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Scattering of one-dimensional bags in the interacting string formalism*

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We apply the interacting string formalism of Mandelstam to the scattering problem of one-dimensional bags. Lorentz covariance together with general properties of the path-integral formalism require that the number of constituent fields be 24 and that the $(\text{mass})^2, -4\pi B\alpha_0$, of the ground state be $-4\pi B$. In this case, the n-point scattering amplitudes are precisely those of the dual resonance model. However, we also write the four-point scattering amplitudes for arbitrary α_0. Then these still possess crossing symmetry but are not Lorentz covariant unless $\alpha_0 = 1$.

I. INTRODUCTION

In this article we wish to study the possibility of introducing interactions into the quark bag theory[1] along the same lines as in the string model as carried out by Mandelstam.[2] In the full three-dimensional theory the exact quantum treatment of the surface motion is probably intractable. The one-dimensional theory, on the other hand, has the virtue that the boundary motion is exactly soluble in light-cone coordinates. Since the method is based on exact knowledge of the spectrum in the noninteracting limit, we restrict our study to the one-dimensional case. As this restriction makes the study academic, we also make the simplifying assumption of scalar constituents.

The formulation is most easily presented in the language of Feynman path integrals. Classically, the possibility of fission corresponds to a motion in which the initial configuration is a single connected bag and the final configuration is two or more bags.[3] At the classical level whenever the conditions for fission are met (that is, the suitable boundary conditions are satisfied at an interior point), there are two possibilities: (a) The bag actually moves as two bags after that instant, or (b) the bag does not fission and the motion after that instant is that of a single bag. Both possibilities obey the classical equations, the first being distinguished from the second by a singularity at the fission point. Quantum mechanically, these two alternatives can be assigned different probability amplitudes, which are the functional integrals of $e^{i\,[\text{action}]}$ over the two topologically distinct domains. The relative measure of these two functional integrals is infinite, but this infinity can be absorbed into an overall coupling constant λ (which is then arbitrary) multiplying the amplitude for fission. The closure property of functional integrals together with the reality of the coupling ensure that S-matrix elements satisfy perturbative unitarity in λ. It will become clear that there is a very close analogy between the one-dimensional bag with g fields and a string embedded in $g+1$ dimensions. In the formulation of Goldstone, Goddard, Rebbi, and Thorn[4] (GGRT) the latter system is characterized (in light-cone variables) by g transverse coordinates $x^i(\sigma,\tau)$, and $x^-(\sigma,\tau)$ is a function of these. In the one-dimensional bag, the end points x_i^- are a very similar function of the g fields $\phi^a(\sigma,\tau)$. One of the difficulties with the string model is that in other than 25 space dimensions the Lorentz group cannot be represented by the usual quantum generators. That is, $[M^{i-},M^{j-}]\neq 0$ unless $d=25$ and α_0, the intercept of the leading trajectory, is unity. In the one-dimensional bag theory there are no transverse Lorentz generators so this is not a problem. In a sense, the Lorentz group is too trivial in one dimension to yield restrictions.

To investigate the self-consistency of interactions in the one-dimensional bag model we must consider the scattering amplitudes in more detail. It has already been remarked that the string formalism can be taken over almost intact to this problem. The fact that for the bag $\phi=0$ at the ends whereas for the string $\partial x^i/\partial\sigma = 0$ has no effect on the three-point function derived by Mandelstam[2]

$$V_3 = \left[\frac{1}{(2P_1^+ 2P_2^+ 2P_3^+)^{1/2}}\right]^{g/24} \langle 0|\mathcal{V}|1\rangle|2\rangle|3\rangle, \quad (1.1)$$

where $\mathcal{V}$ is Mandelstam's expression with all transverse momenta $=0$. The appearance of the power $g/24$ is essential to maintain the closure property of the functional integral. (This factor arises solely from the measure.) For three ground states, $\langle 0|\mathcal{V}|1\rangle|2\rangle|3\rangle = \text{const} = \lambda$. It is therefore obvious that the three-point function is not covariant except possibly when $g=24$, and in that case, we know from string results that the amplitudes are just those of the dual resonance model.

