Verification of virtual Compton-scattering sum rules in quantum electrodynamics*

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Two new sum rules for virtual-photon forward Compton scattering derived by Schwinger are shown to be satisfied in the case of quantum electrodynamics by explicit calculation of the lowest-order contributions.

In a recent investigation of deep-inelastic scattering of electrons on polarized nucleons, Schwinger[1] derived two new sum rules for virtual Compton scattering. As a modest verification of the basic theoretical ideas used by Schwinger, we here will investigate whether these results hold in the known case of electrodynamics. We will show that, to fourth order in e, the sum rules are true, which then lends support to Schwinger's framework.[2]

We will start with a brief review of the derivation of the new sum rules. The deep-inelastic process is viewed as the absorption of polarized virtual photons by a polarized target. The various cross sections are related to the imaginary part of the forward Compton scattering amplitude, which for a polarized target nucleon can be expressed as ($d\omega_P=[(d\vec{\mathrm{P}})/(2\pi)^3](1/2P^0)$, P is the nucleon momentum)

$$1+i\,Vd\omega_P 4e^2A^\mu(-q)A^\nu(q)\sum_{a=1}^{4}T_{a\mu\nu}H_a, \tag{1}$$

where the first term refers to the situation of noninteraction, V is the space-time interaction volume, $A^\mu(q)$ is the field of the virtual photon, and $T_{a\mu\nu}$ is an appropriate basis tensor. A convenient but unconventional choice, which is free of kinematic singularities and zeros, is

$$T_{1\mu\nu}=m^2(q_\mu q_\nu-q^2g_{\mu\nu}), \tag{2}$$

$$T_{2\mu\nu}=q^2P_\mu P_\nu-qP(q_\mu P_\nu+P_\mu q_\nu)+(qP)^2g_{\mu\nu}+m^2(q^2g_{\mu\nu}-q_\mu q_\nu), \tag{3}$$

$$T_{3\mu\nu}=-2m^3i\epsilon_{\mu\nu\kappa\lambda}q^\kappa s^\lambda, \tag{4}$$

$$T_{4\mu\nu}=m(qs)i\epsilon_{\mu\nu\kappa\lambda}q^\kappa P^\lambda, \tag{5}$$

where s^μ is a unit pseudovector which covariantly describes the spin of the target nucleon and satisfies

$$P_\mu s^\mu=0. \tag{6}$$

The coefficients, H_a, are functions of the two Lorentz scalars, q^2 and $qP=-m\nu$. Because of crossing symmetry, it is to be noted that H_4 is antisymmetric in ν, while $H_{1,2,3}$ are symmetric.

Schwinger proceeds by assuming that the H_a may be represented by double-spectral forms[3]:

$$H_a(q^2,qP)=\int\frac{dM_+^2}{M_+^2}\frac{dM_-^2}{M_-^2}\frac{2h_a(M_+^2,M_-^2)}{[(P+q)^2+M_+^2][(P-q)^2+M_-^2]}. \tag{7}$$

The imaginary part of this amplitude corresponding to the production of a real intermediate state through the absorption of the photon by the nucleon is

$$\frac{1}{\pi}\mathrm{Im}\,H_a(q^2,qP)=\frac{1}{M_+^2}\int\frac{dM_-^2}{M_-^2}\frac{h_a(M_+^2,M_-^2)}{q^2+\frac{1}{2}(M_+^2+M_-^2)-m^2}, \tag{8}$$

where now M_+ is the mass of the intermediate state,

$$M_+^2=m^2+2m\nu-q^2. \tag{9}$$

By considering the contribution due to a single nucleon in the intermediate state, we determine the elastic contributions to Im H_a:

$$m^2q^2\frac{1}{\pi}\mathrm{Im}\,H_{1,2}=\delta\left(\frac{M_+^2}{m^2}-1\right)\times\left\{G_E{}^2,\ \frac{G_E{}^2+(q^2/4m^2)G_M{}^2}{1+q^2/4m^2}\right\} \tag{10}$$

