How quark confinement solves the $\eta \to 3\pi$ problem

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Gauge theories of quarks which have chiral SU(3)×SU(3) symmetries necessarily also have a ninth axial-vector symmetry. Empirically this symmetry is not manifest in the particle spectrum in either the conventional or Goldstone form. We argue on the basis of an exactly solvable model that the apparent lack of the ninth axial-vector symmetry is due to the same long-range forces which confine quarks in infrared-unstable Yang-Mills theories. A related problem is the violation of Sutherland's low-energy theorem for the decay $\eta \to \pi^+\pi^-\pi^0$. We show how the same phenomenon naturally ruins the validity of Sutherland's conclusion for soft neutral pions. In the course of this work we discuss the Schwinger model with massive fermions. On the basis of mass perturbation theory and simple energy considerations, it appears that this is a well-behaved theory of confinement.

I. INTRODUCTION

Of all the theoretical methods which have been invented to describe hadrons, only the algebra of conserved and partially conserved quark currents has led to precise connections between underlying fundamental degrees of freedom and observed hadron properties.[1] Of the 18 vector and axial-vector currents forming the algebra U(3)×U(3), nature seems to have utilized only 17 to define the exact and approximate conservation laws of the strong interactions. The conspicuous absence of the ninth axial-vector current, j_μ^5, as a partially conserved current is puzzling and without a natural explanation. The puzzle escalates into a true dilemma in those theories which utilize pure vector forces to bind and scatter hadrons.[2] In these theories the conservation of the 17 SU(3) ×SU(3)× (baryon number) currents implies the conservation of j_μ^5. In particular, the exciting possibility of a non-Abelian Yang-Mills theory[3] involving asymptotic freedom,[4] infrared slavery,[5] and color falls into this class of theories. The Lagrangian is

$$\mathcal{L} = \int d^3r[\bar\psi(i\partial\!\!\!/ + gA\!\!\!/_\alpha C^\alpha)\psi - \tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha + \bar\psi M\psi] \tag{1.1}$$

where C_α are the eight color matrices, $F^\alpha_{\mu\nu}$ is the field strength constructed from the non-Abelian potential A^α_μ, and M is a color-invariant quark mass matrix. In order that the usual eight chiral currents

$$j^\mu_{5,i} = \bar\psi\gamma^\mu\gamma^5\lambda_i\psi \tag{1.2}$$

be conserved, the mass matrix M must vanish. But in this case the additional ninth axial-vector current,

$$j_5^\mu = \bar\psi\gamma^\mu\gamma^5\psi, \tag{1.3}$$

is also conserved.

Under ordinary circumstances, invariance under the symmetry generated by j^5 would require degenerate parity doublets (with the possible exception of massless fermions) or massless pseudoscalar Goldstone bosons. Since both are empirically ruled out, we are faced with one of two possibilities: Reject the pure vector theory and introduce other objects which can break j_μ^5 invariance without spoiling SU(3)×SU(3) (Ref. 6); or search for new modes of symmetry which do not entail either degenerate multiplets or massless pseudoscalars. This paper is about just such a new symmetry mode and its application to the ninth axial-vector symmetry.

We call the new mechanism *seizing of the vacuum* or just plain "seizing" for short. The effect is caused by long-range forces which prevent distant parts of the vacuum from behaving independently. In fact we shall show that the seizing of the vacuum is caused by exactly those long-range forces which prevent quarks from being produced in high-energy collisions.

Long-range forces of the required kind can only be present in infrared-unstable theories. Unfortunately, a realistic Yang-Mills theory is far too intractable for us to make rigorous conclusions about either quark confinement or seizing. However, an exactly solvable infrared-unstable gauge theory exists. It is the (1 + 1)-dimensional Schwinger model (QED in one space dimension).[7] Quarks are confined in this model and a spontaneously broken chiral symmetry is *not* accompanied by massless Goldstone bosons.[8,9] Accordingly, the first part of this article concerns the axial-vector current and vacuum seizing in the Schwinger model.

In the second part we assume that quarks are confined by infrared instabilities in 3 + 1 dimensions. It is shown that the ninth axial-vector current should misbehave just as in one dimension, eliminating the need for a ninth Goldstone boson.

It has long been felt that a proper solution to the puzzle of j_μ^5 should also solve another nagging current-algebra problem. Sutherland[10] has pointed out that an application of conventional current algebra and PCAC (partial conservation of axial-vector current) methods fails to describe the decay $\eta\to\pi^+\pi^-\pi^0$. Sutherland's theorem says that the amplitude is small where experiment says it is big. However, the usual analysis does not account for the misbehavior of j_μ^5 in seized theories. Seizing provides an ideal mechanism for spoiling the Sutherland theorem.

II. THE AXIAL-VECTOR CURRENT IN THE SCHWINGER MODEL

A. The loophole

The evasion of Goldstone's theorem for the ninth axial-vector current is only possible because of the gauge freedom of the theories we consider. We may either carry out the discussion in a gauge with no unphysical degrees of freedom (Coulomb gauge) or in a covariant gauge such as the Lorentz gauge. In the first case, the loophole to the Goldstone theorem is the long-range action-at-a-distance forces which occur.

Gauge theories may also be formulated in the manifestly relativistically invariant Lorentz gauge in which the interactions are local. In this formulation there is no way out of the Goldstone theorem. Nevertheless, there is still a loophole, and it is that the full space of states, needed to represent the degrees of freedom of the Lorentz gauge, is much larger than the space of physical states. Subsidiary conditions define the physical subspace and all other objects decouple. It can and does happen in some cases that the expected Goldstone boson is not in the physical subspace.

B. The Schwinger model in the Coulomb gauge

The Schwinger model[7] is defined by a Lagrangian

$$\mathcal{L} = \int dz(i\bar\psi\not\nabla\psi - g\bar\psi\not A\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}) - \mu\int dz\bar\psi\psi . \tag{2.1}$$

When the mass term $\mu\int\bar\psi\psi$ is absent the model is exactly solvable and chirally invariant.

The solvability of the model is due to a remarkable property of one-dimensional fermion systems, namely, that they can be completely described in terms of canonical one-dimensional boson fields.[11] The main correspondences are listed below. In Eqs. (2.2)–(2.6) ϕ is a canonical boson field.

$$j_\mu = \bar\psi\gamma_\mu\psi = \epsilon_{\mu\nu}\partial_\nu\phi/\sqrt{\pi}, \tag{2.2}$$

$$j_\mu^5 = \bar\psi\gamma_5\gamma_\mu\psi = \partial_\mu\phi/\sqrt{\pi}, \tag{2.3}$$

$$\bar\psi\psi = K:\cos 2\sqrt{\pi}\,\phi: , \tag{2.4}$$

$$\bar\psi\gamma_5\psi = K:\sin 2\sqrt{\pi}\,\phi: , \tag{2.5}$$

$$i\bar\psi\not\nabla\psi = \tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi , \tag{2.6}$$

where K is a simple constant.

The skeptical reader may be helped by applying these correspondences to a few examples from massless free theory. According to (2.6) the Lagrangian for the massless free fermion and massless free boson are the same. The vector and axial-vector currents are both conserved for massless free fermions,

$$\partial_\mu j_\mu = 0 , \tag{2.7}$$

$$\partial_\mu j_\mu^5 = 0 . \tag{2.8}$$

Furthermore, in one dimension[12]

$$j_\mu = \epsilon_{\mu\nu}j_\nu^5 \tag{2.9}$$

so that (2.7) and (2.8) imply

$$\Box j_\mu = 0 . \tag{2.10}$$

Since j_μ is conserved, it can always be written in the form of Eq. (2.2). From (2.2) and (2.10) we find

$$\partial_\mu\Box\phi = 0 . \tag{2.11}$$

Thus $\Box\phi=\text{const}$. Sensible boundary conditions on ϕ imply

$$\Box\phi = 0 . \tag{2.12}$$

Furthermore, the Schwinger commutation relation

$$[j_0(z), j_z(z')] = -\frac{i}{\sqrt{\pi}}\,\delta'(z-z') \tag{2.13}$$

is equivalent to the canonical commutation relation

$$[\dot\phi(z), \phi(z')] = -i\delta(z-z') . \tag{2.14}$$

The correspondence between $\bar\psi(1\pm i\gamma_5)\psi \equiv S_\pm$ and the equivalent-boson expressions

$$:\exp(\pm i2\sqrt{\pi}\,\phi): \tag{2.15}$$

can be verified by computing matrix elements of these quantities and comparing them with one another.

The simplest way to set up and solve the Schwinger model is to apply these correspondences to the Lagrangian in the Coulomb gauge. The Coulomb gauge is defined by

$$A_z = 0 \tag{2.16}$$

and

$$\frac{\partial^2 A_t}{\partial z^2} = g\psi^\dagger \psi . \tag{2.17}$$

Equation (2.17) may be integrated to give

$$A_t = \tfrac{1}{2} g \int |z - z'| \psi^\dagger(z')\psi(z') dz' . \tag{2.18}$$

The Lagrangian

$$\mathcal{L} = \int dz (\bar\psi i \not\nabla \psi - g\bar\psi \not A \psi - \tfrac{1}{4} F_{\mu\nu}F^{\mu\nu}) \tag{2.19}$$

is replaced by

$$\mathcal{L} = \int dz (\bar\psi i \not\nabla \psi) - \tfrac{1}{4}\int dz\, dz' \psi^\dagger(z)\psi(z)V(z - z')\psi^\dagger(z')\psi(z'), \tag{2.20}$$

where $V(z - z')$ is the one-dimensional Coulomb potential

$$V(z - z') = \tfrac{1}{2} |z - z'| . \tag{2.21}$$

Using (2.2)–(2.6) we may replace $\mathcal{L}$ by

$$\mathcal{L} = \tfrac{1}{2} \int [\dot\phi^2 - (\partial_z \phi)^2] dz - \tfrac{1}{4} \frac{g^2}{\pi} \int \partial_1 \phi |z - z'| \partial_1' \phi dz\, dz' . \tag{2.22}$$

Our main concern is with the chiral structure of the theory. The chiral charge density is

$$\psi^\dagger \gamma_5 \psi = \frac{1}{\sqrt{\pi}} \partial_t \phi = \frac{1}{\sqrt{\pi}} \Pi_\phi , \tag{2.23}$$

where Π_ϕ is the canonical momentum conjugate to ϕ. The total chiral charge is

$$Q_5 = \int \psi^\dagger \gamma_5 \psi dz = \frac{1}{\sqrt{\pi}} \int \Pi_\phi dz . \tag{2.24}$$

Thus it follows that

$$[\phi(z), Q_5] = i/\sqrt{\pi} , \tag{2.25}$$

and if α is a constant,

$$e^{i\alpha\sqrt{\pi} Q_5} \phi e^{-i\alpha\sqrt{\pi} Q_5} = \phi + \alpha . \tag{2.26}$$

Equation (2.26) defines the action of chiral transformations on the equivalent boson field. This action is to *translate* $\phi(z)$ *by a constant.*

The field ϕ should be interpreted as an angle with periodic boundary conditions. This follows from the fact that the chiral group is compact. In particular it is evident in the fermion representation that

$$\exp(in\pi Q_5)$$

is the unit operator for any integer n. Accordingly, the wave functional in the equivalent boson representation must be invariant under

$$\phi \to \phi + n\sqrt{\pi} . \tag{2.27}$$

This can be ensured by allowing only those states for which Q_5 is an even integer.

