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Spin structure of dual-resonance spectra*

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A study is made of certain aspects of the spin structure of dual-resonance spectra. Generating functions and recurrence relations are derived which give the multiplicities of internal physical states of definite spin and parity in the conventional dual resonance model, the model of Neveu and Schwarz, and that of Ramond. Certain regularities common to these models are noted.

I. INTRODUCTION

In this article a method is developed for finding the number of resonances of given parity, spin, and excitation number in the internal spectra of dual resonance models. The method can be applied to any model which possesses an operator formalism, and, in fact, is applied to the conventional dual resonance model with intercept 1 (Ref. 1) as well as to the models of Neveu and Schwarz[2] and of Ramond.[3]

It is a first step toward a better understanding of the spin structure in general, or rather, if one wants to break the bounds of four-dimensional space-time, of the structure of little-group representations. We shall, however, restrict ourselves to four-dimensional space-time.

Although all calculations could be done by conventional counting procedures, these methods are tedious and have sometimes led to wrong results. On the contrary, our procedure is very simple, mathematically elegant, and useful insofar as one wants to get an insight into the properties of the spectra at higher excitations.

It can be rigorously proved, in all models considered, that in order to count the "physical states" of a particular spin l, parity P, and excitation level N, it is sufficient to count the number of states with the same quantum numbers l, P, and N, which can be obtained by just applying the usual spacelike creation operators to the vacuum state. For counting purposes it is immaterial whether the vacuum state represents a particle at rest or not, except for the ground state (which is a scalar or pseudoscalar anyway) and the first excited states (which are lightlike and need a separate treatment). The "physical states" include certain zero-norm states which are not coupled to the physical system and must be removed. This can be done easily, as is well known. A proof of the above statement is given in the Appendix for the conventional dual model.

Once one has convinced oneself of the truth of the statement (which would probably not come as a surprise anyway), one has to develop a systematic way of counting, which is greatly helped by deriving generating functions. In contrast to the extremely complicated structure of the spin eigenstates themselves, these generating functions turn out to be simply related to θ functions, which also play a role elsewhere in dual theory. These generating functions then lead to simple recursion relations, enabling one to find the asked-for multiplicities directly.

In Sec. II we derive these generating functions for the conventional dual model, and the Neveu-Schwarz (NS) and Ramond models, while in Sec. III recursion relations are derived and results are given. Finally a general discussion follows in Sec. IV.

II. GENERATING FUNCTIONS

A. The conventional model

Let us first turn our attention to the conventional dual resonance model with intercept 1.[1] Let $|\psi\rangle$ be an eigenstate of four-momentum with eigenvalue p_0^μ. The total number $T^3(N)$ of independent states $|\psi\rangle$ at the Nth excitation level which satisfy the gauge conditions

$$L_n|\psi\rangle = 0 \quad (n \geq 1) \tag{2.1}$$

and the mass-shell condition

$$(L_0 - 1)|\psi\rangle = 0 \tag{2.2}$$

is given by the generating function

$$f^3(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-3} = \sum_{N=0}^{\infty} T^3(N) x^N . \tag{2.3}$$

So $T^3(N)$ is the number of states that can be constructed using three-dimensional operators instead of the usual four-dimensional set.

The momentum eigenstates satisfying (2.1) and (2.2) will from now on be denoted by $|\psi\rangle$. They include zero-norm states with which we deal separately. The states $|\psi\rangle$ with nonzero norm shall be called physical.

For the sake of argument, let us confine ourselves to states representing particles or resonances at rest. We have to deal with the ground and first excited states separately because of their tachyonic and lightlike character.

For counting purposes we can now make use of a theorem, proved in the Appendix, which says that there exists a complete set of orthonormal states $|\psi\rangle$, labeled as

$$|i_1^{(1)},\ldots,i_{n_1}^{(1)},i_1^{(2)},\ldots,i_{n_2}^{(2)},\ldots\rangle,\quad i_1^{(1)},\ldots=1,2,3 \tag{2.4}$$

with the following properties:

(a) The symmetry and the transformation properties under space rotations and reflections are the same as those of the corresponding, not necessarily physical states

$$\mathfrak{N}\, a_1^{\dagger i_1^{(1)}}\cdots a_1^{\dagger i_{n_1}^{(1)}}\, a_2^{\dagger i_1^{(2)}}\cdots a_2^{\dagger i_{n_2}^{(2)}}\cdots|\rangle\ , \tag{2.5}$$

where $|\rangle$ is a "vacuum" state, behaving as a scalar under rotations, $\mathfrak{N}$ is a normalization constant, and the $a^\dagger$ are creation operators satisfying the usual commutation relations.

(b) There exists a one-to-one relation between the states (2.4) and (2.5).

(c) The excitation level N is equal to $\sum_{i=1}^{\infty} i n_i$.

