Chiral confinement: An exact solution of the massive Thirring model

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We investigate the possibility of fermion confinement in a manifestly chiral-invariant theory. In particular we study the nonlinear σ model in one time and one space dimension, and demonstrate that it is equivalent to the massive Thirring model plus a free massless scalar field. We find an exact, time-independent, classical solution to the massive Thirring model. This solution is characterized by a fermion confined in a self-generated potential. In the σ-model analog of this solution, the chiral phase changes rapidly in the region of the confined fermion, and has two different constant limits on either side of this region. We also consider the case in which the mass of the pseudoscalar meson is small but finite, and find an approximate solution which displays both partial conservation of axial-vector current and fermion confinement.

I. INTRODUCTION

In the quark confinement schemes recently proposed and studied at MIT and SLAC,[1] the origin of the partial conservation of axial-vector current (PCAC) and the unique role of pions as Goldstone bosons are obscure.[2] It seems in fact apparent that these models are incompatible with the notion of spontaneously broken chiral symmetry as the foundation of current algebra and the related lore.

This paper reports on our initial results in studying the possibility of fermion confinement in a theory which is manifestly chiral-symmetric. To simplify the mathematics involved in such a complex physical problem, we chose to study first a truncated σ model[2] in one time, one space dimension. This model turns out to be equivalent to the massive Thirring model.[3] To our surprise, an exact, static solution corresponding to a confined fermion is obtainable by elementary means in the classical version of this model. We give here a brief description of these results.

The chiral-symmetric model we consider is a pseudoscalar, isoscalar meson θ coupled gradiently to a massive fermion ψ. The equations of motion of this model are

$$\partial^\mu\left(\partial_\mu\theta - \frac{1}{2f}\bar{\psi}\gamma_5\gamma_\mu\psi\right) = 0\,,$$

$$\left(i\partial\!\!\!/ - m - \frac{1}{2f}\gamma_5\gamma_\mu\partial^\mu\theta\right)\psi = 0\,.$$

This model, with isospin, was studied by Norton and Watson,[4] Feynman,[5] Gell-Mann and Lévy,[6] in connection with PCAC. In two dimensions, it turns out to be equivalent to the massive Thirring model (with the fermion field χ) and a free massless scalar field s:

$$\partial^2 s = 0\,,$$

$$\left[i\partial\!\!\!/ - m + \left(\frac{1}{2f}\right)^2(\bar{\chi}\gamma^\mu\chi)\gamma_\mu\right]\chi = 0\,.$$

The nature of the classical static solution may best be described in terms of the chiral model variables θ and ψ. The variable θ approaches two different constants as $x\to+\infty$ and $-\infty$; the transition from one value to the other occurs rapidly in a region of spatial extent of order $1/m$. Thus,

$$\frac{d}{dx}\theta = -\frac{1}{2f}\psi^\dagger\psi$$

is substantial only in this region. This in turn creates a potential in which the fermion ψ becomes trapped, so that the wave function ψ also has a range of order $1/m$. The total energy of the system is finite and less than m, while the energy of the trapped fermion goes to zero as

$$\left(\frac{1}{2f}\right)^2 \to \pi\,.$$

We have also considered the case in which the mass of the pseudoscalar meson is finite, so that we have the PCAC relation

$$\partial^\nu A_\nu = f\mu^2\theta\,,\quad A_\nu = \tfrac{1}{2}\bar{\psi}\gamma_5\gamma_\nu\psi + f\partial_\nu\theta\,.$$

While the model is no longer soluble exactly, and is not equivalent to the massive Thirring model plus a scalar field, we have established that the nature of the fermion confinement is not altered for small μ. In particular, we are able to give an exact expression for the change in the total energy valid to order μ.

The rest of this paper is organized as follows. In Sec. II we discuss the connection between the σ model, pseudoscalar gradient coupling model, and massive Thirring model. Section III gives the exact classical static solution of the massive

Thirring model, which is interpreted in terms of the chiral models in Sec. IV. Section V discusses the case of a small, but finite pseudoscalar mass. In Sec. VI we give a list of problems and avenues for future research. The appendixes deal with issues outside the central theme of this paper. Appendix A includes a discussion of the connection between the σ model and the massive Thirring model in the quantum-mechanical limit, whereas Appendix B gives a more precise description of what we mean by the classical limit in a theory with fermions plus a discussion of the extension of the model to the several fermion sector. A brief review of the application of variational techniques to the present problem is given in Appendix C.

II. σ MODEL AND ITS RELATION TO THE MASSIVE THIRRING MODEL

As a starting point we consider the Lagrangian of the chirally symmetric σ model[2],

$$\mathcal{L}_\sigma = \bar\psi[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\psi + \tfrac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2] - \frac{H}{4}(\sigma^2+\pi^2-f^2)^2 , \tag{1}$$

where the isospin dependence has been neglected for simplicity, and unless otherwise specified, we will confine our analysis to the classical theory. The Lagrangian is invariant under the chiral transformation:

$$\begin{aligned} &\sigma \to \sigma\cos\alpha + \pi\sin\alpha ,\\ &\pi \to -\sigma\sin\alpha + \pi\cos\alpha ,\\ &\psi \to \exp(i\alpha\gamma_5/2)\psi ,\\ &\alpha \text{ constant}. \end{aligned} \tag{2}$$

It is useful to redefine the fields as $\sigma + i\pi = \rho e^{i\theta/f}$ and $\psi' = \exp(i\gamma_5\theta/2f)\psi$, so that the Lagrangian is now

$$\mathcal{L}_\sigma = \bar\psi'\left(i\partial\!\!\!/ - g\rho - \frac{1}{2f}\gamma_5\gamma^\mu(\partial_\mu\theta)\right)\psi' + \tfrac{1}{2}(\partial_\mu\rho)^2 + \tfrac{1}{2}\left(\frac{\rho}{f}\right)^2(\partial_\mu\theta)^2 - \frac{H}{4}(\rho^2-f^2)^2 . \tag{3}$$

The equation of motion for the field $\rho(x)$ is

$$\partial^2\rho + H\rho(\rho^2-f^2) = -g\bar\psi'\psi' + \left(\frac{\rho}{f^2}\right)(\partial_\mu\theta)^2 . \tag{4}$$

In the limit $H\to\infty$, $\rho^2 = f^2$, and we obtain formally the "nonlinear" σ model[7]

$$\mathcal{L}'_\sigma = \bar\psi'\left(i\partial\!\!\!/ - m - \frac{1}{2f}\gamma_5\gamma^\mu(\partial_\mu\theta)\right)\psi' + \tfrac{1}{2}(\partial_\mu\theta)^2 , \tag{5}$$

where $m = gf$. The nomenclature "nonlinear" is readily understood if we write the above Lagrangian in terms of the ψ field:

$$\mathcal{L}'_\sigma = \bar\psi(i\partial\!\!\!/ - me^{i\gamma_5\theta/f})\psi + \tfrac{1}{2}(\partial_\mu\theta)^2 . \tag{5$'$}$$

The Lagrangian (5) is invariant under the transformation

$$\theta \to \theta - \alpha , \tag{6}$$

from which follows the conservation of the axial-vector current:

$$\partial^\mu A_\mu = 0 , \quad A_\mu = \tfrac{1}{2}\bar\psi\gamma_5\gamma_\mu\psi - f\partial_\mu\theta . \tag{7}$$