Evidently the reason why the free massless Lagrangian and the Lagrangian (2.22) are chirally invariant is because they only contain derivatives of ϕ and are therefore invariant under (2.26). A mass term

$$\mu\bar\psi\psi$$

which breaks chiral symmetry would be represented by

$$K\mu : \cos 2\sqrt{\pi}\, \phi :$$

in equivalent-boson language. This is obviously not invariant under arbitrary chiral transformations although it is invariant under (2.27). The invariance under (2.27) is of course required by the angular character of ϕ.

Now let us attempt to simplify the Lagrangian in (2.22) by integrating the potential-energy term by parts:

$$\begin{aligned} \frac{1}{4} \frac{g^2}{\pi} \int \partial_z \phi |z - z'| \partial_{z'} \phi dz\, dz' &= \frac{1}{4} \frac{g^2}{\pi} \int \phi(z)(\partial_z \partial_{z'} |z - z'|)\phi(z') dz\, dz' \\ &= \frac{1}{2} \frac{g^2}{\pi} \int \phi(z)\phi(z')\delta(z - z') dz\, dz' \\ &= \frac{1}{2} \frac{g^2}{\pi} \int dz \phi^2(z) . \end{aligned} \tag{2.28}$$

This gives a mass term in the Lagrangian for a massive free boson. However, Eq. (2.28) cannot be correct because it no longer has invariance under $\phi \to \phi + \alpha$. The trouble is in the integration by parts which assumes $\phi \to 0$ as $|z| \to \infty$. But since a chiral transformation shifts the value of ϕ at infinity it costs no energy to violate this boundary condition.

The correct procedure is to write ϕ as the sum of a massive field $\hat{\phi}$ and a constant field θ.[9] The chiral transformation acts on θ,

$$e^{i\alpha\sqrt{\pi}Q_5}\hat{\phi}e^{-i\alpha\sqrt{\pi}Q_5}=\hat{\phi}\,, \tag{2.29}$$

$$e^{i\alpha\sqrt{\pi}Q_5}\theta e^{-i\alpha\sqrt{\pi}Q_5}=\theta+\alpha\,. \tag{2.30}$$

Obviously θ is an angular variable. In terms of $\hat{\phi}$ and its canonical momentum $\hat{\Pi}$ the Hamiltonian is

$$H=\frac{1}{2}\int\left[\hat{\Pi}^2+(\partial_z\hat{\phi})^2+\frac{g^2}{\pi}\hat{\phi}^2\right]dz\,. \tag{2.31}$$

The total chiral charge Q_5 is the canonical conjugate to θ:

$$Q_5=\frac{1}{\sqrt{\pi}}\Pi_\theta\,. \tag{2.32}$$

The degrees of freedom θ and Π_θ specify the boundary conditions at infinity. They are independent both dynamically and kinematically of the massive free Klein-Gordon field $\hat{\phi}$. The space of states is a tensor product of a conventional Fock space for the massive field $\hat{\phi}$ and a space spanned by vectors labeled by integer eigenvalues of $\frac{1}{2}Q_5$. The Fock space vectors will be denoted by the symbol $|\Psi)$ and vectors in the "ladder" space by $|n\}$. Vectors in the product space are denoted by $|\Psi\rangle$.

The variables $\hat{\phi}$, $\hat{\Pi}$ act on the space $|\Psi)$ in the usual manner. The quantities θ and Π_θ operate

$$\Pi_\theta|n\}=\sqrt{\pi}\,Q_5|n\} \\ =2\sqrt{\pi}\,n|n\}\,. \tag{2.33}$$

The operators $e^{\pm i2\sqrt{\pi}\,\theta}$ will be called $d^\pm$ and act as raising and lowering operators[9]:

$$d^\pm|n\}=|n\pm 1\}\,.$$

We shall also be interested in the eigenvectors of θ. These are

$$|\theta_0\}=\frac{1}{\pi^{1/4}}\sum_n e^{-2i\sqrt{\pi}n\theta_0}|n\}\,. \tag{2.34}$$

The occurrence of the discrete space $|n\}$ is associated with spontaneous symmetry breaking when long-range forces are present. Since the Hamiltonian does not contain θ or Π_θ the vacuum state is completely ambiguous in the choice of the factor $|\}$. In the original Schwinger solution the vacuum state was chosen to be

$$|0\rangle=|0)|n=0\}\,. \tag{2.35}$$

This is the unique chirally symmetric vacuum and corresponds to the formal perturbation series summation. However, we shall see that with this choice of vacuum matrix elements of local operators violate the cluster property. As an example we can compute matrix elements of $\bar{\psi}(1\pm i\gamma_5)\psi\equiv S_\pm$. According to Eqs. (2.4) and (2.5) $S_\pm$ may be identified with $:e^{\pm i2\sqrt{\pi}\,\hat{\phi}}:d^\pm$. Let us consider

$$\langle 0|TS^+(z)S^-(0)|0\rangle\,. \tag{2.36}$$

This equals

$$(0|T{:}e^{i2\sqrt{\pi}\,\hat{\phi}(z)}{:}\,{:}e^{-2\sqrt{\pi}\,i\hat{\phi}(0)}{:}\,|0)\{0|d^+d^-|0\}\,. \tag{2.37}$$

The first factor is given by

$$\exp[4\pi i\Delta_F(m^2,z^2)]\,, \tag{2.38}$$

where Δ_F is the Feynman propagator which tends to zero as $z\to\infty$. The second factor is unity. Hence the entire matrix element tends to 1 as $z\to\infty$,

$$\lim_{z\to\infty}\langle 0|TS^+(z)S^-(0)|0\rangle=1\,. \tag{2.39}$$

The cluster property requires

$$\lim\langle 0|TS^+(z)S^-(0)|0\rangle=\langle 0|S^+|0\rangle\langle 0|S^-|0\rangle\,. \tag{2.40}$$

Let us evaluate $\langle 0|S^\pm|0\rangle$ to check Eq. (2.40):

$$\langle 0|S^\pm(z)|0\rangle=(0|{:}e^{\pm i2\sqrt{\pi}\,\hat{\phi}}{:}\,|0)\{0|d^\pm|0\}\,. \tag{2.41}$$

Since $d^\pm|0\}=|\pm 1\}$, it is evident that the right-hand side of the equation vanishes.

This violation of the cluster property indicates that a spontaneous symmetry breakdown has occurred and that the vacuum was not wisely chosen. To remedy this, Lowenstein and Swieca[9] prescribe a chirally asymmetric vacuum of the form

$$|\theta_0\rangle=|0)|\theta_0\}\,, \tag{2.42}$$

where $|\theta_0\}$ is an eigenvector of θ. The value of θ_0 is arbitrary and vacua with different θ_0 are related by chiral transformations.

With the new vacuum Eq. (2.41) reads

$$\langle 0|S^\pm|0\rangle=(0|{:}e^{\pm 2i\sqrt{\pi}\,\hat{\phi}}{:}\,|0)e^{\pm i2\sqrt{\pi}\,\theta_0}\,. \tag{2.43}$$

The cluster property is repaired.

C. The effects of long-range forces

Although the vacuum is degenerate, there is an energy gap between the degenerate family of vacua $|\theta\rangle$ and the state with a low-momentum scalar boson. To see how this comes about as a consequence of a long-range force, let us replace the potential energy in the Lagrangian of Eq. (2.20) by a short-range Yukawa potential

$$\frac{g^2}{2}\int\psi^\dagger(z)\psi(z)V(z-z')\psi^\dagger(z')\psi(z')dz\,dz'\,, \tag{2.44}$$

where $V(z-z')$ is given by

$$V(z-z')=\frac{1}{2m}(1-e^{-m|z-z'|}), \tag{2.45}$$

which becomes equal to the Coulomb potential when $m\to 0$. In terms of equivalent boson fields the potential is

$$\frac{1}{2}\frac{g^2}{\pi}\int \partial_z\phi V(z-z')\partial_{z'}\phi dz\,dz' . \tag{2.46}$$

The ground-state expectation value of ϕ may be arbitrarily set equal to zero. A chiral transformation adds a constant to ϕ but clearly does not change the potential energy. We shall call such a transformation a *rigid chiral rotation* in order to indicate that the magnitude of the chiral rotation is constant throughout space.

In considering the possibilities of Goldstone bosons we must estimate the cost in energy of a nonrigid chiral rotation, in which the parameter of the transformation is slowly varied as a function of z. This is accomplished with a generator

$$\sqrt{\pi}\int j_0^5(z)\alpha(z)dz, \tag{2.47}$$

where α is slowly varying. In this case ϕ transforms as

$$\phi(z)\to\phi(z)+\alpha(z) \tag{2.48}$$

and the potential energy has a term

$$\frac{g^2}{2\pi}\int dz\,dz'\partial_z\alpha V(z-z')\partial_{z'}\alpha . \tag{2.49}$$

For example, if α is a plane wave with momentum p, then the potential-energy density becomes

$$V\sim\frac{g^2}{2\pi}p^2V(p^2), \tag{2.50}$$

where

$$V(p^2)=\int e^{ipz}V(z)dz$$
$$=-(p^2+m^2)^{-1}. \tag{2.51}$$

In particular Eqs. (2.50) and (2.51) show that the potential energy of a long-wavelength excitation tends to zero as p^2.

If, on the other hand, $m=0$ the potential energy stored in the long-wavelength excitation is nonzero. The long-range potential prevents distant regions from acting independently and allows only the rigid chiral rotation at a vanishing cost of energy. To the experts in this field the phenomenon is called "seizing."[14] We believe that the seizing of the vacuum with respect to the ninth axial-vector current is caused by the same long-range forces needed to confine quarks in the real world.

D. The algebra of bilinears[15]

The physical space of states of the Schwinger model does not include the entire product space $|)|\}$. The value of θ_0 corresponding to the boundary conditions at spatial infinity should be chosen once and for all. Thereafter no physical operation should change this value. The physical space then consists of all states of the form

$$|\Psi)|\theta_0\},$$

where θ_0 is some arbitrary but fixed angle.