In the bag theory we shall find that consistency demands $\alpha_0 = g/24$ as well as $g=24$. We find it attractive to separate these two conditions. It is clearly possible to define functional integrals for any value of g. However, α_0 is fixed in terms of g by a general feature of functional integrals: a continuous dependence of the functional integral on the domain on which the functions integrated are defined. This same property demands the power $g/24$ in (1.1). It is then *Lorentz* covariance which demands that this power, $g/24$, be unity. It is amusing to observe that even for $g \neq 24$, the amplitudes possess a kind of crossing symmetry or "duality."[5] The advantage of bag theories is that the degrees of freedom giving rise to the excitation spectrum of the hadron are fields, not coordinates, so the restrictions are on the number of fields *not* on the dimension of space-time.

Finally we wish to emphasize an assumption made in this approach. We assume a limit in which bags do not interact (even when they overlap in space). Thus, classically, two bags can occupy the same space-time region and be independent of each other. The interactions occur only at the boundary of the bag and at a single point in the interior of the bag (the interchange interaction).

In the remainder of this paper we shall, as a pedagogical example, work out the four-point function in detail and show how the condition $\alpha_0 = g/24$ emerges. When this condition is met, dramatic simplifications occur. The resulting amplitudes are in general not those of the dual model, but we shall show that they nonetheless possess crossing symmetry.

II. THE THREE-BAG VERTEX FUNCTION

A bag containing a set of scalar fields is described by an action

$$W = \int_R d^2x(\phi_a \partial_\mu \partial^\mu \phi_a + \tfrac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - B), \qquad (2.1)$$

where the index a labels the field component. The variation of this action yields the equations of motion

$$\Box \phi_a = 0 \text{ (inside)}, \qquad (2.2)$$

$$\phi_a = 0 \text{ (on the boundary)}, \qquad (2.3)$$

$$(\partial_\mu \phi_a)^2 = 2B \text{ (on the boundary)}. \qquad (2.4)$$

Equation (2.4) can be viewed as a constraint, implicitly defining the boundary surface in terms of the independent variables ϕ_a. It is analogous to the equation of constraint used to eliminate the longitudinal degrees of freedom in the string theory.[4] The form of field equation (2.2) enables us to use the scattering formalism of Mandelstam in treating bag interactions.[2] The boundary condition (2.3) leads to a different choice of the Neumann function. We find [compare with Eq. (2.9) of Ref. 2]

$$N(z,z') = \ln|z-z'| - \ln|z^* - z'|. \qquad (2.5)$$

This choice of Green's function in the functional integral and the convention in the definition of momenta p_ϕ result in the modification of the vertex function.[6] Instead of Eq. (6.15) of Ref. 2 the bag three-point function (Fig. 1) is

$$\langle 1,2|\mathcal{V}|3\rangle = \left[\frac{1}{(2P_1^+ 2P_2^+ 2P_3^+)^{1/2}}\right]^{g/24} [\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}(-\alpha_3)^{\alpha_3}]^{\sum_{r=1}^3 P_r^-} \left\langle 0 \right| \exp\left(-\tfrac{1}{2}\sum_{r,s}\sum_{m,n=1}^{\infty} \tilde{N}_{mn\,rs}\, a_{mr}^i a_{ns}^i\right) \prod_r \left| r \right\rangle . \qquad (2.6)$$

The coefficients $\tilde{N}_{mn\,rs}$ are, except for the sign, equal to the ones for the scattering of strings and are given by

$$\tilde{N}_{mn\,rs} = +\frac{\alpha_1\alpha_2\alpha_3}{\alpha_r\alpha_s}\frac{nm}{m\alpha_s + n\alpha_r} \times a_m\left(-\frac{\alpha_{r+1}}{\alpha_r}\right) a_n\left(-\frac{\alpha_{s+1}}{\alpha_s}\right),$$

with

$$a_n(j) = \frac{1}{n}\binom{nj-1}{n}. \qquad (2.7)$$

The two sign changes[7] just cancel and so the bag vertex is the same as the string one with all transverse momenta $=0$. Knowing the three-point function we can invoke the closure property to obtain four-point functions.