Equation (7) is just the equation of motion for the field θ. The fermion equation of motion is given by

$$\left(i\partial\!\!\!/ - m - \frac{1}{2f}\gamma_5\gamma^\mu(\partial_\mu\theta)\right)\psi' = 0 . \tag{8}$$

In the following we shall consider only one time, one space dimension. With this restriction, it turns out that Eq. (5) is equivalent to the massive Thirring model.[3] To see this connection, we introduce a new scalar field $s(x)$ by

$$\begin{aligned} A^\mu &= \tfrac{1}{2}\bar\psi'\gamma_5\gamma^\mu\psi' - f\partial^\mu\theta \\ &= \epsilon^{\mu\nu}\partial_\nu s . \end{aligned} \tag{9}$$

This "curl" representation of A^μ is always possible because the axial-vector current is conserved. Since the curl of the axial-vector current in this model is simply related to the divergence of the conserved vector current $\bar\psi'\gamma^\mu\psi'$, we have

$$\epsilon_{\mu\nu}\partial^\mu A^\nu = \partial^2 s = 0 , \tag{10}$$

i.e., the field s describes a free massless scalar field. We may substitute Eq. (9) for $\partial_\mu\theta$ in Eq. (8) and obtain, using the identity $(\gamma_5\gamma^\mu)(\gamma_5\gamma_\mu) = -(\gamma^\mu)(\gamma_\mu)$,

$$\left[i\partial\!\!\!/ - m + \frac{1}{4f^2}(\bar\psi'\gamma_\mu\psi')\gamma^\mu\right]\psi' + \frac{1}{2f^2}\gamma^\mu(\partial_\mu s)\psi' = 0 . \tag{11}$$

The last term $(\partial_\mu s)\gamma^\mu\psi'$ can be eliminated by the transformation $\psi' \to \chi = \exp[-is(x)/2f^2]\psi'$, and we are left with the equation of motion of the massive Thirring model:

$$(i\partial\!\!\!/ - m + \lambda^2\bar\chi\gamma^\mu\chi\gamma_\mu)\chi = 0 , \quad \lambda = \frac{1}{2f} . \tag{12}$$

The free scalar field s and the Thirring fermion χ are completely decoupled. That is, the chiral model of Eq. (5) is equivalent to the Thirring model and a free boson term. The equivalence of the two theories in the quantized form is elaborated in Appendix A.

Since we focus on confined states of a fermion in the following, we choose $s(x) = 0$ and $\psi' = \chi$.

III. EXACT CLASSICAL SOLUTION OF THE MASSIVE THIRRING MODEL

We shall now present an exact, time independent solution of Eq. (12) in the fermion number one sector in the classical limit (See Appendix B for a discussion of the classical limit), which is, to our knowledge, the only known exact solution of the massive Thirring model. The solution we will present corresponds to a localized (confined) fermion field.

The Lagrangian which corresponds to Eq. (12) is

$$\mathcal{L}=\bar{\chi}(i\partial\!\!\!/ - m)\chi + \frac{\lambda^2}{2}(\bar{\chi}\gamma_\mu\chi)^2 ,$$

with the Hamiltonian density

$$\mathcal{H}=\bar{\chi}(i\gamma^1\partial^1+m)\chi - \frac{\lambda^2}{2}(\bar{\chi}\gamma_\mu\chi)^2 .$$

Assuming the existence of a static solution, the two component spinor $\chi(x,t)$ will be written as $\chi(x)\exp(-iEt)$, and with

$$\chi(\pm\infty)=0 , \quad \bar{\chi}\gamma^1\chi=0$$

[recall that $\partial_\mu(\bar{\chi}\gamma^\mu\chi)=0$], the equation of motion may be written as[8]

$$(i\gamma^0\gamma^1\partial^1+m\gamma^0-\lambda^2\chi^\dagger\chi)\chi=E\chi . \tag{13}$$

The following explicit representation of the 2×2 γ matrices will be used:

$$\gamma^0=\sigma_3 , \quad i\gamma^1=\sigma_1 , \text{ and } \gamma^5=\gamma^0\gamma^1=\sigma_2 .$$

With this choice of γ's, it is consistent to describe a bound state χ, satisfying $\bar{\chi}\gamma^1\chi=0$, as a real spinor, i.e.,

$$\chi=\begin{pmatrix}u\\ v\end{pmatrix}, \quad u, v \text{ real}.$$

In this representation we have

$$\chi^\dagger\chi=u^2+v^2 ,$$

and

$$\begin{aligned}&[m-\lambda^2(u^2+v^2)]u+\frac{dv}{dx}=Eu ,\\ &-\frac{du}{dx}+[-m-\lambda^2(u^2+v^2)]v=Ev .\end{aligned} \tag{14}$$

To simplify these coupled equations, we transform u, v into η, ϕ through

$$u=\eta\cos\phi ,$$

$$v=\eta\sin\phi .$$

Equations (14) then reduce to

$$\frac{d\phi}{dx}=E-m\cos2\phi+\lambda^2\eta^2 , \tag{14$'$}$$

$$\frac{d\eta}{dx}=-m\eta\sin2\phi . \tag{14$''$}$$

Eliminating the variable x, one finds

$$\frac{d}{d\phi}\left(\frac{1}{2}E\eta^2+\frac{\lambda^2}{4}\eta^4-\frac{m}{2}\eta^2\cos2\phi\right)=0 , \tag{15}$$

from which it follows, for a localized solution $[\eta(\pm\infty)=0]$, that

$$E+\frac{\lambda^2}{2}\eta^2-m\cos2\phi=0 . \tag{16}$$

Substituting Eq. (16) into Eq. (14$'$) and integrating, one finds

$$\phi(x)=\tan^{-1}(\sqrt{\beta}\tanh\kappa x) , \tag{17a}$$

$$\eta^2(x)=\frac{2(m-E)/\lambda^2}{\cosh^2\kappa x}\frac{1}{1+\beta\tanh^2\kappa x}, \tag{17b}$$

where

$$\beta=\frac{m-E}{m+E} \text{ and } \kappa=(m^2-E^2)^{1/2}$$

(note that $E<m$ corresponds to a confined solution). The normalization condition appropriate to the fermion number $=1$ sector (we consider the question of more fermions in Appendix B)

$$\int_{-\infty}^{\infty}dx\,\chi^\dagger\chi=1$$

leads to the single eigenvalue

$$E=m\cos\frac{\lambda^2}{2}, \quad \beta=\tan^2\frac{\lambda^2}{4}, \quad \kappa=m\sin\frac{\lambda^2}{2}. \tag{18}$$

Note that, in order to restrict ourselves to only the positive energy fermion states, we must require $\lambda^2\leqslant\pi$. This constraint is a standard result in the usual Thirring model.

