Scattering theory for quantum electrodynamics. II. Reduction and cross-section formulas

Daniel Zwanziger*
Department of Physics, New York University, New York, New York 10003

(Received 21 January 1974; revised manuscript received 24 September 1974)

The quantum-electrodynamical S matrix is obtained as the set of on-mass-shell values of the renormalized momentum-space Green's functions multiplied by $C_i(m_i^2 - p_i^2)^{1+\zeta_i}$ for each particle i, where ζ_i is proportional to the fine-structure constant and C_i is a constant. A photon mass is not needed to eliminate virtual infrared divergences. Instead the parameters $\delta_i = m_i^2 - p_i^2$ regularize Feynman integrals in the infrared region, and the dependence on the δ_i is canceled against the expansion of $(m_i^2 - p_i^2)^{\zeta_i}$ multiplying lower-order Green's functions. Exact cross-section formulas are developed which express transition rates in terms of this S matrix. They account for radiation damping nonperturbatively, whereas the S matrix must be calculated perturbatively as a power series in α. It is seen that in processes with very small energy loss to unobserved photons individual elements of the quantum-electrodynamical S matrix are directly observable. Rules for practical calculations are summarized.

I. INTRODUCTION

The present article is devoted to a traditional concern, namely, how to pass from the renormalized Green's functions of quantum electrodynamics to convenient cross-section formulas, without encountering infrared divergences.[1] As a practical matter, physicists have long known how to get around infrared divergences,[2-12] and it seems that any actual physical situation can be dealt with by previously developed methods. The most prevalent one[6,12] involves regularization of Feynman integrals by introducing a small photon mass λ, which ultimately disappears from cross sections. In the present approach Feynman integrals are regularized by taking the external charged particles slightly off the mass shell. The small parameters $\delta_i = m_i^2 - p_i^2$ cancel out of the reduction formula which expresses the quantum-electrodynamical S matrix in terms of Green's functions. It remains to be seen what are the comparative computational advantages of the two methods in actual experimental situations. Radiative correction calculations are crucial to the interpretation of high-energy electron scattering experiments, so it is important to find out if the present method allows any simplification in practice. This can only be known by trying it out on real problems.

The present approach may be summarized in two formulas. The first is the reduction formula which relates the S matrix to Green's functions,

$$S(p_i, k_i) \sim \prod_i \lim_{p_i^2 \to m_i^2} (m_i^2 - p_i^2)^{1+\zeta_i} \times \prod_j \lim_{k_j^2 \to 0} k_j^2 G(p_i, k_j). \tag{1.1}$$

Here p_i represents the charged-particle momenta, and k_i the photon momenta. The second is the cross-section formula. It expresses cross sections in terms of the S matrix by means of the infrared-renormalized inner product developed in the accompanying article.[1] In its dependence on final photon variables it looks like

$$\sigma \sim e^{-K(\Delta)} \sum_{n=0}^{\infty} \frac{1}{n!} \times \prod_{j=1}^{n} \left[-\int d\hat{k}_j \int_0^\infty d\omega_j \ln\left(\frac{\omega_j}{\Delta}\right) \frac{\partial}{\partial \omega_j} \omega_i^2 \right] \times |S(k_i)|^2 \delta^4\Big(\sum_i k_i - P\Big) \rho_n(k_i). \tag{1.2}$$

Here $K(\Delta)$ is an infrared damping factor, and $\rho_n(k)$ is the density matrix representing the acceptance of the final-state detector. The integral over photon variables appears to arise from a partial integration on ω of the usual phase-space integral $\int d\hat{k} \int (d\omega/\omega)\omega^2$, with neglect of a divergent boundary term at $\omega = 0$. The quantity Δ is an arbitrary parameter which cancels between the coefficient $e^{-K(\Delta)}$ and the infinite series. It may be assigned any value, in particular one which makes the infinite series converge rapidly.

In Sec. II A the S matrix is defined as an operator on the asymptotic-state space and it is shown to be transverse and infrared-coherent. (An infrared-coherent state is one which is an eigenstate of the zero-frequency annihilation operator.) In Sec. II B an exact cross-section formula is presented. It is applied in Sec. II C to radiation-exclusive processes. Such a process is one in which the measured energies of the observed particles are very nearly conserved, so only a small amount is lost to the infrared radiation field. In Sec. II D a cross-section formula is developed for radiation-inclusive processes. Such a process is inclusive in the usual sense; the momentum of only one, or some, final particle is measured, and it is not known how the remaining energy is shared between the other hard particles and the radiation field. In this case the total cross section is expressed as an integral over the bremsstrahlung cross section.[13]

In Sec. III the reduction formula is derived. It expresses the S matrix in terms of renormalized momentum-space Green's functions. The starting point is the definition of the S-matrix element as an inner product between vectors in the retarded and advanced spaces. These are subspaces of the asymptotic-state space which have finite numbers of incoming and outgoing photons, respectively. Although infinite numbers of particles are emitted in any scattering process involving charged particles, such an S-matrix element is directly observable in a radiation-exclusive process. An unexpected result is the discovery of superselection sectors for states with incident charged particles. It is found that the most complete description of an incoming state containing charged particles is a density matrix which is diagonal in the numbers and momenta of the charged particles.

