Dynamics of symmetry breaking

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We show that the SU(n)-symmetry breaking is obtained from $d_{ijk}x_jx_k = \lambda x_i$. Two derivations of this equation are given: (a) using a bootstrap approach, and (b) minimizing an SU(n)-invariant potential, leading to spontaneous breakdown of the symmetry. We discuss the implementation of the second approach for the groups SU(3) and SU(4).

I. INTRODUCTION

One of the most puzzling problems in particle physics is the introduction of an internal symmetry group in order to immediately break that perfect symmetry. The central question is whether the violation of these symmetries must be obtained from *ad hoc* postulates for every type of interaction and every possible group or whether one can write a basic equation (or equations) *in the group space* so that by choosing a particular group, defined on a specific Hilbert space, we have already chosen the symmetry breaking.

It is assumed that the Hamiltonian which describes the strong interactions of hadrons has the form

$$H = H_0 + X , \quad (1.1)$$

where H_0 is invariant under a group G and X transforms as an irreducible representation of G. It has been suggested by several authors that the breaking of SU(3) symmetry of strong interaction physics may be of a dynamical origin[1-6] and that this leads to the specific equation[5,6]

$$X_S X = \lambda X . \quad (1.2)$$

On the left-hand side of this equation, one has to take the decomposition of the tensor product of irreducible representation $X \otimes X$ into a direct sum of irreducible representations and to project out the X representation (see Appendix). Equation (1.2) with $\lambda = 0$ was derived[6] for the group $SU(3)^+ \times SU(3)^-$ ($\pm$ are denoting the chirality). Moreover, in a very elegant approach Michel and Radicati[6] analyzed the relation between the solutions of this equation and the geometry of the groups SU(3) and $SU(3)^+ \times SU(3)^-$.

The effects of weak and electromagnetic interactions are represented by Y in the Hamiltonian

$$H = H_0 + X + Y . \quad (1.3)$$

Using the approach of Michel and Radicati[6] for the group $SU(3)^+ \times SU(3)^-$ the following equation was derived[7]:

$$P^X(X_S Y) = 0 , \quad (1.4)$$

where P^X projects out the X representation from the direct product of $X \otimes Y$.

Recently,[8] it was shown that with the introduction of an SU(4) symmetry for the known leptons (e, ν_e, ν_μ, μ), Eq. (1.4) predicts two massless neutrinos, a small ratio for m_e/m_μ, and a Weinberg angle $\sin^2\theta_w = \frac{1}{4}$. We shall not discuss the effects of the weak and electromagnetic interaction.

In this paper we suggest that for any group SU(n), the simplest possible equation (1.2) in the group space yields the phenomenologically assumed breaking symmetry. In Sec. II we give two derivations of Eq. (1.2) for the group SU(n) for every $n \geq 3$ (for $n = 2$ $X_S X = 0$ in a trivial way): (a) using a bootstrap approach, and (b) minimizing an SU(n)-invariant potential, in which case we have a spontaneous breakdown of the symmetry. In Sec. III we give two examples: the groups SU(3) and SU(4). We end with an appendix summarizing the symmetrical and antisymmetrical algebras defined on the $n^2 - 1$ vector space.

II. SU(n) SYMMETRY BREAKING

A. A bootstrap approach

We assume a bootstrap type of equation

$$M = f(M) , \quad (2.1)$$

where M is a mass matrix, or any other physical quantity of interest, transforming as the $n^2 - 1$ multiplet of the group SU(n). The covariance of Eq. (2.1) under the group SU(n) implies

$$f(uMu^{-1}) = uf(M)u^{-1} , \quad (2.2)$$

where u is an SU(n) transformation. The most general $n \times n$ Hermitian matrix can be described as

$$M = X_0 \cdot 1 + \sum_{i=1}^{n^2-1} X_i \lambda_i , \quad (2.3)$$

where λ_i are defined in the Appendix and X_0 and X_i ($i = 1, \ldots, n^2 - 1$) are real numbers. Substituting

