The U(1) problem*

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A detailed analysis of the problems associated with the conserved U(1) axial-vector current in quark-gluon models is presented. It is shown that such models involve a light isoscalar pseudoscalar boson, with a mass less than $\sqrt{3}\, m_\pi$. The existence of this boson would produce a strong off-shell variation in the $\eta \rightarrow 3\pi$ matrix element, thus invalidating the usual conclusions about the rate and energy dependence of this decay. Following Kogut and Susskind, it is proposed that the light Goldstone boson is actually a dipole, with positive- and negative-metric parts, which cancel in matrix elements of gluon-gauge-invariant operators but not in operators such as the U(1) current. It is shown that the masses of the observable pseudoscalar bosons and the η decay rate are then just as they would be in a theory without the U(1) symmetry, and in fair agreement with experiment. The application of current algebra to theories with charmed quarks is briefly discussed.

I. INTRODUCTION

It has become popular to suppose that the elementary strongly interacting particles include only quarks and vector gluons, with no elementary strong-interacting spin-zero fields. For one thing, these seem to be the only satisfactory theories with asymptotic freedom.[1] Also, theories of this general type are the natural "effective field theories" which arise in unified gauge theories of the weak, electromagnetic, and strong interactions.[2] One more attraction of this picture, which particularly concerns us here, is the way that it deals with the order-α weak corrections to the symmetries of the strong interactions.[3] Emission and reabsorption of an intermediate vector boson will produce such corrections solely in the form of a change in the bare quark mass matrix. With a suitable redefinition of the quark fields, this mass shift conserves parity, strangeness, charm, etc., and provides just the "u_3 tadpole" which is needed to account for hadron mass differences[4] and to evade the Sutherland suppression[5] of $\eta \rightarrow 3\pi$ decay.

The one great puzzle presented by all such theories has to do with the status of the η particle. Quark-gluon models always entail a U(1) axial-vector current A_S^μ, whose conservation is broken only by the $\mathcal{P}$ and $\mathcal{N}$ quark masses. In consequence, the usual arguments of current algebra require an isoscalar pseudoscalar Goldstone boson with a mass of the same order of magnitude as the pion mass. Indeed, we shall show in Sec. II that under plausible assumptions the mass of this light particle L must be less than $\sqrt{3}\, m_\pi$,[6] so that it cannot be identified with the observed η particle.

In addition, it has been remarked[7] that the amplitude for $\eta \rightarrow 3\pi$ is proportional to the matrix element of the divergence of the U(1) axial-vector current between states of equal four-momenta, and therefore still vanishes, despite the u_3 tadpole. As we shall see in Sec. III, this argument is not strictly correct, because an internal L-particle line would give the η decay amplitude a rapid variation off the pion mass shell.[8] However, this result is cold comfort indeed, for the η decay rate is saved only at the cost of a completely fictitious new particle.

It is possible to avoid these problems if we suppose that there is no conserved U(1) axial-vector current. One approach is to introduce elementary strongly interacting spin-zero fields,[9] but then we must give up the advantages of the pure quark-gluon models. Another approach is to suppose that the conservation of the axial-vector U(1) current is violated by some sort of anomaly. There are of course current-gluon-gluon triangle anomalies of the Adler-Bell-Jackiw type,[10] which modify the conservation law to read

$$\partial_\mu A_S^\mu = g_0^2 \epsilon_{\mu\nu\lambda\rho} F_\alpha^{\mu\nu} F_\alpha^{\lambda\rho} , \tag{1.1}$$

where $F_\alpha^{\mu\nu}$ is the gauge-covariant curl of the gluon gauge field G_α^μ, and g_0 is a bare strong coupling parameter. But it is well known that we can define a new axial-vector current

$$\tilde{A}_S^\mu \equiv A_S^\mu - g_0^2 \epsilon^\mu{}_{\nu\lambda\rho} F_\alpha^{\lambda\rho} G_\alpha^\nu , \tag{1.2}$$

which *is* conserved, and furthermore has the same commutators with quark mass terms as did A_S^μ. Thus, the triangle anomaly does not immediately provide a way out of the U(1) problem. It has been suggested that there are additional anomalies[11] which cannot be seen in perturbation theory, but this has not been demonstrated, and one wonders whether in this case the theory would have any predictive power at all.

A different approach to the U(1) problem, which does not deny the U(1) invariance of the underlying theory, is to suppose that the light Goldstone boson L appears as a pole only in matrix elements of the second non-gauge-invariant term in (1.2); in this case, it could be hoped that the L particle is trapped like quarks and gluons, and could not be produced in collisions of ordinary particles.[12] However, this proposal runs into trouble if it is interpreted to mean that the L-particle pole does not appear in matrix elements of (1.2) between ordinary hadron states. The U(1) axial charge of ordinary hadrons would then have to vanish; this apparently leads to difficulties in the saturation of equal-time commutation relations,[13] and leads to other problems discussed in Sec. IV. A related difficulty is that if the L pole does not appear in the matrix element $\langle\pi\pi|\tilde{A}_5^\mu|\eta\rangle$ then it cannot help us to escape Sutherland's theorem.[14]

Instead, Kogut and Susskind[15] have recently suggested that the Goldstone boson associated with the U(1) current is actually a dipole, consisting of a positive-metric and a negative-metric boson, which couple equally to physical states and gauge-invariant operators, but unequally to gauge-non-invariant operators. The singularities due to the Goldstone dipole would cancel in any physical matrix element, but not in matrix elements of gauge-noninvariant operators such as the U(1) axial-vector current (1.2).

It will be shown in Sec. V that the Goldstone-dipole hypothesis does eliminate the problems associated with the η particle. Taking into account first-order effects of the $\mathcal{P}$ and $\mathcal{N}$ quark masses, it is found that the dipole remains at zero mass, while the η is at the Gell-Mann–Okubo value $\approx (4m_K{}^2/3)^{1/2}$. Also, using the formalism developed in Sec. III, it is found that the partial width for $\eta \to \pi^+\pi^-\pi^0$ is 65 eV. This is smaller than the observed value of 204 ± 22 eV, but it is definitely not suppressed by the factor $m_\pi{}^4/m_\eta{}^4$ expected from the Sutherland theorem.[5] Suggestions are offered as to the source of the remaining discrepancy.

There does not now appear to be any serious objection to the hypothesis of a $U(3)\otimes U(3)$ symmetry, broken spontaneously down to nonchiral U(3), and also broken more weakly by quark masses. It is therefore natural to try to see if this hypothesis gives reasonable results when applied to the $U(4)\otimes U(4)$ symmetry of theories with a charmed quark.[16] As shown in Sec. V, if $U(4)\otimes U(4)$ is spontaneously broken to SU(4), then the SU(4) mass formulas for pseudoscalar mesons are valid if interpreted in terms of squared masses. In particular, a charmed pseudoscalar triplet is expected at about 2.1 GeV, with the isotopic doublet about 50 MeV below the singlet. On the other hand, if $U(4)\otimes U(4)$ is spontaneously broken to SU(3), then there must be both scalar and pseudoscalar charmed triplets, with singlet-doublet mass square splittings which are approximately equal and opposite.

This work leaves two major questions unanswered:

(1) How does the underlying gluon-gauge invariance enforce the equal coupling of the positive- and negative-metric Goldstone bosons to gauge-invariant operators?

(2) What are the conditions under which it is valid to treat the various quark masses as small perturbations?

II. THE MASS PROBLEM

In this section we shall show that under what are usually regarded as reasonable assumptions, the mass of the lightest isoscalar pseudoscalar particle must be less than $\sqrt{3}m_\pi$.

We assume that the only strongly interacting particles are quarks and vector gluons. The quarks are either the original $\mathcal{P}, \mathcal{N}, \lambda, \ldots$ or perhaps the colored triplet $\mathcal{P}_R, \mathcal{P}_W, \mathcal{P}_B, \mathcal{N}_R, \mathcal{N}_W, \mathcal{N}_B$, etc., including any number of charmed quarks; the gluons are gauge fields governed by either an Abelian U(1) gauge group or an SU(3) gauge group acting only on color indices.