and

$$m^2q^2\frac{1}{\pi}\mathrm{Im}\,H_{3,4}=\delta\left(\frac{M_+^2}{m^2}-1\right)\times\left\{\frac{q^2}{4m^2}G_EG_M,\ \frac{(q^2/4m^2)G_M(G_M-G_E)}{1+q^2/4m^2}\right\}, \tag{11}$$

where G_E and G_M are the familiar Sachs form factors

$$G_E(q^2)=F_1(q^2)-\frac{q^2}{4m^2}F_2(q^2),\qquad G_M(q^2)=F_1(q^2)+F_2(q^2) \tag{12}$$

and $F_{1,2}$ are the electric and magnetic form factors, respectively.

The antisymmetry of H_4 in ν implies that $h_4(M_+{}^2, M_-{}^2)$ is antisymmetric, which, in turn, leads to the sum rule

$$\int dM_+{}^2 \frac{1}{\pi} \operatorname{Im} H_4 = 0, \tag{13}$$

regarding $(1/\pi)\operatorname{Im} H_4$ as the function of $M_+{}^2$ and q^2 given by Eq. (8). Explicitly exhibiting the elastic contribution given by Eq. (11), this result can be written as

$$\int_{>m^2}^{\infty} dM_+{}^2 \frac{1}{\pi} \operatorname{Im} H_4 = -\frac{G_M(G_M - G_E)}{4m^2 + q^2}. \tag{14}$$

A second sum rule can be derived for H_3 from the double-spectral representation and the low-energy theorem for Compton scattering. We consider the case of $q^2 \ll m^2$, $-Pq \ll m^2$ for which the nucleon intermediate state is dominant. A direct calculation yields

$$H_3 \simeq \frac{l(\mu_a + l)}{4m^2} \left(\frac{1}{q^2 - 2m\nu} + \frac{1}{q^2 + 2m\nu} \right) + \frac{l\,\mu_a}{8m^4}, \tag{15}$$

where l is the electric charge (in units of e) and μ_a is the anomalous magnetic moment of the nucleon. The double-spectral form, Eq. (7), can be written as

$$H_3(q^2, -m\nu) = \int d\nu' \left(\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu} \right) \times \frac{1}{\pi} \operatorname{Im} H_3(q^2, -m\nu'). \tag{16}$$

Under the prescribed conditions, the elastic contribution, Eq. (11), reproduces the first two terms of Eq. (15). The remaining term must result from larger values of ν', so that in the limit $\nu, q^2 \to 0$ we deduce the new sum rule

$$\frac{2}{\pi} \int_{\nu'>0}^{\infty} \frac{d\nu'}{\nu'} \operatorname{Im} H_3(0, -m\nu') = \frac{l\,\mu_a}{8m^4}. \tag{17}$$

Finally, we remark that the known Drell-Hearn sum rule[4] can be obtained from Eqs. (14) and (17) by using the relation between the real photo-absorption cross sections $\sigma_\pm$ (where $\pm$ refers to the cases of parallel and antiparallel photon helicity and nucleon spin) and the imaginary parts of H_3 and H_4:

$$\tfrac{1}{2}(\sigma_+ - \sigma_-) = -16\pi\alpha m^2 \left(\operatorname{Im} H_3 + \frac{\nu}{2m} \operatorname{Im} H_4 \right). \tag{18}$$

This concludes our review of Schwinger's work.