(2.3) into (2.2) we have n^2 equations

$$X_0 = f_0(X_0, X_j)\ ,\qquad X_i = f_i(X_0, X_j)\ . \tag{2.4}$$

Eliminating X_0 from these equations gives

$$X_i = f_i(X_1, X_2, \ldots, X_N)\ , \tag{2.5}$$

with $i=1,\ldots,N$ and $N=n^2-1$. f_i must be a vector in the n^2-1 Euclidean space R^{n^2-1} (see Appendix). The most general vector in R^{n^2-1} is given by

$$f_i = A(\alpha,\beta)X_i + B(\alpha,\beta)\,d_{ijk}X_jX_k\ , \tag{2.6}$$

where

$$\alpha = \sum_{i=1}^{n^2-1} X_iX_i\ ,\qquad \beta = \sum_{i,j,k=1}^{n^2-1} d_{ijk}X_iX_jX_k\ . \tag{2.7}$$

and the coefficients d_{ijk} are defined in the Appendix. Equations (2.5) and (2.6) imply

$$d_{ijk}X_iX_j = \lambda X_k\ . \tag{2.8}$$

This is also the simplest possible equation in the group space (a polynomial equation of second order). As mentioned in the Introduction, Eq. (2.8) was derived by many authors for the group SU(3).

B. Spontaneous symmetry breaking

The spontaneous breakdown of a symmetry group G is displayed in the appearance of nonzero vacuum expectation values of a multiplet of scalar fields ϕ_i. The vacuum expectation value of $\langle\phi_i\rangle_0 = \eta_i$ is determined (to zeroth order) by the condition

$$\frac{\partial V(\phi)}{\partial \phi_i} = 0 \quad \text{at } \phi_i = \eta_i\ , \tag{2.9}$$

where $V(\phi)$ corresponds to the classical potential energy density function of the scalar field ϕ. $V(\phi)$ is assumed to be invariant for the group under consideration, SU(n). In general, the solutions $\phi_i = \eta_i$ might be invariant under some subgroup H of SU(n). By defining new scalar fields

$$\Phi_i = \phi_i - \eta_i \tag{2.10}$$

so that $\langle\Phi_i\rangle_0 = 0$, we get a potential $V(\Phi)$ which is (at least) invariant under the group $H \subset$ SU(n). Thus, by starting with a Lagrangian $\mathcal{L}_0(\phi,\ldots)$ which is invariant under the group SU(n), we end with a Lagrangian $\mathcal{L}(\Phi,\ldots)$ which is invariant only under a subgroup H of SU(n). This results from the spontaneous breakdown suggested above, requiring redefinition of the scalar fields $\phi_i \to \Phi_i$.

The renormalizability requirement of the theory implies that $V(\phi)$ has to be at most quartic (with canonical dimension 4). If ϕ_i transforms as the adjoint representation of SU(n), then the most general potential $V(\phi)$ invariant under SU(n) in a renormalizable theory is

$$V(\phi) = \tfrac{1}{2}m_0^{\,2}\phi_i\phi_i + \tfrac{1}{4}\chi(\phi_i\phi_i)^2 + g d_{ijk}\phi_i\phi_j\phi_k\ . \tag{2.11}$$

Condition (2.9) gives

$$m_0^{\,2}\eta_i + \chi\eta^2\eta_i + 3g d_{ijk}\eta_j\eta_k = 0\ . \tag{2.12}$$

For $g \neq 0$ Eq. (2.12) reduces to Eq. (2.8), the equation for symmetry breaking derived in the bootstrap approach. Making the transformation (2.10) we end with a potential

$$V(\Phi) = \{\tfrac{1}{2}(m_0^{\,2} + \chi\eta^2)\delta_{ij} + 3g d_{ijk}\eta_k + \chi\eta_i\eta_j\}\Phi_i\Phi_j + \{g d_{ijk} + \chi\delta_{jk}\delta_{il}\eta_l\}\Phi_i\Phi_j\Phi_k + \tfrac{1}{4}\chi(\Phi^2)^2 + \text{const.} \tag{2.13}$$