The various possible densities $\bar{\chi}\chi$, $\bar{\chi}\gamma^0\chi=\chi^\dagger\chi$, and $i\bar{\chi}\gamma^5\chi$ (recall $\bar{\chi}\gamma^1\chi=0$) are

$$\bar{\chi}\chi=\eta^2\cos2\phi=\frac{4m\sin^2(\lambda^2/4)(1-\tan^2(\lambda^2/4)\tanh^2\kappa x)}{\lambda^2\cosh^2\kappa x(1+\tan^2(\lambda^2/4)\tanh^2\kappa x)^2}, \tag{19a}$$

$$\chi^\dagger\chi=\eta^2=\frac{4m\sin^2(\lambda^2/4)}{\lambda^2\cosh^2\kappa x(1+\tan^2(\lambda^2/4)\tanh^2\kappa x)^2}, \tag{19b}$$

$$\bar{\chi}i\gamma_5\chi=\eta^2\sin2\phi=\frac{8m\sin^2(\lambda^2/4)\tan(\lambda^2/4)\tanh\kappa x}{\lambda^2\cosh^2\kappa x(1+\tan^2(\lambda^2/4)\tanh^2\kappa x)^2}. \tag{19c}$$

In order to confirm that this exact, classical solution is consistent with our initial assumption that it describes a "bound" state, we must exhibit the expectation value of the classical Hamiltonian,

$$\langle H\rangle = \int dx\,\mathcal{H}(x) = E + \frac{\lambda^2}{2}\int dx(\chi^+\chi)^2\,. \tag{20}$$

With the results given in Eq. (19) it is straightforward to evaluate $\langle H\rangle$ and find

$$\langle H\rangle = \frac{2m}{\lambda^2}\sin\lambda^2/2\,. \tag{21}$$

Hence for $\lambda^2 \to 0$, $\langle H\rangle \to m$ (i.e., a free, massive fermion), but for finite values of λ^2, $\langle H\rangle < m$ as required for a bound state. The existence of this exact, time independent classical solution is, in itself, a new and interesting result deserving further study within the context of the massive Thirring model.[8] However, in this paper we prefer to return to the original σ model and discuss the implications of this solution in that framework.

IV. CONFINEMENT SOLUTION IN THE σ MODEL

It is straightforward to translate the fermion solution presented in the previous section into the language of the "nonlinear" σ model of Eq. (5). Since our centered topic is the existence of minimum energy, localized states, we shall, as mentioned earlier, set the free scalar field, $s(x)$, to zero for all x. In this case $\psi'(x) = \chi(x)$ and we can solve for the chiral phase, $\theta(x)$, from Eq. (9). This simplifies to give

$$\frac{d\theta}{dx} = -\frac{1}{2f}(\psi')^\dagger\psi' = -\frac{1}{2f}\chi^\dagger\chi\,. \tag{22}$$

Integrating and choosing $\theta(x)$ to be antisymmetric about $x=0$, we find ($\lambda = 1/2f$)

$$\theta = -4f\tan^{-1}[\tan(1/16f^2)\tanh\kappa x]\,, \tag{23}$$

where $\kappa = gf\sin(1/8f^2)$. Note that $\theta = -4f\phi$ as seen from Eq. (17a). According to Eq. (9),

$$\partial^\mu\theta = \frac{1}{2f}\bar\psi'\gamma_5\gamma^\mu\psi' - \frac{1}{f}\epsilon^{\mu\nu}\partial_\nu s\,,$$

where s is a free field. Since we have let $s=0$, it follows that θ is not an independent variable and is given in terms of the fermion variables. The fact that the chiral model (5) describes an interacting fermion field and a free boson field persists in the quantized case, as discussed in Appendix A.

From Eq. (23) we see that $(\theta/f)(\pm\infty) = \mp 1/4f^2$ and the full shift in the chiral phase, $\Delta(\theta/f) = 1/2f^2$ (due to a single fermion) is bounded from above by 2π since $1/2f^2 = 2\lambda^2 \le 2\pi$ (the requirement that $E \ge 0$).

The various fermion densities [Eq. (19)] can also be expressed in terms of the initial fermion fields, $\psi = e^{-i\gamma_5\theta/2f}\psi' = e^{-i\gamma_5\theta/2f}\chi$. Owing to the simple relationship between θ and ϕ, these densities are almost unchanged in going from one fermion basis to the other and we shall not repeat the formulas here. One needs to know only that $\psi^\dagger\psi = \chi^\dagger\chi$, $\bar\psi\psi = \bar\chi\chi$, and $\bar\psi\gamma_5\psi = -\bar\chi\gamma_5\chi$, with λ replaced by $1/2f$ and m by gf.

We illustrate the behavior of the chiral phase θ/f and the various fermion densities in Fig. 1 for the values $1/4f^2 = 3.0$, 1.5, and 0.5 and $m=1$. Here the abscissa x is measured in units of $1/m$. Note that the primary effect of varying $1/f^2$ is to change the "size" of the confinement region in the spatial variable x. The quantities $\psi^\dagger\psi$ and $\bar\psi\psi$ are very similar, with $\bar\psi\psi$ only slightly more narrow.

In the σ-model language the classical energy for this solution is given by

$$\langle H\rangle = E + \tfrac{1}{2}\int dx(\partial_x\theta)^2$$
$$= 8gf^3\sin\frac{1}{8f^2}\,. \tag{24}$$

Also note that the fraction of the total energy residing in the chiral field [i.e., the second term in Eq. (24)] is

$$\frac{E_\theta}{\langle H\rangle} = \frac{\langle H\rangle - E}{\langle H\rangle} = \frac{\tan(1/8f^2) - 1/8f^2}{\tan(1/8f^2)}\,, \tag{25}$$

which varies from 0 to 1 as $1/4f^2$ varies from 0 to π.

Returning briefly to the limit, $H\to\infty$, taken following Eq. (4), we see that, for $\psi^\dagger\psi \propto gf^3$, the specific limit discussed in this paper is

$$H \gg g^2\,, \tag{26}$$

so that $|\rho - f| \ll f$.

To summarize, the exact solution to the nonlinear σ model presented in this section describes the classical, static confinement of a single fermion within a region where the chiral phase is varying. Said another way, the chiral phase variation induced by the presence of the fermion in turn produces an axial-vector potential in which the fermion is bound ($\langle H\rangle < m$). Furthermore, the chiral phase change is guaranteed to be just such as to ensure the local conservation of the axial-vector current. In the next section we discuss the effects of breaking this chiral symmetry.