This would be a trivial statement if every state of the form (2.5), provided it has the right momentum eigenvalue, could uniquely be supplemented with a term with the same tensor character as (2.5) to form a state $|\psi\rangle$. This is probably true but hard to prove in general. Anyway, it is not a necessary condition. The theorem enables one to deal exclusively with states of the form (2.5) for the purposes of this article. The construction of the generating function for the multiplicities of states of definite N and l is done in three steps:

(i) At the Nth level, characterized by $\langle L_0 - p^2\rangle = N$, the number P_N of possible configurations $\{n_i;\sum i n_i = N\}$ is given by

$$F(x)=\prod_{n=1}^{\infty}\frac{1}{1-x^n} = \sum_{N=0}^{\infty} P_N x^N\ . \tag{2.6}$$

(ii) The states (2.5) are symmetric under the exchange of the indices $i_1^{(k)}\cdots i_{n_k}^{(k)}$ for each k. The reduction into a product of SO(3) irreducible representations (irreps) can therefore immediately be carried out. If the number of ways a spin-l representation is contained in the totally symmetric combination of n spin-1 objects is $K_n(l)$, the generating function is

$$g(x,n)=\frac{x^{[1-(-)^n]/2}-x^{n+2}}{1-x^2} = \sum_{l=0}^{\infty} K_n(l)x^l\ . \tag{2.7}$$

Each oscillator's contribution is of this form. In this way we obtain a sum of product tensors of the form

$$T_{m_1}^{(l_1)}(1)\,T_{m_2}^{(l_2)}(2)\cdots T_{m_q}^{(l_q)}(q)\ , \tag{2.8}$$

where q is the number of nonvanishing numbers n_k. The l_i are angular momentum quantum numbers, and the m_i the corresponding magnetic quantum numbers.

(iii) Products of the form (2.8) must now be rewritten in terms of eigenstates of total angular momentum. Let $S^l_{\{l_i\}} = S^l_{(l_1,\ldots,l_q)}$ be the number of tensors with spin l obtained by multiplying distinguishable representations of spins $l_1,\ldots,l_q$. If we define

$$S^{-l-1}_{\{l_i\}} = -S^l_{\{l_i\}} \tag{2.9}$$

then we have

$$S^l_{(l_1,\ldots,l_q)} = \sum_{l'=|l-l_q|}^{l+l_q} S^{l'}_{(l_1,\ldots,l_{q-1})} = \sum_{l'=l-l_q}^{l+l_q} S^{l'}_{(l_1,\ldots,l_{q-1})}\ . \tag{2.10}$$

If now the generating function $f_{\{l_i\}}$ is defined by

$$f_{\{l_i\}}(x)=\sum_{l=0}^{\infty} S^l_{\{l_i\}}\,\frac{x^{l+1}-x^{-l}}{x-1} \tag{2.11}$$

then one finds immediately by repeated use of (2.10)

$$f_{\{l_i\}}(x)=\prod_{i=1}^{\infty}\frac{x^{l_i+1}-x^{-l_i}}{x-1}\ . \tag{2.12}$$

Now the three steps should be combined. We try to find a function $G(x,y)$ which generates the number $M_l^{(N)}$ of tensors with spin l and excitation level N:

$$G(x,y)=\sum_{N=0}^{\infty}\sum_{l=0}^{\infty} M_l^{(N)} x^N\,\frac{y^{l+1}-y^{-l}}{y-1}\ . \tag{2.13}$$

Let $M_l^{\{n_i\}}$ be the number of tensors of spin l which appear when a tensor of the form (2.5) with the configuration $\{n_i\}$ is reduced. Then we have

$$G(x,y)=\sum_{N=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\{n_i\}}\delta_{N,\Sigma i n_i} M_l^{\{n_i\}}x^N\frac{y^{l+1}-y^{-l}}{y-1}$$
$$=\sum_{\{n_i\}}\sum_{l=0}^{\infty}\sum_{\{l_i\}}\left[\prod_{i=1}^{\infty}K_{n_i}(l_i)\right]S^l_{\{l_i\}}\times x^{\Sigma i n_i}\,\frac{y^{l+1}-y^{-l}}{y-1}\ .$$

Interchanging $\sum_l$ and $\sum_{\{l_i\}}$ and using (2.11) and

(2.12) we obtain

$$G(x,y)=\sum_{\{n_i\}} x^{\Sigma i n_i}\prod_{i=1}^{\infty}\sum_{l_i} K_{n_i}(l_i)\frac{y^{l_i+1}-y^{-l_i}}{y-1} .$$

Now making use of (2.7) we find

$$G(x,y)=\prod_{i=1}^{\infty}\frac{1}{(1-x^i)(1-yx^i)[1-(1/y)x^i]} , \tag{2.14}$$

which is the generating function we asked for. Note that when $y\to 1$ the expressions (2.13) and (2.14) reduce to (2.3), as they should.

For finding the number of different physical "particles" of spin l and excitation level N, occurring in the spectrum, we have to remove the null-norm states. This can be done by multiplication with a factor $1-x$.