In such a theory we may construct a set of axial-vector currents

$$A_a^\mu = -i \sum_{\text{color}} \bar{\psi}\gamma^\mu\gamma_5\lambda_a\psi\,, \tag{2.1}$$

where λ_a are a complete set of Hermitian matrices commuting with all generators of the strong gauge group. It will be understood that gauge-noninvariant terms are added where needed to currents with $\mathrm{Tr}\lambda_a \neq 0$, so that all these currents are conserved in the absence of quark masses. [However, the tilde in Eq. (1.2) will be dropped in what follows.]

In the limit of vanishing quark mass, the spontaneous breakdown of the symmetries associated with these currents gives rise to a set of pseudoscalar Goldstone bosons. The coupling of the nth Goldstone boson π_n to the ath axial-vector current is characterized by a coefficient F_{an}, defined by

$$\langle 0|A_a^\mu(x)|\pi_n\rangle = (2\pi)^{-3/2}(2E)^{-1/2}e^{ip\cdot x}p^\mu F_{an}\,. \tag{2.2}$$

If a perturbation $\delta\mathcal{L}$ produces only a small correction to the symmetries represented by a particular pair of currents A_a^μ and A_b^μ, then the pseudoscalar particles associated with these currents pick up a mass-squared matrix $\mathfrak{M}^2{}_{nm}$, given to first order in $\delta\mathcal{L}$ by

$$\sum_{nm} F_{an}F_{bm}\mathfrak{M}^2{}_{nm} = \int d^4x\, d^4y\, \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\langle T\{A_a^\mu(x)A_b^\nu(y)\delta\mathcal{L}(0)\}\rangle_0\,,$$

because only the mass-zero singularities survive the four-dimensional integration. Standard current-algebra arguments then give[17]

$$\sum_{nm} F_{an}F_{bm}\mathfrak{M}^2{}_{nm} = \int d^3x\, d^3y \langle [A_a^0(\vec{x},0),[A_b^0(\vec{y},0),\delta\mathcal{L}(0)]]\rangle_0 \,. \tag{2.3}$$

In this section we will apply this familiar formalism to the currents that can be constructed from just the $\mathcal{P}$ and $\mathfrak{N}$ quark fields. As usual, we define

$$\lambda_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix},\quad \lambda_2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix},\quad \lambda_3 = \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix}, \tag{2.4}$$

and also introduce the isotopic singlet

$$\lambda_S = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ & & 0 \end{pmatrix} \,. \tag{2.5}$$

If we assume that the perturbation $\delta\mathcal{L}$ is an isoscalar bilinear in the quark fields, then it must take the form

$$\delta\mathcal{L} = \epsilon_S \bar{\psi}\lambda_S\psi + \cdots \,, \tag{2.6}$$

with the additional terms commuting with A_1^0, A_2^0, A_3^0, and A_S^0. Also, with isotopic spin not spontaneously broken, the Goldstone bosons associated with these four currents cannot mix with each other. As usual, we assume that there is only one light isovector pseudoscalar triplet, π_1, π_2, π_3, in which case

$$F_{11} = F_{22} = F_{33} = F_\pi \simeq 190 \text{ MeV} \,, \tag{2.7}$$

$$F_{S1} = F_{S2} = F_{S3} = 0 \,. \tag{2.8}$$

However, we can leave open for the moment the question of how many light isoscalar pseudoscalar particles are present, distinguishing among them with an index σ. Equation (2.3) then yields

$$F_\pi{}^2 m_\pi{}^2 = 4\epsilon_S \langle\bar{\psi}\lambda_S\psi\rangle_0 \,, \tag{2.9}$$

$$\sum_{\sigma\sigma'} F_{S\sigma}\, F_{S\sigma'}\mathfrak{M}^2{}_{\sigma\sigma'} = 4\epsilon_S\langle\bar{\psi}\lambda_S\psi\rangle_0 \,. \tag{2.10}$$

The lowest eigenvalue of any positive Hermitian matrix is always less than any expectation value of the matrix, so the light isoscalar pseudoscalar L with *lowest* mass has

$$m_L{}^2 \leq \frac{\sum_{\sigma\sigma'} F_{S\sigma}F_{S\sigma'}\mathfrak{M}^2{}_{\sigma\sigma'}}{\sum_\sigma F_{S\sigma}{}^2} \,,$$

and therefore (2.9) and (2.10) give

$$m_L{}^2 \leq \left(\frac{F_\pi{}^2}{\sum_\sigma F_{S\sigma}{}^2}\right) m_\pi{}^2 \quad . \tag{2.11}$$

We see that there must be at least one isoscalar pseudoscalar with a mass of the order of the pion mass, unless *all* of the coefficients $F_{S\sigma}$ are much smaller than F_π.

It seems unlikely that all $F_{S\sigma}$ would be much smaller than F_π, because that would mean that the symmetry associated with A_S^μ suffers a much weaker spontaneous breakdown than that associated with $\vec{A}^\mu$. Further, to the extent that SU(3) is a good symmetry, we can use it to put a lower bound in the denominator in Eq. (2.11). We can write the current A_S^μ as the sum of a unitary octet and singlet

$$A_S^\mu = \frac{1}{\sqrt{3}} A_8^\mu + (\tfrac{2}{3})^{1/2} A_0^\mu \,, \tag{2.12}$$

with

$$\lambda_8 \equiv \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & -2 & \\ & & & 0 \end{bmatrix},\quad \lambda_0 \equiv (\tfrac{2}{3})^{1/2} \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ & & & 0 \end{bmatrix}. \tag{2.13}$$

In the limit of vanishing quark masses there must be a massless pseudoscalar unitary octet, with an eighth member π_8 having the same coupling to A_8^μ that the pion has to $\vec{A}^\mu$, and zero coupling to A_0^μ:

$$F_{88} = F_\pi, \quad F_{08} = 0 \,. \tag{2.14}$$

But then

$$F_{S8} = \frac{1}{\sqrt{3}} F_\pi \tag{2.15}$$

and Eq. (2.11) gives[6]

$$m_L{}^2 \leq 3m_\pi{}^2 \,. \tag{2.16}$$

There may of course be any number of unitary singlet pseudoscalar particles which also contribute to the denominator in Eq. (2.11), but that would make m_L even smaller.

III. U(3) ⊗ U(3) THEORIES

The result of the last section certainly casts doubt on the viability of any quark-gluon theory. Nevertheless, we will find it useful to go on pursuing this problem for a while, under somewhat more restricted assumptions, if only as a preparation for our discussion of η decay.

We will now assume that there are just three quarks $\mathcal{P}, \mathfrak{N}, \lambda$, or perhaps three colored triplets. The axial-vector currents will be defined as in Eq. (2.1) (including gluon terms where needed to make the currents conserved for zero quark mass), with the matrices λ_i defined to have the usual Gell-Mann form for $i=1, 2, \ldots, 8$, with λ_0 defined as $(\frac{2}{3})^{1/2}1$. It will also be assumed that nonchiral U(3) [i.e., SU(3) plus baryon conservation] is not spontaneously broken, so that the Goldstone bosons associated with the breakdown of the chiral symmetries form SU(3) multiplets. Specifically, we assume that there is just one Goldstone boson π_a for each of the nine axial-vector currents A_a^μ. Since SU(3) is not supposed to be spontaneously broken, the coefficients F_{ab} in (2.2) are

$$\begin{aligned} &F_{ij}=F_\pi\delta_{ij} \quad (i,j=1,2,\ldots,8)\,, \\ &F_{i0}=F_{0i}=0\,, \\ &F_{00}=F_0\neq 0\,. \end{aligned} \tag{3.1}$$