The operator Q_5 has physical matrix elements given by

$$\langle\Psi|Q_5|\Psi'\rangle=(\Psi|\Psi')\{\theta_0|Q_5|\theta_0\} . \tag{2.52}$$

It is tempting to call the expectation value $\{\theta_0|Q_5|\theta_0\}$ zero. In fact, it is given by

$$\{\theta_0|Q_5|\theta_0\}=\sum_{n,m}\frac{1}{\sqrt{\pi}}e^{i(n-m)\theta_0}\{m|Q_5|n\}$$
$$=\frac{2}{\sqrt{\pi}}\sum_{-\infty}^{\infty}n . \tag{2.53}$$

By symmetry the sum in (2.53) may be called zero. However, this can lead to serious contradictions and it is generally better to admit that there is equal probability for all possible values of Q_5. To see the kinds of inconsistencies that can result from calling $\langle Q_5\rangle$ zero we will consider the algebra of bilinear operators. In particular we are interested in the commutation relation

$$[S^+,Q_5]=2S^+, \tag{2.54}$$

which becomes

$$[:e^{i2\sqrt{\pi}\hat{\phi}}:d^+,Q_5]=2:e^{i2\sqrt{\pi}\hat{\phi}}:d^+ . \tag{2.55}$$

When evaluated between physical states the left-hand side of Eq. (2.55) reads

$$(\psi|:e^{i2\sqrt{\pi}\hat{\phi}}:|\psi')[\{\theta_0|d^+Q_5|\theta_0\}-\{\theta_0|Q_5d^+|\theta_0\}]. \tag{2.56}$$

Since $|\theta_0\}$ is an eigenvector of d^+ with eigenvalue $e^{i\theta_0 2\sqrt{\pi}}$, the two terms in square brackets would appear to cancel. On the other hand, the right-hand side of Eq. (2.55) reads

$$2(\psi|:e^{i\hat{\phi}2\sqrt{\pi}}:|\psi')e^{2i\sqrt{\pi}\theta_0}\{\theta_0|\theta_0\} . \tag{2.57}$$

Calling $\{\theta_0|Q_5|\theta_0\}$ any unambiguous value such as zero, evidently leads to contradiction. The contradiction can only be removed by dealing with the vectors $|\theta\}$ using continuum normalization. We write

$$\{\theta'|[Q_5,d^+]|\theta\}=2\{\theta'|d^+|\theta\} . \tag{2.58}$$

The right-hand side is

$$2e^{i2\sqrt{\pi}\,\theta}\,\delta(\theta-\theta')\,. \tag{2.59}$$

The left-hand side is

$$\int d\theta''\{\theta'|Q_5|\theta''\}\{\theta''|\theta\}[e^{2i\sqrt{\pi}\,\theta}-e^{2i\sqrt{\pi}\,\theta'}]\,. \tag{2.60}$$

Using

$$\{\theta'|Q_5|\theta''\}=\frac{1}{i\sqrt{\pi}}\,\delta'(\theta'-\theta'') \tag{2.61}$$

we get

$$\int d\theta''\frac{1}{i\sqrt{\pi}}\,\delta'(\theta'-\theta'')\delta(\theta''-\theta)[e^{i2\sqrt{\pi}\,\theta}-e^{i2\sqrt{\pi}\,\theta'}] = 2e^{i2\sqrt{\pi}}\,\delta(\theta-\theta')\,. \tag{2.62}$$

The point is that the operator Q_5 can not really be represented on the physical subspace since when it acts on $|\theta_0\}$ it changes the boundary conditions.

E. The Schwinger model with mass

In the real hadronic world chiral symmetry is broken by small quark masses. It is therefore important to test the stability of such ideas as quark confinement and seizing with respect to the addition of a small, bare fermion mass in the original Schwinger model. Our main concern is to show that no almost-massless pseudoscalar or charged fermion reappears in the physical spectrum. Unfortunately, very little is rigorously known about the Schwinger model with mass. It is possible, however, to construct a perturbation theory with the exact solution of the massless theory as a starting point. There is nothing in the perturbation series analysis which in any way suggests that the spectrum of states radically changes once the fermion mass is allowed to be nonzero. The mass of the boson field $\hat{\phi}$ appears to vary continuously and the theory has nonvanishing scattering amplitudes which are well behaved. The most direct method to derive the perturbation rules is to continue using the equivalent-boson method.

The additional term $\mu\bar{\psi}\psi$ in the Lagrangian is represented in terms of equivalent boson fields by

$$\mathcal{L}_{\text{int}}=\int\mu K:\cos 2\sqrt{\pi}\,\phi:dz = \mu K\int\tfrac{1}{2}(:e^{2i\sqrt{\pi}\,\hat{\phi}}:d^{+}+\text{H.c.})dz\,. \tag{2.63}$$

Since the momentum conjugate to θ does not enter the Hamiltonian, the stationary states will be eigenvectors of θ. Once the energy has been minimized with respect to θ, we can replace θ by its eigenvalue θ_0 in all subsequent discussion:

$$H=\frac{1}{2}\int\left[\hat{\Pi}_\phi^2+(\partial_z\hat{\phi})^2+\frac{g^2}{\pi}\phi^2\right]dz - \mu K\int:\cos[2\sqrt{\pi}\,(\hat{\phi}+\theta_0)]:dz\,. \tag{2.64}$$

On symmetry grounds alone the value of θ_0 which minimizes H must be $\theta_0=0$ or $\frac{1}{2}\sqrt{\pi}$. For sufficiently small μ the classical approximation is adequate and gives

$$\theta_0=0\,. \tag{2.65}$$

Henceforth θ may be set equal to zero and $d^{\pm}$ to unity.

The term $\mu K\cos\hat{\phi}$ may be treated by ordinary Feynman graph methods. It is convenient to separate the quadratic term from $\mu K\cos\hat{\phi}$ and include it with the boson mass:

$$m^2=g^2/\pi+4\pi\mu K\,. \tag{2.66}$$

The vertices can have any even number of external lines (except two). The rules are shown in Fig. 1. Furthermore, it is convenient to group together Feynman graphs which differ only in the number of quanta exchanged between definite vertices. For example, the graphs describing the four-point function in order μ^2 should be organized as in Fig. 2. Individual graphs of this type are generally infrared-convergent due to the mass term in the propagators. They are not usually convergent in the ultraviolet. The superrenormalizability of the underlying fermion theory

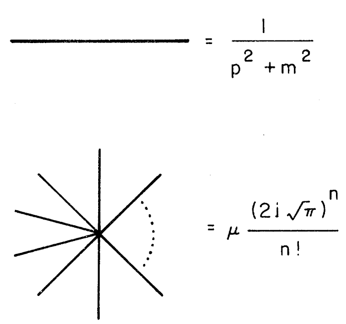

FIG. 1. Feynman rules for the equivalent bosons of the massive Schwinger model (Coulomb gauge).

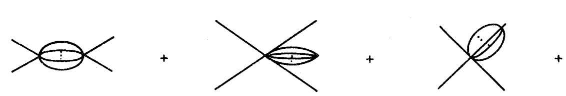

FIG. 2. Second-order mass perturbation theory rules for the massive Schwinger model.

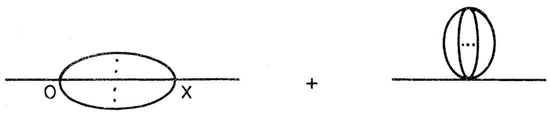

FIG. 3. Second-order mass perturbation corrections to the massive boson propagator.

strongly suggests that all ultraviolet divergences cancel in any given order of μ.

As an example we will calculate the order-μ^2 correction to the boson self-energy. The two contributing graphs are shown in Fig. 3.

It is generally easy to sum all graphs in a given order by working in configuration space. The propagator is replaced by $\Delta_F(m^2, x^2)$.

In coordinate space the first graph is given by

$$\mu^2 \sum_{n=3,5,\ldots} [4\pi i \Delta_F(m^2, x^2)]^n/n!$$

$$= \mu^2[\sinh(4\pi i\Delta_F) - 4\pi i \Delta_F] \ . \quad (2.67)$$

In momentum space this becomes

$$\mu^2 \int e^{ip\cdot x}[\sinh(4\pi i \Delta_F(m^2, x^2)) - 4\pi i\Delta_F(m^2, x^2)]\, d^2x \ . \quad (2.68)$$

For $x \to \infty$ the expression in (2.67) vanishes exponentially so that (2.68) is infrared-convergent. However, for $x \to 0$ it behaves like $1/x^2$, giving rise to a logarithmic divergence.

The second graph is given by

$$-\mu^2 \int d^2x \sum_{n=2,4,\ldots} [4\pi i \Delta_F(m^2, x^2)]^n/n!$$

$$= \mu^2 \int [1 - \cosh(4\pi i \Delta_F)]\, d^2x \ . \quad (2.69)$$

Again, Eq. (2.69) is infrared-convergent and ultraviolet-divergent. For $x \to 0$, it behaves like $1/x^2$. Combining the two graphs, the result for small x behaves similar to

$$\tfrac{1}{2}\mu^2 \int \frac{d^2x}{x^2}(e^{ip\cdot x} - 1)\,, \quad (2.70)$$

which is ultraviolet-convergent. We believe that the cancellation of divergences is a general feature of the theory.

The convergence of the entire perturbation series in μ is not insured by these calculations. To examine this question the Hamiltonian in Eq. (2.64) must be studied for arbitrary complex values of μ near the origin. To do this, one considers the effective potential, which reads

$$V(\hat{\phi}) = \tfrac{1}{2}m^2\hat{\phi}^2 - \mu K \cos 2\sqrt{\pi}\,\hat{\phi}$$
$$+ \text{ quantum corrections.} \quad (2.71)$$

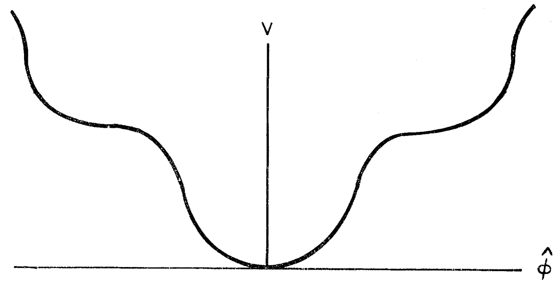

FIG. 4. Classical potential of the massive Schwinger model for small μ.

The classical part of the $V(\Phi)$ is plotted in Fig. 4. In particular, for small μ the curvature at the bottom of the well is positive, independent of the sign of μ.

This suggests that the theory behaves analytically for choices of μ near zero. For sufficiently negative μ the potential can take the form in Fig. 5. This type of potential is usually believed to cause a spontaneous symmetry breakdown. Thus, it is likely that the theory has a singularity for some negative value of μ, but is analytic in the neighborhood of $\mu = 0$. In view of this we feel reasonably confident that the confining and seizing features of the massless theory are shared by the massive theory for sufficiently small μ.

When a mass term is included into the Schwinger model neither of the axial-vector currents

$$j_\mu^5 = \frac{1}{\sqrt{\pi}}\, \partial_\mu \phi$$

or

$$\hat{j}_\mu^5 = \frac{1}{\sqrt{\pi}}\, \partial_\mu \hat{\phi}$$

are conserved. From the form of Eq. (2.64) it is evident that

$$\partial^\mu \hat{j}_\mu^5 = -\frac{g^2}{\pi}\,\hat{\phi} + 2\sqrt{\pi}\,\mu K \sin(2\sqrt{\pi}\,\hat{\phi})\,. \quad (2.72)$$

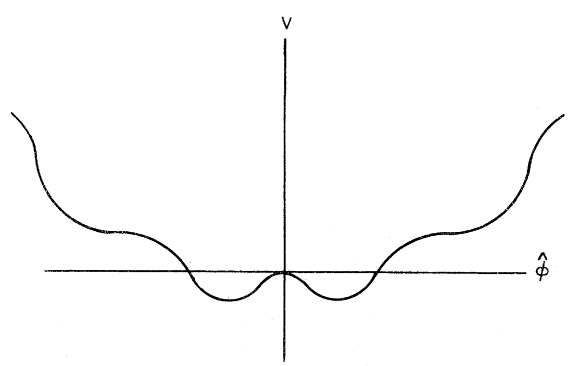

FIG. 5. Same as Fig. 4 except μ is large.