Parity can be introduced by assigning a negative parity to every spacelike creation operator and a positive parity to the ground state. The function

$$\tilde{G}(x,y)=\prod_{n=1}^{\infty}\frac{1}{(1+x^n)(1+yx^n)[1+(1/y)x^n]} = \sum_{N,l}\tilde{M}_l^{(N)}x^N\frac{y^{l+1}-y^{-l}}{y-1} \tag{2.15}$$

then generates the difference $\tilde{M}_l^{(N)}$ of the number of positive- and negative-parity tensors, so that we have

$$M_l^{(N)\pm}=\tfrac{1}{2}[M_l^{(N)}\pm\tilde{M}_l^{(N)}] . \tag{2.16}$$

B. The Neveu-Schwarz model

In the Neveu-Schwarz model (complications due to isospin, etc. are not taken into account) a similar theorem can be proved as for the conventional model. We shall work in the shifted Fock space $\mathfrak{F}_2$ in which the ground state has $m^2=-\frac{1}{2}$.[4] We now have to consider states of the form

$$\mathfrak{N}b_{1/2}^{\dagger j_1^{(1)}}\cdots b_{1/2}^{\dagger j_{s_1}^{(1)}}b_{3/2}^{\dagger j_1^{(2)}}\cdots b_{3/2}^{\dagger j_{s_2}^{(2)}}\cdots a_1^{\dagger i_1^{(1)}}\cdots|\rangle$$

$$(j_1^{(1)},\ldots,i_1^{(1)},\ldots=1,2,3) . \tag{2.17}$$

The operators $b^\dagger$ here satisfy anticommutation relations. Now the generating function $G^{\rm NS}(x,y)$ can be written as

$$G^{\rm NS}(x,y)=G^{(b)}(x,y)G^{(a)}(x,y) , \tag{2.18}$$

where $G^{(a)}$ generates the multiplicities of tensors constructed with a operators and $G^{(b)}$ generates the multiplicities of tensors constructed with b operators only. Thus $G^{(a)}$ is just equal to the corresponding expression (2.14) for the conventional model. The excitation level is

$$N=\sum_{i=1}^{\infty} i n_i+\sum_{j=1}^{\infty}(j-\tfrac{1}{2})s_j .$$

Using the same reasoning as for $G^{(a)}$ we now find

$$G^{(b)}(x,y)=\prod_{i=1}^{\infty}(1+x^{i-1/2})(1+yx^{i-1/2})\times\left(1+\frac{x^{i-1/2}}{y}\right) . \tag{2.19}$$

Therefore, combining (2.14) and (2.19) we obtain from (2.18)

$$G^{\rm NS}(x,y)=\prod_{n=1}^{\infty}\frac{(1+x^{n-1/2})(1+yx^{n-1/2})[1+(1/y)x^{n-1/2}]}{(1-x^n)(1-yx^n)[1-(1/y)x^n]} = \sum_{N=0,1/2,\ldots}\sum_{l=0}^{\infty}N_l^{(N)}x^N\frac{y^{l+1}-y^{-l}}{y-1} . \tag{2.20}$$

Again, in the limit $y\to 1$ we get back the generating function for the total number of irreducible tensors at level N. The null-norm states are removed by multiplication with $1-x^{1/2}$.

Parity is introduced by assigning a negative parity to the a operators, a positive parity to the b operators and a negative parity to the vacuum state.

Consequently, we have for $\tilde{G}^{\rm NS}$, in analogy to $\tilde{G}$,

$$\tilde{G}^{\rm NS}=-\sum_{n=1}^{\infty}\frac{(1+x^{n-1/2})(1+yx^{n-1/2})[1+(1/y)x^{n-1/2}]}{(1+x^n)(1+yx^n)[1+(1/y)x^n]} . \tag{2.21}$$

C. The Ramond model

In the Ramond model, the states to be considered are of the form

$$\mathfrak{N}a_1^{\dagger i_1^{(1)}}\cdots a_1^{\dagger i_{n_1}^{(1)}}a_2^{\dagger i_1^{(2)}}\cdots d_1^{\dagger j_1^{(1)}}\cdots d_1^{\dagger j_{s_1}^{(1)}}d_2^{\dagger j_1^{(2)}}\cdots|\rangle U^k , \tag{2.22}$$

where the $a^\dagger$ operators satisfy commutation relations and the $d^\dagger$ operators anticommutation relations, while U^k ($k=1,2,3,4$) is a four-spinor basis. The excitation level is

$$N=\sum_{i=1}^{\infty} i n_i+\sum_{j=1}^{\infty} j s_j .$$

From (2.22) the "positive-energy solutions" have to be projected out. For each species of tensor this amounts to dividing the number of tensors by 2. It turns out that one can first ignore U^k in (2.22) and afterwards couple the obtained spherical tensors with a spin-$\frac{1}{2}$ system without restrictions.