In Sec. IV A an on-mass-shell ultraviolet renormalization scheme for quantum electrodynamics is described. The Ward identity is used to show that if the renormalized electron propagator in the Feynman gauge is normalized to

$$\lim_{p^2\to m^2} G^{-1}(p) \sim \frac{-i(\not p - m)}{z}\left(\frac{m^2-p^2}{m^2}\right)^{\beta}, \tag{1.3}$$

where $\beta = \alpha/\pi$ and z is an arbitrary finite normalization constant, then the renormalized vertex function near the forward direction satisfies

$$\lim_{p'\to p} \bar u(p)\Gamma^{\mu}(p',p)|_{p^2=m^2}u(p) \sim \frac{1}{z}\left(\frac{m^2-p'^2}{m^2}\right)^{\beta}\bar u(p)\gamma^{\mu}u(p). \tag{1.4}$$

In Sec. IV B as an application, the Schwinger problem,[2] namely the scattering of an electron by a weak external potential, is treated by the present method.

Finally in Sec. V A we summarize the rules for practical calculations. In Sec. V B we indicate how a rigorous scattering theory for quantum electrodynamics might be formulated.

II. S MATRIX AND CROSS SECTIONS

A. Definition and properties of the S matrix

Our scattering theory for quantum electrodynamics is modeled after the theory of bremsstrahlung by the classical external current of a scattered charged particle developed in Sec. I IV B. The reader may wish to review it before proceeding. In that case an exact solution was at hand. In the present case the properties of the representation space and scattering operator might be considered postulates. However, they are susceptible to verification or disproof in perturbative renormalization theory, so they are really *Ansätze*.

Let the Fock space of photon test functions established in Sec. I III B be called $\mathfrak{F}_{\gamma}(\mathcal{E})$. For free charged particles one may make use of either a Hilbert space $\mathcal{H}_q$ or a Fock space $\mathfrak{F}_q$ of test functions defined on the mass shells of the charged particles. Both $\mathcal{H}_q$ and $\mathfrak{F}_q$ possess a positive-definite inner product, and $\mathcal{H}_q$ is simply the completion in norm of $\mathfrak{F}_q$ and provided with the norm topology. For convenience of expression we shall use $\mathfrak{F}_q$. Let $\mathfrak{F}$ be the product space

$$\mathfrak{F} = \mathfrak{F}_{\gamma}(\mathcal{E}) \otimes \mathfrak{F}_q\ , \tag{2.1a}$$

whose elements F are sequences of test functions in the photon and charged-particle on-mass-shell momenta

$$F = \{F_{(n,t)}\}\ ,\quad F_{(n,t)} = F_{(n,t)}(k_1\cdots k_n, p_1\cdots p_t)\ , \tag{2.1b}$$

where polarization indices are suppressed. The inner product in photon variables $(dk)_l$, developed in Sec. I III B, is indefinite, whereas the inner product for charged particles has the usual positive-definite form

$$\langle F|G\rangle = \sum_{t=0}^{\infty}\frac{1}{t!}\sum_{n=0}^{\infty}\frac{1}{n!}\int dp_1\cdots dp_t(dk_1)_l\cdots(dk_n)_l F^*_{(n,t)}(k_1\cdots k_n,p_1\cdots p_t)\,G_{(n,t)}(k_1\cdots k_n,p_1\cdots p_t)\ , \tag{2.2}$$

where $dp_i = d^3p_i(2E_i)^{-1}$. To lighten notation the $n=0$ and $t=0$ terms have been represented only symbolically, polarization indices are suppressed, and we have written $\sum_t (t!)^{-1}$ as if only one type of charged particle were present instead of $\sum_{ta}\prod_a(t_a!)^{-1}$, as is appropriate for several types a. The Dirac notation $F_{(n,t)}(k_1\cdots k_n, p_1\cdots p_t) = \langle k_1\cdots k_n, p_1\cdots p_t|F\rangle$ is convenient to distinguish in and out variables according to $\langle k_1\cdots k_n, p_1\cdots p_t{}^{\text{in}}|$ and $\langle k_1\cdots k_n,p_1\cdots p_t{}^{\text{out}}|$.

Our scattering *Ansatz* is that the asymptotic in-space given by $\mathfrak{F}^{\text{in}} = \mathfrak{F}_{\gamma}^{\text{in}}(\mathcal{E})\otimes\mathfrak{F}_q^{\text{in}}$ and the asymptotic out-space $\mathfrak{F}^{\text{out}} = \mathfrak{F}_{\gamma}^{\text{out}}(\mathcal{E})\otimes\mathfrak{F}_q^{\text{out}}$ are the same space

$$\mathfrak{F}^{\rm in} = \mathfrak{F}^{\rm out} , \tag{2.3}$$

with corresponding states and operators related by the unitary scattering operator

$$F^{\rm in} = SF^{\rm out} , \tag{2.4a}$$

$$O^{\rm in} = SO^{\rm out} S^{\dagger} . \tag{2.4b}$$

Equation (2.4a) reads explicitly

$$\langle k_1 \cdots k_n, p_1 \cdots p_t{}^{\rm out} | F^{\rm in} \rangle = \sum_{s,m=0}^{\infty} \frac{1}{s!m!} \int dp_1' \cdots dp_s' (dk_1')_l \cdots (dk_m')_l S_{n,t;m,s}(k_1 \cdots k_n, p_1 \cdots p_t; k_1' \cdots k_m', p_1' \cdots p_s')$$
$$\times \langle k_1' \cdots k_m', p_1' \cdots p_s'{}^{\rm out} | F^{\rm out} \rangle . \tag{2.5}$$

Unless otherwise stated we always use out variables

$$F_{(n,t)}(k_1 \cdots k_n, p_1 \cdots p_t) \equiv \langle k_1 \cdots k_n, p_1 \cdots p_t^{\rm out} | F \rangle , \tag{2.6}$$

so the equation $F^{\rm in} = SF^{\rm out}$ expresses a given in-state F in terms of out variables. (The S-matrix elements are of course the same in both in and out variables.) The vacuum and one-particle states are invariant under S, and S is Poincaré-invariant.