V. BREAKING CHIRAL SYMMETRY

Since the observed situation in the physical world does not correspond to exact chiral symmetry, it is instructive to consider the results

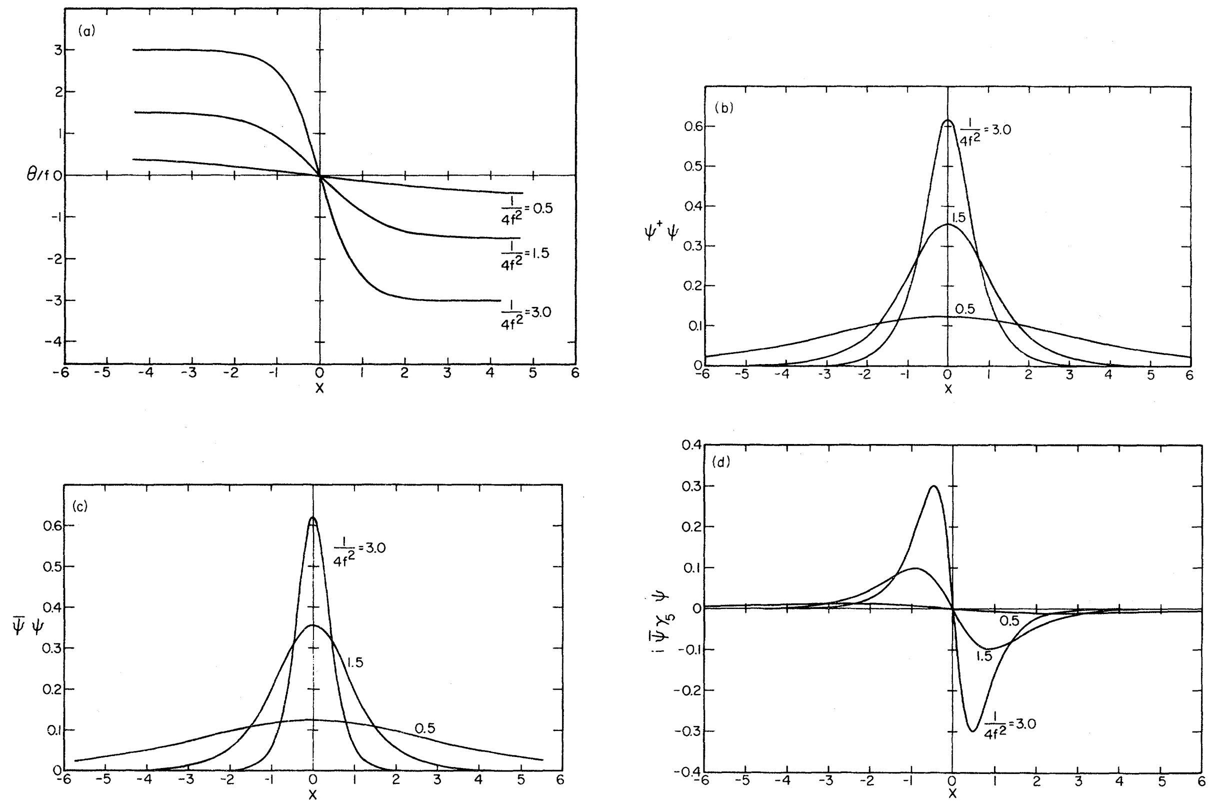

FIG. 1. Typical behavior of σ-model confinement solution. Shown are all curves for $1/4f^2=3.0$, 1.5, and 0.5 with $m=1$. (a) Chiral phase θ/f ; (b) $\psi^\dagger\psi$; (c) $\bar\psi\psi$; (d) $i\bar\psi\gamma_5\psi$.

of breaking chiral symmetry in the present framework. The most straightforward program for accomplishing this is simply to include a chiral breaking ("pion" mass) term in the "nonlinear" σ model given in (5),

$$\mathcal{L}=\bar\psi'\left(i\partial\!\!\!/ - m - \frac{1}{2f}\gamma_5\gamma^\nu\partial_\nu\,\theta\right)\psi' + \tfrac{1}{2}(\partial_\nu\theta)^2 - \tfrac{1}{2}\mu^2\theta^2\,. \tag{27}$$

Conservation of the axial-vector current is now replaced by the PCAC condition,

$$\partial^\nu A_\nu = f\mu^2\theta\,, \tag{28}$$

with

$$A_\nu = \tfrac{1}{2}\bar\psi'\gamma_5\gamma_\nu\psi' - f\partial_\nu\theta \tag{29}$$

as before. In the presence of the mass term, we are not able to separate out a free scalar field by a simple transformation. This indicates that the present theory is no longer equivalent to a massive Thirring model with a factorizable pion field, and is probably not exactly soluble. However, in the limit of small μ^2 (i.e., $\mu^2 \ll \kappa^2$), which is the physically interesting limit, the fermion wave function is only slightly modified in the confinement region. We can still solve for this wave function, and compute the energy of the confinement state approximately.

The fermion field equation obtained from (27) is not affected by the pion mass term, and remains as

$$\left(i\partial\!\!\!/ - m - \frac{1}{2f}\gamma_5\gamma^\mu\partial_\mu\,\theta\right)\psi' = 0\,. \tag{29'}$$

Using the representation of the γ matrices introduced in Sec. III, we find for

$$\psi' = \chi \equiv \eta\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix},\qquad \eta,\phi \text{ real} \tag{30}$$

that

$$\frac{d\phi}{dx} = E - m\cos 2\phi - \frac{1}{2f}\frac{d\theta}{dx}, \tag{31}$$

$$\frac{d\eta}{dx} = -m\eta\sin 2\phi\,, \tag{32}$$

and

$$\left(-\frac{d^2}{dx^2}+\mu^2\right)\theta=\frac{1}{2f}\frac{d}{dx}(\eta^2)\,. \tag{33}$$

Equations (31) and (32) are essentially the same as Eqs. (14′) and (14″), and Eq. (33) implies that θ always damps to zero exponentially outside the confinement region.

By treating the right-hand side of Eq. (33) as a source, we can solve for θ using the Green's function,

$$\theta(x)=\int dx' G(x-x')\frac{1}{2f}\frac{d}{dx'}(\eta(x')^2)\,, \tag{34}$$

with

$$G(x-x')=\frac{1}{2\mu}e^{-\mu|x-x'|}\,, \tag{35}$$

$$\left(-\frac{d^2}{dx^2}+\mu^2\right)G(x-x')=\delta(x-x')\,. \tag{36}$$

Differentiating θ with respect to x, and integrating by parts, we have

$$\frac{d\theta}{dx}=\frac{1}{2f}\int dx'\left[\frac{d^2}{dx^2}G(x-x')\right]\eta(x')^2$$
$$=-\frac{1}{2f}\eta(x)^2+\frac{1}{2f}\mu^2\int dx' G(x-x')\eta(x')^2\,. \tag{37}$$

For small μ and x in the confinement region, we can replace G in (37) by a constant, $1/2\mu$, and obtain

$$\frac{d\theta}{dx}\cong-\frac{1}{2f}\eta^2+\frac{\mu}{4f}\int dx'\eta(x')^2=-\frac{1}{2f}\eta^2+\frac{\mu}{4f}, \tag{38}$$

which is correct to order μ/κ and where we have used the normalization condition for η. Substituting (38) into (31), we have

$$\frac{d\phi}{dx}=\left(E-\frac{\mu}{8f^2}\right)-m\cos 2\phi+\frac{1}{4f^2}\eta^2\,. \tag{39}$$

