may be given (by invoking completeness etc.) in terms of a spectral decomposition,

$$d_c(Q^2) = 1 + Q^2 \int_0^\infty \frac{d\mu^2}{\mu^2} \frac{\rho^{(0)}(\mu^2)}{(\mu^2+Q^2)} \quad (24)$$

(where the factor 1 corresponds to the photon pole for $Q^2 \to 0$), with $Q^2 > 0$ (spacelike), and we have absorbed any discontinuous function that may appear in $\rho^{(0)}(\mu^2)$. The spectral function $\rho^{(0)}(\mu^2)$ is essentially obtained from the vacuum expectation value of the correlation of two electromagnetic currents. The electromagnetic charge (squared) renormalization constant $Z_3$ is formally given by

$$\frac{1}{Z_3} = 1 + \int_0^\infty \frac{d\mu^2}{\mu^2}\,\rho^{(0)}(\mu^2)\,. \quad (25)$$

When $\alpha$ is the (infinite-order) zero[2] of $\tilde{\chi}_0(\alpha)$, $1/Z_3$ is finite.[2] Accordingly, Eq. (25) formally implies the boundary conditions $\rho^{(0)}(\infty) = 0 = \rho^{(0)}(0)$. On dimensional grounds, $d_c$ is a function of the ratio $Q^2/m^2$ and $\rho^{(0)}$ is a function $\mu^2/m^2$. We are interested in the limit $Q^2 \to \infty$; we cannot, however, take the limit $m^2 \to 0$ instead inside the integral (24) in $\rho^{(0)}(\mu^2)$ without the knowledge of the behavior of the latter for $\mu^2 \to 0$ (with $m$ fixed), i.e., at the lower limit of the integration. To bypass this point, we apply the differential operator $\partial/\partial m^2$ to (24), use the boundary conditions just mentioned for $\rho^{(0)}(\mu^2)$ and integrate by parts twice to obtain in an elementary fashion

$$m^2 \frac{\partial}{\partial m^2}\left(m^2 \frac{\partial}{\partial m^2} - 1\right) d_c(Q^2) = 2\int_0^\infty \frac{x\,dx}{(1+x)^3}\,\rho^{(0)}\left(\frac{xQ^2}{m^2}\right). \quad (26)$$

The additional power of $x$ in the numerator improves the behavior of the integrand at the lower limit of the integration without changing its high-energy behavior. From dimensional reasons, we see from Eqs. (23) and (26) that $\rho^{(0)}(Q^2)$ has the form

$$\rho^{(0)}(Q^2) \underset{Q^2\to\infty}{\sim} C_0(\alpha)(m^2/Q^2)^{1+\beta_0(\alpha)/2}\,. \quad (27)$$

Upon substituting (23) and (27) back into (26) we readily obtain[19]

$$\pi \csc(-\tfrac{1}{2}\beta_0\pi) C_0(\alpha) = -q_0(\alpha)\,. \quad (28)$$

The coefficient $q_0(\alpha)$ is easily computed in perturbation theory (cf. Ref. 20) to be

