Transition amplitude for excited particles in the dual-string model*

José L. Torres-Hernández†
University of California, Physics Department, Berkeley, California 94720
(Received 6 February 1975)

In the context of the string description for dual resonance models, the Fourier coefficients of a general function (Neumann function) are determined. These are needed in the calculation of the amplitude for the scattering of excited hadrons.

INTRODUCTION

The formula proposed in Ref. 1 for the transition amplitude in a general strong-interaction collision process involves a functional integral of the type considered by Feynman in his formulation of quantum mechanics in terms of path integrals.[2] It is possible to evaluate this functional integral by a procedure based upon the introduction of the so-called Neumann function for the problem. In fact, what is needed are the coefficients in the Fourier expansion of this function, as these are involved in the formula for the scattering of N arbitrary (i.e., scalar as well as excited) particles in terms of the physical-particle operators.[1] The problem to be treated here is to determine the Fourier coefficients of the general Neumann function.

This work is organized in five sections, under the following headings.

I. Statement of the problem
II. Main results
III. Proof of the main results
IV. The case $\rho=\sum_{r=1}^{N}\alpha_r \ln(z-Z_r)$
V. Comment on the 3- and 4-string cases

In Sec. V the functions $N_{mn,rs}$ for the 4-string case are given in terms of Jacobi polynomials in X, the variable of integration in the 4-point Veneziano formula.

I. STATEMENT OF THE PROBLEM

According to the model proposed in Ref. 1, one considers hadrons as "strings," represented by curves whose parametrization is given in terms of two real variables, τ and σ. One then defines the complex variable $\rho=\tau+i\sigma$ and, for the case of a collision of R (initial) to S (final) strings, with $N=R+S$, one considers the region of the ρ plane indicated in Fig. 1.

The formula for the transition amplitude in the collision process described by Fig. 1 is given by[1]

$$A(\{\alpha_l\})=\int\prod_r{}' dZ_r H(\{\alpha_l\},\{Z_l\}). \tag{1.1}$$

The variables $\{\alpha_l\}$ appearing in this equation are defined by $\alpha_l=2P_l^+$, with $P_l^+=(1/\sqrt{2})(P_l^0+P_l^a)$; P_l^μ, with $\mu=0,1,\ldots,a$, is the momentum of the lth colliding particle (or string). In 4-dimensional space one has, of course, $a=3$, but it is one of the most characteristic features of present dual resonance models that they are consistent only for certain dimensions, different from four[1]; in fact, the amplitude (1.1) is consistent (this meaning relativistically invariant) only for $a=25$.

The variables $\{Z_l\}$ in Eq. (1.1) are related to the interaction times $\{\tau_l\}$ of Fig. 1 by the equation

$$\rho=\sum_r \alpha_r \ln(z-Z_r) \tag{1.2}$$

in the sense that, solving the equation $d\rho/dz=0$, one gets a set of solutions $\{z_l=z_l(\{Z_r\},\{\alpha_r\})\}$; the equations $\rho_l=\sum_r\alpha_r\ln(z_l-Z_r)$ then determine the variables $\{Z_r\}$ as functions of $\{\tau_l\}$ and $\{\alpha_l\}$, with ρ_l the interaction point in Fig. 1 corresponding to $\tau=\tau_l$. The summation in Eq. (1.2) is over all N strings or over $(N-1)$ of them, if one of the Z_r's is chosen to be infinite; this can be done, as, in fact, one can always fix the values of 3 of the variables $\{Z_r\}$ in an arbitrary way, due to the projective invariance of the amplitude (1.1).[1] In accordance with this projective invariance, the product $(\prod'_r)$ in Eq. (1.1) is over the $(N-3)$ nonfixed Z_r's.

The function $H(\{\alpha_l\},\{Z_l\})$ in Eq. (1.1) is given by

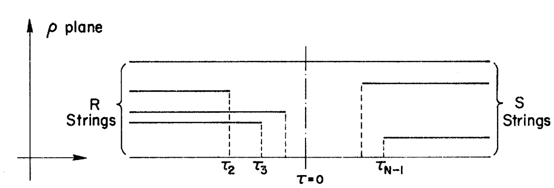

FIG. 1. This region has infinite extension on both sides of the line $\tau=0$. The points $\tau_2,\ldots,\tau_{N-1}$ are the interaction times. The vertical line $\tau=0$ is only included for the sake of concreteness in the discussion on Lagrange's theorem given in Sec. III C; its location is, of course, arbitrary in the model.