On occasion it is convenient to write (k,p) for $(k_1 \cdots k_n, p_1 \cdots p_t)$ and express the inner product (2.2) and matrix multiplication (2.5) symbolically as

$$\langle F | G \rangle = \int F^*(k,p) G(k,p) (dk)_l \, dp , \tag{2.7}$$

$$F^{\rm in}(k,p) = \int S(k,p;k',p') F^{\rm out}(k',p') (dk')_l \, dp' . \tag{2.8}$$

Poincaré invariance leads to the standard invariance properties of the S-matrix elements, in particular

$$S(k,p;\; k',p') = S(k,p;k',p') \delta^4(\textstyle\sum k + \sum p - \sum k' - \sum p') . \tag{2.9}$$

Unitarity $S^{\dagger}S = 1$ reads

$$\int S^*(k,p;k',p') S(k,p;k',p') (dk)_l \, dp = \delta_l (k - k'') \delta(p' - p'') , \tag{2.10}$$

and correspondingly for $SS^{\dagger} = 1$.

The field equations yield some conserved quantities. From the Yang-Feldman equations, $A = A^{\rm in} + \Delta^{\rm ret} J = A^{\rm out} + \Delta^{\rm ad} J$, and current conservation, we have $\partial \cdot A = \partial \cdot A^{\rm in} = \partial \cdot A^{\rm out}$, or in terms of annihilation operators[14]

$$\hat{k} \cdot c^{\rm in}(k) = \hat{k} \cdot c^{\rm out}(k) , \tag{2.11}$$

as follows from the expansion, Eq. (I.4.3c). We have also

$$A_\mu^{\rm out}(x) - A_\mu^{\rm in}(x) = \int \Delta(x-y) J_\mu(y) \, d^4 y , \tag{2.12a}$$

where $\Delta(x) = \Delta^{\rm ret}(x) - \Delta^{\rm ad}(x)$. Again making use of the expansion (I.4.3c) and Eq. (I.4.19) for Δ, one finds

$$c_\mu^{\rm out}(k) - c_\mu^{\rm in}(k) = \frac{i\omega}{(2\pi)^{3/2}} \int e^{ik\cdot y} J_\mu(y) . \tag{2.12b}$$

Upon letting ω approach zero one obtains on the left the zero-frequency annihilation operators, $c(\hat{k}) = \lim_{\omega \to 0} c(k)$,

$$c_\mu^{\rm out}(\hat{k}) - c_\mu^{\rm in}(\hat{k}) = \frac{i}{(2\pi)^{3/2}} \lim_{\omega \to 0} \omega \int e^{ik\cdot y} J_\mu(y) \, d^4 y .$$

As ω approaches zero the explicit factor of ω annihilates contributions from finite regions of integration. Upon taking matrix elements between normalizable states, the only contribution from infinite regions comes from infinite timelike regions for which the asymptotic limit (I.2.5) may be used,

$$c_\mu^{\rm out}(\hat{k}) - c_\mu^{\rm in}(\hat{k}) = \frac{i}{(2\pi)^{3/2}} \lim_{\omega\to 0} \omega \int d^4y\, e^{ik\cdot y} \int d^4p\, p_\mu \left[\int_T^\infty d\tau\, \delta^4(y-p\tau)\rho^{\rm out}(p) + \int_{-\infty}^{-T} d\tau\, \delta^4(y-p\tau)\rho^{\rm in}(p)\right],$$

where T is some large positive number. This gives

$$c_\mu^{\rm out}(\hat{k}) - c_\mu^{\rm in}(\hat{k}) = \frac{i}{(2\pi)^{3/2}} \int d^4p\, p_\mu \lim_{\omega\to 0} \omega \left[\frac{\rho^{\rm out}(p)e^{ik\cdot pT}}{-ik\cdot p} + \frac{\rho^{\rm in}(p)e^{-ik\cdot pT}}{ik\cdot p}\right],$$

$$c_\mu^{\rm in}(\hat{k}) + \int d^4p\, \frac{p_\mu \rho^{\rm in}(p)}{(2\pi)^{3/2} p\cdot\hat{k}} = c_\mu^{\rm out}(\hat{k}) + \int d^4p\, \frac{p_\mu \rho^{\rm out}(p)}{(2\pi)^{3/2} p\cdot\hat{k}}, \tag{2.13}$$

which is the operator form of the low-energy theorem.

Conservation of the quantities (2.11) and (2.13) provides important information about the S matrix. From $\hat{k}\cdot c^{\rm in}(k) = \hat{k}\cdot c^{\rm out}(k)$ and $\hat{k}\cdot c^{\rm in}(k) S = S\hat{k}\cdot c^{\rm out}(k)$ we find

$$[\hat{k}\cdot c^{\rm out}(k), S] = 0, \tag{2.14}$$

and similarly

$$\left[c_\mu^{\rm out}(\hat{k}) + \int d^4p\, \frac{1}{(2\pi)^{3/2}} \frac{p_\mu}{p\cdot\hat{k}} \rho^{\rm out}(p), S\right] = 0. \tag{2.15}$$