$$q_0(\alpha) = \frac{2\alpha}{\pi}\,[1 + O(\alpha)]\,. \quad (29)$$

To our knowledge, the asymptotic form of the photon spectral function in finite quantum electrodynamics at the eigenvalue $\alpha$ has been explicitly given here in Eqs. (27)–(29) for the first time. Positivity requires that $C_0(\alpha) > 0$. It is easily verified, by taking into consideration that $\csc(x\pi)$ has a singularity (among other ones) at $x = 0$, that this positivity condition is indeed satisfied at least to the lowest-order expansion of $C_0(\alpha)$ in $\alpha$. It is interesting to point out that $C_0(\alpha)$ starts at $\alpha^2$ rather than at $\alpha$ and is easily computed from (28) to have the form $C_0(\alpha) = (3\alpha^2/\pi^2)[1+O(\alpha)]$, where we have formally used the value of $\beta_0(\alpha) = (3\alpha/\pi) + \cdots$ in Ref. 7. This coincides with the coefficient of $m^2/\mu^2$ as $\mu^2 \to \infty$ coming from the so-called photon self-energy proper[21] diagrams in the classic work of Källén and Sabry[20] on the evaluation of the spectral function to fourth order in the charge $e$. What we also learn from the fact that $C_0(\alpha) \propto \alpha^2$ is that the coefficient of $\alpha(m^2/\mu^2)$ in $\rho^{(0)}(\mu^2)$ is identically equal to zero for $\mu^2 \to \infty$. When we go back to the work of Ref. 20 we see that this is indeed the case [see, for example, Eq. (6) in this reference]. From this one formally understands why the so-called improper diagrams[21] (to the order $\alpha^2$) do not contribute to $\rho^{(0)}(\mu^2)$ as $\mu^2 \to \infty$. Very roughly speaking, they must yield to an expression of the form $(m^2/\mu^2)^2$ or smaller as $\mu^2 \to \infty$ in the summed-up finite theory, and hence are nonleading. Finally, that from the second-order expression (29) we were able to reproduce the fourth-order expression for $C_0(\alpha)$ directly without further work is certainly interesting. $\csc(-\frac{1}{2}\beta_0\pi)$ has also a singularity at $\frac{1}{2}\beta_0 = 1$. The value of $\frac{1}{2}\beta_0$ may be allowed, however, formally to be extended above this value. It is important to realize that the form in (27) does not imply that the integral expression for $1/Z_3$ in (25) develops a singularity at the lower limit $\mu^2 \sim 0$, since the spectral function may have a completely different behavior there for $m^2 \neq 0$ in contrast to the situation with $\mu^2 \neq 0$ and $m \sim 0$. The substitution of (27) into the expression (26) is justified because of the additional improvement obtained in the low-energy region ($x \sim 0$), in (24), by partial integration. In standard perturbation theory, $\rho^{(0)}(\mu^2)$ goes like a constant plus terms which increase with powers of $\ln(\mu^2)$ plus terms which vanish like $m^2/\mu^2$ up to powers of logs.[20] In the summed finite theory, at the eigenvalue $\alpha$, we obtain a leading (convergence) factor $m^2/\mu^2$ to $\rho^{(0)}(\mu^2)$ and an additional factor $(m^2/\mu^2)^{\beta_0/2}$ from summing up expressions with arbitrary powers of the logarithm of $\mu^2$.

In the correlated dynamics, the leading (energy-dependent) contribution to $d_c(Q^2)$, as $Q^2 \to \infty$, as we have seen in Sec. III comes from the $g^2$ coupling [Eq. (14)]. It is important to notice the difference in $d_c(Q^2)$ between the situation in pure electrodynamics [Eq. (23)] and in the correlated dynamics [Eq. (14)]. The reason for this is that the just-mentioned energy-dependent leading term in pure electrodynamics comes from the inhomogeneous part of the Callan-Symanzik equation at $\alpha$. In the correlated dynamics [Eq. (14)], the leading energy-dependent part comes from the homogeneous part. It may be verified (as done above) that the inhomogeneous part for this latter case vanishes with an additional $1/Q^2$ factor to the homogeneous (energy-dependent) solution. Thus we clearly distinguish between the above two cases. A similar analysis as above yields, formally for the correlated dynamics[22] (up to logs),

$$\rho(Q^2) \underset{Q^2\to\infty}{\sim} C(m^2/Q^2)^{\beta_0(\alpha)/2}, \tag{30}$$

with $C \propto g^2$.[23] As emphasized above, the expressions (27) and (30) are presumably to be taken as truly asymptotic ones. At lower energies a completely different behavior is not ruled out. The difference in the behavior of (27) and (30) in our study has been discussed above. The damping of $\rho^{(0)}(Q^2)$ and $\rho(Q^2)$ at truly asymptotic $Q^2$ for a *finite* theory is, of course, expected and is not surprising.