Finally, it is assumed that the perturbation $\delta\mathcal{L}$ which breaks U(3)×U(3) in the Lagrangian is of the form

$$\delta\mathcal{L}=\sum_a \epsilon_a u_a\,, \tag{3.2}$$

where u_a forms a set of scalar densities which, together with a corresponding set of pseudoscalar v_a, have the same chiral transformation properties as $\bar\psi\lambda_a\psi$ and $\bar\psi i\lambda_a\gamma_5\psi$, respectively. That is, if we define a set of coefficients d_{abc} by

$$\{\lambda_a,\lambda_b\}=2\sum_c d_{abc}\lambda_c\,, \tag{3.3}$$

then

$$\begin{aligned} &[A_a^0(\vec{x},t),u_b(\vec{y},t)]=2i\sum_c d_{abc}v_c(\vec{x},t)\delta^3(\vec{x}-\vec{y})\,, \\ &[A_a^0(\vec{x},t),v_b(\vec{y},t)]=-2i\sum_c d_{abc}u_c(\vec{x},t)\delta^3(\vec{x}-\vec{y})\,. \end{aligned} \tag{3.4}$$

[The d_{abc} are totally symmetric, and have the usual Gell-Mann values when a, b, and c run from 1 to 8, while d_{ab0} is defined as $(\frac{2}{3})^{1/2}\delta_{ab}$.] The only u_a which conserve charge, strangeness, baryon number, and parity are u_3, u_8, and u_0, so we set all ϵ_a equal to zero except ϵ_3, ϵ_0, and ϵ_8; the u_3 term is included as a possible nonelectromagnetic correction to isospin conservation. Our assumption that SU(3) is not spontaneously broken implies that the only u_a with a nonvanishing vacuum expectation value is u_0.

With these assumptions, the elements of the pseudoscalar mass matrix are given by (2.3) as

$$\mathfrak{M}^2{}_{ij}=\frac{4}{F_\pi^{\,2}}(\tfrac{2}{3})^{1/2}\langle u_0\rangle_0\sum_a d_{ija}\epsilon_a\,, \tag{3.5}$$

$$\mathfrak{M}^2{}_{i0}=\frac{4}{F_\pi F_0}(\tfrac{2}{3})\langle u_0\rangle_0\epsilon_i\,, \tag{3.6}$$

$$\mathfrak{M}^2{}_{00}=\frac{4}{F_0^{\,2}}(\tfrac{2}{3})\langle u_0\rangle_0\epsilon_0\,. \tag{3.7}$$

(Recall that i,j, etc. run from 1 to 8, while a runs from 0 to 8.)

We use the results here to determine the quantities ϵ_a in terms of the masses of particles with nonvanishing isospin, i.e.,

$$\begin{aligned} &m_{\pi^+}{}^2=\mathfrak{M}^2{}_{11}=\mathfrak{M}^2{}_{22}, \quad m_{\pi^0}{}^2=\mathfrak{M}^2{}_{33}\,, \\ &m_{K^+}{}^2=\mathfrak{M}^2{}_{44}=\mathfrak{M}^2{}_{55}, \quad m_{K^0}{}^2=\mathfrak{M}^2{}_{66}=\mathfrak{M}^2{}_{77}\,. \end{aligned}$$

With ϵ_3 neglected, we find the familiar results[17]

$$\frac{4}{F_\pi^{\,2}}\langle u_0\rangle_0\epsilon_0=m_K{}^2+\tfrac{1}{2}m_\pi{}^2\,, \tag{3.8}$$

$$\frac{4}{F_\pi^{\,2}}\langle u_0\rangle_0\epsilon_8=-\sqrt{2}\,(m_K{}^2-m_\pi{}^2)\,. \tag{3.9}$$

Also, to the extent that u_3 provides the only correction to isospin conservation, we would have the π^+ degenerate with the π^0, and[4]

$$\frac{4}{F_\pi^{\,2}}\langle u_0\rangle_0\epsilon_3=(\tfrac{3}{2})^{1/2}(m_{K^+}{}^2-m_{K^0}{}^2)_{\text{tadpole}}\,.$$

Actually, electromagnetism makes an approximately equal contribution to both the π and K mass splittings,[18] so this can be rewritten in terms of the *total* mass splittings[19]

$$\frac{4}{F_\pi^{\,2}}\langle u_0\rangle_0\epsilon_3\simeq(\tfrac{3}{2})^{1/2}\delta m^2\,, \tag{3.10}$$

with

$$\delta m^2\equiv m_{K^+}{}^2-m_{K^0}{}^2-m_{\pi^+}{}^2+m_{\pi^0}{}^2\,. \tag{3.11}$$

We can also rederive the results of the last section. The mass matrix for states of isospin zero has elements given by (3.5)–(3.9) as

$$\mathfrak{M}^2{}_{88}=\tfrac{1}{3}(4m_K{}^2-m_\pi{}^2)\,, \tag{3.12}$$

$$\mathfrak{M}^2{}_{80}=-\frac{2\sqrt{2}}{3z}(m_K{}^2-m_\pi{}^2)\,, \tag{3.13}$$

$$\mathfrak{M}^2{}_{00}=\frac{2}{3z^2}(m_K{}^2+\tfrac{1}{2}m_\pi{}^2)\,, \tag{3.14}$$

where

$$z\equiv F_0/F_\pi\,.$$

The eigenvalues of this matrix are

$$m^2=\begin{cases} \dfrac{3m_\pi{}^2}{1+2z^2}+O\left(\dfrac{m_\pi{}^4}{m_K{}^2}\right)\,, \\[2ex] \dfrac{4}{3}m_K{}^2\left(1+\dfrac{1}{2z^2}\right)+O(m_\pi{}^2)\,. \end{cases} \tag{3.15}$$

The heavier particle has a mass which approaches the Gell-Mann–Okubo value for $z\gg 1$, so this particle might be identified with the observed η. However, we see again that there is a light particle with mass less than $\sqrt{3}\,m_\pi$. As shown in the last section, this result becomes even stronger if more quarks are added.

IV. η DECAY

We now calculate the rate of the decay $\eta \to \pi^+\pi^-\pi^0$ in the $U(3)\times U(3)$ picture described in the last section. The results we obtain will also serve as a basis for practical calculations in the improved picture of Sec. V.

It is rather complicated to calculate the rate for $\eta \to 3\pi$ by a direct use of current algebra, because when we let two of the pion four-momenta vanish, either the η or the other pion must be far off its mass shell. (This complication does not arise, for instance, in calculations of pion-pion scattering.[20]) It is very much easier to carry out the calculation using a phenomenological Lagrangian[21] expressed in terms of a set of pseudoscalar fields transforming according to a minimal nonlinear realization of $U(3)\otimes U(3)$. Such a Lagrangian is guaranteed to give the same results as current algebra for any amplitude predicted uniquely by current algebra; furthermore, if the amplitude is insensitive to nonminimal terms in the Lagrangian, we can be sure that it is in fact determined by current algebra.

It will be convenient to make use of a set of 0^- Goldstone boson fields π_a ($a=0,\ldots,8$) defined as an "exponential parameterization"[22] of the cosets of $U(N)\otimes U(N)/U(N)$. That is, if the quark fields are subjected to a general chiral transformation

$$\psi \to \psi' \equiv U\psi \tag{4.1}$$

(with U a unitary matrix containing terms proportional to γ_5 as well as to 1) then the pseudoscalar fields undergo the transformation

$$\phi_a \to \phi_a' , \tag{4.2}$$

with ϕ' defined by

$$U\exp\left[i\gamma_5\left(\frac{\lambda_i\phi_i}{F_\pi}+\frac{\lambda_0\phi_0}{F_0}\right)\right]=\exp\left[i\gamma_5\left(\frac{\lambda_i\phi_i'}{F_\pi}+\frac{\lambda_0\phi_0'}{F_0}\right)\right]\times\exp[i\lambda_a\mu_a(\phi)] , \tag{4.3}$$

where λ_i are again the usual eight Gell-Mann matrices, $\lambda_0\equiv(\frac{2}{3})^{1/2}1$, the F's are the coefficients defined in the last section, and the μ_a are a set of functions of the ϕ_a which need not be calculated here.