$$H(\{\alpha_l\},\{Z_l\})=f(\{\alpha_l\},\{Z_l\})\left\langle 0\left|\exp\left[\tfrac{1}{2}\sum_{r,s}\sum_{m,n}{}' N_{mn,rs}a^i{}_{m,r}a^i{}_{n,s}\right]\prod_{r'}\right|r'\right\rangle, \tag{1.3a}$$

with $f(\{\alpha_l\},\{Z_l\})$ defined by

$$f(\{\alpha_l\},\{Z_l\})=VI\prod_{r\neq s}(Z_r-Z_s)^{(\alpha_s/\alpha_r)(P_r\cdot P_r+1)+(\alpha_r/\alpha_s)(P_s\cdot P_s+1)}, \tag{1.3b}$$

$$V=|Z_{N-1}-Z_1|\prod_{r=2}^{N-2}|Z_{r+1}-Z_r|^{-1}, \tag{1.3c}$$

$$I=\prod_{r\neq s}|Z_r-Z_s|^{-2P_r\cdot P_s}\prod_{r''}|Z_{r''+1}-Z_{r''}|. \tag{1.3d}$$

In Eqs. (1.3a)–(1.3d), the indices r, r'', and s always refer to strings, and are to be taken from 1 to $N-1$, unless otherwise indicated. The indices m and n in Eq. (1.3a) refer to the normal modes of the strings. The $a^i{}_{m,r}$ are annihilation operators (the physical-particle operators referred to in the Introduction). $\prod_{r'}|r'\rangle$ is an arbitrary (scalar or excited) hadron state. The sum $(\sum'_{m,n})$ excludes the term with $m=n=0$. The $N_{mn,rs}$ are the Fourier coefficients of the Neumann function for the region of Fig. 1. They are functions of $\{\alpha_l\}$ and $\{Z_l\}$. The aim of this work is the calculation of these functions $N_{mn,rs}$.

II. MAIN RESULTS

For concreteness, and to keep all the results in parallel with those of Ref. 1, Eq. (1.2) will be taken as

$$\rho=\sum_{r=1}^{N-1}\alpha_r\ln(z-Z_r). \tag{2.1}$$

The case $\rho=\sum_{r=1}^{N}\alpha_r\ln(z-Z_r)$ is much more symmetrical and will be treated in Sec. IV.

The functions $N_{mn,rs}$ are given by the following expressions, with $m,n\geq 1$:

$$N_{mn,rs}=\frac{mn}{m\alpha_s+n\alpha_r}\sum_{q=1}^{N-1}\alpha_q A_{mqr}A_{nqs}, \tag{2.2}$$

$$N_{m0,rs}=\frac{1}{\alpha_s}\sum_{q=1}^{N-1}\alpha_q\delta'_{qs}A_{mqr}, \tag{2.3}$$

$$N_{0n,rs}=\frac{1}{\alpha_r}\sum_{q=1}^{N-1}\alpha_q\delta'_{qr}A_{nqs}. \tag{2.4}$$

In these equations

$$\delta'_{qp}\equiv\begin{cases}\delta_{qp}, & \text{for } p\neq N\\ -1, & \text{for } p=N\end{cases}, \tag{2.5}$$

$$A_{nqp}=\mathrm{Re}[B_{nqp}e^{in\sigma_{op}/\alpha_p}] \tag{2.6}$$

for $p=1,2,\ldots,N-1$, and

$$A_{nqN}=\mathrm{Re}(B_{nqN}e^{in\pi}) \tag{2.7}$$

with the following expressions for the B_{nqp} and B_{nqN}:

$$B_{npp}=\frac{1}{n!n}\left[\frac{d^n}{du^n}\prod_{\substack{r=1\\(r\neq p)}}^{N-1}(u+Z_p-Z_r)^{-n\alpha_r/\alpha_p}\right]_{u=0}, \tag{2.8}$$

$$B_{nqp}=B_{npp}-\frac{Z_p-Z_q}{n!n}\left\{\frac{d^n}{du^n}\left[(u+Z_p-Z_q)^{-1}\prod_{\substack{r=1\\(r\neq p)}}^{N-1}(u+Z_p-Z_r)^{-n\alpha_r/\alpha_p}\right]\right\}_{u=0};\quad q\neq p,\quad p\neq N; \tag{2.9}$$

$$B_{nqN}=\frac{-1}{n!n}\prod_{\substack{r=1\\(r\neq q)}}^{N-1}(Z_q-Z_r)^{-n\alpha_r/\alpha_N}\left[\frac{d^n}{du^n}\prod_{\substack{s=1\\(s\neq q)}}^{N-1}\left(u+\frac{1}{Z_q-Z_s}\right)^{-n\alpha_s/\alpha_N}\right]_{u=0}. \tag{2.10}$$

In Eq. (2.6), σ_{op} corresponds to the lower end of the pth string in Fig. 1, if it is incoming, or to the upper end if it is outgoing.

The symmetry $N_{mn,rs}=N_{nm,sr}$ is apparent from Eqs. (2.2) to (2.4).