Let $\Pi(x,y)$ be the generating function for the number $V_l^{(N)}$ of tensors at excitation level N and spin l (integer) which would result if U^k is ignored. Then we find in complete analogy with the first and second examples

$$\Pi(x,y)=\prod_{n=1}^{\infty}\frac{(1+x^n)(1+yx^n)[1+(1/y)x^n]}{(1-x^n)(1-yx^n)[1-(1/y)x^n]} = \sum_{N=0}^{\infty}\sum_{l=0}^{\infty}V_l^{(N)}x^N\frac{y^{l+1}-y^{-l}}{y-1} . \tag{2.23}$$

Now coupling each tensor with the spin-$\frac{1}{2}$ system and calling $W_j^{(N)}$ the number of resulting tensors of spin j and excitation level N, we have

$$W_j^{(N)} = V_{j-1/2}^{(N)} + V_{j+1/2}^{(N)} ; \tag{2.24}$$

so it follows that $\Pi(x, y)$ generates $W_j^{(N)}$ in the following way:

$$\Pi(x, y) = \sum_{N=0}^{\infty} \sum_{j=1/2}^{\infty} W_j^{(N)} x^N \frac{y^{j+1/2} - y^{-(j+1/2)}}{y - 1/y} , \tag{2.25}$$

so that (2.23) together with (2.25) enables one to compute $W_j^{(N)}$.

In order to be sure that the positive-energy states are eigenstates of parity one must assign a negative parity to the d as well as the a operators. We find

$$\tilde{\Pi}(x, y) = \pm \prod_{n=1}^{\infty} \frac{(1 - x^n)(1 - yx^n)[1 - (1/y)x^n]}{(1 + x^n)(1 + yx^n)[1 + (1/y)x^n]} = \sum_{N=0}^{\infty} \sum_{j=1/2}^{\infty} \tilde{W}_j^{(N)} x^N \frac{y^{j+1/2} - y^{-(j+1/2)}}{y - 1/y} , \tag{2.26}$$

where the $\pm$ sign is a matter of convention.

III. RECURRENCE RELATIONS

In this section we will exploit an important property of θ functions, namely that they can be written both as an infinite product and as an infinite sum, in order to derive some recurrence relations for the spin multiplicities $M_l^{(N)}$, $N_l^{(N)}$, and $W_j^{(N)}$ and the corresponding $\tilde{M}_l^{(N)}$, $\tilde{N}_l^{(N)}$, and $\tilde{W}_j^{(N)}$.

These properties are[5]

$$\prod_{n=1}^{\infty} (1 - x^n)(1 - yx^n)(1 - x^n/y) = \frac{1}{y - 1} \sum_{n=-\infty}^{+\infty} (-1)^{n+1} x^{n(n-1)/2} y^n \tag{3.1}$$

and

$$\prod_{n=1}^{\infty} (1 - x^n)(1 + yx^{n-1/2}) \left(1 + \frac{x^{n-1/2}}{y}\right) = \sum_{n=-\infty}^{+\infty} x^{n^2/2} y^n . \tag{3.2}$$

Moreover, use will be made of the property[6]

$$\prod_{n=1}^{\infty} \frac{1 - x^n}{1 + x^{n-1/2}} = \sum_{n=0}^{\infty} (-)^{n(n+1)/2} x^{n(n+1)/4} . \tag{3.3}$$

In the following we shall also make the substitution

$$y = e^{2\pi i \nu} . \tag{3.4}$$

A. The conventional model

Here we insert (3.1) into (2.14), making use of (2.13) and (3.4). This gives

$$\sin^2 \pi\nu = \sum_{n, l, N=0}^{\infty} M_l^{(N)} x^{N+n(n+1)/2} (-)^n \sin[(2n+1)\pi\nu] \times \sin[(2l+1)\pi\nu] . \tag{3.5}$$

If we define $M_{-l-1}^{(N)} = -M_l^{(N)}$, multiply both sides of (3.5) by $\cos 2\pi k\nu$, integrate over ν from 0 to 1, and compare powers of x we obtain the expression

$$\sum_{l=-\infty}^{+\infty} (-)^{l+k} M_{l+k}^{(N-l(l+1)/2)} = \delta_{N,0}(2\delta_{k,0} + \delta_{k,1} + \delta_{k,-1}) . \tag{3.6}$$

With the help of (3.6) the $M_l^{(N)}$ can easily be computed. An even simpler recurrence relation can be obtained for the "integrated" multiplicities $P_j^{(N)}$:

$$P_j^{(N)} = \sum_{i=0}^{\infty} M_{j+i}^{(N)} \tag{3.7}$$

by manipulation of (3.6), namely

$$P_l^{(N)} - P_{l-1}^{(N-1)} = P_l^{(N-l)} - P_{l+1}^{(N-l-1)} . \tag{3.8}$$

Unfortunately, (3.8) is a trivial identity when $l = 0$. Therefore, this has to be supplemented with another expression obtained from (3.6):

$$\sum_{l=-\infty}^{+\infty} (-)^l P_l^{(N-l(l+1)/2)} = \delta_{N,0} . \tag{3.9}$$