Now, Eqs. (32) and (39) are identical to (14′) and (14″), with the energy eigenvalue E replaced by $E-\mu/8f^2$ (and λ by $1/2f$). In particular, the fermion wave function ψ', and equivalently η, ϕ, are not affected at all to this order of μ. The fermion energy E and the total classical energy $\langle H\rangle$ are modified slightly to

$$E\cong m\cos\frac{1}{8f^2}+\frac{\mu}{8f^2}, \tag{40}$$

$$\langle H\rangle\cong 8f^2 m\sin\frac{1}{8f^2}+\frac{\mu}{16f^2}\,. \tag{41}$$

Knowing η, we can compute the chiral phase from (34). For $\mu\ll\kappa$, we have

$$\theta(\kappa)=\frac{1}{2f}\int dx'\left[\frac{d}{dx}G(x-x')\right]\eta(x')^2$$
$$=-\frac{1}{4f}\int dx'\epsilon(x-x')e^{-\mu|x-x'|}\eta(x')^2$$
$$\approx -e^{-\mu|x|}\frac{1}{4f}\int dx'\epsilon(x-x')\eta(x')^2$$
$$=\theta(x)|_{\mu=0}e^{-\mu|x|}\,, \tag{42}$$

where we have approximated $e^{-\mu|x-x'|}$ by $e^{-\mu|x|}$, and $\theta(x)|_{\mu=0}$ is the chiral phase obtained in Eq. (23), Sec. IV. Equation (42) reveals that the chiral phase damps exponentially to zero for $\mu|x|\gg 1$, and varies rapidly, with scale $1/\kappa\ll 1/\mu$, in the region of the confined fermion. A typical chiral phase variation is shown in Fig. 2, where the values $m=1$, $1/4f^2=3$, and $\mu=0.14$ were used.

If one uses a chiral symmetry breaking term of the more conventional form, $\mathcal{L}_1=c\sigma=cf\cos(\theta/f)$, the results are substantially the same as those above for small $c(\mu^2\sim c/f)$. The major difference appears in the form of the corrections to the energies, which in this case are

$$E\cong m\cos\frac{1}{8f^2}+\left(\frac{c}{f}\right)^{1/2}\sin\frac{1}{8f^2} \tag{43a}$$

and

$$\langle H\rangle\cong\left[8f^2 m+\frac{1}{2}\left(\frac{c}{f}\right)^{1/2}\right]\sin\frac{1}{8f^2}\,. \tag{43b}$$

VI. CONCLUDING COMMENTS

We found that the chiral-symmetric model (5) is equivalent to the massive Thirring model plus

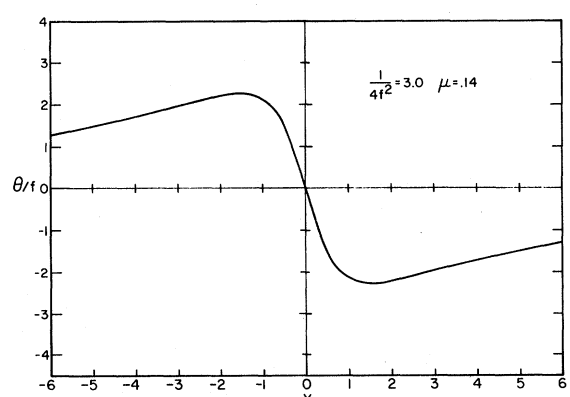

FIG. 2. Typical behavior of chiral phase when a small mass is included for the "pion" field. Shown is the case $m=1$, $1/4f^2=3.0$, $\mu=0.14$.

a decoupled massless scalar field. As shown in Appendix A, this is true even in quantum theory. The spontaneously broken chiral model does contain a Goldstone boson, but it is a free field (as can be checked in perturbation theory); the infrared problem associated with massless bosons in one space dimension[9] is thereby avoided.

In a broken chiral model, the massive pseudoscalar field is no longer decoupled from the fermion (this has also been verified in perturbation theory), but the confinement of the fermion field persists.

Before we can establish contact with reality, however, the following questions must be answered:

1. Does the same kind of confinement occur in three spatial dimensions with a nontrivial internal symmetry? Does it confine the right kind and number of quarks? Realistic chiral confinement models we can envisage are in some respects very similar to the SLAC bag model, except for pions and except with respect to PCAC.

2. How does our classical solution emerge in a quantized version? The recent papers of Goldstone and Jackiw,[10] and Dashen, Hasslacher, and Neveu[11] are important in answering this general question, but we have not pursued this problem in this paper. Similarly, we have not fully explored the implications for our solution of Coleman's work on the connection between the (quantized) massive Thirring and sine-Gordon models.[8]

3. Are there solutions similar to the present one but in different fermion number sectors (other than the trivial extension discussed in Appendix B)?

Note added in proof. A. Zee pointed out to us that the massive Thirring model is formally equivalent to the $(\bar\psi\psi)^2$ interaction theory, namely,

$$\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi - \tfrac{1}{2}g(\bar\psi\gamma_\mu\psi)^2 \tag{44a}$$

$$= \bar\psi(i\partial\!\!\!/ - m)\psi + g(\bar\psi\psi)^2 + \text{mass counterterms}. \tag{44b}$$

This is due to a special property of the one-dimensional theory which states that there is only one independent four-fermion local interaction. To demonstrate that the two Lagrange functions (44a) and (44b) are indeed equivalent, one can show explicitly that both Lagrange functions give rise to the same Feynman rules. We have shown in this paper that for $g=-\lambda^2<0$ the massive Thirring model (44a) contains a confined (and, as is understood, bounded) one-fermion solution. For $g>0$ the original Thirring Lagrange function (44a) no longer leads to any confined solution. On the other hand, the Lagrange function (44b) does not contain any confined solution for $g<0$, but it has a confined solution for $g>0$. Viewing the complementary nature of these two descriptions, we conjecture that the true quantum-mechanical one-fermion state of the massive Thirring model is described alternatively by these two confined solutions in their respective regions of validity, i.e., for $g>0$ the fermion is described approximately by the confinement solution associated with the Lagrange function (44b), and for $g<0$ it is represented by the confinement state associated with the Lagrange function (44a). To test our conjecture, we have computed the one-fermion-loop quantum corrections to these confinement states and found that, at small $|q|$, the corrections are always small. Thus, in the weak coupling limit, the classical confinement solution studied in this paper should provide a good description for the one-fermion state even after the one-loop quantum correction is taken into account. We should present the details of these calculations in a separate publication. The authors wish to thank A. Zee for discussions and for informing us of his results.

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APPENDIX A

To show the equivalence of the quantized Lagrangian (5) and the massive Thirring model, we proceed from the path-integral formula for the generating functional for Green's functions of the former,

$$Z_{\text{ch}}(\xi,\xi^\dagger) = \int [d\phi_\mu\, d\theta\, d\psi\, d\psi^\dagger] \exp\left\{ i\int dx [\phi_\mu \partial^\mu\theta - \tfrac{1}{2}\phi_\mu{}^2 + \bar\psi(i\partial\!\!\!/ - m + \lambda\gamma_5\gamma_\mu\partial^\mu\theta)\psi + \xi^\dagger\psi + \psi^\dagger\xi]\right\}, \tag{A1}$$

where ξ and $\xi^\dagger$ are anticommuting fermion source functions.