The time derivative of the other axial charge,

$$Q_5=\int j_0^5\,dz\,,$$

can be obtained by commuting it with $H_{\rm int}$ given in Eq. (2.63):

$$\begin{aligned}\dot{Q}_5&=i[H_{\rm int}, Q_5]\\&=\mu K i\int dz[:e^{2i\sqrt{\pi}\,\hat\phi}:d^+-:e^{-2i\sqrt{\pi}\,\hat\phi}:d^-],\end{aligned}\qquad(2.73)$$

where we have used

$$[Q_5, d^\pm]=\pm 2d^\pm\,.\qquad(2.74)$$

Now we may compute physical matrix elements of $\dot{Q}_5$ by setting $d^\pm=1$:

$$\langle\psi|\dot{Q}_5|\psi'\rangle=-2\mu K\left(\psi\left|\int dz{:}\sin 2\sqrt{\pi}\,\hat\phi{:}\right|\psi'\right).\qquad(2.75)$$

This agrees with the fact that in the underlying fermion theory

$$\partial^\mu j_\mu^5=2\mu\bar\psi\gamma_5\psi\,.\qquad(2.76)$$

F. Massive Schwinger model in the Lorentz gauge

In the Lorentz gauge there are no long-range action-at-a-distance forces. The loophole in the Goldstone theorem in this case is that the massless Goldstone boson may exist in the decoupled unphysical sector. This is possible because the conserved current which generates the symmetry is not gauge-invariant. To see this we begin with the axial charge density defined with careful point separating,

$$j_0^5(x)=\text{sym}\lim_{\epsilon\to 0}\psi^\dagger(x)\gamma_5\psi(x+\epsilon)\,.\qquad(2.77)$$

This is obviously not gauge-invariant for finite ϵ. If, however, the limit $\epsilon\to 0$ is sufficiently smooth, the gauge dependence of j_0^5 would disappear. However, the quantity $\psi^\dagger(x)\gamma_5\psi(x+\epsilon)$ has a leading short-distance expansion which goes like ϵ^{-1} which prevents a smooth limit. According to Schwinger j_0^5 can be replaced by a nonconserved but gauge-invariant current $\hat{j}_0^5$,

$$\begin{aligned}\hat{j}_0^5&=\lim_{\epsilon\to 0}\psi^\dagger(x)\gamma_5\exp\left(ig\int_x^{x+\epsilon}A_\mu\,dx^\mu\right)\psi(x+\epsilon)\\&=\lim_{\epsilon\to 0}\psi^\dagger(x)\gamma_5\psi(x+\epsilon)-\frac{g}{\pi}A_z(x)\,.\end{aligned}\qquad(2.78)$$

Equation (2.78) is the one-dimensional analog of the Adler-Bell-Jackiw-Schwinger anomaly.[16] Thus, it is evident that the conserved axial-vector current j^5 is gauge-non-invariant while the gauge-invariant current $\hat{j}^5$ is not conserved. Our loophole involves the fact the gauge-noninvariant operators can sometimes create decoupled states in the Lorentz gauge.

In the Lorentz gauge A_μ is divergence-free and can be expressed in the form

$$A_\mu=\epsilon_{\mu\nu}\partial^\nu\Phi\,.\qquad(2.79)$$

In the massless model the Dirac field ψ satisfies

$$(i\partial\!\!\!/-gA\!\!\!/)\psi=0\,.\qquad(2.80)$$

The theory is solved by the introduction of a free Dirac field χ which satisfies[8,9]

$$\begin{aligned}\chi&=e^{ig\alpha\Phi}\psi\\&=e^{ig\gamma_5\Phi}\psi\,.\end{aligned}\qquad(2.81)$$

This assertion is easily verified by direct substitution into the Dirac equation using the definition of Φ in Eq. (2.79).

When the fermion mass is no longer zero, the field χ is no longer free. It is still useful, nonetheless, in formulating the theroy. In the presence of a mass term the equations of motion for χ become

$$\begin{aligned}i\partial\!\!\!/\chi&=\mu e^{-2ig\alpha\Phi}\chi\\&=\mu e^{-ig\alpha\Phi}\psi\,.\end{aligned}\qquad(2.82)$$

The currents satisfy the continuity equations

$$\partial_\mu\hat{j}^\mu=0\,,\qquad(2.83\text{a})$$

$$\partial_\mu\hat{j}_5^\mu=\frac{g}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}+2\mu\bar\psi\gamma_5\psi\,,\qquad(2.83\text{b})$$

$$\partial_\mu j_5^\mu=2\mu\bar\psi\gamma_5\psi\,,\qquad(2.83\text{c})$$

$$\partial_\mu\bar\chi\gamma^\mu\chi=0\,,\qquad(2.83\text{d})$$

$$\partial_\mu\bar\chi\gamma^\mu\gamma^5\chi=2\mu\bar\psi\gamma_5\psi\,.\qquad(2.83\text{e})$$

Equations (2.83a) and (2.83d) allow us to represent $\bar\chi\gamma_\mu\chi$ and $\hat{j}_\mu$ in the form

$$\bar\chi\gamma^\mu\chi=\frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\phi_2,\quad \hat{j}_\mu=\frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\hat\phi\,.\qquad(2.84)$$

Equations (2.83a)–(2.8 e) are then summarized in terms of equivalent boson fields,

$$\begin{aligned}(\Box+m^2)\hat\phi&=2\sqrt{\pi}\,\mu\bar\psi\gamma_5\psi\,,\\\Box\phi_2&=2\sqrt{\pi}\,\mu\bar\psi\gamma_5\psi\,.\end{aligned}\qquad(2.85)$$

In addition to Eq. (2.85) Maxwell's equations give

$$\Box\Phi=-\frac{g}{\sqrt{\pi}}\hat\phi\,.\qquad(2.86)$$

It is convenient to define

$$\phi_1=\hat\phi-\frac{g}{\sqrt{\pi}}\Phi\qquad(2.87)$$

so that

$$\Box\phi_1=2\sqrt{\pi}\,\mu\bar\psi\gamma_5\psi\,.\qquad(2.88)$$

The fields $\hat{\phi}$ and ϕ_2 are conventional Bose fields with familiar commutation relations. The field ϕ_1 must be defined with ghost commutation relations in order to guarantee conventional Lorentz-gauge commutation relations,

$$\left[\hat{\phi},\ \frac{\partial}{\partial t}\hat{\phi}\right]=i\delta(z-z'),$$
$$[\phi_1, \dot{\phi}_1]=-i\delta(z-z'), \tag{2.89}$$
$$[\phi_2,\dot{\phi}_2]=i\delta(z-z').$$

Finally we should write $\mu\bar{\psi}\gamma_5\psi$ in terms of equivalent boson fields. We first write

$$\begin{aligned}\bar{\psi}\gamma_5\psi &= \bar{\chi}\, e^{-ig\gamma_5\Phi}\gamma_5\, e^{-ig\gamma_5\Phi}\chi\\ &=\bar{\chi}\gamma_5(\cos 2g\Phi - i\gamma_5 \sin 2g\Phi)\chi\,. \end{aligned}\tag{2.90}$$

Now using the equivalent boson representation for $\bar{\chi}\chi$, and $\bar{\chi}\gamma_5\chi$, we obtain

$$\begin{aligned}\bar{\psi}\gamma_5\psi &=-iK\sin(2\sqrt{\pi}\,\phi_2)\cos(2g\Phi)\\ &\quad -iK\cos(2\sqrt{\pi}\,\phi_2)\sin(2g\Phi)\\ &=-iK\sin(2\sqrt{\pi}\,\phi_2+2g\Phi)\\ &=-iK\sin 2\sqrt{\pi}\,(\hat{\phi}+\phi_2-\phi_1)\,.\end{aligned}\tag{2.91}$$

Thus the equations of motion for $\hat{\phi}$, ϕ_1, ϕ_2 become

$$(\Box+m^2)\hat{\phi}=\Box\phi_1=\Box\phi_2=-2\sqrt{\pi}\,i\mu K\sin 2\sqrt{\pi}\,(\hat{\phi}+\phi_2-\phi_1)\,. \tag{2.92}$$

We note that in the symmetry limit $\mu\to 0$, j^5_μ is given by the gradient of a massless free ghost field. This field can be thought of as a Goldstone boson. Thus in a formal sense, in covariant gauges, chiral symmetry is realized with a mass-zero Goldstone boson. However, we shall see

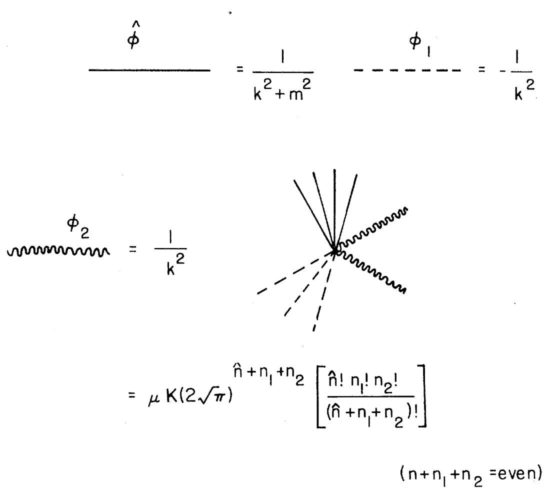

FIG. 6. Feynman rules for the massive Schwinger model in the Lorentz gauge.

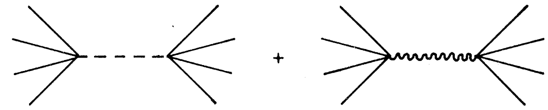

FIG. 7. Example of the cancellation of ϕ_1 and ϕ_2 propagators in gauge-invariant amplitudes.

that in all matrix elements of gauge-invariant quantities the ghost Goldstone boson is identically canceled by the second massless field ϕ_2, leaving only the massive field $\hat{\phi}$ to propagate in intermediate states.

The effective Lagrangian which leads to Eq. (2.92) is

$$\begin{aligned}\mathcal{L}(x)&=\tfrac{1}{2}\left[(\partial_\mu\hat{\phi})^2+(\partial_\mu\phi_2)^2-(\partial_\mu\phi_1)^2-\frac{g^2}{\pi}\hat{\phi}^2\right]\\ &\quad +\mu K\!:\!\cos 2\sqrt{\pi}\,(\hat{\phi}+\phi_2-\phi_1)\!:\,.\end{aligned}\tag{2.93}$$

The subsidiary condition which defines the physical subspace is analogous to the Gupta-Bleuler condition. It states that the positive-frequency part of the free massless field $(\phi_1-\phi_2)$ annihilates the physical subspace.