In a similar way we find, with

$$\tilde{P}_j^{(N)} = \sum_{i=0}^{\infty} \tilde{M}_{j+i}^{(N)} , \tag{3.10}$$

where

$$\tilde{M}_{-l-1}^{(N)} = -\tilde{M}_l^{(N)} ,$$

that

$$\tilde{P}_l^{(N)} + \tilde{P}_{l-1}^{(N-1)} = \tilde{P}_l^{(N-l)} + \tilde{P}_{l+1}^{(N-l-1)}$$

and (3.11)

$$\sum_{l=-\infty}^{+\infty} (-)^l P_{2l}^{(N-l(l+1))} = \delta_{N,0} - \delta_{N,1} .$$

Here the zero-norm states satisfying (2.1) and (2.2) are included. As remarked before, these can easily be removed. If we call $\bar{M}_l^{(N)}$ the number of physical states, then we have

$$\bar{M}_l^{(N)} = M_l^{(N)} - M_l^{(N-1)} .$$

For $N=1$ and $l=0$ the result would be -1. This is because the first excited state is lightlike. The interpretation is now that for $N=1$, $l=1$ a longitudinally polarized state should be eliminated. Similarly, if $\hat{M}_l^{(N)}$ is the difference between the number of physical states of positive and negative parity, we have

$$\hat{M}_l^{(N)} = \tilde{M}_l^{(N)} - \tilde{M}_l^{(N-1)} ,$$

where again for $N=1$ a reinterpretation is necessary. When the condition (2.2) is replaced by

$$(L_0+m_0^2)|\psi\rangle=0,\quad m_0^2>-1 \tag{2.2$'$}$$

the need for elimination of zero-norm states is not there any more. Then all states satisfying (2.1) and (2.2′) are physical and $M_l^{(N)}$ represents the true number of physical tensors.[7]

B. The Neveu-Schwarz model

If $Q_l^{(N)}$ and $\tilde{Q}_l^{(N)}$ are defined by

$$Q_l^{(N)}=\sum_{i=0}^{\infty} N_{l+i}^{(N)}\,,\qquad \tilde{Q}_l^{(N)}=\sum_{i=0}^{\infty}\tilde{N}_{l+i}^{(N)}\,, \tag{3.12}$$

then, following the same procedure as in A, we find

$$Q_l^{(N)}-Q_{l-1}^{(N-1)}=Q_{l-1}^{N-l+1/2}-Q_l^{N-l-1/2}\,,$$
$$\sum_{n=0}^{\infty}\sum_{l=-\infty}^{+\infty}(-)^{l+n(n+1)/2}Q_l^{(N-l(l+1)/2-n(n+1)/4)}=\delta_{N,0}-\delta_{N,1/2}\,, \tag{3.13}$$

and

$$\tilde{Q}_l^{(N)}+\tilde{Q}_{l-1}^{(N-1)}=\tilde{Q}_{l-1}^{(N-l+1/2)}+\tilde{Q}_l^{(N-l-1/2)}\,,$$
$$\sum_{n=0}^{\infty}\sum_{l=-\infty}^{+\infty}(-)^{l+n(n+1)/2}\tilde{Q}_{2l}^{(N-l(l+1)-n(n+1)/4)}=\sum_{m=-\infty}^{+\infty}(-)^{m+1}(\delta_{N,m^2+(1/2)m}+\delta_{N,m^2+(1/2)m+1/2})\,. \tag{3.14}$$

Also here the zero-norm states satisfying the gauge conditions for physical states have to be eliminated. If we call $\bar{N}_l^{(N)}$ the number of physical states we have

$$\bar{N}_l^{(N)}=N_l^{(N)}-N_l^{(N-1/2)}\,,$$

which for $N=\frac{1}{2}$ would again give -1. The interpretation of this result is the same as in the conventional model. The need for elimination of zero-norm states disappears when the condition $(L_0-\frac{1}{2})|\psi\rangle=0$ is replaced by $(L_0+m_0^2)|\psi\rangle=0$, with $m_0^2>-\frac{1}{2}$.