III. PROOF OF THE MAIN RESULTS

A. Method of solution

The familiar representations[3] of dual resonance amplitudes involve a half-plane (z plane) instead of the region of Fig. 1. This z plane is related to the ρ plane (Fig. 1) by Eq. (1.2). In the z plane the Neumann function is known, and it has the simple expression[3]

$$N(z,z')=\ln|z-z'|+\ln|z-z'^*|\,. \tag{3.1}$$

The problem then reduces to writing Eq. (3.1) in terms of the variables ρ, ρ', connected with z and z' by Eq. (1.2), using the conformal invariance of the Neumann function.[1]

Posed in this way, the problem is very complicated, because it implies the direct inversion of Eq. (1.2). The method to be used will be essentially the same as that employed in Ref. 1 for the calculation of the vertex operator. It consists in working with the function

$$M(\rho,\rho')\equiv\frac{\partial}{\partial\tau}N(\rho,\rho')+\frac{\partial}{\partial\tau'}N(\rho,\rho')\,, \tag{3.2}$$

where $N(\rho,\rho')$ is the Neumann function in terms of the variables ρ, ρ', i.e., it is the Neumann function for the region of Fig. 1.

From its definition, one can prove[1] that $M(\rho,\rho')$, taken as a function of ρ (i.e., for fixed ρ'), has the following simple properties [which are the motivation for working with the function $M(\rho,\rho')$ instead of directly with $N(\rho,\rho')$].

1. $M(\rho,\rho')$ satisfies Laplace's equation inside the region of Fig. 1.

2. $M(\rho,\rho')$ has zero normal derivative, except at the joining and separation points of the strings (points $\tau=\tau_2,\tau_3,\ldots,\tau_{N-1}$ in Fig. 1).

3. As $\tau\to\pm\infty$ along the pth string ($+\infty$ for an outgoing, $-\infty$ for an incoming string), $M(\rho,\rho')$ behaves as follows:

$$M(\rho,\rho')\to 2\,\mathrm{Re}\frac{\partial}{\partial\tau'}\ln(z'-Z_p)\,, \tag{3.3}$$

this equation being valid for $p=1,2,\ldots,N-1$. Take $z'=z'(\rho')$.

A hint for the construction of such a function comes from the key observation that the function $M(\rho,\rho')$ constructed in Ref. 1 for the special case of three strings, with $\rho=\alpha_1\ln(z-1)+\alpha_2\ln z$, can be written in the symmetrical form

$$M(\rho,\rho')=2\alpha_1\left[\mathrm{Re}\frac{\partial}{\partial\tau}\ln(z-1)\right]\left[\mathrm{Re}\frac{\partial}{\partial\tau'}\ln(z'-1)\right]+2\alpha_2\left[\mathrm{Re}\frac{\partial}{\partial\tau}\ln z\right]\left[\mathrm{Re}\frac{\partial}{\partial\tau'}\ln z'\right]. \tag{3.4}$$

It is therefore suggested that, in the general case, $M(\rho,\rho')$ can be written as follows:

$$M(\rho,\rho')=\sum_{q=1}^{N-1}2\alpha_q\left[\mathrm{Re}\frac{\partial}{\partial\tau}\ln(z-Z_q)\right]\left[\mathrm{Re}\frac{\partial}{\partial\tau'}\ln(z'-Z_q)\right]. \tag{3.5}$$

As it stands, it is clear that $M(\rho,\rho')$ given by Eq. (3.5) satisfies condition 1 above (it is, as a function of ρ [i.e., of $z(\rho)$], the real part of an analytic function). It will now be proved that condition 3 is satisfied, too.

As $\tau\to\pm\infty$ along string p, it follows from the equation $\rho=\sum_{r=1}^{N-1}\alpha_r\ln(z-Z_r)$ that $\rho=\alpha_p\ln(z-Z_p)+O(\text{constant})$, because $z\to Z_p$ in this limit. This implies that

$$\ln(z-Z_p)=\frac{\rho}{\alpha_p}+O(0) \tag{3.6}$$

$$\Rightarrow \mathrm{Re}\frac{\partial}{\partial\tau}\ln(z-Z_p)=\frac{1}{\alpha_p}+O(0)\,. \tag{3.7}$$

For $q\neq p$ one has

$$\mathrm{Re}\frac{\partial}{\partial\tau}\ln(z-Z_q)=\mathrm{Re}\frac{\partial z/\partial\tau}{z-Z_q}\,. \tag{3.8}$$

From Eq. (3.6), $z\sim e^{\rho/\alpha_p}+Z_p$, so that $\partial Z/\partial\tau\sim(1/\alpha_p)e^{\rho/\alpha_p}$, which approaches zero as $\tau\to\pm\infty$ ($+\infty$ or $-\infty$ as explained in condition 3). Hence

$$\mathrm{Re}\frac{\partial}{\partial\tau}\ln(z-Z_q)\sim 0\,. \tag{3.9}$$

Substituting Eqs. (3.7) and (3.9) into (3.5) one finds that condition (3.3) is satisfied.