III. THE SCATTERING AMPLITUDE FOR FOUR GROUND-STATE BAGS

In this section we shall evaluate the four-point amplitudes for the scattering of unexcited states. The light-cone variables $P_\pm$ are used and our metric is such that $P^2 = 2P_+P_- = m^2$. Also

$$m^2 = -\alpha_0 + R \equiv -\alpha_0 + \sum_{n=1}^{\infty} a_n^{\dagger i} a_n^i, \qquad (3.1)$$

where $-\alpha_0$ is the (mass)^2 of the ground state. The scattering amplitude having poles in a particular pair of channels and constructed by closure has the general form

$$A=\int_0^\infty d\tau \sum_c e^{(P_{\bar{a}}-P_{\bar{c}})\tau}\langle a|\mathcal{V}|c\rangle\langle c|\mathcal{V}|b\rangle, \quad (3.2)$$

where $|a\rangle$ and $|b\rangle$ are the outgoing and incoming two-particle states, $|c\rangle$ is an arbitrary single-particle state, and $\mathcal{V}$ is the vertex function. The variable τ is the difference between the times at which the bags join and separate. As will be shown later, the amplitude (3.2) is a function of α_0 and g (the number of fields inside the bag). The dependence on α_0 can be determined explicitly and is independent of the measure. The g-dependent part of (3.2) can be obtained in the following way: The measure for the vertex (2.6) is known and fixed, and it was shown by Mandelstam that for $g=24$ and $\alpha_0=1$ the use of (2.6) in (3.2) leads to the appearance of the Veneziano β-function integrand.[2] We can then let g and α_0 take on these particular values, compare (3.2) with the appropriate β-function integrand, and thus obtain the four-point function for any value of g and α_0. We will illustrate the above discussion by considering in more detail the contribution with s-channel poles to the scattering amplitude $A(s,t)$. Other contributions are computed analogously and thus only the results will be given.

The graph with s-channel poles contributing to the scattering amplitude $A(s,t)$ is shown in Fig. 2. We label particles as shown and define η by

$$P_{+1}\equiv 2\pi\alpha_1=(1-\eta)P_+, \quad P_{+2}\equiv 2\pi\alpha_2=\eta P_+, \quad 0<\eta<1. \tag{3.3}$$

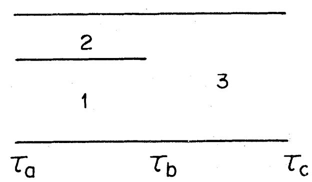

FIG. 1. The three-point function for joining of two bags.

We have for the external particles

$$P_{-r}=\frac{m^2}{2P_{+r}}=-\frac{\alpha_0}{2P_{+r}}. \tag{3.4}$$

Further, from momentum conservation laws $P_{\pm 1}+P_{\pm 2}=P_{\pm 4}+P_{\pm 5}$ and (3.3) we obtain $P_{+4}=\eta P_+$ and $P_{+5}=(1-\eta)P_+$ or vice versa only. We choose $P_{+4}=(1-\eta)P_+$ and $P_{+5}=\eta P_+$ and define

$$s\equiv(P_1+P_2)^2=2(P_{+1}+P_{+2})(P_{-1}+P_{-2}) = -\frac{\alpha_0}{\eta(1-\eta)}, \tag{3.5a}$$

$$t\equiv(P_1-P_5)^2=\frac{(1-2\eta)^2\alpha_0}{\eta(1-\eta)}, \tag{3.5b}$$

$$u\equiv(P_1-P_4)^2=0. \tag{3.5c}$$

The four-point function for elastic scattering of unexcited particles shown in Fig. 2 is given by (3.2). Using (2.6), (3.3), (3.4), and removing the complete set of states $\sum_c|c\rangle\langle c|$ we obtain for (3.2)