The covariant derivatives of the Goldstone boson fields in this formalism are defined by[22]

$$\exp\left[-i\gamma_5\left(\frac{\lambda_i\phi_i}{F_\pi}+\frac{\lambda_0\phi_0}{F_0}\right)\right]\frac{\partial}{\partial x^\mu}\exp\left[i\gamma_5\left(\frac{\lambda_i\phi_i}{F_\pi}+\frac{\lambda_0\phi_0}{F_0}\right)\right]=iF_\pi^{-1}\gamma_5\lambda_iD_\mu\phi_i+iF_0^{-1}\gamma_5\lambda_0D_\mu\phi_0+\text{etc.}, \tag{4.4}$$

with "etc." denoting additional terms proportional to λ_i or λ_0, with no γ_5 factor. It is straightforward to calculate that

$$D_\mu\phi_0=\partial_\mu\phi_0 , \tag{4.5}$$

$$D_\mu\phi_8=\partial_\mu\phi_8+K\text{ terms} , \tag{4.6}$$

where "K terms" indicates terms involving the K-meson field, which will not concern us here because we deal with graphs having no internal or external K-meson lines. Also expanding to third order in the fields,

$$D_\mu\vec{\phi}=\partial_\mu\vec{\phi}+\frac{2}{3F_\pi^{\,2}}[\vec{\phi}(\vec{\phi}\cdot\partial_\mu\vec{\phi})-\vec{\phi}^2\partial_\mu\vec{\phi}] + O(\phi^5)+K\text{ terms}. \tag{4.7}$$

In the absence of symmetry breaking, the minimal Lagrangian is

$$\mathcal{L}_0=-\tfrac{1}{2}D_\mu\phi_iD^\mu\phi_i-\tfrac{1}{2}D_\mu\phi_0D^\mu\phi_0 . \tag{4.8}$$

Apart from terms involving the K-meson field, this gives

$$\mathcal{L}_0=-\tfrac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}-\frac{2}{3F_\pi^{\,2}}[(\vec{\phi}\cdot\partial_\mu\vec{\phi})^2-\vec{\phi}^2(\partial_\mu\vec{\phi})^2] + O(\phi^6)-\tfrac{1}{2}\partial_\mu\phi_8\partial^\mu\phi_8-\tfrac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 . \tag{4.9}$$

We also need to construct functions u_a and v_a which transform like $\bar\psi\lambda_a\psi$ and $\bar\psi i\gamma_5\lambda_a\psi$. That is, we want $u_a(\phi')$ and $v_a(\phi')$ to be the same linear combination of $u_a(\phi)$ and $v_a(\phi)$ that $\gamma_4U^\dagger\gamma_4\lambda_aU$ and $\gamma_4U^\dagger\gamma_4i\gamma_5\lambda_aU$ are of λ_a and $i\gamma_5\lambda_a$. To this end, multiply Eq. (4.3) on the right by γ_4 times its adjoint. This gives

$$U\exp\left[2i\gamma_5\left(\frac{\lambda_i\phi_i}{F_\pi}+\frac{\lambda_0\phi_0}{F_0}\right)\right]\gamma_4U^\dagger =\exp\left[2i\gamma_5\left(\frac{\lambda_i\phi_i'}{F_\pi}+\frac{\lambda_0\phi_0'}{F_0}\right)\right]\gamma_4 ,$$

and therefore, for any matrix Γ,

$$\mathrm{Tr}\left\{\gamma_4U^\dagger\gamma_4\Gamma U\exp\left[2i\gamma_5\left(\frac{\lambda_i\phi_i}{F_\pi}+\frac{\lambda_0\phi_0}{F_0}\right)\right]\right\} =\mathrm{Tr}\left\{\Gamma\exp\left[2i\gamma_5\left(\frac{\lambda_i\phi_i'}{F_\pi}+\frac{\lambda_0\phi_0'}{F_0}\right)\right]\right\} . \tag{4.10}$$

Thus, the functions which transform like $\bar\psi\lambda_a\psi$ and $\bar\psi i\gamma_5\lambda_a\psi$ are respectively

$$u_a=\tfrac{1}{2}\,\mathrm{Tr}\left\{\lambda_a\exp\left[2i\gamma_5\left(\frac{\lambda_i\phi_i}{F_\pi}+\frac{\lambda_0\phi_0}{F_0}\right)\right]\right\}, \tag{4.11}$$

$$v_a=\tfrac{1}{2}\,\mathrm{Tr}\left\{i\gamma_5\lambda_a\exp\left[2i\gamma_5\left(\frac{\lambda_i\phi_i}{F_\pi}+\frac{\lambda_0\phi_0}{F_0}\right)\right]\right\}. \tag{4.12}$$

Again dropping all terms involving the K-meson fields, we now find

$$u_3=-\frac{\phi_3}{\sqrt{\vec{\phi}^2}}\sin\left(\frac{2(\vec{\phi}^2)^{1/2}}{F_\pi}\right)\sin\left[\frac{2}{\sqrt{3}}\left(\frac{\phi_8}{F_\pi}+\frac{\sqrt{2}\,\phi_0}{F_0}\right)\right], \tag{4.13}$$

$$u_8=\frac{1}{\sqrt{3}}\cos\left(\frac{2(\vec{\phi}^2)^{1/2}}{F_\pi}\right)\cos\left[\frac{2}{\sqrt{3}}\left(\frac{\phi_8}{F_\pi}+\frac{\sqrt{2}\,\phi_0}{F_0}\right)\right]-\frac{1}{\sqrt{3}}\cos\left[\left(\frac{8}{3}\right)^{1/2}\left(\frac{\sqrt{2}\,\phi_8}{F_\pi}-\frac{\phi_0}{F_0}\right)\right], \tag{4.14}$$

$$u_0=\left(\frac{2}{3}\right)^{1/2}\cos\left(\frac{2(\vec{\phi}^2)^{1/2}}{F_\pi}\right)\cos\left[\frac{2}{\sqrt{3}}\left(\frac{\phi_8}{F_\pi}+\frac{\sqrt{2}\,\phi_0}{F_0}\right)\right]+\frac{1}{\sqrt{6}}\cos\left[\left(\frac{8}{3}\right)^{1/2}\left(\frac{\sqrt{2}\,\phi_8}{F_\pi}-\frac{\phi_0}{F_0}\right)\right]. \tag{4.15}$$

The quark mass term (3.2) is

$$\delta\mathcal{L}=\epsilon_3u_3+\epsilon_8u_8+\epsilon_0u_0\,. \tag{4.16}$$

With our present normalization of the u_a, the vacuum expectation value of u_0 is given by (4.15) as

$$\langle u_0\rangle_0=(\tfrac{3}{2})^{1/2}\,. \tag{4.17}$$

Using (3.8)–(3.10) and (4.13)–(4.17), we may write the mass term as

$$\begin{aligned}\delta\mathcal{L}=&-\frac{F_\pi^{\,2}}{4}\,\delta m^2\,\frac{\phi_3}{\sqrt{\vec{\phi}^2}}\sin\left(\frac{2(\vec{\phi}^2)^{1/2}}{F_\pi}\right)\sin\left[\frac{2}{\sqrt{3}}\left(\frac{\phi_8}{F_\pi}+\frac{\sqrt{2}\,\phi_0}{F_0}\right)\right]\\&+\frac{F_\pi^{\,2}}{4}\,m_\pi^{\,2}\cos\left(\frac{2(\vec{\phi}^2)^{1/2}}{F_\pi}\right)\cos\left[\frac{2}{\sqrt{3}}\left(\frac{\phi_8}{F_\pi}+\frac{\sqrt{2}\,\phi_0}{F_0}\right)\right]\\&+\frac{F_\pi^{\,2}}{8}\,(2m_K^{\,2}-m_\pi^{\,2})\cos\left[\left(\frac{8}{3}\right)^{1/2}\left(\frac{\sqrt{2}\,\phi_8}{F_\pi}-\frac{\phi_0}{F_0}\right)\right]\\&+K\text{ terms.}\end{aligned} \tag{4.18}$$