As for condition 2, it will be proved in Sec. III B that one can write, for $q,p=1,2,\ldots,N-1$,

$$\ln(z-Z_q)=\delta_{qp}\zeta_p+\sum_{n=1}^{\infty}B_{nqp}e^{n\rho/\alpha_p}+\text{const}\,, \tag{3.10}$$

$$\ln(z-Z_q)=-\zeta_N+\sum_{n=1}^{\infty}B_{nqN}e^{n\rho/\alpha_N}+\text{const}\,, \tag{3.11}$$

with the coefficients B_{nqp} and B_{nqN} given by Eqs. (2.8) to (2.10). Define ζ_r by[1]

$$\zeta_r=\frac{\rho-i\sigma_{\alpha r}}{\alpha_r}=\xi_r+i\eta_r \tag{3.12}$$

for $r=1,2,\ldots,N$.

From Eqs. (3.10) to (3.12) one gets

$$\mathrm{Re}\frac{\partial}{\partial\tau}\ln(z-Z_q)=\delta_{qp}\left(\frac{1}{\alpha_p}\right)+\sum_{n=1}^{\infty}\frac{n}{\alpha_p}A_{nqp}\cos n\eta_p\,e^{n\xi_p}\,, \tag{3.13}$$

$$\mathrm{Re}\frac{\partial}{\partial\tau}\ln(z-Z_q)=-\frac{1}{\alpha_N}+\sum_{n=1}^{\infty}\frac{n}{\alpha_N}A_{nqN}\cos n\eta_N\,e^{n\xi_N}\,, \tag{3.14}$$

with A_{nqp} and A_{nqN} given by Eqs. (2.6) and (2.7).

From the expression for $M(\rho,\rho')$, Eq. (3.5), it is then clear that the normal derivatives at the boundaries, $(\partial/\partial\sigma)M(\rho,\rho')$ and $(\partial/\partial\sigma')M(\rho,\rho')$, will be zero, because they involve terms of the form $\sin n\eta_p$ $(=(\alpha_p/n)(\partial/\partial\sigma)\cos n\eta_p)$ or $\sin n\eta_p'$ as factors of the individual summands, and η_p and η_p' are zero or π at the boundaries (ends) of the pth string. The rule is then to express $M(\rho,\rho')$ as a function of $\zeta_p=\xi_p+i\eta_p$ when studying the normal derivatives on the pth string.

Then, it has been proved that the conjecture for $M(\rho,\rho')$ [Eq. (3.5)] is correct. Of course, Eqs. (3.10) and (3.11) remain to be proved.

Now, from Eqs. (3.5), (3.13), and (3.14) one can obtain $M(\rho,\rho')$ in terms of any of the variables ζ_r, ζ_s', for $r,s=1,\ldots,N$. One needs the quantities $N_{mn,rs}$, which are the Fourier coefficients of $N(\rho,\rho')$ for ρ on the rth string and ρ' on the sth string. These can be obtained by using Eq. (3.2) and expressing $\mathrm{Re}(\partial/\partial\tau)\ln(z-Z_q)$ in terms of ζ_r $\mathrm{Re}(\partial/\partial\tau')\ln(z'-Z_q)$ in terms of ζ_s' by means of Eqs. (3.13) and (3.14). One obtains the result

$$M(\rho,\rho')=\sum_{q=1}^{N-1}2\alpha_q\delta'_{qr}\delta'_{qs}\left(\frac{1}{\alpha_r}\right)\left(\frac{1}{\alpha_s}\right)+\sum_{m=1}^{\infty}\left(\sum_{q=1}^{N-1}\alpha_q\delta'_{qs}A_{mqr}\right)\left(\frac{2m}{\alpha_r\alpha_s}\right)\cos m\eta_r\, e^{m\xi_r}$$
$$+\sum_{n=1}^{\infty}\left(\sum_{q=1}^{N-1}\alpha_q\delta'_{qr}A_{nqs}\right)\left(\frac{2n}{\alpha_r\alpha_s}\right)\cos n\eta_s' e^{n\xi_s'}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{q=1}^{N-1}\alpha_q A_{mqr}A_{nqs}\right)\left(\frac{2mn}{\alpha_r\alpha_s}\right)e^{m\xi_r+n\xi_s'}\cos m\eta_r\cos n\eta_s', \tag{3.15}$$

with δ'_{qp} given by definition (2.5).

The expansion of $N(\rho,\rho')$ in terms of the Fourier coefficients $N_{mn,rs}$ is given by [cf. Eq. (4.3) of Ref. 1]:

$$N(\rho,\rho')=-\delta_{rs}\sum_{n=1}^{\infty}\frac{2}{n}e^{-n|\xi_r-\xi_s'|}\cos n\eta_r\cos n\eta_s'+\sum_{m,n}{}'\,2N_{mn,rs}e^{m\xi_r+n\xi_s'}\cos m\eta_r\cos n\eta_s'+2[\delta_{rs}\max(\xi,\xi')-\xi\delta_{rN}-\xi'\delta_{sN}+b_{rs}] \tag{3.16}$$

($\sum'_{m,n}$ means that the term $m=n=0$ is absent; the range of m and n is from 0 to ∞).