$$\int_0^\infty d\tau\, e^{-(\tau\alpha_0/2P_+)[1/\eta(1-\eta)-1]}\eta^{-2\alpha_0[1/(1-\eta)-\eta]}(1-\eta)^{-2\alpha_0(1/\eta-1+\eta)}$$
$$\times\left\{\left\langle 0\left|\exp\left(\sum_{m,n}\tfrac{1}{2}\tilde{N}_{mn}a_m a_n\right)\exp\left(-\frac{\tau}{2P_+}\sum_n a_n^\dagger a_n\right)[\eta^\eta(1-\eta)^{(1-\eta)}]^{-2\sum n a_n^\dagger a_n}\exp\left(\sum_{m,n}\tfrac{1}{2}\tilde{N}^\dagger_{mn}a_n^\dagger a_m^\dagger\right)\right|0\right\rangle\right\}^g[2P_{+3}]^{-g/24}, \tag{3.6}$$

where the summation on i appearing in (2.6) has been replaced by the power g, where $\tilde{N}_{mn}=-\tilde{N}_{mncc}$, and where the external factors $[(2P_{+i})^{1/2}]^{-g/24}$ were omitted for ease of writing. Defining for convenience

$$y=\eta^{-2\eta}(1-\eta)^{-2(1-\eta)}e^{-\tau/2P_+}, \tag{3.7}$$

(3.6) can be rewritten as

$$\int_0^\infty d\tau\, y^{-\alpha_0[1-1/\eta(1-\eta)]}K_y^g[2P_{+3}]^{-g/24}, \tag{3.8}$$

where

$$K_y=\left\langle 0\left|\exp\left(\sum_{m,n=1}^\infty \tfrac{1}{2}\tilde{N}_{mn}a_m a_m\right)y^{\sum_{n=1}^\infty a_n^\dagger a_n^\dagger}\exp\left(\sum_{m,n=1}^\infty\tfrac{1}{2}\tilde{N}^\dagger_{mn}a_n^\dagger a_m^\dagger\right)\right|0\right\rangle. \tag{3.9}$$

We now transform the integration variable, τ, in (3.8) so that the integrand resembles that of the dual resonance model. This transformation is discussed by Mandelstam[2] and is given by

$$-\tau = 2\eta P_+ \ln\left|\frac{2\eta-1+[1-4\eta x(1-\eta)]^{1/2}}{2\eta-1-[1-4\eta x(1-\eta)]^{1/2}}\,\frac{1-[1-4\eta x(1-\eta)]^{1/2}}{1+[1-4\eta x(1-\eta)]^{1/2}}\right| + 2(\eta-1)P_+ \ln\left|\frac{2x(\eta-1)+1+[1-4\eta x(1-\eta)]^{1/2}}{2x(\eta-1)+1-[1-4\eta x(1-\eta)]^{1/2}}\right|, \tag{3.10}$$

where x is the new integration variable. Using (3.10) one obtains after a short calculation

$$\frac{\partial\tau}{\partial x} = -\frac{2P_+[1-4\eta x(1-\eta)]^{1/2}}{x(1-x)} \tag{3.11}$$

and

$$y = x(1-x)\{\tfrac{1}{2}-\eta x+\tfrac{1}{2}[1-4\eta x(1-\eta)]^{1/2}\}^{-2\eta}[\eta \leftrightarrow (1-\eta)]^{(\eta\to 1-\eta)}. \tag{3.12}$$