C. The Ramond model

If $Z_j^{(N)}$ and $\tilde{Z}_j^{(N)}$ are defined by

$$Z_j^{(N)}=\sum_{i=0}^{\infty}W_{j+i}^{(N)}\,,\qquad \tilde{Z}_j^{(N)}=\sum_{i=0}^{\infty}(-)^i\tilde{W}_{j+i}^{(N)}\,, \tag{3.15}$$

then again following the same procedure we find

$$Z_j^{(N)}=Z_{j-1}^{(N-1)}+Z_{j-1}^{(N-j+1/2)}-Z_j^{(N-j-1/2)}$$

and

$$\sum_{\substack{m=-\infty\\ \text{int}}}^{+\infty}\;\sum_{\substack{j=-\infty\\ \text{half int}}}^{+\infty}(-)^{m+j-1/2}Z_{j+1}^{(N-m^2-(j^2-1/4)/2)}=\delta_{N,1}-\delta_{N,0}\,, \tag{3.16}$$

while

$$\tilde{Z}_j^{(N)}=-\tilde{Z}_{j-1}^{(N-1)}-\tilde{Z}_{j-1}^{(N-j+1/2)}-\tilde{Z}_j^{(N-j-1/2)}$$

and

$$\sum_{\substack{n=-\infty\\ \text{int}}}^{+\infty}\tilde{Z}_{3/2-n}^{(N-n(n-1)/2)}=\sum_{m=-\infty}^{+\infty}(-)^m(\delta_{N,m^2}-\delta_{N,m^2+1})\,. \tag{3.17}$$

IV. DISCUSSION

From the relations (3.8) and (3.11) certain regularities can be obtained which amount to saturation of Regge trajectories in the case of the conventional model. In (3.8) when $l>N/2$, we obtain

$$P_l^{(N-l)}=P_{l+1}^{(N-l-1)}=0\,. \tag{4.1}$$

Therefore, we have

$$P_l^{(N)}=P_{l-1}^{(N-1)}\,, \tag{4.2}$$

so that

$$M_l^{(N)}=M_{l-1}^{(N-1)}\,. \tag{4.3}$$

From (3.11) we also have

$$\tilde{M}_l^{(N)}=-\tilde{M}_{l-1}^{(N-1)}\,. \tag{4.4}$$

The conventional dual-resonance spectrum can apparently be interpreted as an infinite set of infinitely long Regge trajectories with the following properties:

(1) They all start in the region $l\le N/2$ and are not interrupted any more as soon as $l>N/2$. (In fact, for $l\le N/2$ they appear not to be interrupted either, but this is not so obvious.)

(2) At each daughter level, the number of odd-signature trajectories with definite parity is equal to the number of even-signature trajectories with opposite parity.

In the case of the Neveu-Schwarz model the same conclusions can be drawn. All Regge trajectories start in the region $l\le(N+1)/2$ and are uninterrupted as soon as $l>(N+1)/2$. The second statement follows from (3.13) and (3.14). In all cases considered, these conclusions are not changed due to the need for elimination of zero-norm states.

Finally, with minor adaptations, the above statements are true also for the Ramond model, as can be seen from (3.16) and (3.17). The two statements are apparently quite universally true in dual resonance models, although the building in of isospin or other internal-symmetry groups requires the necessary modifications.

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APPENDIX

Here we give a proof of the theorem defined in Sec. II. With minor modifications similar proofs can be given for other models. It will be preceded by two lemmas.

First of all, we extend the definition of the states $|\psi\rangle$ by having them satisfy the conditions

$$L_n|\psi\rangle = 0, \quad n > 0 \tag{A1}$$

$$(L_0 + m_0{}^2)|\psi\rangle = 0, \quad m_0{}^2 \geq -1 \tag{A2}$$

while we restrict ourselves to states satisfying

$$p^\mu|\psi\rangle = p_0^\mu|\psi\rangle \ ,$$

with (A3)

$$p_0^\mu = (m_0', 0, 0, 0) \ ,$$

where $m_0' \geq 0$ if $m_0'^2 \geq 0$ and $\mathrm{Im} m_0' > 0$ if $m_0'^2 < 0$. Thus we have made the mass a continuous variable. The gauge operators L_n are defined by

$$L_n = \tfrac{1}{2} : \sum_{m=-\infty}^{+\infty} \alpha_m \cdot \alpha_{n-m} : \quad (n > 0) \ ,$$

$$L_0 = p^2 + H = p^2 + \sum_0^\infty \alpha_{-m} \cdot \alpha_m \ , \tag{A4}$$

$$L_{-n} = L_n^\dagger \ ,$$

where

$$\alpha_m^\mu = \sqrt{m}\ a_m^\mu \text{ if } m > 0 \ ,$$

$$\alpha_{-m}^\mu = \alpha_m^{\mu\dagger} \ ,$$

$$\alpha_0^\mu = \sqrt{2}\, p_0^\mu \ .$$

They satisfy

$$[L_m, L_n] = (m - n) L_{m+n} + c(n)\delta_{m,-n} \tag{A5}$$

and

$$[L_n, \alpha_m^\mu] = m\alpha_{m+n}^\mu \ . \tag{A6}$$

The excitation level N (= 0, 1, 2, . . .) is an eigenvalue of the operator H in (A4), so that $m_0'^2 = m_0{}^2 + N$.

In what follows we shall need special states $|\phi\rangle$ satisfying (A2) and (A3) for some value of m_0, but which are built up with the help of *timelike operators only*. We now formulate the following lemma:

Lemma I. For every excitation level N, states $|\phi\rangle$ which satisfy the conditions (A1) and (A2) do not exist, except possibly for a finite number of discrete values of m_0.