We will now verify the sum rules, Eqs. (14) and (17), for lowest-order electrodynamics. Since we are interested only in the imaginary parts of the H_a, we need only consider the casual vacuum amplitude for forward Compton scattering. In general, the causal vacuum amplitude is described by

$$\langle 0_+ | 0_- \rangle = -\alpha^2 \int \frac{(dP_1)}{(2\pi)^4} \frac{(dP_2)}{(2\pi)^4} \frac{(dk_1)}{(2\pi)^4} \frac{(dk_2)}{(2\pi)^4} \psi_1(-P_1)\gamma^0 A_1^\mu(-k_1) I_{\mu\nu} A_2^\nu(k_2)\psi_2(P_2)\, dM^2 d\omega_Q \times (2\pi)^4\delta(Q - P_1 - k_1)(2\pi)^4\delta(Q - P_2 - k_2), \tag{19}$$

where, for forward scattering ($k_1 = k_2 = q$, $P_1 = P_2 = P$, $Q = P + q$),

$$I_{\mu\nu} = (4\pi)^2 \int d\omega_p\, d\omega_k (2\pi)^3\delta(Q - p - k) \left(\gamma_\mu \frac{1}{m + \gamma Q} \gamma^\lambda + \gamma^\lambda \frac{1}{m + \gamma(p - q)} \gamma_\mu \right) \times (m - \gamma p) \left(\gamma_\lambda \frac{1}{m + \gamma Q} \gamma_\nu + \gamma_\nu \frac{1}{m + \gamma(p - q)} \gamma_\lambda \right). \tag{20}$$

The various $\operatorname{Im} H_a$ are related to $I_{\mu\nu}$ by

$$\frac{\alpha}{16} \operatorname{Tr} \left[(m - \gamma P) \frac{1 + (\gamma s) i\gamma_5}{2} I_{\mu\nu} \right] = \sum_{a=1}^{4} (\operatorname{Im} H_a) T_{a\mu\nu}. \tag{21}$$

For $H_{1,2,3}$, we are only interested in the case of $q^2 = 0$. In terms of the variable $x = \nu/m$, we find

$$\operatorname{Im} H_1 = \frac{\alpha}{4m^4} \left[\frac{x^3 - 7x^2 - 12x - 4}{x^3(2x + 1)} + \frac{1}{2} \frac{(x + 2)(3x + 2)}{x^4} \ln(2x + 1) \right], \tag{22}$$

$$\operatorname{Im} H_2 = \frac{\alpha}{4m^4}\left[\frac{x^3+9x^2+8x+2}{x^3(2x+1)^2} + \frac{1}{2}\,\frac{x^2-2x-2}{x^4}\ln(2x+1)\right], \tag{23}$$

and

$$\operatorname{Im} H_3 = \frac{\alpha}{32m^4}\left[-2\,\frac{5x+3}{x(2x+1)} + \frac{2x+3}{x^2}\ln(2x+1)\right]. \tag{24}$$

We can now check Eq. (17). We find

$$\frac{2}{\pi}\int_0^\infty \frac{dx}{x}\operatorname{Im} H_3 = \frac{\alpha}{16\pi m^4}, \tag{25}$$

which gives the known value of the anomalous magnetic moment,

$$\mu_a = \frac{\alpha}{2\pi}. \tag{26}$$

As for H_4, we will consider the sum rule, Eq. (14), for arbitrary values of q^2. To lowest order, the sum rule reduces to

$$\int_{y/2}^\infty dx \operatorname{Im} H_4 = -\frac{\pi}{8m^4}F_2(y), \tag{27}$$

where $q^2 = m^2 y$, F_2 is the magnetic form factor

$$F_2(y) = \frac{\alpha}{2\pi}\int_0^1 dv\,\frac{1}{1+\frac{1}{4}(1-v^2)y}, \tag{28}$$

and

$$\operatorname{Im} H_4 = -\frac{\alpha}{16m^4}\left\{\frac{4}{(1+y)(1+2x-y)^2} - \frac{3y[x(3+2y)+(2+3y)]}{(1+y)(x^2+y)^2} - \frac{1}{(1+y)(x^2+y)} - \frac{1+y^2}{(1+y)(1+2x-y)(x^2+y)} + \left(1+2y+\tfrac{3}{2}y\,\frac{5x-2y+2}{x^2+y}\right)\frac{1}{(x^2+y)^{3/2}}\ln\xi\right\}, \tag{29}$$