Let us consider the Feynman diagrams of the theory and show that the singularities of the fields ϕ_1 and ϕ_2 cancel in physical matrix elements. The diagrams and rules are shown in Fig. 6. The propagator for ϕ_1 carries a negative sign due to its ghost commutation relations. The vertex involving $\hat{n}$, n_1, n_2 external lines vanishes if $\hat{n}+n_1+n_2$ is odd and is given by

$$\mu K(2\sqrt{\pi})^{\hat{n}+n_1+n_2}\ \frac{\hat{n}!\,n_1!\,n_2!}{(\hat{n}+n_1+n_2)!}$$

if $\hat{n}+n_1+n_2$ is even.

Consider for example the graphs in Fig. 7. The numerical values of the two graphs are identical except for the sign difference in the ϕ_1 and ϕ_2 propagators. Therefore the sum vanishes. The idea of this example is easily generalized into a proof that the intermediate-state singularities of ϕ_1 and ϕ_2 always cancel in S-matrix elements in which the external lines are all $\hat{\phi}$.

Furthermore, it can be shown that all gauge-in-

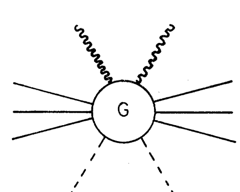

FIG. 8. A general matrix element of a gauge-invariant operator G.

variant operators involve ϕ_1 and ϕ_2 only in the combination $\phi_1-\phi_2$. Let G in Fig. 8 represent a vertex for a gauge-invariant operator. As the reader can easily check, any graph in which G is inserted will also have the cancellation property as long as the external states satisfy the subsidiary condition.

However, gauge-noninvariant operators can contain ϕ_1 and ϕ_2 in different combinations. For example, the axial-vector current j_μ^5 is expressed in terms of ϕ_1 alone:

$$j_\mu^5=\frac{1}{\sqrt{\pi}}\,\partial_\mu\phi_1\,.$$

A typical graph is shown in Fig. 9, and there is no possibility of canceling the ϕ_1 pole. Therefore the gauge-noninvariant operators generally have massless poles associated with ϕ_1 or ϕ_2. This has a peculiar consequence for matrix elements of $\bar\psi\gamma_5\psi$.

Since $2\mu\bar\psi\gamma_5\psi$ is the divergence of j_μ^5, its matrix elements are of the form

$$\langle i|\bar\psi\gamma_5\psi|f\rangle=\frac{i}{2\mu}\,(k_i-k_f)^\mu\langle i|j_\mu^5|f\rangle\ . \tag{2.94}$$

Ordinarily if the physical spectrum is free of massless particles the right-hand side of (2.94) must tend to zero as $k_i-k_f\to 0$. This conclusion would be correct if j_μ^5 were gauge-invariant. However, as we have seen, j_μ^5 contains massless singularities from Fig. 9,

$$\langle i|j_\mu^5|f\rangle\sim\frac{(k_i-k_f)_\mu}{(k_i-k_f)^2}\,,$$

and the right-hand side of (2.94) remains finite as $k_i\to k_f$.

The general conclusion is that low-energy theorems which follow from the fact that some gauge-invariant quantity is a divergence are not generally true unless the current is also gauge-invariant. We shall see later how this allows us to circumvent Sutherland's theorem.

III. THE NINTH AXIAL-VECTOR CURRENT IN FOUR DIMENSIONS

A. How confinement effects j_μ^5

Gauge-theory enthusiasts are hopeful that the theory of strong interactions defined by the Lagrangian in Eq. (1.1) has the property of infrared slavery in real (3 + 1)-dimensional space-time.[5,17] It is hoped that the effective coupling between long wavelengths diverges in the infrared so that it would require an infinite amount of energy to isolate any color nonsinglet state. This hope is inspired by the discovery that the theory is asymptotically free.[4] In other words, the quark-gluon interactions vanish at small distances and show a tendency to grow at longer distances.

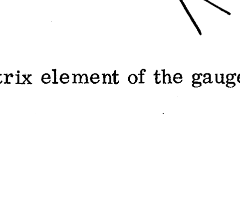

FIG. 9. A matrix element of the gauge-noninvariant current j_μ^5.

Further motivation for infrared slavery comes from a recent formulation of strongly coupled cutoff gauge theories due to Wilson.[18] His approach, which respects the local gauge invariance of the theory at the sacrifice of covariance, offers calculational methods to investigate strongly coupled theories. He finds that quarks do not exist as free objects and that there is a linearly rising potential $V(r)\sim r$ which acts between colored quanta. Thus, the long-distance properties of this theory resemble those of the much simpler Schwinger model.

We will digress briefly to describe the quark-confining mechanism. First, consider the force law between two isolated, static colored quarks. In defining the force law we ignore the virtual and real production of quark pairs. The remaining effects associated with only the Yang-Mills field are computed exactly. In the model theories which exhibit quark confinement the potential energy between spatially isolated quarks grows in proportion to the distance between them. This energy is stored in a deformed Yang-Mills electric field which is collimated into a narrow tube between the quarks.[17] When the production of pairs is allowed, the vacuum between the quarks becomes polarized to such an extent that the long-range field is entirely screened. The original colored quarks are completely neutralized by vacuum polarization currents. The complete screening of the color sources insures that the *long-range field is absent* in the physical finite energy space of states.

The phenomenon we are discussing is similar to the Higgs mechanism[19] in that the massless gauge bosons and long-range forces are removed from the theory. However, there the similarity ends. Unlike the Higgs phenomenon, the confinement mechanism does *not* involve the spontaneous breakdown of the global color group.[17]

The phenomenon of complete screening of long-range fields without the violation of the global symmetry occurs in the (1 + 1)-dimensional Schwinger model. It will be referred to as the Schwinger mechanism.[7]

We shall explore the consequences of the Schwinger mechanism further and see that the ninth Goldstone boson is removed from the spectrum[20] and that Sutherland's soft-pion theorems[10] for $\eta \to \pi^+\pi^-\pi^0$ do not apply to the π^0. In short, we claim that the ninth axial-vector current of these four-dimensional models should, if the Schwinger mechanism operates, behave precisely as the axial-vector current does in (1 + 1)-dimensional quantum electrodynamics. This will lead us to suggest that the vacua of these four-dimensional theories are seized.

Since the arguments are quite technical, we should sketch the logic before delving into details. First, observe that the ninth axial-vector current must be defined as the limit of a point-separated expression. This is essential because of the singular character of the product of interacting fermion fields. The point-separated current is not invariant to local color gauge transformations. However, it can be modified in such a way that it becomes gauge-invariant. Schwinger's traditional method of accomplishing this leads to a local but nonconserved current.[16] To discuss the possible appearance of a Goldstone boson we need a conserved, gauge-invariant current. We construct below a nonlocal current with these properties. The charge constructed from this current generates global γ_5 transformations and thus generates the symmetry of interest. In a weakly coupled gauge theory it would create a Goldstone boson (called η'' in Ref. 20) out of the vacuum. However, the nonlocal conserved current is constructed with an operator which creates a long-range color dipole field. This field is present as a consequence of gauge invariance and is of fundamental significance. We recognize this as the source of the undoing of the η'' because if the Schwinger mechanism operates in the gauge theory, long-range dipole fields are forbidden. Hence, the η'' which must be clothed in such a field must decouple from the physical spectrum.

Now we turn to a detailed review of our argument that the Schwinger mechanism eliminates the η''. Local gauge invariance and the associated subsidiary conditions play a crucial role in this discussion. We have found that the decoupling problem is easiest to understand if the full gauge freedom is restricted to the class of time-independent gauge transformations.[20] The reason is that these gauge transformations can be implemented by conventional, time-independent unitary operators U. The subsidiary conditions defining gauge-invariant states are expressed in the form

$$U|\psi\rangle = |\psi\rangle \ . \tag{3.1}$$

In terms of an infinitesimal generator G of the gauge transformation, this reads

$$G|\psi\rangle = 0 \,. \tag{3.2}$$

The natural class of gauges which admit only the time-independent gauge transformations is defined by $A_0 = 0$. We will illustrate the application of gauge invariance in the simplified Abelian theory.

In the $A_0 = 0$ gauge, the strong-interaction Lagrangian [Eq. (1.1)] becomes

$$\mathcal{L} = \bar\psi(i\not\nabla - m)\psi - g\bar\psi\vec\gamma\,\psi\cdot\vec{A} + \tfrac{1}{2}\left(\frac{\partial}{\partial t}\,\vec{A}\right)^2 - \tfrac{1}{2}(\vec\nabla\times\vec{A})^2 \,. \tag{3.3}$$

Then the momentum conjugate to $\vec{A}$ is the strong "electric" field $\dot{A}_i = -E_i$. The canonical commutation relations are postulated to be

$$[E_i(x), A_j(y)]\,|_{x^0 = y^0} = i\,\delta_{ij}\,\delta^3(\vec{x} - \vec{y}) \,. \tag{3.4}$$

The Hamiltonian is obtained from $\mathcal{L}$ in the usual way:

$$H = \psi^\dagger(-i\vec\alpha\cdot\vec\nabla + \beta m)\psi + g\bar\psi\vec\gamma\,\psi\cdot\vec{A} + \tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}(\vec\nabla\times\vec{A})^2 \,. \tag{3.5}$$

It is easy to verify that Eqs. (3.4) and (3.5) lead to the correct Heisenberg equations of motion. It is also clear that H is invariant under the restricted gauge group which leaves $A_0 = 0$,

$$\psi(x) \to e^{ig\Lambda(x)}\psi(x), \quad \vec{A}(x) \to \vec{A}(x) + \vec\nabla\Lambda(x) \,. \tag{3.6}$$

Here $\Lambda(x)$ is a *time-independent* gauge function. We now observe that these gauge transformations are generated by the operator

$$G_\Lambda = \int \Lambda(x)(g\psi^\dagger\psi - \vec\nabla\cdot\vec{E})\,d^3x \tag{3.7}$$

since

$$\begin{aligned}[G_\Lambda, \psi(x)] &= i\Lambda(x)\psi(x), \\ [G_\Lambda, \vec{A}(x)] &= i\vec\nabla\Lambda(x) \,.\end{aligned} \tag{3.8}$$

In this formulation of the theory, Gauss's law,

$$\vec\nabla\cdot\vec{E} = g\psi^\dagger\psi, \tag{3.9}$$

is the *constraint* equation expressing the gauge invariance of physical states, i.e., $(\vec\nabla\cdot\vec{E} - g\psi^\dagger\psi)$ annihilates any physical state,

$$(\vec\nabla\cdot\vec{E} - g\psi^\dagger\psi)|\psi\rangle = 0 \,. \tag{3.10}$$

Equivalently, the physical subspace can be said to be invariant under transformations generated by G_Λ. It is this subsidiary condition (Gauss's law) which in the weakly coupled theory reduces the number of independent components of $\vec{A}$ from three to two.