We express ϕ_μ in terms of two scalar fields ϕ and σ:

$$\phi_\mu = \partial_\mu \phi + \epsilon_{\mu\nu}\partial^\nu \sigma - \lambda\bar{\psi}\gamma_5\gamma_\mu\psi \ .$$

The generating functional (A1) can be written as

$$Z_{\rm ch}(\xi,\xi^\dagger) = \int [d\phi\, d\sigma\, d\theta\, d\psi\, d\psi^\dagger] \exp\left\{ i\int dx \left[\partial_\mu\phi\partial^\mu\theta - \tfrac{1}{2}(\partial^\mu\phi)^2 + \tfrac{1}{2}(\partial^\mu\sigma)^2 + \lambda A_\mu \partial^\mu\phi - \lambda V_\mu\partial^\mu\sigma - \frac{\lambda^2}{2} A_\mu{}^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi + \xi^\dagger\psi + \psi^\dagger\xi \right]\right\} , \tag{A2}$$

where $A_\mu = \bar{\psi}\gamma_5\gamma_\mu\psi$, $V_\mu = \bar{\psi}\gamma_\mu\psi$. The functional integration over θ is trivially performed, yielding the factor

$$\prod_x \delta(\partial^2\phi(x)) \ .$$

The condition $\partial^2\phi(x)=0$ in general implies

$$\partial^\mu\phi(x) = \epsilon^{\mu\nu}\partial_\nu\,\omega(x) \ , \tag{A3}$$

where

$$\partial^2\omega(x) = 0 \ .$$

If we define a new variable $s(x)$ by

$$s(x) = \sigma(x) + \omega(x)$$

and eliminate $\sigma(x)$ in Eq. (A2) in favor of $s(x)$, we obtain

$$Z_{\rm ch}(\xi,\xi^\dagger) = \int [ds\, d\psi\, d\psi^\dagger] \exp\left[i\int dx \left(\tfrac{1}{2}(\partial_\mu s)^2 - \lambda V_\mu\partial^\mu s + \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{\lambda^2}{2}\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma^\mu\psi + \xi^\dagger\psi + \psi^\dagger\xi \right)\right] . \tag{A4}$$

The transformation $\psi = e^{-i\lambda s(x)}\chi$ eliminates the term $-\lambda V_\mu\partial^\mu s$:

$$Z_{\rm ch}(\xi,\xi^\dagger) = \int [d\chi\, d\chi^\dagger] \exp\left\{ i\int dx \left[\bar{\chi}(i\partial\!\!\!/ - m)\chi + \frac{\lambda^2}{2}\bar{\chi}\gamma_\mu\chi\bar{\chi}\gamma^\mu\chi \right]\right\} \times \int [ds] \exp\left\{ i\int dx \left[\tfrac{1}{2}(\partial_\mu s)^2 + \xi^\dagger\chi e^{-i\lambda s(x)} + \chi^\dagger\xi e^{i\lambda s(x)} \right]\right\} . \tag{A5}$$

Thus, the Green's function $\langle T(\psi(x_1)\cdots\psi(x_n)\psi^\dagger(y_1)\cdots\psi^\dagger(y_n))\rangle_{\rm ch}$ of the chiral model (5) is related to the Green's function $\langle T(\chi(x_1)\cdots\chi(x_n)\chi^\dagger(y_1)\cdots\chi^\dagger(y_n))\rangle_{\rm Th}$ of the massive Thirring model through

$$\langle T(\psi(x_1)\cdots\psi(x_n)\psi^\dagger(y_1)\cdots\psi^\dagger(y_n))\rangle_{\rm ch} = \langle T(\chi(x_1)\cdots\chi(x_n)\chi^\dagger(y_1)\cdots\chi^\dagger(y_n))\rangle_{\rm Th} \times \langle T(e^{-i\lambda s(x_1)}\cdots e^{-i\lambda s(x_n)} e^{i\lambda s(y_1)}\cdots e^{i\lambda s(y_n)})\rangle , \tag{A6}$$

where $s(x)$ is a free massless scalar field.

APPENDIX B

In this appendix we formulate the precise meaning of classical solutions in models containing fermions. We begin with the nonlinear σ model

$$\mathcal{L} = \bar{\psi}_{\rm op}(i\partial\!\!\!/ - m - \lambda\gamma_5\gamma^\mu\partial_\mu\theta)\psi_{\rm op} + \tfrac{1}{2}(\partial_\mu\theta)^2 , \tag{B1}$$

where $\psi_{\rm op}$ represents the fermion field operator. The classical solutions to the massive Thirring model will be introduced through the connection to the nonlinear σ model. The Lagrange function (B1) implies that the field equations

$$(i\partial\!\!\!/ - m - \lambda\gamma_5\gamma^\mu\partial_\mu\theta)\psi_{\rm op} = 0 , \tag{B2}$$

$$\partial_\mu(\partial^\mu\theta - \lambda\bar{\psi}_{\rm op}\gamma_5\gamma^\mu\psi_{\rm op}) = 0 . \tag{B3}$$

By classical solutions to (B1) in the one fermion sector, we mean the following:

1. θ is a c-number function.
2. Express the field operators $\psi_{\rm op}$ and $\psi_{\rm op}^\dagger$ as

$$\psi_{\rm op} = \psi a + \cdots , \tag{B4}$$

$$\psi_{\rm op}^\dagger = \psi^\dagger a^\dagger + \cdots , \tag{B5}$$

where a ($a^\dagger$) is the annihilation (creation operator and ψ is the c-number wave function associated with the lowest energy, localized one fermion state. The wave function ψ obeys the classical equation

$$(i\partial\!\!\!/ - m - \lambda\gamma_5\gamma^\mu\partial_\mu\theta)\psi = 0\ , \tag{B6}$$

$$\int dx\, \psi^\dagger\psi = 1\ . \tag{B7}$$

The dots in (B4) and (B5) represent higher-frequency fermion and antifermion states which we ignore in the classical solution.[12]

3. Replace the bilinear product $\bar\psi_{\text{op}}\psi_{\text{op}}$ in (B3) by

$$\bar\psi_{\text{op}}\Gamma\psi_{\text{op}} \to \langle\bar\psi_{\text{op}}\Gamma\psi_{\text{op}}\rangle_{\text{one-fermion}} = \bar\psi\Gamma\psi\ , \tag{B8}$$

which leads to

$$\partial_\mu(\partial^\mu\theta - \lambda\bar\psi\gamma_5\gamma^\mu\psi) = 0\ . \tag{B9}$$

Equations (B6), (B7), and (B9) specify the classical limits of our models, where θ, ψ, and $\psi^\dagger$ are all c numbers. It is important to note that these equations can be obtained directly from an effective classical Lagrange function

$$\mathcal{L}_{\text{eff}} = \bar\psi(i\partial\!\!\!/ - m - \lambda\gamma_5\gamma^\mu\partial_\mu\theta)\psi + \tfrac{1}{2}(\partial_\mu\theta)^2\ . \tag{B10}$$

In (B10), ψ, $\psi^\dagger$, and θ are all treated as commuting numbers. The precise meaning of ψ ($\psi^\dagger$) is given in (B4) and (B5), and it should *not* be interpreted as the classical limit of the anticommuting ψ_{op}. (In fact, ψ_{op}, as an anticommuting operator, does not have a classical limit.)