Thus now we can determine the g dependence by requiring that upon substitution of (3.11) and (3.12) into (3.8) and for $\alpha_0 = 1$, $g = 24$ the amplitude reduces to an integral over the Veneziano β-function integrand $x^{-s-2}(1-x)^{-t-2}$. Identifying K_y from the resulting equation we obtain

$$K_y = (1-x)^{(1/12)[2-1/\eta(1-\eta)]}\{\tfrac{1}{2}-\eta x+\tfrac{1}{2}[1-4\eta x(1-\eta)]^{1/2}\}^{(1/12)[1/(1-\eta)-\eta]}[\eta \leftrightarrow (1-\eta)]^{(\eta\to 1-\eta)} \times \{[1-4\eta x(1-\eta)]^{1/2}\}^{-1/24}. \tag{3.13}$$

We can now write the amplitude for arbitrary α_0 and g as

$$A_s = \int dx\{2P_+[1-4\eta x(1-\eta)]^{1/2}\}^{1-g/24}x^{-1-\alpha_0-s}(1-x)^{-1-\alpha_0-s+g/6-g/12\eta(1-\eta)} \times\{\tfrac{1}{2}-x\eta+\tfrac{1}{2}[1-4\eta x(1-\eta)]^{1/2}\}^{2\eta(\alpha_0-g/24)[1-1/\eta(1-\eta)]}[\eta \leftrightarrow 1-\eta]^{(\eta\to 1-\eta)}. \tag{3.14}$$

By analyzing the region of integration in the x variable it can be determined that the amplitude of Fig. 2 does not completely cover the range $\{0, 1\} = \{x\}$. However, the whole set is covered by including in $A(s, t)$ the amplitude having t-channel poles, shown in Fig. 3. Proceeding as in the previous case we obtain for the analog of (3.8)

$$\int_0^\infty d\tau\, y^{-\alpha_0[1/(1-2\eta)-1/\eta(1-\eta)]}\eta^{4\alpha_0}(1-\eta)^{4\alpha_0(1-\eta)/\eta} \times\left\{\left\langle 0\left|\exp\left(\sum_{mn}\tfrac{1}{2}\tilde{N}_{mn}a_m a_n\right)y^{[1/(1-2\eta)]\Sigma_n a_n^\dagger a_n}\exp\left(\sum_{mn}\tfrac{1}{2}\tilde{N}_{mn}^\dagger a_n^\dagger a_m^\dagger\right)\right|0\right\rangle\right\}^g[2P_{+3}]^{-g/24}$$

$$\equiv \int_0^\infty d\tau\, y^{-\alpha_0[1/(1-2\eta)-1/\eta(1-\eta)]}\eta^{4\alpha_0}(1-\eta)^{4\alpha_0(1-\eta)/\eta}G_y^g[2P_{+3}]^{-g/24}, \tag{3.8$'$}$$

finally yielding

$$A_t = \int dx\{2P_+[1-4\eta x(1-\eta)]^{1/2}\}^{1-g/24}x^{-1-\alpha_0-s}(1-x)^{-1-\alpha_0-s+g/6-g/12\eta(1-\eta)} \times\{\tfrac{1}{2}-x\eta+\tfrac{1}{2}[1-4\eta x(1-\eta)]^{1/2}\}^{2\eta(\alpha_0-g/24)[1-1/\eta(1-\eta)]}[\eta \leftrightarrow 1-\eta]^{(\eta\to 1-\eta)}\eta^{4\alpha_0-g/6}(1-\eta)^{(4\alpha_0-g/6)[(1-\eta)/\eta]}. \tag{3.14$'$}$$

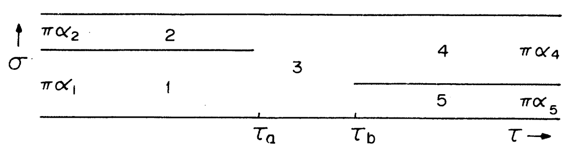

FIG. 2. The graph with s-channel poles contributing to the scattering amplitude $A(s,t)$.

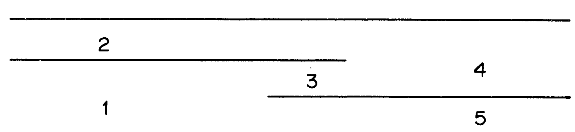

FIG. 3. The graph with t-channel poles contributing to the scattering amplitude $A(s,t)$.