The fact that the physical vacuum of the theory is gauge-invariant implies that matrix elements of certain gauge-*noninvariant* operators must vanish. For example, consider the propagator

$$S(\vec{x}, t) = -i\langle 0|\psi(0)\overline{\psi}(\vec{x}, t)|0\rangle, \quad t>0. \tag{3.11}$$

Inserting $e^{iG\Lambda}e^{-iG\Lambda}=1$ inside Eq. (3.11) implies that

$$S(\vec{x}, t) = -i\langle 0|\psi(0)\overline{\psi}(\vec{x}, t)|0\rangle\, e^{-ig[\Lambda(x)-\Lambda(0)]}. \tag{3.12}$$

Since $g\neq 0$ and Λ is an arbitrary function, it follows that $S(\vec{x}, t)=0$ unless $\vec{x}=0$. This result might appear to imply that the charged particle (electron in quantum electrodynamics, quark in strong interactions) cannot propagate. This is not so, however, because the state $\psi^\dagger(\vec{x}, t)|0\rangle$ is *not* gauge-invariant (it violates the constraint $\vec{\nabla}\cdot\vec{E}=g\psi^\dagger\psi$). To describe the propagation of an interacting charged particle consistently one should use a gauge-*invariant* description of the particle, i.e., one should construct an electron field operator which satisfies Gauss's law. Let us proceed to do this. Since the strong "electric" field E_i is canonically conjugate to A_i, an operator of the form

$$\exp\left[-i\int \vec{A}(x)\cdot\vec{U}(x)d^3x\right] \tag{3.13}$$

creates a state with an average strong electric field,

$$\langle E_i\rangle = U_i(x). \tag{3.14}$$

Unless $\vec{\nabla}\cdot\vec{U}(x)=0$ the operator in Eq. (3.13) will not be gauge-invariant. Now consider the quark field operator $\psi(x)$. We can multiply $\psi(x)$ by an operator having the form of that in Eq. (3.13) to yield a gauge-invariant description of a quark. Define

$$\Psi(x) = \exp\left[i\int \vec{A}(y)\cdot\vec{U}(y-x)d^3y\right]\psi(x). \tag{3.15}$$

Then if $\vec{U}(x)$ satisfies

$$\vec{\nabla}\cdot\vec{U}(x) = -g\delta^3(x). \tag{3.16}$$

$\Psi(x)$ will be gauge-invariant. To verify this, apply the gauge transformation Eq. (3.6) on Eq. (3.15). If Eq. (3.16) is true, then the gauge dependence of $\vec{A}$ and ψ cancel in $\Psi(x)$.

Now let us turn to the real problem at hand—to understand the ninth pseudoscalar Goldstone boson in a gauge-invariant, physical fashion. The first obstacle we meet is the fact that the axial-vector current $\overline{\psi}\gamma^\mu\gamma_5\psi$ requires a careful point-separated definition,

$$\rho_5(x) = \operatorname{sym}\lim_{\epsilon\to 0}\rho_5(x,\epsilon), \tag{3.17}$$

where

$$\rho_5(x,\epsilon) = \psi^\dagger(x+\epsilon)\gamma_5\psi(x). \tag{3.18}$$

The operator $\rho_5(x)$ is the local generator of γ_5 transformations. Its spatial integral Q_5 generates the global transformation

$$\psi\to e^{i\lambda\gamma_5}\psi, \tag{3.19}$$

which is an exact symmetry of $\mathcal{L}$ when $m\to 0$. Furthermore, the density $\rho_5(x)$ satisfies a local continuity equation,

$$\frac{\partial}{\partial t}\rho_5 + \vec{\nabla}\cdot\vec{j}^5 = 0, \tag{3.20}$$

where $\vec{j}_5$ is the appropriate axial-vector flux. Assuming that γ_5 invariance is not realized algebraically, ρ_5 should generate soft Goldstone bosons when applied to the vacuum. Thus, for $k_\mu\approx 0$,

$$|\eta''\rangle = \int e^{ik\cdot x}\rho_5(x)d^3x|0\rangle \tag{3.21}$$

is a candidate for the singlet SU(3) Goldstone boson.

It is important to note that $\rho_5(x,\epsilon)$ is *not* gauge-invariant since $\psi(x)$ and $\psi^\dagger(x+\epsilon)$ transform differently under local gauge transformations. As in the case of the unclothed electron field, one can see that

$$\langle 0|T\rho_5(x,\epsilon)\rho_5(x',\epsilon)|0\rangle$$

must vanish for $\vec{x}\neq\vec{x}'$. In order to compensate the noninvariance, Schwinger[21] multiplies $\rho_5(x,\epsilon)$ by the factor $\exp[ig\vec{A}(x)\cdot\vec{\epsilon}]$. Thus, we define the gauge-invariant (denoted by a caret) density

$$\hat{\rho}_5(x,\epsilon) = \psi^\dagger(x)\gamma_5\psi(x+\epsilon)\exp[ig\vec{A}(x)\cdot\vec{\epsilon}] \tag{3.22}$$

and

$$\hat{\rho}_5(x) = \operatorname{sym}\lim_{\epsilon\to 0}\hat{\rho}_5(x,\epsilon). \tag{3.23}$$

When evaluating the symmetric limit $\epsilon\to 0$ in Eq. (3.23) one cannot ignore the factor $\exp[ig\vec{A}(x)\cdot\vec{\epsilon}]$. This is so because the operator products $\psi^\dagger(x)\gamma_5\psi(x+\epsilon)$ have short-distance singularities which diverge as ϵ^{-1}. The behavior of this singularity was computed originally by Schwinger,[16]

$$\psi^\dagger(x)\gamma_5\psi(x+\epsilon) \underset{\epsilon\to 0}{\sim} \frac{3g}{4\pi^2}\frac{\vec{B}\cdot\vec{\epsilon}}{i\epsilon^2}, \tag{3.24}$$

where B_i is the "magnetic" gluon field $\epsilon_{ijk}F_{jk}$. Using Eq. (3.24) we may expand the factor $\exp[ig\vec{A}(x)\cdot\vec{\epsilon}]$ in Eq. (3.22) and compute $\hat{\rho}_5(x)$,

$$\begin{aligned}\hat{\rho}_5(x) &= \rho_5(x) + \operatorname{sym}\lim_{\epsilon\to 0}\frac{g}{8\pi^2}\frac{\vec{B}\cdot\vec{\epsilon}}{i\epsilon^2}(ig\vec{A}\cdot\vec{\epsilon})\\ &= \rho_5(x) + \frac{g^2}{4\pi^2}\vec{B}(x)\cdot\vec{A}(x).\end{aligned} \tag{3.25}$$

The same construction can be carried out for the axial-vector fluxes $\vec{j}_5(x)$. As a result one has constructed a local, gauge-invariant axial-vector current. However, as can be verified from Eq. (3.25) and its partners, the resulting current is not conserved. Schwinger,[16] in fact, computed

$$\frac{\partial}{\partial t}\hat{\rho}_5+\nabla_i\hat{j}_i^5=\frac{g^2}{2\pi^2}\vec{E}\cdot\vec{B}\,, \tag{3.26}$$

where

$$\hat{j}_5^i=j_5^i+\frac{g^2}{8\pi^2}\epsilon_{i\mu\sigma\tau}A^\mu F^{\sigma\tau}\,,\quad i=1,2,3\,.$$

Since this gauge-invariant current is not conserved, it cannot be used to prove the existence of a massless particle. However, an alternative[20] to Schwinger's construction exists which gives a gauge-invariant, conserved current whose corresponding total charge generates chiral transformations. The construction is analogous to the construction of the gauge-invariant electron field in that the operator $\rho_5(x,\epsilon)$ is clothed with a long-range dipole field.

Consider the operator[20]

$$\tilde{\rho}_5(x)=\operatorname{sym}\lim_{\epsilon\to 0}\tilde{\rho}_5(x,\epsilon)\,, \tag{3.27}$$

where,

$$\tilde{\rho}_5(x,\epsilon)=\psi^\dagger(x)\gamma_5\psi(x+\epsilon)\times\exp\left[-i\int\vec{A}(r)\cdot\vec{V}(r-x)d^3r\right], \tag{3.28}$$

where $\vec{V}$ is a c-number vector field satisfying

$$\vec{\nabla}\cdot\vec{V}(r)=g\delta^3(r)-g\delta^3(r+\epsilon)\underset{\epsilon\to 0}{\sim}-g\vec{\epsilon}\cdot\vec{\nabla}\delta^3(r)\,. \tag{3.29}$$

Therefore, $\tilde{\rho}_5(x,\epsilon)$ is gauge-invariant by construction.

Next we calculate $\tilde{\rho}_5(x)$ explicitly. It is convenient to write

$$V_i(r)=-\vec{\epsilon}\cdot\vec{\nabla}U_i(r) \tag{3.30}$$

where $U_i(r)$ satisfies Eq. (3.16). We may now use Eq. (3.25) to write

$$\tilde{\rho}_5(x,\epsilon)=\psi^\dagger(x)\gamma_5\psi(x+\epsilon)+\frac{g^2}{4\pi^2}\frac{\vec{B}(x)\cdot\vec{\epsilon}}{\epsilon^2}\int\vec{\epsilon}\cdot\vec{\nabla}_r\vec{U}(r-x)\cdot\vec{A}(x)d^3x\,. \tag{3.31}$$

It is straightforward to evaluate the symmetric limit of this expression. The reader is referred to Ref. 20 for the details which lead to

$$\begin{aligned}\tilde{\rho}_5&=\rho_5+\frac{g^2}{4\pi^2}\vec{B}\cdot\vec{A}-\frac{g^2}{4\pi^2}\vec{B}\cdot\int\vec{U}(r-x)\times\vec{B}(r)d^3r\\&=\hat{\rho}_5-\frac{g^2}{4\pi^2}\vec{B}\cdot\int\vec{U}(r-x)\times\vec{B}(r)d^3r\,.\end{aligned} \tag{3.32}$$

Observe that the long-range dipole field modifies ρ_5 in a significant fashion, i.e., the term proportional to $\vec{U}$ survives after the limit $\epsilon\to 0$ is taken. Furthermore, it was shown in Ref. 20 after some algebra that an axial-vector flux $\tilde{j}_5^i$ could be defined such that

$$\frac{\partial}{\partial t}\tilde{\rho}_5+\nabla_i\tilde{j}_{5i}=0 \tag{3.33}$$

as desired. Therefore, the current $\tilde{j}_{5\mu}$ is conserved and may be used to generate a symmetry which might be realized algebraically or through Goldstone bosons. The relevant global symmetry is just the original chiral symmetry generated by $\int\rho_5(x)d^3x$. In fact,

$$\begin{aligned}Q_5&=\int\rho_5(x)d^3x\\&=\int\tilde{\rho}_5(x)d^3x\end{aligned} \tag{3.34}$$

as was shown in Ref. 20.