By choosing the associated free boson field $s(x) = 0$ as in Sec. II, we have, from (B9),

$$\partial^\mu\theta - \lambda\bar\psi\gamma_5\gamma^\mu\psi = 0\ , \tag{B11}$$

and consequently through (B6), an analogous classical equation for the massive Thirring model,

$$(i\partial\!\!\!/ - m + \lambda^2\bar\psi\gamma^\mu\psi\gamma_\mu)\psi = 0\ . \tag{B12}$$

We define Eq. (B12) to be the classical equation of motion for the massive Thirring model. This result can be derived from the effective Lagrange function

$$\mathcal{L}_{\text{eff}} = \bar\psi(i\partial\!\!\!/ - m)\psi + \frac{\lambda^2}{2}(\bar\psi\gamma_\mu\psi)^2 \tag{B13}$$

by treating ψ as a commuting c-number wave function. Equation (B13) is the starting point of Sec. III. We emphasize again that ψ should not be considered as the classical limit of ψ_{op}, but rather it should be interpreted as the c-number wave function appearing in (B4).

The generalization of our formulation to the multifermion sector is straightforward. As an example, we consider the two-fermion sector. Then we have, instead of (B4),

$$\psi_{\text{op}} = \psi_1 a_1 + \psi_2 a_2 + \circ\circ\circ\ , \tag{B14}$$

where ψ_1 and ψ_2 are the wave functions for the individual fermion states in the presence of the other fermion. They obey the orthonormality conditions

$$\int dx\, \psi_i^\dagger\psi_j = \delta_{ij}, \quad i,j = 1,2\ . \tag{B15}$$

Now, we replace $\bar\psi_{\text{op}}\psi_{\text{op}}$ in (B8) by

$$\bar\psi_{\text{op}}\Gamma\psi_{\text{op}} \to \langle\bar\psi_{\text{op}}\Gamma\psi_{\text{op}}\rangle_{\text{two-fermion}} = \bar\psi_1\Gamma\psi_1 + \bar\psi_2\Gamma\psi_2. \tag{B16}$$

Then, Eqs. (B6) and (B9) are changed to

$$(i\partial\!\!\!/ - m\lambda\gamma_5\gamma^\mu\partial_\mu\theta)\psi_i = 0, \quad i = 1,2 \tag{B17}$$

$$\partial_\mu[\partial^\mu\theta - \lambda(\bar\psi_1\gamma_5\gamma^\mu\psi_1 + \bar\psi_2\gamma_5\gamma^\mu\psi_2)] = 0. \tag{B18}$$

In the two-fermion sector the "classical" Thirring equation is

$$[i\partial\!\!\!/ - m + \lambda^2(\bar\psi_1\gamma^\mu\psi_1 + \bar\psi_2\gamma^\mu\psi_2)\gamma_\mu]\psi_i = 0, \quad i = 1,2. \tag{B19}$$

The generalization of these results to the N-fermion state is now obvious.

In a theory with an internal symmetry (e.g., color), the introduction of the N-fermion confined state with N smaller than the degree of the internal symmetry can be handled trivially. In this case, there is no exclusion principle to complicate the problem and all fermions are in the ground state with the same spatial wave function $\psi(x)$,

$$\psi_i(x) = \psi(x)\eta_i\ , \tag{B20}$$

with

$$\int dx\, \psi^\dagger\psi = 1. \tag{B21}$$

The orthonormality conditions on ψ_i is guaranteed by the proper choice of the unit vectors η_i in the internal space. We have, for confinement states,

$$\partial_\mu\theta = \lambda\sum_i \bar\psi_i\gamma_5\gamma_\mu\psi_i = N\lambda\bar\psi\gamma_5\gamma_\mu\psi, \tag{B22}$$

and

$$(i\partial\!\!\!/ - m - \lambda\gamma_5\gamma_\mu\partial^\mu\theta)\psi = (i\partial\!\!\!/ - m + N\lambda^2\bar\psi\gamma_\mu\psi\gamma^\mu)\psi = 0. \tag{B23}$$

The fermion wave function is just as in the single-fermion case but with the replacement $\lambda^2 \to N\lambda^2$. In terms of $f = 1/2\lambda$, we find the energy of the N-fermion state as

$$E_N = m\cos N\lambda^2/2 = m\cos N/8f^2. \tag{B24}$$

Note that we must now require $N/4f^2 \le \pi$ to ensure positive fermion energies. The total energy for the N-fermion case is

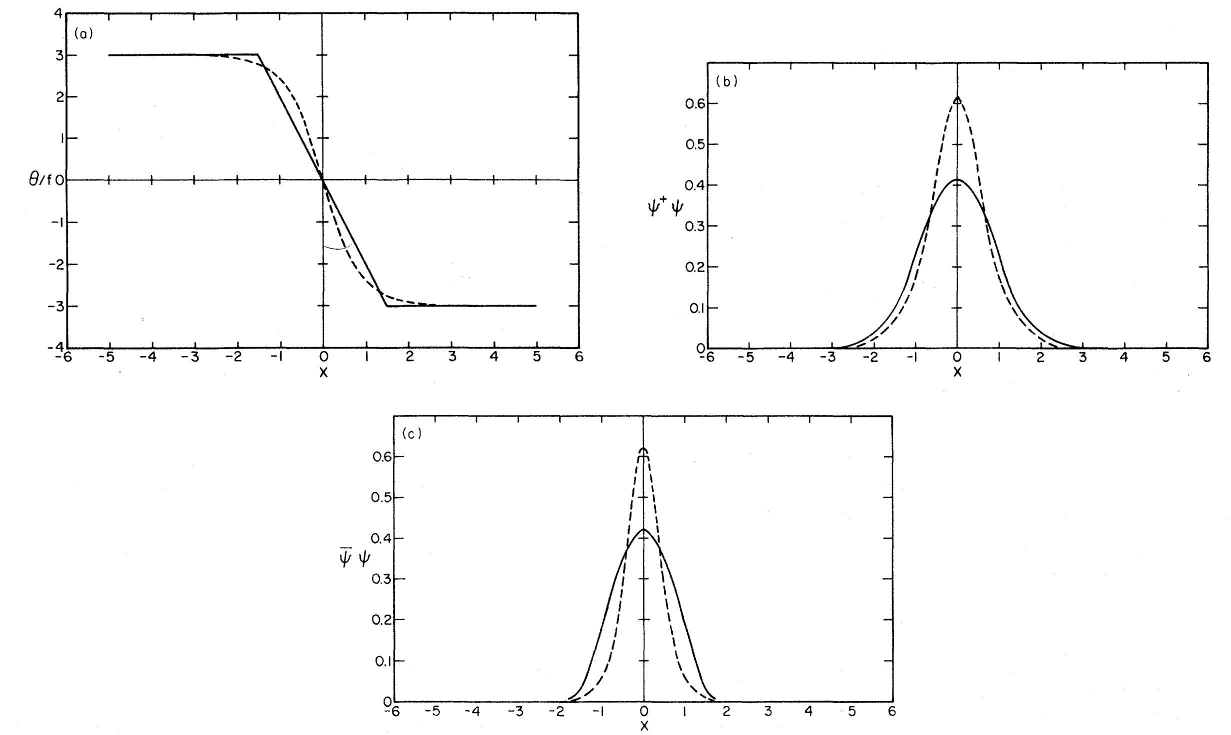

FIG. 3. Comparison of simple trial solution for the σ model with exact results of Fig. 1. The solid line is the trial function and the dashed line is the exact result. (a) Chiral phase, θ/f; (b) $\psi^\dagger\psi$; (c) $\bar{\psi}\psi$.

$$\langle H\rangle_N = m\,8f^2 \sin(N/8f^2). \tag{B25}$$

For $1/f^2 \to 0$, we have $\langle H\rangle_N \to Nm$, as it should, and for the limit $N/8f^2 \to \pi/2$, $\langle H\rangle_N \to 2mN/\pi$.