In terms of the S-matrix elements (2.5) the first relation gives, with suppression of the charged-particle variables,

$$k_1^{\mu_1} S_{n,m}(k_1\cdots k_n; k_1'\cdots k_m')_{\mu_1\cdots\mu_n;\mu_1'\cdots\mu_m'} = k_1^{\mu_1} \sum_P (-g_{\mu_1,\mu_1'})\delta_l(k_1-k_1') S_{n-1,m-1}(k_2\cdots k_n; k_2'\cdots k_m')_{\mu_2\cdots\mu_n;\mu_2'\cdots\mu_m'}, \tag{2.16}$$

where the sum extends over the $m!$ permutations of the initial photons, and $S_{n,-1}=0$. There is a similar relation which holds upon contraction on an initial-photon index. This shows that the part of S which is connected in the photon variables is transverse. Similarly from Eqs. (2.5) and (2.15) one has

$$\lim_{\omega_1\to 0}\left[S_{n,m}(k_1\cdots k_n, p_f; k_1'\cdots k_m', p_i)_{\mu_1\cdots\mu_n;\mu_1'\cdots\mu_m'} - \sum_P (-g_{\mu_1,\mu_1'})\delta_l(k_1-k_1') S_{n-1,m-1}(k_2\cdots k_n, p_f; k_2'\cdots k_m', p_i)_{\mu_2\cdots\mu_n;\mu_2'\cdots\mu_m'}\right]$$
$$= \frac{-1}{(2\pi)^{3/2}}\left(\sum_f \frac{e_f p_{f\mu_1}}{p_f\cdot\hat{k}_1} - \sum_i \frac{e_i p_{i\mu_1}}{p_i\cdot\hat{k}_1}\right) S_{n-1,m-1}(k_2\cdots k_n, p_f; k_2'\cdots k_m', p_i)_{\mu_2\cdots\mu_n;\mu_2'\cdots\mu_m'}. \tag{2.17}$$

The sum on the left-hand side extends over $m!$ permutations of the initial photons; the sum on f extends over final charged particles, and on i over initial charged particles. With respect to photon variables, S has the usual connectivity properties. In terms of the photon connected S-matrix element S^c, the last two properties become

$$k_{1\mu_1} S^c_{n,m}(k_1\cdots k_n, p_f; k_1'\cdots k_m', p_i)^{\mu_1\cdots\mu_n;\mu_1'\cdots\mu_m'} = 0, \tag{2.18a}$$

$$k'_{1\mu_1'} S^c_{n,m}(k_1\cdots k_n, p_f; k_1'\cdots k_m', p_i)^{\mu_1\cdots\mu_n;\mu_1'\cdots\mu_m'} = 0, \tag{2.18b}$$

$$\lim_{\omega_1\to 0} S^c_{n,m}(k_1\cdots k_n, p_f; k_1'\cdots k_m', p_i)^{\mu_1\cdots\mu_n;\mu_1'\cdots\mu_m'} = \frac{-1}{(2\pi)^{3/2}} \sum_a \frac{\eta_a e_a p_a^{\mu_1}}{p_a\cdot\hat{k}_1} S^c_{n-1,m}(k_2\cdots k_n, p_f; k_1'\cdots k_m', p_i)^{\mu_2\cdots\mu_n;\mu_1'\cdots\mu_m'}, \tag{2.19a}$$

$$\lim_{\omega_1\to 0} S^c_{n,m}(k_1\cdots k_n, p_f; k_1'\cdots k_m', p_i)^{\mu_1\cdots\mu_n;\mu_1'\cdots\mu_m'} = \frac{1}{(2\pi)^{3/2}} \sum_a \frac{\eta_a e_a p_a^{\mu_1'}}{p_a\cdot\hat{k}_1'} S^c_{n,m-1}(k_1\cdots k_n, p_f; k_2'\cdots k_m', p_i)^{\mu_1\cdots\mu_n;\mu_2'\cdots\mu_m'}. \tag{2.19b}$$

Here

$$\sum_a \frac{\eta_a e_a p_a}{p_a\cdot\hat{k}} = \sum_f \frac{e_f p_f}{p_f\cdot\hat{k}} - \sum_i \frac{e_i p_i}{p_i\cdot\hat{k}}, \tag{2.20a}$$

$$\eta_f = 1, \quad \eta_i = -1, \tag{2.20b}$$

so a runs over all final and initial charged particles with $\{e_a\}=\{e_f, e_i\}$, $\{p_a\}=\{p_f, p_i\}$, and $\{\eta_a\}=\{\eta_f, \eta_i\}$. Relations (2.18) and (2.19) show that the set of photon connected S-matrix elements is transverse and infrared-coherent in initial and final photon variables.

The kernels $S_{n,t;m,s}$ of the S operator, Eq. (2.5), are very simply related to S-matrix elements between states which have definite numbers of initial and final photons and charged particles. These states lie in the subspaces

$$\mathfrak{F}^{\text{ret}} \equiv \mathfrak{F}_\gamma^{\text{in}}(\mathcal{E}_0) \otimes \mathfrak{F}_q^{\text{in}} \subset \mathfrak{F}^{\text{in}}, \tag{2.21a}$$

$$\mathfrak{F}^{\text{ad}} \equiv \mathfrak{F}_\gamma^{\text{out}}(\mathcal{E}_0) \otimes \mathfrak{F}_q^{\text{out}} \subset \mathfrak{F}^{\text{out}}, \tag{2.21b}$$

characterized by

$$c_\mu^{\text{in}}(\hat{k})\mathfrak{F}^{\text{ret}} = 0, \quad c_\mu^{\text{out}}(\hat{k})\mathfrak{F}^{\text{ad}} = 0. \tag{2.22}$$