Proof. The state $|\phi\rangle$ satisfies (A1) for all $n > 0$ if it satisfies (A1) for $n = 1$ and $n = 2$ as follows from (A5). Let $T(N)$ be the dimension of the space of states $|\phi\rangle$ at level N. Then $T(N)$ satisfies

$$\prod_{n=1}^\infty (1 - x^n)^{-1} = \sum_{N=0}^\infty T(N) x^N \ . \tag{A7}$$

An arbitrary state $|\phi\rangle$ at level N can therefore be expanded in terms of $T(N)$ states with coefficient $\lambda_1, \ldots, \lambda_{T(N)}$. If we apply the operators L_1 and L_2 we obtain linear combinations of states at the levels $N-1$ and $N-2$, so this gives $T(N-1) + T(N-2)$ linear and homogeneous equations for $T(N)$ unknowns. As follows from a formula given by Abramowitz and Stegun[8] we have

$$T(N-1) + T(N-2) \geq T(N) \ , \tag{A8}$$

so there are at least an equal number of equations and unknowns, but not less. Strictly speaking, it is not necessary to make use of this relation. In fact, (A8) follows automatically from the remainder of our proof. Since the coefficients are at most linear functions of m_0' we find that in order for the set to be solvable, m_0' must satisfy a series of equations of the form

$$f_i(m_0') = 0, \quad i = 1, \ldots, k$$

$$k = T(N-1) + T(N-2) - T(N) + 1 \tag{A9}$$

where the $f_i(m_0')$ are polynomials, or else there are no conditions at all. We must now prove that not all $f_i(m_0')$ are identically zero. This we do by taking m_0' sufficiently large and proving that then no solution exists. Let us define the operators $\bar{L}_n$ by leaving out the spacelike operators of L_n. Then

$$\bar{L}_n = -\tfrac{1}{2} \sum_{m=-\infty}^{+\infty} : \alpha_{-k}^0 \alpha_{k+n}^0 :$$

$$= -\sqrt{2}\, m_0' \alpha_n^0 - F_n \ , \tag{A10}$$

where the operator F_n does not depend on m_0'. Next we define

$$\hat{L}_n = -\frac{1}{\sqrt{2}\, m_0'} \bar{L}_n$$

$$= \alpha_n^0 + \frac{1}{\sqrt{2}\, m_0'} F_n \ . \tag{A11}$$

With the help of $\bar{L}_n$, or equivalently with $\hat{L}_n (n < 0)$, the whole space of spurious states $|\phi\rangle$ can be constructed. According to (A11), every state built up with α_n^0 will be the limit of a corresponding state built up with $\hat{L}_n$ when $m_0' \to \infty$. For given N one can therefore always find a sufficiently large value of m_0' such that the dimensions of the space of spurious $|\phi\rangle$ and the space of all $|\phi\rangle$ at that level are the same. Then these spaces must coincide. Apparently no state $|\phi\rangle$ can be found which satisfies (A1) and (A2), which had to be proved. (Note that no trouble arises from zero-norm states.) This also proves the lemma.

For what follows we use the fact that every $|\psi\rangle$ can be written as

$$|\psi\rangle = a|\chi\rangle + b|\eta\rangle \ , \tag{A12}$$

where $|\chi\rangle$ is built up with the help of spacelike operators only and every term of $|\eta\rangle$ contains at least one timelike operator.

We now formulate the following lemma:

Lemma II. For every state $|\psi\rangle$ at level N the coefficient a in expression (A12) is unequal to zero, except possibly for a finite number of discrete values of m_0.

Proof. Suppose that there exists a state $|\psi\rangle$ at level N in which the coefficient a in (A12) is zero. Let $\psi_M^{(i)}$ form a complete set of independent operators, built up out of spacelike creation operators α only and satisfying

$$[H, \psi_M^{(i)}] = M\psi_M^{(i)} \ . \tag{A13}$$

Furthermore, let $\phi_M^{(i)}$ form a complete set of independent operators, built up out of timelike creation operators α only and satisfying

$$[H, \phi_M^{(i)}] = M\phi_M^{(i)} \ . \tag{A14}$$

Then $|\psi\rangle$ can be expanded as follows:

$$|\psi\rangle = \sum_{i,j} C_k^{i,j}\psi_{N-k}^{(i)}\phi_k^{(j)}|p_0\rangle + \sum_{i,j} C_{k+1}^{i,j}\psi_{N-k-1}^{(i)}\phi_{k+1}^{(j)}|p_0\rangle + \cdots + \sum_{i,j} C_{N-1}^{i,j}\psi_1^{(i)}\phi_{N-1}^{(j)}|p_0\rangle + \sum_j C_N^j\phi_N^{(j)}|p_0\rangle \ . \tag{A15}$$

Here k is chosen such that not all coefficients $C_k^{i,j}$ are equal to zero. The case

$$|\psi\rangle = \sum_j C_N^j \phi_N^{(j)}|p_0\rangle$$

is covered by lemma I, so we may assume that a value for k satisfying $1 \le k < N$ can be found.