## V. APPLICATIONS

### A. $\gamma^*$ decay

In this subsection we discuss the decay of a virtual photon $\gamma^*(Q^2) \to$ anything for $Q^2 \to \infty$. As mentioned in the Introduction, this vurtual photon may be considered to be attached to some external charge distribution or it may be "ideally" considered as a single photon contributing to $e^+e^-$ annihilation, with the former decaying in turn into anything. As is well known,[24] the cross section for this latter process is given by $\sigma(e^+e^- \to \text{anything}) \sim \rho(Q^2)/Q^2$. From Eq. (27), we see that finite electrodynamics (at $\alpha$) predicts a very rapid decrease at *truly asymptotic* $Q^2$. For "relatively lower" energies this seems to indicate that improved techniques are to be devised in place of the ones used here or one must simply rely on the lowest-order perturbation-theory result to obtain the experimental result for the object of interest which goes like $1/Q^2$. Similarly, the expression (30) in the correlated dynamics predicts also a very rapid decrease as $Q^2 \to \infty$; however, at a slower rate than (27). Of course these results are not necessarily in contradiction with experiment as they are extrapolated to the truly asymptotic region. At relatively lower energies a completely different behavior for these quantities is not ruled out. Experimentally, at energies of a few orders of the mass of a hadron $\rho(Q^2)$, with strong interaction, seems to increase linearly with $Q^2$. Similarly, when strong interactions are not involved (for example $e^+e^- \to \mu^+\mu^-$), the corresponding $\rho^{(0)}(Q^2)$ is flat at the above-mentioned energies. Accordingly, the object of interest $R(Q^2) = \rho(Q^2)/\rho^{(0)}(Q^2)$ seems to have a linear increase in $Q^2$.[25] Therefore the best we can do is to also consider this object in this work as it is extrapolated to the truly asymptotic region in $Q^2$. We may avoid referring to the dimensional scale parameters in the just-mentioned expression by choosing arbitrarily some point $Q'^2$, with $Q'^2 < Q^2 \to \infty$, to obtain from (27) and (30)

$$R_{Q'^2}(Q^2) \equiv \frac{R(Q^2)}{R(Q'^2)} \underset{Q'^2 < Q^2 \to \infty}{\sim} \frac{Q^2}{Q'^2} , \qquad (31)$$

where $R(Q^2) = \rho(Q^2)/\rho^{(0)}(Q^2)$. How fundamental our result (31) is (or the related expressions are) we do not know; the linear increase in $Q^2$, however, is intriguing. We shall, however, not speculate further on this. [If in $\rho^{(0)}(Q^2)$ we restrict the decay of $\gamma^*$ to a fermion-antifermion pair then the equality in (31) changes to an inequality (see Sec. V B) $R_{Q'^2}(Q^2) \geq Q^2/Q'^2$]. We have also to bear in mind that a single-photon contribution in the above-mentioned object is taken into account.

### B. High-energy estimates for form factors

Our high-energy estimates for the elastic form factors also come from the photon spectral functions. The latter may be written in terms of a unitarity sum,

$$Q^2\rho(Q^2) \sim \sum_{(n)} \langle 0|j_\mu|n\rangle\langle n|j_\mu|0\rangle , \qquad (32)$$

or symbolically as

$$\rho(Q^2) = \sum_{(n)} \rho_n(Q^2) . \qquad (33)$$

The elastic form factors[26] are most conveniently defined from a specific term in (33) with $|n\rangle$ corresponding to a fermion-antifermion state through[27]

$$e\langle \bar{u}(p)|F_1\gamma_\mu + F_2\sigma_{\mu\nu}Q^\nu/2m\,|v(p')\rangle ,$$

where $F_1(Q^2)$ and $F_2(Q^2)$ define the so-called Dirac form factors normalized as $F_1(0)=1$, $F_2(0)=\mu$ (the anomalous magnetic moment of the fermion) in units of $e$ and $e/2m$, respectively. $|u(p)\rangle$ and $|v(p')\rangle$ denote fermion and antifermion states, respectively, with $Q = p + p'$, and $j_\mu$ as defined in (32) is the electromagnetic current. In terms of the experimentally more interesting electric and magnetic form factors,

$$G_E(Q^2) = F_1(Q^2) + \frac{Q^2}{4m^2} F(Q^2) ,$$
$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2) , \qquad (34)$$

the fermion-antifermion contribution to $\rho(Q^2)$ in (33) is given by[27] ($\alpha = e^2/4\pi$)

$$\rho_2(Q^2) = \frac{\alpha}{3\pi}\left(1 - \frac{4m^2}{Q^2}\right)^{1/2} \times\left[|G_M(Q^2)|^2 + \frac{2m^2}{Q^2}|G_E(Q^2)|^2\right] . \qquad (35)$$