They satisfy

$$\mathfrak{F}^{\text{ret}} = S\mathfrak{F}^{\text{ad}}, \quad \mathfrak{F}^{\text{ret}} = \theta\mathfrak{F}^{\text{ad}}, \tag{2.23}$$

where θ is the CPT operator. Let $F_{m,s}^{\text{ret}}$ be a state with m initial photons and s initial charged particles, and wave function $F_{m,s}(k_1 \cdots k_m, p_1 \cdots p_s)$, and correspondingly for $G_{n,t}^{\text{ad}}$. For such states the inner product on photon variables is ordinary integration, and we have, by Eqs. (2.2) and (2.5), with suppression of polarization indices

$$\langle G_{n,t}^{\text{ad}} | F_{m,s}^{\text{ret}} \rangle = \frac{1}{t!n!s!m!} \int G_{n,t}(k_1 \cdots k_n, p_1 \cdots p_t) S_{n,t;m,s}(k_1 \cdots k_n, p_1 \cdots p_t; k_1' \cdots k_m', p_1' \cdots p_s')$$
$$\times F_{m,s}(k_1' \cdots k_m', p_1' \cdots p_s')\, dk_1 \cdots dk_n\, dp_1 \cdots dp_t\, dk_1' \cdots dk_m' dp_1' \cdots dp_s', \tag{2.24a}$$

where $dk = \frac{1}{2} d\hat{k}\, d\omega/\omega$. This may be written formally, in terms of (improper) momentum eigenstates of $\mathfrak{F}^{\text{ret}}$ and $\mathfrak{F}^{\text{ad}}$,

$$\langle k_1 \cdots k_n, p_1 \cdots p_t{}^{\text{ad}} | k_1' \cdots k_m', p_1' \cdots p_s'{}^{\text{ret}} \rangle = S_{n,t;m,s}(k_1 \cdots k_n, p_1 \cdots p_t; k_1' \cdots k_m', p_1' \cdots p_s'). \tag{2.24b}$$

Use of the retarded and advanced spaces which have finite numbers of initial and final photons is essential to obtain the reduction formula, namely an expression for an S-matrix element in terms of a single Green's function, for by definition the Green's function contains a finite number of photon and charged fields and relates directly to an inner product between a finite number of initial and final particles. Each space $\mathfrak{F}^{\text{ret}}$ and $\mathfrak{F}^{\text{ad}}$ contains a physical subspace of non-negative metric

$$\mathfrak{F}^{\text{ret}+} = \mathfrak{F}_\gamma^{\text{in}}(\mathcal{E}_0^+) \otimes \mathfrak{F}_q^{\text{in}} \subset \mathfrak{F}^{\text{ret}}, \tag{2.25a}$$

$$\mathfrak{F}^{\text{ad}+} = \mathfrak{F}_\gamma^{\text{out}}(\mathcal{E}_0^+) \otimes \mathfrak{F}_q^{\text{out}} \subset \mathfrak{F}^{\text{ad}}, \tag{2.25b}$$

characterized by Eqs. (2.22) and

$$\hat{k} \cdot c^{\text{in}}(k)\mathfrak{F}^{\text{ret}+} = 0, \quad \hat{k} \cdot c^{\text{out}}(k)\mathfrak{F}^{\text{ad}+} = 0, \tag{2.26}$$

with

$$\mathfrak{F}^{\text{ret}+} = S\mathfrak{F}^{\text{ad}+}, \quad \mathfrak{F}^{\text{ret}+} = \theta\mathfrak{F}^{\text{ad}+}. \tag{2.27}$$

Each of these may be completed in norm to a physical Hilbert space $\mathcal{H}^{\text{ret}}$ and $\mathcal{H}^{\text{ad}}$ containing finite numbers of incoming and outgoing photons, respectively. There is a unique extension of S and θ to

$$\mathcal{H}^{\text{ret}} = S\mathcal{H}^{\text{ad}}, \quad \mathcal{H}^{\text{ret}} = \theta\mathcal{H}^{\text{ad}}. \tag{2.28}$$

However, because in any scattering process involving charged particles infinite numbers of photons are emitted,

$$\mathfrak{F}^{\text{ret}} \neq \mathfrak{F}^{\text{ad}}, \quad \mathfrak{F}^{\text{ret}+} \neq \mathfrak{F}^{\text{ad}+}, \quad \mathcal{H}^{\text{ret}} \neq \mathcal{H}^{\text{ad}}, \tag{2.29}$$

as may be seen analytically by comparing Eqs. (2.13) and (2.22). Use of two different physical subspaces $\mathfrak{F}^{\text{ret}+}$ and $\mathfrak{F}^{\text{ad}+}$ with S-matrix elements defined as the inner product between an element of one subspace with an element of the other is characteristic of the present approach. It will be seen shortly, Sec. II C, that in processes with small energy loss to the radiation field, the S-matrix elements defined here are directly observable.

B. Cross-section formula

In an ideal collision the system is in a state where the initial particle numbers and momenta $k_{i1}\cdots k_{in}$, $p_{i1}\cdots p_{it}$ are sharp. We suppress the labels of the individual initial particles and write k_i, p_i. In terms of out variables the system is in a state described by the sequence of wave functions ($n, t=0,1,2,\ldots$)

$$F_{n,t}(k_{f1}\cdots k_{fn},p_{f1}\cdots p_{ft})=S_{n,t}(k_{f1}\cdots k_{fn},p_{f1}\cdots p_{ft};k_i,p_i)$$
$$=S_{n,t}(k_{f1}\cdots k_{fn},p_{f1}\cdots p_{ft};k_i,p_i)\delta^4(\sum k_f+\sum p_f-\sum k_i-\sum p_i). \tag{2.30}$$

Here polarization indices are suppressed.