Now we apply L_1 and L_2 and demand that the results be zero. First applying L_1 we obtain

$$\sum_{i,j} C_k^{i,j}\psi_{N-k}^{(i)}[L_1, \phi_k^{(j)}]|p_0\rangle + \sum_{i,j} C_k^{i,j}[L_1, \psi_{N-k}^{(i)}]\phi_k^{(j)}|p_0\rangle + L_1\,(\text{terms which contain } \psi_M \text{ operators with } M<N-k) = 0 \ .$$

Since $[L_1, \psi_l^{(i)}]$ is a linear combination of $\psi_{l-1}^{(i)}$ we can now combine the second and third terms and get

$$\sum_{i,j} C_k^{i,j}\psi_{N-k}^{(i)}[L_1, \phi_k^{(j)}]|p_0\rangle + (\text{terms which contain } \psi_M \text{ operators with } M<N-k) = 0.$$

Since the set $\{\psi_l^{(i)}\}$ is complete and independent we now have

$$\sum_j C_k^{i,j}[L_1, \phi_k^{(j)}]|p_0\rangle = L_1\Big(\sum_j C_k^{i,j}\phi_k^{(j)}|p_0\rangle\Big) = 0 \ .$$

Similarly,

$$L_2\Big(\sum_j C_k^{i,j}\phi_k^{(j)}|p_0\rangle\Big) = 0 \ .$$

The conclusion is that, since some of the $C_k^{i,j}$ are nonzero, there must exist states $|\phi\rangle$ at level k which satisfy

$$L_n|\phi\rangle = 0 \ .$$

According to lemma I this is not possible, except possibly for a finite number of discrete m_0 values. This proves lemma II.

Now we can move on to our main theorem. From lemma II it follows that for a continuous infinity of m_0' values the expansion (A15) needs a term of the form

$$\sum_i C_0^i\psi_N^{(i)}|p_0\rangle \ .$$

Let us now split up the set $\{\psi_M^{(i)}\}$ into independent irreducible tensors of the group of rotations and reflections in three-space. We can concentrate on one kind of tensor and enumerate the different independent tensors of this kind with the index α. We are then sure that irreducibility is not lost by taking linear combinations of these tensors. Suppressing the tensor indices we can now find for a continuous infinity of nonexceptional m_0' values and for every $\psi_{N,\alpha}$ a state $|\psi_{N,\alpha}\rangle$ such that

$$|\psi_{N,\alpha}\rangle = C_{0,\alpha}\psi_{N,\alpha}|p_0\rangle + \sum_{\substack{j,k,\alpha' \\ k\neq 0}} C_{k,\alpha,\alpha'}^j\psi_{N-k,\alpha'}\phi_k^{(j)}|p_0\rangle + \sum_j C_{N,\alpha}^j\phi_N^{(j)}|p_0\rangle \ . \tag{A16}$$

This is so because for given N, independent of m_0', the space of all states $\psi_{N,\alpha}|p_0\rangle$ has the same dimension as the space of all states $|\psi_{N,\alpha}\rangle$. See (2.3).

Due to the conditions (A1) imposed on $|\psi_{N,\alpha}\rangle$ the coefficients C, if not zero, can all be written as polynomials in terms of m_0' which have no common zero. Except for a multiplicative constant, these polynomials are uniquely determined. Clearly, as long as all $C_{0,\alpha}$ are unequal to zero, the number of independent $|\psi_{N,\alpha}\rangle$ is equal to the number of independent states $\psi_{N,\alpha}|p_0\rangle$. When m_0' approaches a critical value m_{cr}' for which some of the $C_{0,\alpha}$ become zero, there exists the possibility that the corresponding $|\psi_{N,\alpha}\rangle$ become dependent. However, by explicit construction one can easily show that it is always possible to find a suitable set of m_0'-dependent linear combinations such that this set remains independent even in the limit of $m_0' \to m_{cr}'$.

Apparently, in order to count the number of independent physical tensors of a certain kind it is unimportant whether $C_{0,\alpha}$ becomes zero or not. This proves the theorem stated in Sec. II.

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[1]See, for example, V. Alessandrini, D. Amati, M. Le Bellac, and D. Olive, Phys. Rep. 1C, 269(1971), and references therein.

[2]A. Neveu and J. H. Schwarz, Nucl. Phys. B31, 86 (1971).

[3]P. Ramond, Phys. Rev. D 3, 2415 (1971).

[4]A. Neveu, J. H. Schwarz, and C. B. Thorn, Phys. Lett. 35B, 529 (1971).

[5]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. II.

[6]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1955), Vol. III.

[7]C. Dullemond and P. M. Vervoort, Nuovo Cimento Lett. 7, 197 (1973).

[8]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).