Now we can see why the η'' does not exist. A physical, gauge-invariant description of the potential Goldstone boson is provided by the expression

$$|\eta''\rangle=\int e^{ik\cdot x}\tilde{\rho}_5(x)d^3x|0\rangle\,,\quad k_\mu\approx 0\,. \tag{3.35}$$

However, local color gauge invariance requires the operator $\tilde{\rho}_5(x)$ to consist of two factors: First, $\psi^\dagger(x)\gamma_5\psi(x+\epsilon)$ which is local but not gauge-invariant, and second, $\exp[-i\int\vec{A}(r)\cdot\vec{V}(r-x)d^3r]$ which is nonlocal, but guarantees that $\tilde{\rho}_5$ is gauge-invariant. In fact, the presence of the dipole field is essential because it appears, for example, in Eq. (3.32) after the limit $\epsilon\to 0$ is taken. Therefore, we must interpret Eqs. (3.28) and (3.35) to mean that local color gauge invariance constrains the η'' to have a long-range gluon gauge field given by

$$\exp\left[-i\int\vec{A}(r)\cdot\vec{V}(r-x)d^3r\right]|0\rangle\,.$$

But, if the Schwinger mechanism operates in the field theory, such long-range fields are forbidden—depending on the gauge used in formulating the theory, they either have infinite energy or violate subsidiary conditions. Therefore, since gauge invariance requires a long-range field for the η'', it (like colored quarks and gluons) must be *absent* from the physical spectrum.

The elegant argument, however, appears to generate a curious paradox. How can it be that $\tilde{\rho}_5$ creates an unphysical state when it acts on the vacuum while its integral Q_5 generates a symmetry? It appears that the behavior of the zero-momentum component of $\tilde{\rho}_5$ is not smoothly connected to the remaining momentum components of the operator. This suggests that the space of gauge-invariant states is constructed as in the (1 + 1)-dimensional examples, as a product of a ladder space and a conventional S-matrix Hilbert space. The ladder space would again be labeled by eigenvalues of Q_5. The physical vacuum should be chosen to be an eigenvector of θ, the angular variable conjugate to Q_5. Then any operator of the form

$$\int \tilde{\rho}_5(x) F(x) d^4x, \qquad (3.36)$$

where $F(x)$ is a finite-range test function, creates states of infinite energy while the total integrated charge Q_5 creates unphysical states by changing the boundary conditions at infinity.

Other remedies for the j^5 problem have been proposed. One which was entertained by Fritzsch and Gell-Mann[22] is the possibility that Q_5 annihilates the physical spectrum of states, i.e., $Q_5|\psi\rangle = 0$. However, it was pointed out by Fritzsch and Gell-Mann that this possibility leads to difficulties with the usual equal-time algebra of quark bilinears. [We have already seen in the (1 + 1)-dimensional examples that the conjecture $Q_5|\psi\rangle = 0$ is inconsistent with the equal-time algebra of the densities $\overline{\psi}\gamma_5\psi$, $\psi^\dagger\gamma_5\psi$, and $\overline{\psi}\psi$.] These authors consider the SU(6) algebra generated by the usual vector and axial-vector currents and the tensor bilinears $\overline{\psi}\sigma^{\mu\nu}\lambda_\alpha\psi$. In particular, it follows from canonical commutation relations that

$$[Q_5, \overline{\psi}\sigma^{ij}\lambda_\alpha\psi] = \epsilon^{ijk}\overline{\psi}\sigma^{0k}\lambda_\alpha\psi. \qquad (3.37)$$

A difficulty appears now when one attempts to saturate this relation with physical states. If $Q_5|\psi\rangle = 0$, then Eq. (3.37) implies that physical matrix elements of $\overline{\psi}\sigma^{0k}\lambda_\alpha\psi$ must also be zero. This possibility is unattractive because it would undermine the applications of SU(6) algebra to the hadron spectrum.

However, if the vacuum of the (3 + 1)-dimensional gauge theory seizes, then no difficulties with local commutation relations and the properties of the ninth axial charge arise. We follow the (1 + 1)-dimensional examples to illustrate how this occurs. First, consider linear combinations of the quark bilinear operators such that they cause only transitions of definite ΔQ_5. Following the (1 + 1)-dimensional models we assign to these bilinears a product of operators,

$$(\text{bilinear})_i \to F_i (d^\pm)^{\Delta Q_5^\frac{1}{5}/2}, \qquad (3.38)$$

where F_i acts in the conventional field theory space of states and $d^\pm$ are raising and lowering operators in the ladder space introduced earlier. The operators F_i obey the naive commutation relations of the quark bilinears. Now consider Eq. (3.37) again and recall that Q_5 has infinite dispersion in an acceptable seized vacuum state. Saturation of the commutation relation in Eq. (3.37) can then be carried out without implying the vanishing matrix elements of $\overline{\psi}\sigma^{0k}\lambda_\alpha\psi$. The details of an illustration of this fact parallel the same discussion in the (1 + 1)-dimensional model.

Before concluding this section we remark that the properties of the other eight axial-vector currents,

$$j^\mu_{5,\alpha} = \overline{\psi}\gamma^\mu\gamma_5\lambda_\alpha\psi,$$

are not effected by the quark-confining long-range forces. This is because the $\epsilon \to 0$ limit of the point-separated currents,

$$j^\mu_{5,\alpha}(x, \epsilon) = \overline{\psi}(x)\gamma^\mu\gamma_5\lambda_\alpha\psi(x+\epsilon),$$

is smooth. They are therefore color gauge-invariant and do not require long-range fields.

B. The $\eta \to 3\pi$ problem

Another serious difficulty of the quark-gluon theory is found when one attempts to calculate the electromagnetic decay of the η into three pions π^+, π^-, and π^0. The problem was originally pointed out by Sutherland,[10] who attempted to calculate the matrix element of interest using conventional current algebra and partially conserved axial-vector current methods. He was able to show that the amplitude should vanish whenever *any* of the pions become soft ($k_\mu \to 0$). Experimentally it is known that the $\eta \to 3\pi$ amplitude does vanish when the momentum of the π^+ or π^- is taken to zero. However, the amplitude is large and unsuppressed in that region of the Dalitz plot where the π^0 is soft. In fact, the amplitude is usually parametrized in the form[23]

$$A(\eta \to \pi^+\pi^-\pi^0) = \overline{A}_\eta[1 - m_\pi^{-2} a(S_0 - \overline{S}_\eta)], \qquad (3.39)$$

where

$$S_0 = (q_\eta - q_{\pi^0})^2,$$
$$\overline{S}_\eta = m_\pi^2 + \tfrac{1}{3} m_\eta^2,$$
$$a = -0.2 \pm 0.015.$$

Letting the momentum of the π^+ (or π^-) vanish, one can check that the amplitude falls to $\approx 0.1\overline{A}_\eta$. However, letting the four-momentum of the π^0

vanish gives an amplitude $\approx 3\bar{A}_\eta$ which is 30 times greater than one would expect on the basis of Sutherland's analysis. Apparently conventional PCAC and current-algebra methods apply reliably to the currents $\bar{\psi}\gamma^\mu\gamma^5\lambda_{1\pm i2}\psi$, but do not apply to $\bar{\psi}\gamma^\mu\gamma^5\lambda_3\psi$.

It has been guessed for some times that the resolution of the problem lies in a better understanding of the ninth axial-vector current. We shall review the derivation of the $\eta\to 3\pi$ amplitude in the soft π^0 limit below and see that it is controlled by a simple matrix element of the ninth axial-vector current. In fact, we shall argue that in gauge theories of quark confinement, Sutherland's theorem no longer applies to the π^0, although the experimentally successful application of his analysis to the π^+ and π^- remains unchanged.

The $\eta\to\pi^+\pi^-\pi^0$ decay is a second-order electromagnetic effect which is described by the effective interaction Lagrangian,

$$\mathcal{L}_I(0)=e^2\int T J_\mu^{\rm em}(x)J_\nu^{\rm em}(0)\mathfrak{D}^{\mu\nu}(x)d^4x, \tag{3.40}$$

where $J_\mu^{\rm em}(x)$ is the hadronic electromagnetic current and $\mathfrak{D}^{\mu\nu}(x)$ is the photon propagator. Experimental analysis indicates that η decay is primarily a $\Delta I=1$ transition, so we concentrate just on the $\Delta I=1$ piece of Eq. (3.40). Wilson[24] has argued that matrix elements of $\mathcal{L}_I(0)$ are in general expected to be logarithmically divergent at short distances so a $\Delta I=1$, $\Delta I_3=0$ operator must be inserted into the full Lagrangian to act as a counterterm. In a quark theory this term is just a $\mathcal{P}$ quark–$\mathfrak{N}$ quark electromagnetic mass difference,

$$\delta m\bar{\psi}\lambda_3\psi\,.$$

So, Eq. (3.40) is replaced by

$$\mathcal{L}_I(0)=e^2\int [T J_\mu^{\rm em}(x)J_\nu^{\rm em}(0) - C_{\mu\nu}(x)\bar{\psi}\lambda_3\psi]\,\mathfrak{D}^{\mu\nu}(x)d^4x + \delta m\int\bar{\psi}\lambda_3\psi d^4x, \tag{3.41}$$

where $C_{\mu\nu}(x)$ is a singular function ($\sim x^{-2}$) at short distances and insures that matrix elements of the first term of Eq. (3.41) are finite. Of course, the form of Eq. (3.41) is just that dicated by renormalized perturbation theory. We remind the reader of a few facts about Eq. (3.41): Its second term is often referred to as "an electromagnetic tadpole." In properly constructed gauge theories of electromagnetic and weak interactions it is even computable. A proper numerical evaluation of this term constitutes a piece of the calculation of the neutron-proton mass difference.

Let us now turn to a detailed analysis of the $\eta\to 3\pi$ amplitude,

$$\mathfrak{M}=e^2\left\langle\pi^+\pi^-\pi^0\right|\int[T J_\mu^{\rm em}(x)J_\nu^{\rm em}(0)\mathfrak{D}^{\mu\nu}(x)-C_{\mu\nu}(x)\bar{\psi}\lambda_3\psi\mathfrak{D}^{\mu\nu}(x)]d^4x\left|\eta\right\rangle+\delta m\left\langle\pi^+\pi^-\pi^0\right|\int\bar{\psi}\lambda_3\psi d^4x\left|\eta\right\rangle. \tag{3.42}$$

The first term in Eq. (3.42) has finite matrix elements and Sutherland's theorem applies to it. In particular, the theorem indicates that its contribution to the amplitude will be suppressed when the π^0 momentum is small. To evaluate the second term we reduce in the π^0 and apply standard PCAC techniques.[23] Since Eq. (3.42) is explicitly of order $\alpha=e^2/4\pi$, we can ignore electromagnetic corrections to further operator equations and states. Therefore, we use the naive PCAC definition of the pion field $\phi_\alpha(x)$,

$$\partial_\mu j_{5,\alpha}^\mu(x)=m_\pi^{\,2}f_\pi\phi_\alpha(x), \tag{3.43}$$

where we have used conventional notation. In the quark model the axial-vector SU(3) current is

$$j_{5,\alpha}^\mu(x)=\bar{\psi}\gamma^\mu\gamma^5\lambda_\alpha\psi\,. \tag{3.44}$$