Positivity requires, for example, that $\rho(Q^2) \geq \rho_2(Q^2)$.[28] Because of the $1/Q^2$ factor multiplying $|G_E(Q^2)|^2$ in (35) one may assume that the $G_M$ term will dominate[29] over the other one as $Q^2 \to \infty$ in (35). Equation (35) together with Eqs. (27) and (30) then give the interesting estimates[30]

$$|G_M(Q^2)| \leq O((m^2/Q^2)^{[1+\beta_0(\alpha)/2]/2}), \quad Q^2 \to \infty \qquad (36)$$

in pure finite electrodynamics and

$$|G_M(Q^2)| \leq O((m^2/Q^2)^{\beta_0(\alpha)/4}), \quad Q^2 \to \infty \qquad (37)$$

in the correlated dynamics. [The scale parameters may be taken to be different in the expressions (36) and (37).] As mentioned before, it is not clear how high in energy one has to go in the estimates in (36) and (37) and they presumably are to be taken as truly asymptotic expressions as done earlier. It may be possible, for example, that at lower energies the expression corresponding to (27) varies slower than the one corresponding to (30) [and hence (36) varies slower than (37)]. It is very interesting that finite electrodynamics makes an explicit prediction of the form in (36). Whether expressions of the sort in (36) will be useful in settling such questions as "Does nature pick $\alpha$ or the unrenormalized $\alpha_0$ as the zero of the ABJ eigenvalue condition?" in the sense of Refs. 2 and 3 we do not know. This might be an interesting point to consider further but we shall not dwell upon it here. Elastic form factors[31] may be defined independently of which process they participate in and a similar study to the above may be carried out with the external photon replaced by a pion, for example.

## VI. DISCUSSION

We have first quickly reviewed some of the key points involved in our earlier study of the stability of the eigenvalue condition for $\alpha$. We have then studied the high-energy behavior of our correlated dynamics including the $\phi^4$ term with $\alpha$ as the (infinite-order) zero of $\tilde{\chi}_0(\alpha)$ (see also Sec. II). We have argued that the effective $\gamma_5$ and $\phi^4$ couplings in this Abelian gauge theory may become small at very high energies and finally vanish, interestingly even faster than the presently estimated *non*-Abelian gauge theory ones. A general basic Green's function has been also studied at high energies. Our method of summation has been clearly stated. No attempt, however, has been made to study convergence problems and make estimates for the allowed magnitude of the renormalized couplings $g^2$ and

$\lambda$—problems which are certainly beyond what may be tackled by our presently available techniques. *Truly asymptotic* statements have been made for $\gamma^*$ decay and for the well-known object $R(Q^2)$, which we have seen to increase linearly in $Q^2$ (up to logs). Truly high-energy estimates have also been made for the magnetic form factors both in pure (finite) electrodynamics and in the correlated dynamics. The explicit asymptotic form of the photon spectral function has been also obtained.

Finally, we wish to make an interesting remark concerning our study of this correlated dynamics. Very roughly and perhaps naively, electrodynamics seems as if it "reduces" the space-time dimension of the pion-fermion "sector" of the theory from 4 to $4-\epsilon$ [the latter in the language of Wilson[32]—see in particular Eq. (4.14) in Ref. 32], with $\epsilon \sim \beta_0(\alpha)$. Here, of course, we do not have to (and we cannot, since $\alpha$ is fixed) take the limit $\epsilon \to 0$, since $\epsilon$ (which is not an artifice) emerges naturally from the physics as quantum radiative corrections and is a measure of how fast[1] the $\gamma_5$ interaction, for example, damps out at high energies.[33] Seemingly, electrodynamics acts as if to "reduce" the pion-fermion "sector" of the theory to a superrenormalizable one.

More complicated applications of this work, such as to deep-inelastic scattering, are under study.

ACKNOWLEDGMENT

It is a pleasure to thank the Dublin Institute for Advanced Studies for the very kind hospitality.

---

[1]E. B. Manoukian, Phys. Rev. D 10, 1883 (1974); 10, 1894 (1974).

[2]S. L. Adler, Phys. Rev. D 5, 3021 (1972); 7, 1984(E) (1973).

[3]M. Baker and K. Johnson, Phys. Rev. D 3, 2541 (1971); K. Johnson and M. Baker, *ibid.* 8, 1110 (1973).

[4]C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

[5]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, *ibid.* 30, 1346 (1973); S. Coleman and D. J. Gross, *ibid.* 31, 851 (1973).

[6]For a convenient discussion of this, see the Appendix of the second paper in Ref. 1.