Now we include a pseudoscalar multiplet P_i ($i=1,\ldots,n^2-1$) transforming as the adjoint representation of SU(n). The SU(n)-invariant, renormalizable, and parity-conserving Lagrangian $\mathcal{L}_0(P,\phi)$ of the pseudoscalar and scalar multiplets is

$$\mathcal{L}_0(P,\phi) = -\tfrac{1}{2}\partial_\mu P_i\partial_\mu P_i - \tfrac{1}{2}M_0^{\,2}P_iP_i - G d_{ijk}P_iP_j\phi_k - \tfrac{1}{4}\mu(P_iP_i)^2 - \tfrac{1}{2}\nu(P_i\phi_i)^2 - \tfrac{1}{2}\partial_\mu\phi_i\partial_\mu\phi_i - V(\phi)\ , \tag{2.14}$$

where $M_0^{\,2}$, G, μ, and ν are parameters and $V(\phi)$ is given in Eq. (2.11). Performing the transformation (2.10), we obtain a new Lagrangian (due to the spontaneous breakdown) which in general is not invariant under SU(n), and can be written

$$\mathcal{L}_0(P,\phi) \to \mathcal{L}(P,\Phi) = -\tfrac{1}{2}\partial_\mu P_i\partial_\mu P_i - \tfrac{1}{2}(M_0^{\,2}\delta_{ij} + 6G d_{ijk}\eta_k + \nu\eta_i\eta_j)P_iP_j - (G d_{ijk} + \nu\delta_{ik}\eta_j)P_iP_j\Phi_k - \tfrac{1}{4}\mu(P_iP_i)^2 - \tfrac{1}{2}\nu(P_i\Phi_i)^2 - \tfrac{1}{2}\partial_\mu\Phi_i\partial_\mu\Phi_i - V(\Phi)\ , \tag{2.15}$$

where $V(\Phi)$ is given in Eq. (2.13). To summarize: The dynamics of symmetry breaking are Eqs. (2.8) and (2.15).

III. THE GROUPS SU(3) AND SU(4)

Now we discuss the implementation of Eq. (2.8) and the Lagrangian (2.15) for the groups SU(3) and SU(4).

A. The group SU(3)

Phenomenologically, the SU(3) symmetry is broken by (semi-) strong interactions preserving the U_{2y} [the SU(2) group of isospin rotation and the U group of hypercharge] subgroup of SU(3). This can be described by the Lagrangian[9]

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_8 , \tag{3.1}$$

where $\mathcal{L}_0$ is invariant under SU(3) and $\mathcal{L}_8$ transforms as the eighth component of an SU(3) octet.

A possible solution of Eq. (2.8),

$$d_{ijk} X_j X_k = \lambda X_i , \tag{3.2}$$

is given by

$$X_i = 0 \quad (i = 1, 2, \ldots, 7), \quad X_8 = \eta_8 \neq 0 . \tag{3.3}$$

Substituting this solution into Eq. (2.15) we get

$$\begin{aligned}\mathcal{L}(P, \Phi) = & -\tfrac{1}{2}\partial_\mu P_i \partial_\mu P_i - \tfrac{1}{2}(M_0{}^2\delta_{ij} + 6\eta_8 G d_{8ij} + \nu\eta_8{}^2\delta_{i8}\delta_{j8})P_i P_j - (G d_{ijk} + \nu\eta_8\delta_{ik}\delta_{j8})P_j P_k \Phi_k - \tfrac{1}{4}\mu(P_i P_i)^2 \\ & -\tfrac{1}{2}\nu(P_i\Phi_i)^2 - \tfrac{1}{2}\partial_\mu\Phi_i\partial_\mu\Phi_i - \tfrac{1}{2}(m_0{}^2\delta_{ij} + \chi\eta_8{}^2\delta_{i8}\delta_{j8} + 6\eta_8 g d_{8ij})\Phi_i\Phi_j - (g d_{ijk} + \chi\eta_8\delta_{jk}\delta_{i8})\Phi_i\Phi_j\Phi_k + \tfrac{1}{4}\chi(\Phi_i\Phi_i)^2 ,\end{aligned} \tag{3.4}$$

where ϕ_i is a scalar octet ($J^P = 0^+$) and P_i is a pseudoscalar octet ($J^P = 0^-$).

It is interesting to point out that to first order in η_8 the Lagrangian (3.4) has the same transformation properties as (3.1). Thus, neglecting the $\eta_8{}^2$ terms, e.g., assuming $\eta_8 \ll 1$ or $\nu \ll 1$, one has Gell-Mann–Okubo mass formulas for the scalar and pseudoscalar octets.