Now standard manipulations give

$$\begin{aligned}\mathfrak{M}&=\delta m(-k^2+m_\pi^{\,2})\left\langle\pi^+\pi^-\right|\int d^4y\,e^{ik\cdot y}\int d^4x\,T\phi_3(y)\bar{\psi}(x)\lambda_3\psi(x)\left|\eta\right\rangle\\
&=(\delta m)m_\pi^{\,-2}f_\pi^{\,-1}(-k^2+m_\pi^{\,2})\left\langle\pi^+\pi^-\right|\int d^4y\,e^{ik\cdot y}\int d^4x\,T\partial_\mu j_{5,3}^\mu(y)\bar{\psi}(x)\lambda_3\psi(x)\left|\eta\right\rangle\\
&=(\delta m)m_\pi^{\,-2}f_\pi^{\,-1}(-k^2+m_\pi^{\,2})\\
&\quad\times\left\{-ik_\mu\left\langle\pi^+\pi^-\right|\int d^4y\,e^{ik\cdot y}\int d^4x\,Tj_{5,3}^\mu(y)\bar{\psi}(x)\lambda_3\psi(x)\left|\eta\right\rangle\right.\\
&\qquad\left.-\left\langle\pi^+\pi^-\right|\int d^4y\,e^{ik\cdot y}\int d^4x\,\delta(x^0-y^0)[j_{5,3}^0(y),\bar{\psi}(x)\lambda_3\psi(x)]\left|\eta\right\rangle\right\}.\end{aligned} \tag{3.45}$$

Again, the first term in Eq. (3.45) satisfies Sutherland's theorem. The commutator in the second term is evaluated using canonical commutation relations,

$$\delta(x^0-y^0)[j^0_{5,3}(y), \bar\psi(x)\lambda_3\psi(x)] = -2\delta^4(x-y)\bar\psi\gamma_5\psi\,. \tag{3.46}$$

The operator on the right-hand side of Eq. (3.46) can be identified as the divergence of the ninth axial-vector current j_5^μ divided by twice the quark mass, m_q.[25] Thus, the equal-time commutator term contributes

$$\begin{aligned}2\left\langle\pi^+\pi^-\right|&\int d^4y\, e^{ik\cdot y}\bar\psi(y)\gamma_5\psi(y)\left|\,\eta\right\rangle\\ &=\frac{1}{m_q}\left\langle\pi^+\pi^-\right|\int d^4y\, e^{ik\cdot y}\,\partial_\mu j_5^\mu(y)\left|\,\eta\right\rangle\\ &=\frac{ik_\mu}{m_q}\left\langle\pi^+\pi^-\right|\int d^4y\, e^{ik\cdot y}j_5^\mu(y)\left|\eta\right\rangle\,. \end{aligned}\tag{3.47}$$

Naively, in the absence of a physical massless pseudoscalar particle coupled to j_5^μ, one would expect this term to vanish in the soft π^0 limit, $k_\mu\to 0$. However, the ninth axial-vector current j_5^μ is a local, gauge-*noninvariant* bilinear. Assuming that the vacuum seizes, then in the Lorentz gauge the description of the gauge-noninvariant ninth axial-vector current should closely resemble that of the axial-vector current in the Schwinger model. The symmetry should be realized by the formal presence of a Goldstone boson and a second massless compensating excitation. As in the $(1+1)$-dimensional case the two massless excitations should exactly cancel in all intermediate states of gauge-invariant matrix elements. Following the $(1+1)$-dimensional example, we expect to find the massless singularities uncanceled in gauge-noninvariant matrix elements such as j_μ^5. Then Eq. (3.47) becomes

$$\begin{aligned}\lim_{k\to 0}\frac{ik_\mu}{m_q}\langle\pi^+\pi^-|j_5^\mu(k)|\eta\rangle &= \lim_{k\to 0}\frac{ik_\mu}{m_q}\frac{k^\mu}{k^2}\Gamma_{\pi\pi\eta}\\ &=\frac{i}{m_q}\Gamma_{\pi\pi\eta}\,,\end{aligned}\tag{3.48}$$

where Γ is the coupling of the massless gauge-noninvariant excitation to the η and $\pi^+\pi^-$ systems. Of course, Γ itself is gauge-invariant. Throughout this article we have proposed that the ninth axial-vector current of gauge theories with infrared slavery in $3+1$ dimensions has similar behavior to the axial-vector current in $(1+1)$ dimensional quantum electrodynamics. If this is true, we then have a simple evasion mechanism for Sutherland's theorem. It is clear from Eq. (3.48) that we expect the $\eta\to 3\pi$ matrix element to be large and unsuppressed in the soft π^0 region of the Dalitz plot. One can also check that our arguments do not effect the Sutherland theorems for soft π^+ and π^- emission in $\eta\to 3\pi$, so these zeros which are observed experimentally are left untouched by quark confinement.

IV. DISCUSSION AND SUMMARY

One of the fundamental ingredients of the Goldstone phenomenon is that spatially distant regions of the vacuum are dynamically uncoupled. If this is true, then gradual spatial variations of the parameter $\alpha(x)$ of a symmetry transformation will not introduce appreciable energy into the vacuum state. One can think of an interaction length l which determines the minimum distance for which degrees of freedom are dynamically uncoupled. Only when the variations of $\alpha(x)$ are small over this length will the cost in energy be correspondingly small. When long-range forces are present, it is generally possible that the length l is infinite so that no variation in the parameter $\alpha(x)$ can be tolerated. This phenomenon is "seizing" of the vacuum.

The vacuum may seize with respect to some symmetry transformations and not others. For example, it is possible that the long-range forces couple to one current but not to a second. In the case of the strong interactions the vacuum must seize with respect to the ninth axial-vector current, but not the other eight. We have seen in the text how this can happen.

The long-range forces which we have postulated to cause seizing play a second role in strong-interaction dynamics. Namely, they prevent free quarks from escaping the environment of a hadron. Although the existence of such forces have not yet been established in realistic theories, it is encouraging to see that they solve two independent puzzles of hadron physics.

We believe that our work casts considerable light on two problems of the quark model. However, there are other related problems which we have not discussed. An outstanding one concerns the differences between the pseudoscalar and vector-meson nonets. Recall that according to conventional wisdom the physical η behaves essentially as part of an SU(3) octet, while the physical η' behaves as an SU(3) singlet. This should be contrasted with the nine vector mesons which are well described by the ideal mixing scheme of the naive quark model. Perhaps the considerations of this article can be used to explain (at least in part) why the η and η' do not mix significantly.

In order for such an explanation to be convincing, it would certainly have to be quantitative. We have not discovered how to do a numerically trustworthy calculation. In addition, we would like to have a deeper understanding of the success of the ideal mixing scheme for the vector mesons. These are interesting and important problems to study.

Note added in proof. M. Peskin (unpublished work) has pointed out to us that $\rho_5(x, \epsilon)$ of (3.18) should contain a Schwinger line integral with the "wrong" sign. Only then does (3.20) follow, and the charge $\int \rho_5(x) d^3x$ generates global γ_5 transformations. Accordingly, $\tilde{\rho}_5(x, \epsilon)$ should be the product of the corrected $\rho_5(x, \epsilon)$ and the exponential of (3.28) with $-i \to -2i$. Then $\tilde{\rho}_5(x)$ has the properties stated in the text. These corrections should also be made in Ref. 20.

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[1]M. Gell-Mann, Physics 1, 63 (1964).

[2]This point has been recognized by many authors and was emphasized recently by H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973).

[3]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[4]G. 't Hooft, unpublished discussion remarks at the Marseilles Conference on Yang-Mills Fields, 1972. H. Politzer, Phys. Rev. Lett. 30, 1346 (1973); D. J. Gross and F. Wilczek, *ibid.* 30, 1343 (1973).

[5]This term was introduced by H. Georgi and S. Glashow, Phys. Rev. Lett. 32, 438 (1974). The possibility of quark confinement in non-Abelian gauge theories of strong interactions was suggested by many authors. See, for example, S. Weinberg, *ibid.* 31, 494 (1973).

[6]If the theory contains scalar fields, this is easy to do. See, for example, P. Carruthers and R. W. Haymaker, Phys. Rev. D 4, 1808 (1971). Many other authors have considered possible solutions of this general type.

[7]J. Schwinger, Phys. Rev. 128, 2425 (1962).

[8]A. Casher, J. Kogut, and Leonard Susskind, Phys. Rev. Lett. 31, 792 (1973); Phys. Rev. D 9, 706 (1974).

[9]J. Lowenstein and J. Swieca, Ann. Phys. (N.Y.) 68, 172 (1971).

[10]D. G. Sutherland, Phys. Lett. 23, 384 (1966).

[11]This fact is well-known in statistical mechanics. See, for example, D. Mattis and E. Lieb, J. Math. Phys. 6, 304 (1965). References 8 and 9 also employ these methods in part.

[12]There are many kinematic facts about one dimension which prove useful in analyzing (1 +1)-dimensional field theories. See, for example, B. Klaiber, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. 10, p. 141.

[13]L. S. Brown, Nuovo Cimento 29, 617 (1963).

[14]In a strict sense the term "seizing" applies to the behavior of a piece of machinery when, due to overheating or severe stresses, its moving parts fuse. Seizing is a technical term in the vocabulary of mechanics, etc. Used in the context of this paper it means that the vacuum cannot support long-wavelength, zero-energy excitations.

[15]This investigation was provoked by remarks of H. Fritzsch.

[16]A comprehensive review and bibliography of anomalies has been given by S. L. Adler, in *Lectures on Elementary Particles and Quantum Field Theory,* proceedings of the Brandeis University Summer Institute, 1970, edited by S. Deser, M. Grisaru, and H. Pendleton (M.I.T. Press, Cambridge, Mass., 1970), Vol. I.

[17]J. Kogut and Leonard Susskind, Phys. Rev. D 9, 3501 (1974).

[18]K. G. Wilson, Phys. Rev. D 10, 2445 (1974).

[19]P. W. Higgs, Phys. Lett. 12, 132 (1964); F. Englert and R. Brout, Phys. Rev. Lett. 13, 321 (1964); G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, *ibid.* 13, 585 (1964); T. W. B. Kibble, Phys. Rev. 155, 1554 (1967).

[20]J. Kogut and Leonard Susskind, Phys. Rev. D 10, 3468 (1974). Other solutions to the U(3)× U(3) problem have been proposed within the contex of theories other than the one considered here. See, for example, P. Langacker and H. Pagels, Phys. Rev. D 9, 3413 (1974), and I. Bars and M. B. Halpern, *ibid.* 9, 3430 (1974).

[21]J. Schwinger, Phys. Rev. Lett. 3, 296 (1959).

[22]H. Fritzsch and M. Gell-Mann, in *Proceedings of the International Conference on Duality and Symmetry in Hadron Physics*, edited by E. Gotsman (Weizmann Science Press, Jerusalem, 1971).

[23]S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968).

[24]K. G. Wilson, Phys. Rev. 179, 1499 (1969).

[25]R. N. Mohapatra and J. C. Pati, Phys. Rev. D 8, 4212 (1973). The discussion that we present ignores strangeness, which is inessential. The reader can consult this reference for more general formulas.