The complete amplitude is given by $A(s,t)=A_s+A_t$; however, comparing (3.14) and (3.14′) we note that the two integrands are in general not continuous at $\tau=0$. This property will hold only when

$$\alpha_0=\frac{g}{24}\,. \tag{3.15}$$

This requirement of continuity is merely the statement that if functional integration has any meaning at all, Fig. 2 should equal Fig. 3 when $\tau=0$. With (3.15) satisfied the amplitude $A(s,t)$ becomes not only well behaved but also much simplified, resulting finally in

$$A(s,t)=\int_0^1 dx\left[2P_+\left(1+\frac{4\alpha_0 x}{s}\right)^{1/2}\right]^{1-\alpha_0} x^{-1-\alpha_0-s}\times(1-x)^{-1-\alpha_0-t}\,. \tag{3.16}$$

The complete scattering amplitude is of course given by $A=A(s,t)+A(s,u)+A(t,u)$. The amplitudes $A(s,u)$, shown in Fig. 4, and $A(t,u)$, shown in Fig. 5, can be computed analogously. For the t,u amplitude a direct four-bag coupling is also needed.[9] Defining for convenience in the case of s-channel scattering $2P_{+s}\equiv 2P_{+3}=2P_+$ and for t-channel scattering $2P_{+t}\equiv 2P_{+3}=(1-2\eta)2P_+$ and noting

$$(1-2\eta)^2=-\frac{t}{s}\,, \tag{3.17}$$

we can write the resulting scattering amplitudes as

$$A(s,u)=A(s,0)=\int_0^1 dx\left\{2P_{+s}\left[(1-x)^2-\frac{4\alpha_0(1-x)x}{s}\right]^{1/2}\right\}^{1-\alpha_0}\times x^{-1-\alpha_0-s}(1-x)^{-1-\alpha_0} \tag{3.18}$$

and

$$A(t,u)=A(t,0)=\int_0^1 dx\left\{2P_{+t}\left[(1-x)^2-\frac{4\alpha_0(1-x)x}{t}\right]^{1/2}\right\}^{1-\alpha_0}\times x^{-1-\alpha_0-t}(1-x)^{-1-\alpha_0}\,. \tag{3.19}$$

IV. DISCUSSION AND SUMMARY

The scattering amplitudes obtained in Sec. III are reminiscent of dual models. For a particular value of α_0 ($\alpha_0=1$) they become the Veneziano β functions.[10] The complete four-point scattering amplitude, however, possesses crossing symmetry, i.e., has the same functional form in the s as well as the t variables, for arbitrary value of α_0. To verify this statement it is sufficient to show that the functional form of the square bracket in (3.16) remains unchanged (up to the substitution $x\leftrightarrow 1-x$) when $s\leftrightarrow t$. Using (3.17) in (3.16) we easily obtain

$$2P_{+s}\left(1+\frac{4\alpha_0 x}{s}\right)^{1/2}=2P_{+t}\left[1+\frac{4\alpha_0(1-x)}{t}\right]^{1/2}, \tag{4.1}$$

which is the required relation.

It is not surprising that the one-dimensional scalar bag leads to string results when $\alpha_0=g/24=1$. It is interesting that, although Lorentz covariance does not restrict α_0 and g for noninteracting bags, the usual restrictions arise when interactions are introduced. A by-product of our work is one more illustration of how a quantization via functional integrals leads to the same consistency conditions as operator quantization. In operator quantization the consistency conditions arise from the requirements on the commutation relations of dynamical variables. In path-history

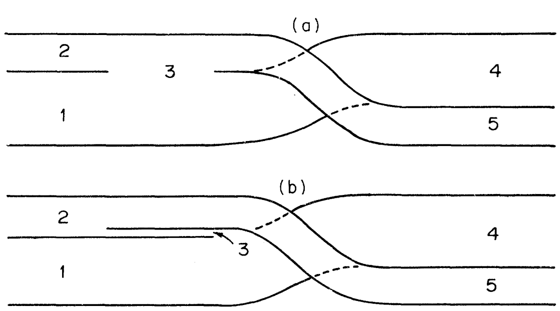

FIG. 4. (a) The graph with s-channel poles contributing to the scattering amplitude $A(s,u)$. (b) The graph with u-channel poles contributing to the scattering amplitude $A(s,u)$. In the limit $P_3^+\to 0$ its contribution vanishes.