The reader may wish to be reminded here that

$$\delta m^2\equiv m_{K^+}^{\,2}-m_{K^0}^{\,2}-m_{\pi^+}^{\,2}+m_{\pi^0}^{\,2}\,.$$

From the quadratic terms in (4.18) we may read off the same isoscalar mass matrix as given by Eqs. (3.12)–(3.14). Once again, we see that there are two 0^- isoscalars, one of which may be identified with the observed η particle, with

$$m_\eta^{\,2}\simeq\frac{4}{3}m_K^{\,2}\left(1+\frac{1}{2z^2}\right)\,,$$

and the other an unobserved light particle L, with mass

$$m_L^{\,2}=3m_\pi^{\,2}(1+2z^2)^{-1}+O\left(\frac{m_\pi^{\,4}}{m_K^{\,2}}\right)\,,$$

where again

$$z\equiv F_0/F_\pi\,.$$

The fields corresponding to these two particles are linear combinations of ϕ_0 and ϕ_8:

$$\eta=-\phi_8\cos\alpha+\phi_0\sin\alpha\,,$$

$$L=\phi_8\sin\alpha+\phi_0\cos\alpha\,,$$

with

$$\tan\alpha=\frac{1}{\sqrt{2}\,z}+O\left(\frac{m_\pi^{\,2}}{m_K^{\,2}}\right)\,.$$

In particular, the linear combination appearing in the first two terms of (4.18) can be written

$$\phi_8+\left(\frac{\sqrt{2}\,F_\pi}{F_0}\right)\phi_0=a\eta+bL\,, \tag{4.19}$$

where

$$a=-\cos\alpha+\left(\frac{\sqrt{2}}{z}\right)\sin\alpha=[z^{-2}-1][1+\tfrac{1}{2}z^{-2}]+O\left(\frac{m_\pi^{\,2}}{m_K^{\,2}}\right)\,, \tag{4.20}$$

$$b=\sin\alpha+\left(\frac{\sqrt{2}}{z}\right)\cos\alpha=\frac{3}{\sqrt{2}\,z}\,[1+\tfrac{1}{2}z^{-2}]^{1/2}+O\left(\frac{m_\pi^{\,2}}{m_K^{\,2}}\right)\,. \tag{4.21}$$

We are now equipped to calculate the amplitude for $\eta\to\pi^+\pi^-\pi^0$. The relevant Feynman diagrams are shown in Fig. 1. To calculate diagram (a), we need the η-π interaction in (4.18),

$$\mathcal{L}_{\eta\pi}=-\frac{a}{\sqrt{3}}\,\delta m^2\phi_3\eta \tag{4.22}$$

and the π^4 interaction in (4.9)

$$\mathcal{L}_{\pi\pi\pi\pi}=-\frac{2}{3F_\pi^{\,2}}\,[(\vec{\phi}\cdot\partial_\mu\vec{\phi})^2-\vec{\phi}^2(\partial_\mu\vec{\phi})^2]\,. \tag{4.23}$$

The resulting matrix element (neglecting terms of order $m_\pi^{\,2}/m_\eta^{\,2}$) is

$$T^{(a)}\simeq\frac{8a\delta m^2}{3\sqrt{3}\,F_\pi^{\,2}}\left(1-\frac{3E_0}{m_\eta}\right)\,, \tag{4.24}$$

where E_0 is the energy of the final neutral pion in the rest frame of the η. To calculate diagram (b), we need only the η-π^3 interaction in (4.18),

$$\mathcal{L}_{\eta\pi\pi\pi}=\frac{2a\delta m^2}{3\sqrt{3}\,F_\pi^{\,2}}\,\eta\phi_3\vec{\phi}^2\,. \tag{4.25}$$

The resulting matrix element is

$$T^{(b)} \simeq \frac{4a\delta m^2}{3\sqrt{3}F_\pi^{\ 2}} \quad . \tag{4.26}$$

Together, (a) and (b) give an amplitude with the expected spectrum shape[22′]

$$T^{(b)} + T^{(b)} \simeq \frac{4a\delta m^2}{\sqrt{3}F_\pi^{\ 2}}\left(1 - \frac{2E_0}{m_\eta}\right) \quad . \tag{4.27}$$

However, there is one more diagram, Fig. 1(c). To calculate this diagram we need the $\eta L\pi^2$ and $L\pi_3$ terms in (4.18),

$$\mathcal{L}_{\eta L\pi\pi} = \frac{2m_\pi^{\ 2}ab}{3F_\pi^{\ 2}}\,\eta L\vec{\phi}^2 \ , \tag{4.28}$$

$$\mathcal{L}_{L\pi} = -\frac{\delta m^2 b}{\sqrt{3}}\,L\phi_3 \ . \tag{4.29}$$

The $\eta L\pi\pi$ interaction is suppressed by a factor $m_\pi^{\ 2}$, but this diagram is nevertheless of the same order of magnitude as the others, because the L mass is of order m_π. (This is why we drop the corresponding graph with an internal η line.) The resulting matrix element is

$$T^{(c)} \simeq -\frac{4m_\pi^{\ 2}\delta m^2 ab^2}{3\sqrt{3}F_\pi^{\ 2}}\left(\frac{1}{m_L^{\ 2} - m_\pi^{\ 2}}\right) \quad . \tag{4.30}$$

But (4.21) shows that

$$b^2 \simeq \frac{3m_L^{\ 2}}{m_\pi^{\ 2}} \quad , \tag{4.31}$$

so the total matrix element is

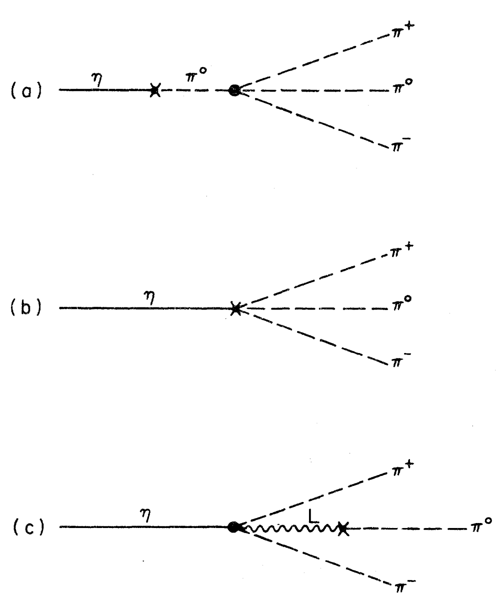

FIG. 1. Diagrams for the decay $\eta \to \pi^+\pi^-\pi^0$. The crosses indicate a u_3 tadpole vertex, while darkened circles indicate a strong interaction.

$$T \simeq \frac{4a\delta m^2}{\sqrt{3}F_\pi^{\ 2}}\left(1 - \frac{2E_0}{m_\eta} - \frac{m_L^{\ 2}}{m_L^{\ 2} - m_\pi^{\ 2}}\right) \quad . \tag{4.32}$$

The last term arising from diagram (c) is comparable to the other two, and completely changes the spectrum shape, in disagreement with the experimentally verified partially conserved axial-vector current (PCAC) prediction that the matrix element should vanish when $E_0 = m_\eta/2$.

Why should PCAC fail here? The reason is just that the small mass of the L particle gives the T matrix a rapid variation with the π^0 four-momentum, and thus invalidates the smoothness assumptions usually invoked in applications of PCAC. Indeed, we note that if instead of putting the π^0 on its mass shell, we set the π^0 four-momentum equal to zero, then E_0 and the pion mass must be set equal to zero in (4.32), so the matrix element vanishes. Of course, the fact that the matrix element vanishes when the π^0 four-momentum vanishes is an accident, depending on the particular definition of the pion field being used here. The only point of physical significance is that the L particle produces a major change in the decay amplitude.