[7]K. Johnson, M. Baker, and R. Willey, Phys. Rev. 136, B1111 (1964); M. Baker and K. Johnson, Phys. Rev. D 3, 2516 (1971).

[8]For the explicit statement of this in the literature, see Ref. 2, p. 3029, and the second paper in Ref. 3, p. 1118.

[9]M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954); N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959).

[10]The effective electromagnetic coupling in *pure* electrodynamics, at the eigenvalue $\alpha$, is given by $\alpha(\kappa)=\alpha$ (Ref. 2). As an illustration, consider typically an interesting example for $\frac{1}{2}\alpha(\kappa)\tilde{\chi}_0(\alpha(\kappa))$ due to Adler (Ref. 2) [vs Eq. (101) with $p=1$]:

$$\tfrac{1}{2}\alpha(\kappa)\tilde{\chi}_0(\alpha(\kappa))=[\alpha-\alpha(\kappa)]^2\exp\{-1/[\alpha-\alpha(\kappa)]\}.$$

The solution to

$$d\alpha(\kappa)/d\kappa=\tfrac{1}{2}\alpha(\kappa)\tilde{\chi}_0(\alpha(\kappa))$$

is

$$\alpha(\kappa)-\alpha=-1/\ln(\kappa-\kappa'+\exp\{1/[\alpha-\alpha(\kappa')\})|_{\kappa'\to 0},$$

and hence $\alpha(\kappa)=\alpha$ for all $\kappa$. More generally, from the equation $[-(\partial/\partial\kappa)+\frac{1}{2}\tilde{\chi}_0(\partial/\partial\alpha)]\alpha(\kappa)=0$, by a loopwise summation procedure [by following Eqs. (121)–(127) in Ref. 2], for example, it is easily seen inductively that $\alpha(\kappa)=\alpha$ for *all* $\kappa$ by finally invoking the boundary condition $\alpha(0)=\alpha$. It is also easily seen by the same procedure from (i) and (ii) in Sec. II (see the second paper in Ref. 1, p. 1898) that $(\partial/\partial\alpha)^s\alpha(\kappa)$ is independent of $\kappa$ for *all* $\kappa$, with $s=1,2,\ldots$ *at* the eigenvalue $\alpha$. This last property is used *unambiguously* when the $\tilde{\chi}_1(\partial/\partial\alpha)$ term is taken into account in our expansion procedure in powers of the $\gamma_5$ and $\phi^4$ couplings.

[11]As often argued in the literature, this is the correct sign (see, e.g., last paper in Ref. 5).

[12]The relation $dg^2(\kappa)\propto g^2(\kappa)d\kappa$ rather than $\propto g^4(\kappa)d\kappa$ is of great importance in this work and should be kept in mind. As a matter of fact, if this relation were not realized, then the solutions presented would not have necessarily been admitted. This relation also distinguishes our dynamics from other ones (see, for example, Ref. 5).

[13]Needless to say, it is understood, here and throughout, that the scaling of the couplings by $\xi$ is made, unambiguously, only after having carried out (disentangled) the differentiation with respect to $g^2$ and $\lambda$, in the Callan-Symanzik equations, of the relevant objects under study when expanded in powers of $g^2$ and $\lambda$.

[14]The finite part in general depends also on $\xi$.

[15]One is inserting the composite object $\overline{\Psi}\Psi$ in the photon self-energy part. Accordingly, one extracts the object $Z_2$ to define renormalized fields and also extracts a scalar-vertex renormalization constant $Z_S{}^{-1}$ to make the composite object in turn finite in perturbation theory. Thus the over-all constant multiplying the resulting renormalized object is $m_0Z_2/Z_S$. The latter is well known to be cutoff-independent [see Appendix A in the paper by S. L. Adler and W. A. Bardeen, Phys. Rev. D 4, 3045 (1971); 6, 734(E) (1972), for a complete discussion, for example].

[16]S. Weinberg, Phys. Rev. 118, 838 (1960).

[17]We are emphasizing the statement at $\alpha$. Needless to say, for example, in the JBW (Johnson-Baker-Willey) (Refs. 3 and 7, and references therein) solution to electrodynamics, the energy-dependent rapid damping we derived below is not necessarily expected there (Refs. 2 and 3).

[18]For a similar situation occurring elsewhere, see Eqs. (29) and (30) in Ref. 15.