From Eq. (3.16) one obtains

$$\frac{\partial}{\partial\tau}N(\rho,\rho')+\frac{\partial}{\partial\tau'}N(\rho,\rho')=2\left(\frac{1}{\alpha_r}\delta_{rs}-\frac{1}{\alpha_N}\delta_{rN}-\frac{1}{\alpha_N}\delta_{sN}\right)+\sum_{m=1}^{\infty}\frac{2m}{\alpha_r}N_{m0,rs}e^{m\xi_r}\cos m\eta_r+\sum_{n=1}^{\infty}\frac{2n}{\alpha_s}N_{0n,rs}e^{n\xi_s'}\cos n\eta_s'+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}2\left(\frac{m}{\alpha_r}+\frac{n}{\alpha_s}\right)N_{mn,rs}e^{m\xi_r+n\xi_s'}\cos m\eta_r\cos n\eta_s'. \tag{3.17}$$

Results (2.2) to (2.4) then follow directly from Eqs. (3.2), (3.15), and (3.17).

B. Proof of equations (3.10) and (3.11)

These equations [i.e., results (2.8) to (2.10)] will be proved by a method based on the use of Lagrange's theorem[4] for the expansion of an analytic function in powers of another analytic function. This theorem will be quoted from Ref. 5, for the sake of completeness.

Let $f(u)$ and $\phi(u)$ be functions of u analytic on and inside a contour C surrounding a point a, and let t be such that the inequality

$$|t\phi(u)|<|u-a| \tag{3.18}$$

is satisfied at all points u on the perimeter of C; then the equation

$$\zeta=a+t\phi(\zeta), \tag{3.19}$$

regarded as an equation in ζ, has one root in the interior of C; and further any function of ζ analytic on and inside C can be expanded as a power series in t by the formula

$$f(\zeta)=f(a)+\sum_{n=1}^{\infty}\frac{t^n}{n!}\left\{\frac{d^{n-1}}{du^{n-1}}\left[\left(\frac{df(u)}{du}\right)[\phi(u)]^n\right]\right\}_{u=a}. \tag{3.20}$$

1. The case $q=p$ in Eq. (3.10)

Start from Eq. (2.1) and write it in the form

$$\rho=\alpha_p\ln(z-Z_p)+\sum_{\substack{r=1\\(r\neq p)}}^{N-1}\alpha_r\ln(Z_p-Z_r)+\sum_{\substack{r=1\\(r\neq p)}}^{N-1}\alpha_r\ln[1+(Z_p-Z_r)^{-1}(z-Z_p)]. \tag{3.21}$$

This equation can be rewritten as

$$\zeta=t\sum_{\substack{r=1\\(r\neq p)}}^{N-1}(1-\Omega_{rp}\zeta)^{-\alpha_r/\alpha_p}, \tag{3.22}$$

with the definitions

$$t=e^{\rho/\alpha_p}, \tag{3.23}$$

$$\Omega_{rp}=(Z_r-Z_p)^{-1}\prod_{\substack{s=1\\(s\neq p)}}^{N-1}(Z_p-Z_s)^{-\alpha_s/\alpha_p}, \tag{3.24}$$

$$\zeta = (z - Z_p) \prod_{\substack{r=1 \\ (r\neq p)}}^{N-1} (Z_p - Z_r)^{\alpha_r/\alpha_p} . \tag{3.25}$$

Define the functions

$$\phi(u) = \prod_{\substack{r=1 \\ (r\neq p)}}^{N-1} (1 - \Omega_{rp}u)^{-\alpha_r/\alpha_p} , \tag{3.26}$$

$$f(u) = \ln\phi(u) . \tag{3.27}$$

Equation (3.22) is then equivalent to

$$\zeta = t\phi(\zeta) . \tag{3.28}$$

Apply now Lagrange's theorem with the functions (3.26) and (3.27) and with $a=0$ (cf. Sec. III C):

$$f(\zeta) = \sum_{n=1}^{\infty} t^n \left\{ \frac{1}{n!n} \left[\frac{d^n}{du^n} \prod_{\substack{r=1 \\ (r\neq p)}}^{N-1} (1 - \Omega_{rp}u)^{-n\alpha_r/\alpha_p} \right]_{u=0} \right\} . \tag{3.29}$$

From Eqs. (3.27) and (3.28), $f(\zeta) = \ln(\zeta/t)$. Using definitions (3.23) to (3.25), one immediately obtains (3.10) for the case $q=p$, with B_{npp} given by Eq. (2.8).