B. The group SU(4)

The increasing interest in gauge theories of leptons and hadrons together with the experimental evidence of the existence of strangeness-conserving neutral currents and the nonexistence of strangeness-changing neutral currents has renewed the interest in the SU(4) group as an approximate symmetry of hadrons.[10-12] Moreover, the recently discovered narrow resonances[13] [ψ(3100) and ψ(3700)] are causing even a greater interest in the group SU(4).

In this case, the scalar and the pseudoscalar multiplets transform as a "15" multiplet (the fifteenfold way). A possible solution of Eq. (2.1) is (this solution is not unique)

$$X_i = 0 \quad (i = 1, \ldots, 7, 9, \ldots, 14), \quad X_8 = \eta_8 , \quad X_{15} = \eta_{15} . \tag{3.5}$$

d_{ijk} for the group SU(4) are given in Ref. 12. Substituting this solution into Lagrangian (2.15) we get for the mass of the pseudoscalar particles the following expressions:

$$\begin{aligned} m_\pi{}^2 &= M_0{}^2\left(1 + \frac{1}{\sqrt{3}}\,\epsilon_8 + \frac{1}{\sqrt{6}}\,\epsilon_{15}\right) , \\ m_K{}^2 &= M_0{}^2\left(1 - \frac{1}{2\sqrt{3}}\,\epsilon_8 + \frac{1}{\sqrt{6}}\,\epsilon_{15}\right) , \\ m_D{}^2 &= M_0{}^2\left(1 + \frac{2}{2\sqrt{3}}\,\epsilon_8 - \frac{1}{\sqrt{6}}\,\epsilon_{15}\right) , \\ m_F{}^2 &= M_0{}^2\left(1 - \frac{1}{\sqrt{3}}\,\epsilon_8 - \frac{1}{\sqrt{6}}\,\epsilon_{15}\right) , \end{aligned} \tag{3.6}$$

where $\epsilon_8 \equiv 6\eta_8 G/M_0{}^2$, $\epsilon_{15} \equiv 6\eta_{15} G/M_0{}^2$, and D and F are the isospin-doublet and isospin-singlet charmed pseudoscalar particles. Under the group SU(3), (DF) transforms as the triplet. For the 8th and the 15th [and possibly the SU(4)-singlet] members of this multiplet, one has to diagonalize a 2×2 (or 3×3) mass matrix in order to obtain the physical masses of these neutral particles. An immediate sum rule which follows from Eq. (3.6) is

$$m_K{}^2 - m_\pi{}^2 = m_F{}^2 - m_D{}^2 . \tag{3.7}$$

The SU(3) symmetry is violated by the ϵ_8 term, while SU(4) is broken down to SU(3) by the ϵ_{15} term. The parameters ϵ_8 and ϵ_{15} are given in terms of the pseudoscalar masses:

$$\epsilon_8 = -\frac{4}{\sqrt{3}}\left(\frac{m_K{}^2 - m_\pi{}^2}{m_F{}^2 + m_\pi{}^2}\right) \simeq -\frac{4}{\sqrt{3}}\frac{m_K{}^2}{m_F{}^2} , \tag{3.8}$$

$$\epsilon_{15} = -\sqrt{6}\,\frac{3m_F{}^2 + m_\pi{}^2 - 4m_K{}^2}{3(m_F{}^2 + m_\pi{}^2)} \simeq -\sqrt{6}\left(1 - \frac{4}{3}\frac{m_K{}^2}{m_F{}^2}\right). \tag{3.9}$$

For a value of $m_F \simeq 2$ GeV we get

$$\frac{\epsilon_8}{\epsilon_{15}} \simeq 0.065\,, \qquad (3.10)$$

suggesting that SU(3) is a better symmetry than SU(4).