We are interested in the transition rate $\sigma(\Omega)$ into some volume Ω of final-state phase space. Assume for simplicity that polarizations are not measured, and let the volume Ω refer to a fixed number t of final charged particles. The set of momenta $p_{f1}\cdots p_{ft}$ will be represented by p_f and $(t!)^{-1}\int dp_{f1}\cdots dp_{ft}$ by $\int dp_f$, henceforth. Our attention is centered on the final photons. The outgoing phase-space volume Ω is a set of volumes Ω_n, $\Omega=\{\Omega_n\}$, $n=0,1,2,\ldots$ in n-photon phase space, symmetric in $k_{f1}\cdots k_{fn}$. Let $\chi_n(k_{f1}\cdots k_{fn},p_f)$ be a function which is one for $(k_{f1}\cdots k_{fn},p_f)\in\Omega_n$ and zero otherwise.[15] Because zero-frequency photons are unobservable, the χ_n satisfy

$$\lim_{\omega_1\to 0}\chi_n(k_1\cdots k_n,p_f)=\chi_{n-1}(k_2\cdots k_n,p_f). \tag{2.31}$$

The projector onto Ω is represented by

$$(P_\Omega F)_{(n)}(k_1\cdots k_n,p)=\chi_n(k_1\cdots k_n,p)F_{(n)}(k_1\cdots k_n,p). \tag{2.32}$$

The projection does not disturb infrared coherence because of condition (2.31), $[P_\Omega,c_\mu(\hat{k})]=0$.

The transition rate $\sigma(\Omega)$ into Ω is given, with a convenient normalization, by

$$\sigma(\Omega)=\int dp_f\Big[\chi_0(p_f)|s_0(p_f;k_i,p_i)|^2\delta^4(\sum p_f-\sum k_i-\sum p_i)$$
$$+\sum_{n=1}^{\infty}\frac{1}{n!}\int(dk_{f1})_l\cdots(dk_{fn})_l\,\chi_n(k_{f1}\cdots k_{fn},p_f)|s_n(k_{f1}\cdots k_{fn},p_f;k_i,p_i)|^2\delta^4(\sum k_f+\sum p_f-\sum k_i-\sum p_i)\Big], \tag{2.33}$$

where $\int(dk)_l\ldots$ is the inner product developed in Sec. I III A. It is understood that s_n is the photon connected part of the S-matrix element. The infrared-coherence condition on the S matrix, Eq. (2.19a), allows us to make use of the explicit form of the inner product on photon variables, Eq. (I.3.62),

$$\sigma(\Omega)=\int dp_f\,e^{-K(p,\Delta)}\Big[\chi_0(p_f)|s_0(p_f;k_i,p_i)|^2\delta^4(\sum p_f-\sum k_i-\sum p_i)$$
$$+\sum_{n=1}^{\infty}\frac{1}{n!}\int(dk_{f1})_\Delta\cdots(dk_{fn})_\Delta\chi_n(k_{f1}\cdots k_{fn},p_f)|s_n(k_{f1}\cdots k_{fn},p_f;k_i,p_i)|^2$$
$$\times\delta^4(\sum k_f+\sum p_f-\sum k_i-\sum p_i)\Big]. \tag{2.34}$$

Here we have

$$(dk)_\Delta=-\tfrac{1}{2}d\hat{k}\,d\omega\ln\Big(\frac{\omega}{\Delta}\Big)\frac{\partial}{\partial\omega}, \tag{2.35}$$

$$\{p_a\}=\{p_f,p_i\},\quad \{e_a\}=\{e_f,e_i\}, \tag{2.36a}$$

$$\{\eta_a\}=\{\eta_f,\eta_i\},\quad \eta_f=+1,\quad \eta_f=-1, \tag{2.36b}$$

$$K(p,\Delta)=-\sum_{a,b}\eta_a e_a\eta_b e_b\langle p_a|p_b\rangle,$$

$$K(p,\Delta)=\frac{1}{(2\pi)^3}\int\frac{d\hat{k}}{2}\Big(\sum_a\frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\Big)\cdot\sum_a\frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\ln\Big(\frac{p_a\cdot\hat{k}\Delta}{m_a l}\Big). \tag{2.37}$$

The cross section (2.34) is independent of Δ, which may be chosen to be any function of $\hat{k}$ and of the momenta. The derivatives implicit in $(dk)_\Delta$ act on all the factors on the right, i.e., $\chi_n|s_n|^2\delta^4$.

The damping factor $e^{-K(p,\Delta)}$ has a dependence on the infrared renormalization constant l given by l^{-B} where

$$B=\frac{-1}{(2\pi)^3}\int\frac{d\hat{k}}{2}\Big(\sum_a\frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\Big)^2\geq 0. \tag{2.38a}$$

B is non-negative because the integrand is the square of a vector which is orthogonal to $\hat{k}$ and is therefore spacelike. Use of Eq. (I.3.10) gives easily

$$B = -\sum_{a,b} \frac{\eta_a e_a \eta_b e_b}{(2\pi)^2} \psi_{ab} \coth\psi_{ab}, \tag{2.38b}$$

where $\psi_{ab} \geq 0$ is the hyperbolic angle between p_a and p_b, $p_a \cdot p_b = m_a m_b \cosh\psi_{ab}$. As will be seen in the reduction formula, each S-matrix element $s_{n,m}(k_{f1} \cdots k_{fn}, p_f; k_{i1} \cdots k_{im}, p_i)$ contains a factor of l^{-Z}, with $\mathrm{Re}Z = -B/2$, so the cross section is independent of l. Since S is calculated perturbatively, it is convenient to remove this factor first and have an exact cancellation. However, l has the dimension of mass, and if l^{-Z} alone were factored out, the S-matrix element would acquire an anomalous dimension. One may in fact view the radiation-damping factor ω_0^B as arising from the anomalous dimension introduced by the infrared renormalization. However, for a perturbative calculation of S it is convenient to maintain normal engineering dimensions. If there is only one mass m in the theory, one could factor out $(l/m)^{-Z}$. If there are several masses m_a, a convenient factor may be chosen as follows. As will be seen later, Eq. (3.45), Z is given by