APPENDIX C

We give here a brief summary of a variational estimate of the confinement solution which preceded the exact solution presented in the text. The following discussion is really an appraisal of the variational technique in a problem such as this.

We begin with the Hamiltonian

$$H = \int dx \left[\frac{1}{2}\left(\frac{d\theta}{dx}\right)^2 + \psi^\dagger\left(i\gamma^0\gamma^1 \frac{d}{dx} + m\gamma^0 + \lambda\frac{d\theta}{dx}\right)\psi\right], \tag{C1}$$

with the constraint

$$\int dx\, \psi^\dagger\psi = 1. \tag{C2}$$

By the variational principle, we have

$$\frac{d}{dx}\left(\frac{d\theta}{dx} + \lambda\psi^\dagger\psi\right) = 0 \tag{C3}$$

and

$$\left[i\gamma^0\gamma^1 \frac{d}{dx} + m\gamma^0 + \left(\lambda\frac{d\theta}{dx} - E\right)\right]\psi = 0, \tag{C4}$$

where the energy of the fermion E appears as the Lagrange multiplier. We seek a trial function which minimizes the total energy $H > 0$.

From Eq. (C2) we find that

$$\frac{d\theta}{dx} + \lambda\psi^\dagger\psi = \text{const} = 0, \tag{C5}$$

since for a confined solution the left-hand side must go to zero as $x \to \pm\infty$. We choose as our trial function

$$\theta(x) = \begin{cases} -\theta_0, & x > x_0 \\ -\theta_0\dfrac{x}{x_0}, & -x_0 < x < x_0 \\ \theta_0, & x < -x_0 \end{cases} \tag{C6}$$

and determine the x_0 which minimizes $H > 0$. The parameter θ_0 is determined from the integrated form of (C5):

$$\theta(\infty) - \theta(-\infty) = -\lambda,$$

or

$$2\theta_0 = \lambda. \tag{C7}$$

The eigenvalue E is determined from the Dirac equation (C4). It is a root of

$$E=-\frac{\lambda^2}{2x_0}+\left\{m^2+\left[\frac{1}{x_0}\arctan\left(\frac{m-E}{m+E}\,\frac{E+\lambda^2/2x_0+m}{E+\lambda^2/2x_0-m}\right)^{1/2}\right]^2\right\}^{1/2}. \tag{C8}$$

The total energy H is

$$H=E+\tfrac{1}{2}\int dx\left(\frac{d\theta}{dx}\right)^2$$

$$=E(x_0)+\frac{\lambda^2}{4x_0}. \tag{C9}$$

The solution of Eq. (C8) and the minimization of Eq. (C9) may be effected numerically.

For $\lambda^2=3$, we find that the miminum of $H>0$ occurs at $mx_0=1.5$, where $E=0.26m$, and

$$(H)_{\text{variational}}=0.76\,m, \tag{C10}$$

to be compared with $(H)_{\text{exact}}=0.67m$. Some of the characteristics of the variational solution are plotted in Fig. 3, and are compared with the exact results from Fig. 1. The variational approach certainly works as well as might be expected with such a simple trial function.

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[1]A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn, Phys. Rev. D 10, 2599 (1974); W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein, and T.-M. Yan, *ibid.* 11, 1094 (1975). See also P. Vinciarelli, Nuovo Cimento Lett. 4, 905 (1972); CERN Reports Nos. TH1934, 1974 (unpublished), and TH 1952, 1974 (unpublished); M. Creutz, Phys. Rev. D 10, 1749 (1974); M. Creutz and K. S. Soh, *ibid.* (to be published).

[2]For a summary of the σ model and PCAC see B. W. Lee, *Chiral Dynamics* (Gordon and Breach, New York, 1972), and references cited therein.

[3]W. Thirring, Ann. Phys. (N.Y.) 3, 91 (1958).

[4]R. E. Norton and W. K. R. Watson, Phys. Rev. 110, 996 (1958).

[5]R. P. Feynman (unpublished).

[6]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[7]W. A. Bardeen and B. W. Lee, Phys. Rev. 177, 2389 (1969).

[8]For completeness we note that there exists a simple change of variables which takes the exact solution of the massive Thirring model discussed here into the "soliton" solution of the sine-Gordon equation. In particular, the substitutions $\tan\phi=\sqrt{\beta}\tan\omega/4$ and $\alpha=2\kappa$ cast the equation of motion for ϕ into

$$\partial_x{}^2\omega+\alpha^2\sin\omega=0.$$

However, this simple connection between the two models appears to be valid only for the single fermion confinement solution discussed here. The connection between the Thirring model and the sine-Gordon equation is also discussed by S. Coleman, Phys. Rev. D 11, 2088 (1975).

[9]S. Coleman, Commun. Math. Phys. 31, 259 (1973).

[10]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).

[11]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); 10, 4130 (1974); 10, 4138 (1974); this issue, Phys. Rev. D 11, 3424 (1975).

[12]This line of argument may be understood in the framework of R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4130 (1974), Sec. IV, as follows. Their equation (4.18) may be translated into

$$\frac{d^2\theta}{dx^2}-\frac{\lambda}{2\pi i}\int_c dE\,\mathrm{Tr}\gamma_5\gamma^1\frac{d}{dx}\left\langle x\left|\left[\gamma^0E-i\gamma^1\frac{d}{dx}-m-\lambda\gamma_5\gamma^\mu\partial_\mu\theta\right]^{-1}\right|x\right\rangle=0,$$

where the contour c is chosen appropriately to the $Q=1$ sector, and Tr refers to the trace over spinor indices. As they explain, it can be written as

$$\frac{d^2\theta}{dx^2}-\lambda\frac{d}{dx}\bar{U}_0(x)\gamma_5\gamma_1U_0(x)+\text{"fermion loop"}=0,$$

where

$$\text{"fermion loop"}=-\frac{\lambda}{2\pi i}\int_{c_0}dE\,\mathrm{Tr}\gamma_5\gamma^1\frac{d}{dx}\left\langle x\left|\left[\gamma^0E-i\gamma^1\frac{d}{dx}-m-\lambda\gamma_5\gamma^\mu\partial_\mu\theta\right]^{-1}\right|x\right\rangle,$$

and $U_0(x)$ is the commuting c-number eigenvector of the operator

$$\gamma^0E-i\gamma^1\frac{d}{dx}-m-\lambda\gamma_5\gamma^\mu\partial_\mu\theta$$

with the least positive eigenvalue E_0:

$$\left(\gamma^0E_0-i\gamma^1\frac{d}{dx}-m-\lambda\gamma_5\gamma^\mu\partial_\mu\theta\right)U_0(x)=0,$$

with the normalization

$$\int dx\,U_0^\dagger(x)\,U_0(x)=1.$$

The classical system of equations of motion for $U_0(x)$ and $\theta(x)$ is obtained by ignoring the term "fermion loop."