APPENDIX: THE SYMMETRICAL AND ANTISYMMETRICAL SU(n) ALGEBRAS ON THE n^2-1 VECTOR SPACE

The group SU(n) is the group of $n\times n$ unitary unimodular matrices

$$U^{-1}=U^\dagger\,, \quad \det U=1\,. \qquad (A1)$$

Every $U\in$ SU(n) can be written in the form

$$U=e^{iX}\,, \qquad (A2)$$

where X is an $n\times n$ Hermitian traceless matrix

$$X=X^\dagger, \quad \mathrm{tr}X=0\,. \qquad (A3)$$

The set of all X form an (n^2-1)-dimensional real vector space R^{n^2-1}. The Euclidean scalar product of two vectors X and Y is defined by

$$(X,Y)=(Y,X)=\tfrac{1}{2}\mathrm{tr}(XY), \quad X, Y\in R^{n^2-1}\,. \qquad (A4)$$

The action of SU(n) on R^{n^2-1} is given by

$$X\rightarrow UXU^\dagger \qquad (A5)$$

for every $U\in$ SU(n) and every $X\in R^{n^2-1}$. This transformation leaves invariant the scalar product defined in (A4). [A function $f(x)$ of n vectors of R^{n^2-1} is invariant under SU(n) if for every $U\in$ SU(n) $f(UX_1U^{-1},\ldots,UX_nU^{-1})=f(X_1,\ldots,X_n)$.] The scalar product (A4) is the only SU(n)-invariant bilinear function of two vectors. For three vectors X, Y, Z, there are two linearly independent trilinear invariants

$$\{X,Y,X\}=\tfrac{1}{2}\sqrt{n}\ \mathrm{tr}((XY+YX)Z)\,, \qquad (A6)$$

$$[X,Y,Z]=-\tfrac{1}{2}i\ \mathrm{tr}((XY-YX)Z)\,, \qquad (A7)$$

where (A6) [(A7)] is completely symmetric (antisymmetric) in the three vectors. Therefore, there are only two linearly independent algebras on R^{n^2-1}:

(i) the Lie algebra

$$X_A Y=-\tfrac{1}{2}i(XY-YX)\,, \qquad (A8)$$

(ii) the symmetrical algebra

$$X_S Y=\tfrac{1}{2}\sqrt{n}\,(XY+YX)-\frac{1}{\sqrt{n}}\ \mathrm{tr}(XY)\,. \qquad (A9)$$

It is convenient to describe the vector R^{n^2-1} space by the Gell-Mann matrices λ_i $(i=1,2,\ldots,n^2-1)$ satisfying

$$(\lambda_i,\lambda_j)=\tfrac{1}{2}\mathrm{tr}(\lambda_i\lambda_j)=\delta_{ij}\,. \qquad (A10)$$

Since λ_i, $i\lambda_i$, 1, and $i1$ span the space of all complex $n\times n$ matrices, it follows that

$$\lambda_i\lambda_j=\frac{2}{n}\,\delta_{ij}+(d_{ijk}+if_{ijk})\lambda_k\,, \qquad (A11)$$

where

$$[\lambda_i,\lambda_j]=2if_{ijk}\lambda_k\,, \qquad (A12)$$

$$\{\lambda_i,\lambda_j\}=\frac{4}{n}\,\delta_{ij}+2d_{ijk}\lambda_k\,. \qquad (A13)$$

f_{ijk} and d_{ijk} are respectively totally antisymmetric and totally symmetric in i, j, and k.

Every $n\times n$ Hermitian matrix $X\in R^{n^2-1}$ can be described in the λ_i basis as

$$X=X_i\lambda_i\,, \qquad (A14)$$

where X_i $(i=1,\ldots,n^2-1)$ are real numbers and transform according to the adjoint representation of SU(n). The scalar product (A4) of two vectors X_i and Y_i $(i=1,\ldots,n^2-1)$ is given by

$$(X,Y)=X_iY_i\,, \qquad (A15)$$

and the antisymmetric and symmetric algebras (A8) and (A9) are

$$X_AY=f_{ijk}X_iY_j\lambda_k \qquad (A16)$$

or

$$X_{iA}Y_j=f_{kij}X_iY_j \qquad (A16')$$

and

$$X_SY=\sqrt{n}\ d_{ijk}X_iY_j\lambda_k \qquad (A17)$$

or

$$X_{iS}Y_j=\sqrt{n}\ d_{kij}X_iY_j\,. \qquad (A17')$$

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