with

$$\xi = \frac{x+1+(x^2+y)^{1/2}}{x+1-(x^2+y)^{1/2}}. \tag{30}$$

By explicit integration of $\operatorname{Im} H_4$, Eq. (27) is verified.[5] As we noted above, the Drell-Hearn sum rule[4,6] (to the order considered)

$$\int_0^\infty \frac{dx}{x}\left[x \operatorname{Im} H_4 + 2\operatorname{Im} H_3\right](q^2=0) = 0 \tag{31}$$

is a direct consequence of Eq. (25) and the $y=0$ form of Eq. (27).

Two final points deserve comment. In terms of cross sections, $\operatorname{Im} H_3$ is determined by means of an interference between longitudinal and transverse polarizations.[1] As such, it satisfies an inequality which, for $q^2=0$, is

$$x^2 \operatorname{Im} H_1 \operatorname{Im} H_2 \geq 4(\operatorname{Im} H_3)^2. \tag{32}$$

It is easy to see that this is satisfied for $x \sim 0$ and for $x \to \infty$. Numerically, one can verify that the inequality is very easily satisfied for any other value of x.

It is interesting to investigate the behavior of the polarization asymmetry,

$$P(\nu, q^2) = \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-}, \tag{33}$$

in the deep-inelastic region, as a function of $\omega = 2m\nu/q^2$. Schwinger[1] has conjectured that P ranges from -1 for ω near unity to $+1$ for ω large, with a single sign change. What is the situation in pure electrodynamics? With the information at hand we can make only the following immediate remarks. In the elastic region, where $\omega = 1$, it is evident indeed from Eqs. (10) and (11) that $P = -1$. In the inelastic region, we can use Eqs. (23), (24), and (29), at $q^2=0$, to compute P in the two limits $\nu/m \to 0$ and $\nu/m \to \infty$. The results are, respectively, $P(0,0)=0$ and $P(\infty,0) = -1$. To calculate P in the deep-inelastic region we would have to extend our calculations to arbitrary values of ν and q^2 for all the $\operatorname{Im} H_a$. This is just one example of the interesting questions which deserve further study in this area.

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[1]J. Schwinger, Proc. Natl. Acad. Sci. USA 72, 1559 (1975).
[2]J. Schwinger, Proc. Natl. Acad. Sci. USA 72, 1 (1975).

[3]In general, double-spectral forms are accompanied by single-spectral forms (SSF). The assumption that SSF do not appear is suggested by the experimental observation of the dipole fit for the elastic electromagnetic form

factors. (See Ref. 2.) There is as yet no proof that this representation is valid in electrodynamics, nor is there an alternative derivation of the sum rule (13). The fact that the sum rule holds in electrodynamics leads us to suspect that the spectral representation (7) is valid there. [In particular, it is interesting to ask whether Eq. (29) for $\mathrm{Im}H_4$ can be recast in the form of Eq. (8).]

[4]S. D. Drell and A. C. Hearn, Phys. Rev. Lett. 16, 908 (1966). The Drell-Hearn sum rule states that

$$\int_0^\infty \frac{d\nu}{\nu}[\sigma_+(\nu)-\sigma_-(\nu)]=\frac{2\pi^2\alpha}{m^2}\mu_a{}^2.$$

[5]For the $\mathrm{Im}H_4$ term, we first integrate by parts on the $\ln\xi$ part. The lower limit of the integrated term yields the correct result. The sum of what remains then integrates to zero.

[6]This sum rule may also be directly verified from the invariant amplitudes for real photon Compton scattering presented in papers by K. A. Milton, Wu-yang Tsai, and L. L. DeRaad, Jr. [Phys. Rev. D 6, 1411 (1972); 6, 1428 (1972)]. To this order, using the notation presented there, the Drell-Hearn sum rule becomes the easily verified statement

$$\int_0^\infty d\nu\, \mathrm{Im}M_6(\nu)=0.$$