2. The case $q\neq p$ in Eq. (2.10)

Rewrite Eq. (2.1) in the form

$$\rho = \alpha_p \ln(z - Z_p) + \alpha_q \ln(z - Z_q) + \sum_{\substack{r=1 \\ (r\neq p,q)}}^{N-1} \alpha_r \ln(z - Z_r) . \tag{3.30}$$

Using definitions (3.24) and (3.25), this equation can be rewritten in the following way:

$$\ln(z - Z_q) - \ln(Z_p - Z_q) = \ln(1 - \Omega_{qp}\zeta) . \tag{3.31}$$

Use now the expansion $\ln(1-\nu) = -\sum_{n=1}^{\infty}\nu^n/n$, whose validity will be examined in Sec. III C. One obtains from Eq. (3.31)

$$\ln(z - Z_q) - \ln(Z_p - Z_q) = \sum_{n=1}^{\infty} \frac{(-\Omega_{qp}{}^n)\zeta^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-\Omega_{qp}{}^n)t^n(\zeta/t)^n}{n} . \tag{3.32}$$

Using Lagrange's theorem, a power series in t will now be obtained for $(\zeta/t)^n$. Define $f(u)$ by the equation

$$f(u) = [\phi(u)]^n , \tag{3.33}$$

with $\phi(u)$ given by (3.26). Apply Lagrange's theorem with the functions (3.33) and (3.26) and with $a=0$ (cf. Sec. II C) to obtain [after using Eq. (3.28)]

$$\left(\frac{\zeta}{t}\right)^n = f(\zeta) = \sum_{m=0}^{\infty} \beta_{mnp} t^m , \tag{3.34}$$

with

$$\beta_{mnp} = \frac{n}{n+m} \frac{1}{m!} \left[\frac{d^m}{du^m} \prod_{\substack{r=1 \\ (r\neq p)}}^{N-1} (1 - \Omega_{rp}u)^{-(m+n)\alpha_r/\alpha_p} \right]_{u=0} . \tag{3.35}$$

Insertion of Eq. (3.35) into (3.32) gives Eq. (3.10) for the case $q\neq p$, with (cf. Sec. III C)

$$B_{nqp} = \sum_{m=0}^{n-1} \frac{-\Omega_{qp}{}^{n-m}}{n-m} \beta_{m,n-m,p}$$

$$= \sum_{m=0}^{n-1} \left\{ \frac{-\Omega_{qp}{}^{n-m}}{n} \frac{1}{m!} \left[\frac{d^m}{du^m} \prod_{\substack{r=1 \\ (r\neq p)}}^{N-1} (1 - \Omega_{rp}u)^{-n\alpha_r/\alpha_p} \right]_{u=0} \right\} . \tag{3.36}$$

To prove the equivalence of (3.36) and (2.9), use the fact (from the theory of power series) that, for a function $g(u)$ analytic in a region that contains the origin

$$\frac{1}{m!}\left[\frac{d^m}{du^m} g(u)\right]_{u=0} = \frac{1}{2\pi i} \oint \frac{du}{u^{m+1}} g(u) , \tag{3.37}$$

the integration being over a closed curve around the origin. One then obtains [using Eq. (3.24)], after a simple change of variables, a factor $\sum_{l=1}^{n}[u/(Z_q - Z_p)]^l$ in the integrand. From the relation $\sum_{l=0}^{n}\nu^l = (\nu^{n+1}-1)/(\nu-1)$ and a second application of (3.37), the equivalence of (3.36) and (2.9) is proved.

3. The case $p=N$ [Eq. (2.11)]

Start again from Eq. (2.1) and rewrite it in the form

$$\rho = \sum_{r=1}^{N-1} \alpha_r \ln(z - Z_q)$$

$$+ \sum_{\substack{r=1 \\ (r\neq q)}}^{N-1} \alpha_r \ln[1 + (Z_q - Z_r)(z - Z_q)^{-1}] . \tag{3.38}$$

Use now the condition (from momentum conservation[1])

$$\sum_{r=1}^{N} \alpha_r = 0 \Rightarrow \sum_{r=1}^{N-1} \alpha_r = -\alpha_N . \tag{3.39}$$

One can then rewrite Eq. (3.38) in the form

$$\zeta' = t_1 \prod_{\substack{r=1 \\ (r\neq q)}}^{N-1} (1 - \lambda_{rq}\zeta_1)^{-\alpha_r/\alpha_N} , \tag{3.40}$$

with the definitions

$$t_1 = e^{\rho/\alpha_N}$$

$$\lambda_{rq} = Z_r - Z_q \tag{3.41}$$

$$\zeta' = (z - Z_q)^{-1}$$

Applying again Lagrange's theorem with

$$\phi(u) = \prod_{\substack{r=1 \\ (r\neq q)}}^{N-1} (1 - \lambda_{rq}u)^{-\alpha_r/\alpha_N}$$

and $f(u)=\ln[\phi(u)]$ and with $a=0$ (cf. Sec. III C) one gets

$$-\ln(z-Z_q)-\frac{\rho}{\alpha_N} =\sum_{n=1}^{\infty}\frac{t_1{}^n}{n!\,n}\left[\frac{d^n}{du^n}\prod_{\substack{r=1\\(r\neq q)}}^{N-1}(1-\lambda_{rq}u)^{-n\alpha_r/\alpha_N}\right]_{u=0}. \qquad (3.42)$$

Then, from (3.44) one immediately obtains Eq. (3.11), with B_{nqN} given by (2.10).