By choosing F_0 close to F_π we can make m_L close to m_π, so that the rate for η decay can be made as large as we like. The real problem is not so much with η decay as with the L particle itself.

V. THE DIPOLE GOLDSTONE BOSON

We have seen that all the problems associated with the η particle really boil down to the problem of the nonexistence of the light L particle. What then can be done about it?

As discussed in the Introduction, we would like to believe that the L particle is a trapped particle, which cannot be produced in collisions among ordinary hadrons.[12] However, this idea runs into trouble if it means that the amplitudes $i \to f + L$ vanish for any initial and final hadron state i, f, because then (by the same argument that accounts for the Goldberger-Treiman relation) the axial-vector coupling must vanish at zero momentum transfer in matrix elements $\langle f|A_0^\mu|i\rangle$ of the unitary singlet axial-vector current. This may be possible for some hadrons, such as baryonic states, for which the unitary singlet axial-vector coupling is an unknown parameter. However, the $\mathrm{U}(3)\otimes\mathrm{U}(3)$ group structure will not allow the unitary singlet axial-vector couplings to vanish when i and f consist of Goldstone bosons such as η and π, any more than the ordinary isovector axial-vector coupling could vanish for multipion states. Indeed, we see from Eq. (4.18) that the amplitude for $\eta \to 2\pi + L$ is a definite nonzero quantity, proportional to $m_\pi^{\ 2}$.

One possible solution[15] to this problem is that the Goldstone boson field ϕ_0 associated with the unitary singlet axial-vector current is actually the sum of two pseudoscalar fields, an ordinary positive-metric field ϕ_+ and a negative-metric field ϕ_-. In order to accomplish this, we make in Eq. (4.18) the replacement

$$\phi_0 \to \phi_+ + \phi_- , \tag{5.1}$$

and in Eq. (4.9)

$$-\tfrac{1}{2}\partial_\mu \phi_0 \partial^\mu \phi_0 \to -\tfrac{1}{2}\partial_\mu \phi_+ \partial^\mu \phi_+ + \tfrac{1}{2}\partial_\mu \phi_- \partial^\mu \phi_- . \tag{5.2}$$

It is the change of sign of the ϕ_- term in (5.2) that gives this field a negative metric.

The probabilities for producing a ϕ_+ or a ϕ_- particle in any reaction will cancel, just as for the longitudinal and timelike photons in the Gupta-Bleuler formulation of quantum electrodynamics. However, the $\phi_\pm$ do show up as poles in certain operators, such as the unitary singlet axial-vector current A_0^μ. In order to give $\phi_+ + \phi_-$ the correct transformation properties for a U(1) Goldstone boson field, we suppose that under the chiral transformation

$$\delta\psi = i\gamma_5 \epsilon \psi \tag{5.3}$$

the fields $\phi_\pm$ undergo changes

$$\delta\phi_\pm = \epsilon F_\pm , \tag{5.4}$$

with constants $F_\pm$ such that

$$F_+ + F_- = F_0 \neq 0 . \tag{5.5}$$

The unitary singlet axial-vector current will then contain terms linear in the ϕ fields

$$\begin{aligned}[A_0^\mu]_\phi &= iF_+\partial^\mu\phi_+ - iF_-\partial^\mu\phi_- \\ &= \frac{i}{2}(F_+ - F_-)\partial^\mu(\phi_+ + \phi_-) + \frac{i}{2}F_0\partial^\mu(\phi_+ - \phi_-) .\end{aligned} \tag{5.6}$$

The first term again involves only the combination $\phi_+ + \phi_-$, and therefore has a zero matrix element between hadronic states, the $1/q^2$ propagator of ϕ_- canceling the $1/q^2$ propagator of ϕ_+. However, the second term involves $\phi_+ - \phi_-$, and therefore has nonvanishing hadronic matrix elements, as required if the chiral U(1) symmetry is to be spontaneously broken.

Of course, we would really like to show that the underlying strong gauge invariance forces the cancellation of $\phi_\pm$ poles in matrix elements of gauge-invariant operators, while allowing the presence of these poles in non-gauge-invariant operators such as A_0^μ. This may perhaps be explained in terms of the rapid growth of gluon propagators at zero four-momentum, producing poles in matrix elements of operators such as $\epsilon^{\mu\nu\lambda\rho}G_\nu F_{\lambda\rho}$, but not in matrix elements of operators formed from $F_{\mu\nu}$ alone. However, for the present this is only wishful thinking.

Let us now see what the introduction of a dipole Goldstone boson does to predictions of pseudoscalar masses. We suppose that there are other spontaneously broken chiral symmetries [such as SU(3)⊗SU(3)] besides chiral U(1), so in the absence of any quark mass perturbation there will be a set of massless pseudoscalar particles with fields ϕ_i as well as the dipole Goldstone bosons with fields $\phi_\pm$. When we turn on the quark mass perturbation, the ϕ_i acquire a mass matrix $\mathfrak{M}^2_{ij}$, which is the same as we would have in the absence of the chiral U(1) symmetry. In addition, there will be a 2×2 $\phi_\pm$ mass matrix and a $\phi_\pm - \phi_i$ mass mixing matrix. Since the symmetry-breaking terms in the effective Lagrangian involve only $\phi_+ + \phi_-$, these take the form

$$\mathfrak{M}^2_{++} = \mathfrak{M}^2_{+-} = \mathfrak{M}^2_{-+} = \mathfrak{M}^2_{--} = \mathfrak{M}^2_{00} , \tag{5.7}$$

$$\mathfrak{M}^2_{+i} = \mathfrak{M}^2_{-i} = \mathfrak{M}^2_{i+} = \mathfrak{M}^2_{i-} = \mathfrak{M}^2_{i0} . \tag{5.8}$$

To find the particle masses, we do *not* diagonalize the full matrix $\mathfrak{M}^2_{AB}$; rather, we must look for poles in the propagator

$$\begin{aligned}\Delta'(q) &= \frac{\eta}{q^2} - \frac{\eta}{q^2}\mathfrak{M}^2\frac{\eta}{q^2} + \cdots \\ &= \eta[q^2 + \mathfrak{M}^2\eta]^{-1} ,\end{aligned} \tag{5.9}$$

where η is the metric matrix

$$\begin{aligned}&\eta_{ij} = \delta_{ij}, \quad \eta_{++} = +1, \quad \eta_{--} = -1 , \\ &\eta_{AB} = 0 \text{ for } A \neq B .\end{aligned} \tag{5.10}$$

It is straightforward to calculate that

$$\Delta'_{ij}(q) = (q^2 + \overline{\mathfrak{M}}^2)^{-1}{}_{ij} , \tag{5.11}$$

$$\begin{aligned}\Delta'_{i\pm}(q) &= \Delta'_{\pm i}(q) \\ &= \mp \sum_j (q^2 + \overline{\mathfrak{M}}^2)^{-1}{}_{ij}\mathfrak{M}^2_{j0}\frac{1}{q^2} ,\end{aligned} \tag{5.12}$$

$$\begin{aligned}\Delta'_{++}(q) &= \Delta'_{--}(q) \\ &= \frac{1}{q^2} - \frac{\mathfrak{M}^2_{00}}{(q^2)^2} \\ &\quad + \frac{1}{(q^2)^2}\sum_{jk}\mathfrak{M}^2_{j0}\mathfrak{M}^2_{k0}(q^2 + \overline{\mathfrak{M}}^2)^{-1}{}_{jk} ,\end{aligned} \tag{5.13}$$

$$\begin{aligned}\Delta'_{+-}(q) &= \Delta'_{-+}(q) \\ &= \frac{1}{(q^2)^2}\left[\mathfrak{M}^2_{00} - \sum_{jk}\mathfrak{M}^2_{j0}\mathfrak{M}^2_{k0}(q^2 + \overline{\mathfrak{M}}^2)^{-1}{}_{jk}\right]\end{aligned} \tag{5.14}$$