[19]We have become aware that a relation between the photon spectral function and the photon propagator for $Q^2 \to \infty$ has also been given by J. D. Bjorken at the Bonn Conference, 1973 (see, for example, the review article by C. H. Llewellyn Smith on $e^+e^-$ annihilation [CERN Report No. TH. 1849, 1973 (unpublished)]). Unfortunately, the work of Bjorken was not available to us at the time of the writing of this work.

[20]See for example, G. Källén and A. Sabry, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 29, no. 17, 1 (1955).

[21]For a discussion of the work of Källén and Sabry (Ref. 20) in terms of proper and improper diagrams, see B. E. Lautrup and E. de Rafael, Phys. Rev. 174, 1835 (1968).

[22]The mass of the pion does not cause problems. The dimensional scale parameters in Eqs. (23), (27) vs the one in Eq. (30) may in general be taken not to be equal. Since we shall choose later our reference scale parameter at some $Q'^2$ with $Q'^2 < Q^2$, a distinction between the above-mentioned scale parameters is not necessary for a qualitative study.

[23]The coefficient of $g^2$ in here has been worked out to lowest order in $\alpha$, and reduced, however, only to a sum of integrals with four (and some with five) Feynman parameters and is quite cumbersome to write down here. The algebra involved in this parallels the one in the evaluation of the fourth-order *finite* part of $Z_3$ in pure electrodynamics and hence is extremely tedious [see, for example, Refs. 20 and 21 (and references therein) for a summary of the study of this latter object].

[24]N. Cabibbo and R. Gatto, Phys. Rev. 124, 1577 (1961); J. D. Bjorken, *ibid.* 148, 1467 (1966).

[25]A. Litke *et al.*, Phys. Rev. Lett. 30, 1189 (1973); G. Tarnopolsky *et al.*, *ibid.* 32, 432 (1974).

[26]Elastic form factors cannot in general be rigorously defined for charged particles. The reason associated with this is the well-known infrared-divergence problem in quantum electrodynamics [see, e.g., D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961)]. Accordingly, either a nonzero small mass must be supplied to the photon in perturbation theory or one may define "inelastic" form factors which admit the creation also of very soft photons. For a lucid discussion of the former point see, e.g., R. Barbieri, J. A. Mignaco, and E. Remiddi, Nuovo Cimento 11A, 824 (1972). In this subsection we shall ignore these points.

[27]See, e.g., S. D. Drell and F. Zachariasen, Phys. Rev. 119, 463 (1960).

[28]This basic positivity condition together with Eq. (35) have been used to obtain the very interesting result relating the finiteness of $1/Z_3$ and the vanishing of the form factors $F_1, F_2$ asymptotically in Ref. 27. We were inspired by this work in this subsection.

[29]Such an attitude has also been taken in other recent works [A. De Rújula, Phys. Rev. Lett. 32, 1143 (1974); D. J. Gross and S. B. Treiman, *ibid.* 32, 1145 (1974)]. Experimentally [see, e.g., P. N. Kirk *et al.*, Phys. Rev. D 8, 63 (1973)] because of this $1/Q^2$ factor multiplying $G_E^2(Q^2)$, it is more difficult to obtain as good information on $G_E$ as on $G_M$ for large $Q^2$. In *perturbation theory* one has $F_2 \sim (1/Q^2) \times$ logarithmic growth and $F_1 \sim$ logarithmic growth, and hence $G_{E,M} \sim$ logarithmic growth and this assumption is correct there.

[30]Our estimates are strictly valid for $Q^2 > 0$, i.e., in the timelike region, and we cannot *a priori* say anything about their continuation elsewhere.

[31]High-energy behavior of elastic form factors has also been recently studied (by renormalization-group techniques) in a class of theories [S.-S. Shei, Phys. Rev. D 11, 164 (1975); G. C. Marques, *ibid.* 9, 386 (1974)]. Unfortunately, in this class of theories the strong dynamics may not be admitted to damp out at high energies (according to the last paper in Ref. 5).

[32]K. G. Wilson, Phys. Rev. D 7, 2911 (1973).

[33]As already emphasized in Ref. 1, our work introduces the very interesting property of the damping of strong interaction as the origin of (electromagnetic) radiative corrections. This idea has also been mentioned there generally in other field theories as well, and is under study to be explored and sharpened in the general case.