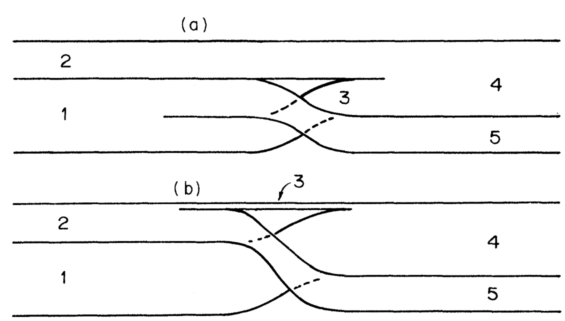

FIG. 5. (a) The graph with t-channel poles contributing to the scattering amplitude $A(t,u)$. (b) The graph with u-channel poles contributing to the scattering $A(t,u)$. In the limit $P_3^+\to 0$ its contribution vanishes.

quantization the conditions arise from the fundamental geometrical properties of closure and a continuous dependence of the functional integral on the shape of the domain of definition of the integrated functions.

Finally we should emphasize that the system studied here is highly unrealistic. Not only have we restricted ourselves to one space dimension, but we have also ignored spin and interactions among the constituents. Low has argued[11] that a reasonable model of the Pomeron in bag-like models can only be achieved if the constituents interact. The scattering mechanism described in the present article can at best describe nondiffractive scattering. Also it is not clear that in three dimensions it makes sense to formulate interactions among bags in the perturbative way described here: The essential difference is that in three dimensions fission can proceed via a continuous deformation of the surface so that the measure for interacting diagrams may be uniquely prescribed in terms of that for noninteracting diagrams.

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[3]These classical motions have been discussed by T. T. Wu, B. McCoy, and H. Cheng, Phys. Rev. D 9, 3495 (1974).

[4]J. Goldstone, P. Goddard, C. Rebbi, and C. B. Thorn, Nucl. Phys. B56, 109 (1973).

[5]This has also been suggested by Brink and Nielsen, who find $g/24=\alpha_0$ but not $g=24$ by a duality argument: L. Brink and H. B. Nielsen, Phys. Lett. 43B, 319 (1973). They also arrive at $g/24=\alpha_0$ from considerations of the quantum zero-point energy of the dual string: L. Brink and H. B. Nielsen, Phys. Lett. 45B, 332 (1973).

[6]The notation used is defined and explained in Ref. 2. We select units so that $4\pi B=1$.

[7]The momenta conjugate to ϕ are defined in such a way that the conformal generators L_n for bags are identical to the ones for strings. This definition leads to the appearance of a minus sign in the exponential of the vertex function.

[8]Figures 4(b) and 5(b) are drawn for completeness and are to be understood in the limit as $P_{+3}=0$. However, in this limit they do not contribute to the scattering amplitudes.

[9]The analysis of Kaku and Kikkawa goes through for bags as well. The direct four-bag coupling is an analytic continuation in the σ direction of the four-point function built by closure from two three-point functions (at least in the case $\alpha_0=1$; it has not been discussed for other values of α_0). The analysis of this question for strings is given in M. Kaku and K. Kikkawa, Phys. Rev. D 10, 1110 (1974).

[10]In this case the amplitudes (3.18) and (3.19) are separately not well defined; however, their sum $A(s,u)+A(t,u)$ is. Internal symmetries can therefore not be accommodated when $\alpha_0=1$.

[11]F. E. Low, Phys. Rev. D (to be published).