where $\overline{\mathfrak{M}}^2$ is the submatrix with elements $\mathfrak{M}^2_{ij}$. We see that the dipole poles at $q^2 = 0$ remain at zero mass even after the perturbation is turned on. In addition, the propagator has poles at the eigenvalues m^2 of the submatrix $\overline{\mathfrak{M}}^2$, with

$$\Delta'_{AB}(q) \underset{q^2 \to -m^2}{\simeq} \frac{u_A u_B}{q^2+m^2} , \tag{5.15}$$

where u_i is the normalized eigenvector of $\overline{\mathfrak{M}}^2$ with eigenvalue m^2

$$\sum_j \mathfrak{M}^2{}_{ij} u_j = m^2 u_i , \quad \sum_i u_i u_i = 1 , \tag{5.16}$$

and

$$u_+ = -u_- = \frac{1}{m^2} \sum_j \mathfrak{M}^2{}_{j0} u_j . \tag{5.17}$$

We see that the masses of the ϕ_i particles are unaffected by any mixing between them and the $\phi_\pm$. In particular, Eq. (3.12) shows that the η mass should be given in U(3)$\otimes$U(3) theories by the quadratic Gell-Mann–Okubo formula

$$m_\eta{}^2 = \mathfrak{M}^2{}_{88} = \tfrac{1}{3}(4m_K{}^2 - m_\pi{}^2) \tag{5.18}$$

in reasonably good agreement with experiment. (Also, the fact that $u_+ = -u_-$ eliminates any contribution from graphs in which a pseudoscalar particle is created by a $\phi_+ + \phi_-$ field and destroyed by a ϕ_i field.)

The η decay problem is also now easily dealt with. Because of the cancellation of the ϕ_+ and ϕ_- poles, there are now no diagrams of the form 1(c). Also, the η pole now appears solely in the ϕ_8 field, so the constant a in Eq. (4.19) is unity. Thus, the matrix element for the decay $\eta \to \pi^+\pi^-\pi^0$ in U(3)$\otimes$U(3) theories is now given by just the first two terms in Eq. (4.32) with $a=1$:

$$T(\eta \to \pi^+\pi^-\pi^0) \simeq \frac{4\delta m^2}{\sqrt{3}F_\pi{}^2}\left(1 - \frac{2E_0}{m_\eta}\right) , \tag{5.19}$$

which is just the same answer as would be given by an SU(3)$\otimes$SU(3) theory with a u_3 tadpole.[23] This result definitely does not suffer from the difficulty pointed out by Sutherland[5]: The matrix element does not vanish like $m_\pi{}^2$ for $m_\pi \to 0$. On the other hand, the rate predicted by (5.19) is

$$\Gamma(\eta \to \pi^+\pi^-\pi^0) \simeq 65 \text{ eV} . \tag{5.20}$$

For several years the actual value has been quoted[24] as 630$\pm$140 eV, in which case the theoretical result would be considerably too small, although nowhere near so small as expected if Sutherland's theorem applied here. Recently, a remeasurement of the Primakoff effect has led to a new value for the $\eta \to 2\gamma$ width,[25] with a consequent reduction in the value of the $\eta \to \pi^+\pi^-\pi^0$ width to 204$\pm$22 eV. It may not be utterly rash to hope for further reductions in the observed width, especially in view of the great difficulty in measuring and interpreting the Primakoff effect.

Also, the theoretical calculation might be improved, either by taking into account logarithmic correction terms[26] in the estimate of ϵ_3 from δm^2, or by including the leading edge of the ϵ(700 MeV) enhancement of π-π scattering, or both.

VI. U(4)$\otimes$U(4) THEORIES

We have not found any insuperable objection to the hypothesis that chiral U(3)$\otimes$U(3) is spontaneously broken down to nonchiral SU(3) and baryon conservation, with SU(3) broken only by a "small" quark mass perturbation. It is therefore natural, in theories with a fourth charmed quark,[16] to consider the possibility that chiral U(4)$\otimes$U(4) is a good approximate symmetry of the Lagrangian, broken in the Lagrangian only by small quark masses.

This of course runs counter to the general expectation that charmed hadrons should be much heavier than ordinary hadrons, but we do not really understand what conditions on observable masses are required for symmetry-breaking terms in the Lagrangian to be considered small. For instance, from the point of view of nuclear physics, the pion mass might be considered to be so much larger than nuclear energy level differences that it would be hopeless to derive soft-pion theorems, and this opinion would be supported by the general failure of current-algebra predictions for pion-nucleus scattering lengths. Yet there is a large domain of high-energy experiments in which the quark mass terms which break SU(2)$\times$SU(2) can be considered very small. It is therefore at least worth trying to see whether there is any sense in which the charmed quark mass is also small.[27]

In applying this hypothesis, we must first decide what symmetry would be preserved by the spontaneous symmetry breakdown in the absence of quark masses. The two most attractive possibilities are nonchiral SU(3) or SU(4), plus baryon conservation. Let us consider each of these in turn. (Isospin breaking will be ignored in what follows.)

(a) SU(3). If U(4)$\otimes$U(4) is spontaneously broken down to SU(3), the Goldstone bosons will form SU(3) multiplets: specifically, a pseudoscalar $\underline{8}$, $\underline{3}$, $\underline{\overline{3}}$, and $\underline{1}$, and a scalar $\underline{3}$ and $\underline{\overline{3}}$. Each of these couples to the corresponding broken U(4)$\otimes$U(4) current with its own F parameter, so there will be no relations connecting the magnitudes of the Goldstone boson masses in different multiplets. We can, however, find relations between the mass *ratios* in different multiplets. Specifically, we find that

$$\frac{m_K^2}{m_\pi^2} = \frac{m'+m}{2m}\,, \tag{6.1}$$

$$\frac{m_D^2}{m_F^2} = \frac{m''+m'}{m''+m}\,, \tag{6.2}$$

$$\frac{m_d^2}{m_f^2} = \frac{m''-m'}{m''-m}\,, \tag{6.3}$$

where D and F are respectively the isotopic doublet and singlet part of the 0^- charmed triplet, d and f are the corresponding 0^+ particles, and m, m, m', and m'' are the masses of the $\mathcal{P}$, $\mathcal{N}$, λ, and $\mathcal{P}'$ quarks.[27] From these formulas we may obtain one relation free of quark mass ratios: This relation is a bit complicated, but in the limit $m_\pi^2 \ll m_K^2$ it becomes

$$\frac{m_D^2}{m_F^2} + \frac{m_d^2}{m_f^2} = 2\,. \tag{6.4}$$

That is, the mass splitting is equal and opposite in the scalar and pseudoscalar triplets.

(b) SU(4). If U(4)⊗U(4) is spontaneously broken down to SU(4), with SU(4) broken only by small quark mass perturbations, then the Goldstone bosons will form SU(4) multiplets, specifically, a pseudoscalar $\underline{15}$ and $\underline{1}$. If the singlet is interpreted as a Goldstone dipole, then the mass relations for the remaining $\underline{15}$ are the same as if the original symmetry were SU(4)⊗SU(4) (plus baryon conservation) rather than U(4)⊗U(4). These relations can be summarized in the statement that the SU(4) extensions of the Gell-Mann–Okubo formula are obeyed by the *squares* of the pseudoscalar boson masses. Of course, such mass relations can be derived from broken SU(4) alone,[28] but this derivation does not reveal whether these relations are obeyed by the masses themselves or their squares; indeed, the naive quark model suggests that the mass formulas should be satisfied by the masses themselves, not their squares.