C. Conditions for the use of Lagrange's Theorem

It will be proved that for parts 1 and 2 in Sec. III B the conditions for Lagrange's theorem were met, provided one chooses the contour C to be any contour which is inside the circle of analyticity for $\phi(u)$ [cf. Eq. (3.26)] and which contains the origin in its interior. A similar discussion is valid for part 3, with t_1, λ_{rq}, and ζ' taking the place of t, Ω_{rq}, and ζ, respectively.

Take C to be a contour contained in the circle of analyticity of $\phi(u)$ [i.e., inside the circle where a power series expansion of $\phi(u)$ about the origin is valid]. It will be proved in the following paragraph that $\phi(u)$ is different from zero in C. Then, $f(u)$, given in terms of $\phi(u)$ by Eq. (3.27) or (3.33), $f(u)=\ln\phi(u)$ or $f(u)=[\phi(u)]^n$, will be analytic inside C and on its perimeter. As C is a compact set (cf. next paragraph), $|\phi(u)|$ is bounded in C, i.e., $|\phi(u)|<M$, for M a positive number. Let R_m be the smallest of the numbers $\{|u|\}$ such that u is on the perimeter of C. Choose t such that $|t|<R_m/M$. Then, for every u on the perimeter of C,

$$|t\phi(u)|<\frac{R_m}{M}|\phi(u)|<|u|\,. \qquad (3.43)$$

It follows that, for t such that $|t|<R_m/M$, all the hypotheses of Lagrange's theorem are satisfied, with $a=0$ [cf. expression (3.18)]. This theorem then implies that Eq. (3.28) is valid for ζ in the interior of C (one ζ for each value of t such that $|t|<R_m/M$) and that expansions (3.29) and (3.34) are correct.

From definition (3.26),

$$\phi(u)=\prod_{\substack{r=1\\(r\neq p)}}^{N-1}(1-\Omega_{rp}u)^{-\alpha_r/\alpha_p}\,,$$

it is clear that the circle of analyticity of $\phi(u)$ is given by the region where $|\Omega_{rp}u|<1$, for $r=1,\ldots,N-1$; $r\neq p$. Then, this region is determined by the points u such that $|u|<|\tilde{\Omega}_{rp}|^{-1}$, with $|\tilde{\Omega}_{rp}|$ the maximum of $\{|\Omega_{rp}|\}$, for $r=1,\ldots,N-1$; $r\neq p$. In this region, moreover, $\phi(u)\neq 0$.

From the above discussion, it is clear too that the power series expansion used in going from (3.31) to (3.32) is justified, for ζ inside the contour C.

The condition that $|t|$ must be "small enough" (i.e., $|t|<R_m/M$) imposes the restrictions [from Eqs. (3.28) and (3.25)] that $|z-Z_p|$ must be sufficiently small. Then, the results of Sec. III will be valid [cf. Eq. (2.1) and Fig. 1] if τ is sufficiently large and negative for $\alpha_p>0$ (i.e., an incoming string), and sufficiently large and positive for $\alpha_p<0$ (i.e., an outgoing string). This causes no trouble, as the Fourier coefficients of the Neumann function can be calculated anywhere in the region of Fig. 1.

IV. THE CASE $\rho=\sum_{r=1}^{N}\alpha_r\ln(z-Z_r)$

In this case, in which no ζ_r is chosen to be infinite, one writes, corresponding to Eq. (3.5),

$$M(\rho,\rho')=\sum_{r=1}^{N}2\alpha_r\left[\mathrm{Re}\frac{\partial}{\partial\tau}\ln(z-Z_r)\right]\left[\mathrm{Re}\frac{\partial}{\partial\tau'}\ln(z'-Z_r)\right] \qquad (4.1)$$

and obtains the analogous expressions to Eqs. (3.10), using the same methods as in Sec. III. These equations are sufficient to give $\ln(z-Z_q)$ in terms of ζ_p, for $1\le q\le N$, $1\le p\le N$. This solves the problem.