$$Z = \tfrac{1}{2}\sum_a \zeta_{aa} + \sum_{a>b} \zeta_{ab}, \tag{2.39a}$$

$$\zeta_{aa} = \frac{e_a^2}{(2\pi)^2}, \tag{2.39b}$$

$$\zeta_{ab} = \frac{\eta_a e_a \eta_b e_b}{(2\pi)^2 \tanh\psi_{ab}} [\psi_{ab} - i\pi\theta(\eta_a\eta_b)], \quad a \neq b. \tag{2.39c}$$

Introduce a scattering amplitude F independent of l according to

$$s_{n,m}(k_{f1} \cdots k_{fn}, p_f; k_{i1} \cdots k_{im}, p_i) = \prod_a \left(\frac{2l_1}{m_a}\right)^{-\zeta_{aa}/2} \prod_{a>b} \left(\frac{4l_1^2}{m_a m_b}\right)^{-\zeta_{ab}/2} F_{n,m}(k_{f1} \cdots k_{fn}, p_f; k_{i1} \cdots k_{im} p_i), \tag{2.40}$$

where $l_1 = le^C$ and C is Euler's constant. Anticipating later convenience we have also factored out powers of $2e^C$. Note that the ζ_{ab} are independent of the photon numbers m and n, so the $F_{n,m}$ has the same infrared-coherence properties as the $s_{n,m}$. This implies

$$|s_{n,m}|^2 = \exp\left\{\frac{1}{(2\pi)^3}\int \frac{d\hat{k}}{2}\left[\sum_a \frac{\eta_a e_a p_a}{p_a \cdot \hat{k}} \ln\left(\frac{2l_1}{m_a}\right)\right]\cdot\left[\sum_a \frac{\eta_a e_a p_a}{p_a \cdot \hat{k}}\right]\right\} |F_{n,m}|^2. \tag{2.41}$$

We thus obtain as an exact expression for the counting rate

$$\sigma(\Omega) = \int dp_f \exp[-J(p,\Delta)] \Big[\chi_0(p_f)|F_0(p_f; k_i, p_i)|^2 \delta^4(\textstyle\sum p_f - \sum k_i - \sum p_i)$$
$$+ \sum_{n=1}^{\infty} \frac{1}{n!}\int (dk_{f1})_\Delta \cdots (dk_{fn})_\Delta \chi_n(k_{f1} \cdots k_{fn}, p_f)|F_n(k_{f1} \cdots k_{fn}, p_f; k_i, p_i)|^2$$
$$\times \delta^4(\textstyle\sum k_f + \sum p_f - \sum k_i - \sum p_i)\Big], \tag{2.42}$$

where

$$J(p,\Delta) = \frac{1}{(2\pi)^3}\int \frac{d\hat{k}}{2}\left[\sum_a \frac{\eta_a e_a p_a}{p_a \cdot \hat{k}} \ln\left(\frac{p_a \cdot \hat{k}\, 2e^C \Delta}{m_a^2}\right)\right]\cdot\left[\sum_a \frac{\eta_a e_a p_a}{p_a \cdot \hat{k}}\right], \tag{2.43}$$

and $(dk)_\Delta = -\int d\hat{k}\, d\omega \ln(\omega/\Delta)\partial/\partial\omega$. The amplitudes F are suitable for perturbative calculation or phenomenological parametrization.

The cross section is independent of Δ provided the F_n satisfy the infrared-coherence condition. All quantities in Eq. (2.42) may be expanded in a power series in α, and to each order in α, $\sigma(\Omega)$ is independent of Δ. Successive terms in the sum correspond to emission of an additional photon and decrease by order α. In the next two subsections it will be seen how the cross-section formula (2.42) may be used to account for radiation damping nonperturbatively, while retaining a perturbative expansion for the scattering amplitudes F.

C. Radiation exclusive cross section

Here we pose the traditional question: "What is the rate $\sigma(\Omega_h, \omega_0)$ for observing a number of final particles called hard in a volume of phase space Ω_h, accompanied by an undetected energy loss to the radiation field no greater than a small number ω_0?" Call such a rate "radiation exclusive" because loss of much energy to unobserved photons is excluded. The quantity ω_0 is the resolution of the apparatus which measures the energy of the hard particles, and may depend on their momenta $\omega_0 = \omega_0(p_f)$. Suppose for simplicity that none of the hard particles are photons.[16] In this case the phase-space volume is a product of the charged-particle phase space and the photon phase space

$$\chi_n(k_{f1}\cdots k_{fn}, p_f) = \chi(p_f)\,\theta(\omega_0 - \omega_{f1} - \cdots - \omega_{fn}) . \tag{2.44}$$