For the pseudoscalar particles with nonvanishing isospin, strangeness, or charm, the methods of Sec. II give

$$m_\pi^2 = 2Am\,, \tag{6.5}$$

$$m_K^2 = A(m+m')\,, \tag{6.6}$$

$$m_D^2 = A(m+m'')\,, \tag{6.7}$$

$$m_F^2 = A(m'+m'')\,, \tag{6.8}$$

where A is a common constant, depending on the common coupling F_{15} of the 15 Goldstone bosons to the associated axial-vector currents and the common vacuum expectation value Z of bilinear products of quark fields:

$$A = 4Z/F_{15}^2\,.$$

There are no 0^+ Goldstone bosons here, but now we get a single relation between these four masses,[28]

$$m_F^2 - m_D^2 = m_K^2 - m_\pi^2\,. \tag{6.9}$$

This relation may be of some practical importance, because if m_F is much larger than m_K then m_D will be very close to m_F; for instance, if $m_F = 2$ GeV then the D-F mass splitting is 56 MeV.

Aside from the singlet Goldstone dipole, there is also one Goldstone boson with zero isospin, strangeness, and charm. The methods of Sec. III give its mass as

$$m_{15}^2 = \tfrac{1}{6}A(2m + m' + 9m'')\,. \tag{6.10}$$

However, there are two physical particles which provide candidates for this Goldstone boson: the observed η'(958) and the expected "paracharmonium" (p-Ch) at about 3 GeV.[29] In general, the fifteenth Goldstone boson might be any linear combination of η' and paracharmonium, and m_{15} would be somewhere in between the masses of η' and paracharmonium. However, if paracharmonium is pure $\bar{\mathcal{P}}'\mathcal{P}'$ and the η' is the orthogonal quark-model state, then[30]

$$m_{15}^2 = \tfrac{1}{4}m_{\eta'}^2 + \tfrac{3}{4}m_{p\text{-Ch}}^2\,. \tag{6.11}$$

We can use this in conjunction with (6.5)–(6.8) and (6.10) to obtain a formula for m_D^2,

$$m_D^2 = \tfrac{1}{2}m_{p\text{-Ch}}^2 + \tfrac{1}{6}m_{\eta'}^2 - \tfrac{1}{9}m_K^2 + \tfrac{4}{9}m_\pi^2\,. \tag{6.12}$$

For a paracharmonium mass of 3 GeV, this gives $m_D = 2.15$ GeV, and (6.9) then gives $m_F = 2.20$ GeV.

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[1]H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); D. J. Gross and F. Wilczek, *ibid.* 30, 1343 (1973); Phys. Rev. D 8, 3497 (1973); 9, 980 (1974); G. 't Hooft (unpublished); S. Coleman and D. J. Gross, Phys. Rev. Lett. 31, 851 (1973); etc.

[2]H. Georgi, H. R. Quinn, and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).

[3]S. Weinberg, Phys. Rev. Lett. 31, 494 (1973); Phys.

Rev. D 8, 605 (1973); 8, 4482 (1973). These order-α effects of the weak interactions play essentially the same role as the counterterms which were suggested by K. Wilson, Phys. Rev. 179, 1449 (1969), as a means of canceling the infinities in the purely electromagnetic corrections.

[4]S. Coleman and S. L. Glashow, Phys. Rev. Lett. 6, 423 (1961); Phys. Rev. 134, B671 (1964).

[5]D. G. Sutherland, Phys. Lett. 23, 384 (1966); J. S. Bell and D. G. Sutherland, Nucl. Phys. B4, 315 (1968).

[6]S. Weinberg, in *Proceedings of the XVII International Conference on High Energy Physics, London, 1974,* edited by J. R. Smith (Rutherford Laboratory, Chilton, Didcot, Berkshire, England, 1974), p. III-59.

[7]S. L. Glashow, R. Jackiw, and S. Shei, Phys. Rev. 187, 1416 (1969); R. N. Mohapatra and J. C. Pati, Phys. Rev. D 8, 4212 (1973); H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973).

[8]A rapid variation of the η decay matrix element was also proposed by P. C. McNamee and M. D. Scadron, Phys. Rev. D 11, 226 (1975). Our point of view here is that such rapidly varying terms must be considered in the light of the pseudoscalar mass spectrum.

[9]See, e.g., W. Hudnall and J. Schechter, Phys. Rev. D 9, 2111 (1974); I. Bars and M. Halpern, *ibid.* 9, 3430 (1974); 11, 956 (1975); etc.

[10]S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60, 47 (1969).

[11]P. Langacker and H. Pagels, Phys. Rev. D 9, 3413 (1974).

[12]J. Kogut and L. Susskind, Phys. Rev. D 10, 3468 (1974); K. Lane (unpublished); S. Weinberg, Ref. 6.

[13]M. Gell-Mann and H. Fritzsch (unpublished).

[14]L. Susskind (unpublished).

[15]J. Kogut and L. Susskind, following paper, Phys. Rev. D 11, 3594 (1975).

[16]S. L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D 2, 1285 (1970).

[17]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968); S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968). It can be shown that despite the apparent difference in these two formalisms, they yield precisely the same results for Goldstone boson masses.

[18]R. Dashen, Phys. Rev. 183, 1245 (1969). It has been pointed out by P. Langacker and H. Pagels, Phys. Rev. D 8, 4620 (1973), that the $m^2 \ln m^2$ corrections to chiral symmetry can make the electromagnetic self-energy of the K considerably different from that of the π. However, the pion mass term in δm^2 is only a 30% correction, so our results would not be changed very much if there were even a serious error in this term.

[19]See, e.g., P. Dittner, P. H. Dondi, and S. Eliezer, Phys. Rev. D 8, 2253 (1973).

[20]S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).

[21]S. Weinberg, Phys. Rev. Lett. 18, 188 (1967); Phys. Rev. 166, 1568 (1968).

[22]S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177, 2239 (1968); C. G. Callan, S. Coleman, J. W. Wess, and B. Zumino, *ibid.* 177, 2247 (1968).

[22′]G. Barton and S. P. Rosen, Phys. Rev. Lett. 8, 414 (1962).

[23]I believe that $\eta \rightarrow 3\pi$ was first calculated in SU(3)$\otimes$SU(3) theories with a u_3 tadpole using nonlinear phenomenological Lagrangians by A. J. Cantor, Harvard University Ph.D. thesis, 1969 (unpublished). Also see Ref. 9 and other references quoted therein.

[24]Particle Data Group, Phys. Lett. 50B, 1 (1974).

[25]A. Browman *et al.*, Phys. Rev. Lett. 32, 1067 (1974).

[26]P. Langacker and H. Pagels, Ref. 18.

[27]It should be emphasized that as used here, "quark mass" means the coefficient of a quark bilinear in the Lagrangian, *not* the mass of hadron constituents in a phenomenological quark model. It may well be that even in the limit of perfect chiral symmetry (zero "quark mass") the hadrons may be treated as composed of more or less free massive quarks. Chiral symmetry would not constrain the masses of these phenomenological quarks any more than it does that of the nucleon.

[28]See, e.g., M. K. Gaillard, B. W. Lee, and J. L. Rosner, Rev. Mod. Phys. 47, 277 (1975); S. Okubo, V. S. Mathur, and S. Borchardt, Phys. Rev. Lett. 34, 236 (1975).

[29]T. Appelquist and H. D. Politzer, Phys. Rev. Lett. 34, 43 (1975); A. De Rújula and S. L. Glashow, *ibid.* 34, 46 (1975); T. Appelquist, A. De Rújula, H. D. Politzer, and S. L. Glashow, *ibid.* 34, 365 (1975); E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K. D. Lane, and T.-M. Yan, *ibid.* 34, 369 (1975); C. G. Callan, R. L. Kingsley, S. B. Treiman, F. Wilczek, and A. Zee, *ibid.* 34, 52 (1975); B. Harrington, S. Y. Park, and A. Yildiz, *ibid.* 34, 168 (1975).

[30]In Ref. 6, I assumed that m_{15} was just equal to the η' mass, and found that in this case D and F are much too light to have escaped detection. The inclusive of paracharmonium in Eq. (6.11) invalidates the conclusion reached in Ref. 6, that SU(4) *must* be spontaneously broken.