V. COMMENT ON THE 3- AND 4-STRING CASES

Formulas (2.2) to (2.10) reproduce the results found in Ref. 1 for the special case of three strings. In this case, the coefficients B_{nqr} [cf. Eqs. (2.8) to (2.10)] can be put in terms of beta functions (or, equivalently, of gamma functions) of the variables $\{\alpha_r\}$ and $\{z_r\}$. Hence the functions $N_{mn,rs}$ [cf. Eqs. (2.2) to (2.4)] are given in terms of simple combinations of these beta functions.[9]

In the 4-string case one again finds that the coefficients B_{nqr} can be expressed in closed form, namely in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (or, equivalently, in terms of the hypergeometric function)[6,7]:

$$B_{npp}=\frac{(Z_{r_1}-Z_{r_2})^n}{n}(Z_p-Z_{r_1})^{\lambda_1}(Z_p-Z_{r_2})^{\omega_1}P_n{}^{(\lambda_1,\omega_1)}(X_1)\,, \qquad (5.1)$$

with $p\neq 4$, $r_1\neq r_2$, $r_1,r_2\neq p,4$, and

$$\lambda_1=-n\left(\frac{\alpha_{r_1}}{\alpha_p}+1\right),\quad \omega_1=-n\left(\frac{\alpha_{r_2}}{\alpha_p}+1\right),\quad X_1=\frac{2Z_p-Z_{r_1}-Z_{r_2}}{Z_{r_1}-Z_{r_2}}\,; \qquad (5.2)$$

$$B_{nqp}=\frac{(Z_q-Z_s)^n}{n}(Z_p-Z_q)^{\lambda_2}(Z_p-Z_s)^{\omega_2}\times[P_n^{(\lambda_2,\omega_2)}(X_2)-(Z_p-Z_q)^{-1}P_n^{(\lambda_2-1,\omega_2)}(X_2)]\,, \tag{5.3}$$

with $s\neq q,p,4$, $q\neq p$, $q,p\neq 4$, and

$$\lambda_2=-n\left(\frac{\alpha_q}{\alpha_p}+1\right)\,,\quad \omega_2=-n\left(\frac{\alpha_s}{\alpha_p}+1\right)\,,\quad X_2=\frac{2Z_p-Z_q-Z_s}{Z_q-Z_s}\,; \tag{5.4}$$

$$B_{nq4}=-\frac{(Z_{r_2}-Z_{r_1})^n}{n}P_n^{(\lambda_3,\omega_3)}(X_3)\,, \tag{5.5}$$

with $r_1\neq r_2$, $r_1,r_2\neq q,4$, $q\neq 4$, and

$$\lambda_3=-n\left(\frac{\alpha_{r_1}}{\alpha_4}+1\right)\,,\quad \omega_3=-n\left(\frac{\alpha_{r_2}}{\alpha_4}+1\right)\,,\quad X_3=\frac{2Z_q-Z_{r_2}-Z_{r_1}}{Z_{r_2}-Z_{r_1}}\,. \tag{5.6}$$

The coefficients $N_{mn,rs}$ [cf. Eqs. (2.2) to (2.4)] are then obtained as combinations of these Jacobi polynomials.

Equations (5.1) to (5.6) can be put in terms of the variable of integration X used in the ordinary 4-point Veneziano amplitude, with the definition[8]

$$X=\frac{Z_2-Z_1}{Z_3-Z_1} \tag{5.7}$$

valid for $z_4=\infty$. If one chooses the values of Z_1 and Z_3 to be $Z_1=0$ and $Z_3=1$ (using the projective invariance of the amplitude[8]), Eq. (5.7) reduces to the simple identification $X=Z_2$.

ACKNOWLEDGMENTS

I am very grateful to Professor S. Mandelstam for his invaluable interest and advice throughout this work. I thank H. Arfaei and Lisa Douglass for useful conversations.

*Work supported by the National Science Foundation, under Grant No. GP-42249X.

†Consejo Nacional de Ciencia y Tecnología (México) Fellow. On leave from Escuela Superior de Física y Matemáticas del I. P. N., Mexico City.

[1]S. Mandelstam, Nucl. Phys. B64, 205 (1973).

[2]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York 1965).

[3]J. L. Gervais and B. Sakita, Phys. Rev. D 4, 2291 (1971).

[4]The use of Lagrange's theorem for the special case of 3 strings was suggested by S. Fenster (S. Fenster, private communication to S. Mandelstam).

[5]E. T. Whittaker and G. N. Watson, *Course of Modern Analysis* (Cambridge Univ. Press, New York, 1952), 4th edition, p. 133.

[6]I am grateful to Prof. S. Mandelstam for pointing this out to me.

[7]W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems of Special Functions* (Springer, Berlin, Germany, 1966), pp. 210, 212, 213.

[8]S. Mandelstam, in *Lectures on Elementary Particles and Quantum Field Theory*, ed. by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass, 1971), Vol. 1, p. 197.

[9]The result of Ademollo *et al*. (see Ref. 1) for the 3-string vertex is obtained here in a simple way. A different formula for the general Neumann function is given in a recent work [E. Cremmer and J. L. Gervais, Nucl. Phys. B90, 410 (1975)]. Considerable effort is needed to obtain the expression of Ademollo *et al.* from the result of Cremmer and Gervais.