For a first orientation assume that ω_0 is sufficiently small that the infrared-coherence property of the S matrix may be used,

$$F_n(k_{f1}\cdots k_{fn}, p_f; k_i, p_i)^{\mu_{f1}\cdots\mu_{fn}} \approx \left(-\sum_a \frac{\eta_a e_a p_a^{\mu_{f1}}}{(2\pi)^{3/2}\, p_a\cdot \hat{k}}\right)\cdots\left(-\sum_a \frac{\eta_a e_a p_a^{\mu_{fn}}}{(2\pi)^{3/2}\, p_a\cdot \hat{k}_{fn}}\right) F_0(p_f; k_i, p_i) , \tag{2.45}$$

and that the recoil due to the radiation may be neglected

$$\delta^4\left(\sum k_f + \sum p_f - \sum k_i - \sum p_i\right) \approx \delta^4\left(\sum p_f - \sum k_i - \sum p_i\right) . \tag{2.46}$$

In this case the cross-section formula becomes

$$\sigma(\Omega_h, \omega_0) = \int dp_f\, R(\omega_0)\, \chi(p_f)\, | F_0(p_f; k_i, p_i)|^2 F_0(p_f; k_i, p_i)|^2\, \delta^4\left(\sum p_f - \sum k_i - \sum p_i\right) , \tag{2.47}$$

$$R(\omega_0) = e^{-J(p,\Delta)} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int (dk_1)_\Delta \cdots (dk_n)_\Delta \left[\sum_a \frac{\eta_a e_a p_a}{(2\pi)^{3/2}\, p_a\cdot\hat{k}}\right]^2 \cdots \left[\sum \frac{\eta_a e_a p_a}{(2\pi)^{3/2}\, p_a\cdot\hat{k}}\right]^2 \times \theta(\omega_0 - \omega_1 - \cdots - \omega_n) \right\} . \tag{2.48}$$

The radiation damping factor $R(\omega_0)$ may be evaluated by setting

$$\theta(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dt}{t - i\epsilon}\, e^{i\omega t} , \tag{2.49}$$

which gives

$$R(\omega_0) = e^{-J(p,\Delta)} \left\{ 1 + \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \frac{dt}{\epsilon + it}\, e^{i\omega_0 t} \left[e^{I(t)} - 1\right] \right\} , \tag{2.50}$$

$$I(t) = \frac{-1}{(2\pi)^3} \int (dk)_\Delta \left(\sum_a \frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\right)^2 e^{-i\omega t} . \tag{2.51}$$

From Eq. (2.35) this gives

$$I(t) = \frac{1}{2}\frac{1}{(2\pi)^3} \int d\hat{k} \left(\sum_a \frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\right)^2 \int_0^{\infty} d\omega \ln\left(\frac{\omega}{\Delta}\right) \frac{\partial}{\partial\omega}\, e^{-(\epsilon + it)\omega} .$$

An ϵ has been introduced to make the integral on ω well defined. It has been evaluated previously, Eqs. (I.4.60)–(I.4.63),

$$I(t) = \frac{1}{(2\pi)^3} \int \frac{d\hat{k}}{2} \left(\sum_a \frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\right)^2 \ln\left[(\epsilon + it)\, \Delta e^C\right] , \tag{2.52}$$

where C is Euler's constant. With B defined in Eq. (2.38) we have

$$I(t) = -B \ln(\epsilon + it) + L(\Delta) , \tag{2.53}$$

$$L(\Delta) = \frac{1}{2}\frac{1}{(2\pi)^3} \int d\hat{k} \left(\sum_a \frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\right)^2 \ln(\Delta e^C) . \tag{2.54}$$

Equation (2.50) becomes

$$R(\omega_0) = e^{-J(p,\Delta)} \left[1 - \theta(\omega_0) + \frac{e^{L(\Delta)}}{(2\pi)} \int_{-\infty}^{\infty} dt\, e^{i\omega_0 t} (\epsilon + it)^{-B-1}\right] ,$$
$$R(\omega_0) = e^{-[J(p,\Delta) - L(\Delta)]} \frac{\omega_0^{\,B}}{\Gamma(1+B)} . \tag{2.55}$$

The expected dependence $\omega_0^{\,B}$ on the experimental resolution ω_0 has made its appearance. From Eqs. (2.43) and (2.54) we find

$$J(p,\Delta)-L(\Delta)=\frac{1}{2}\frac{1}{(2\pi)^3}\int d\hat{k}\left[\sum_a \frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\right]\cdot\left[\sum_a \frac{e_a\eta_a p_a}{p_a\cdot\hat{k}}\ln\left(\frac{2p_a\cdot\hat{k}}{m_a{}^2}\right)\right] \quad . \tag{2.56}$$

The arbitrary quantity Δ has canceled out. Because $J(p,\Delta)$ satisfies

$$J(p,\Delta)=J(p,1)-B\ln\Delta\ , \tag{2.57}$$

we may write

$$R(\omega_0)=\exp[-J(p,e^{-C})]\ \frac{\omega_0{}^B}{\Gamma(1+B)}\ , \tag{2.58}$$

$$R(\omega_0)=\exp[-J(p,\omega_0/D)]\ , \tag{2.59}$$

where

$$D=e^C[\Gamma(1+B)]^{1/B}=1+O(B^2)\ . \tag{2.60}$$

[Note that B, Eq. (2.38), is proportional to α.] We thus obtain for the cross section (2.47)

$$\sigma(\Omega_h,\omega_0)=\int dp_f \exp[-J(p,\omega_0/D)]\chi(p_f)\,|F_0(p_f;k_i,p_i)|^2\,\delta^4(\textstyle\sum p_f-\sum k_i-\sum p_i)\ . \tag{2.61}$$

The dependence of the cross section on the experimental resolution has been obtained many times.[17] In the massive-photon method it emerges from a cancellation between real and virtual infrared divergences. The novel feature here is that it multiplies the modulus squared of one scattering amplitude. Thus in experiments in which the observed hard particles account for almost all energy except a small amount lost to radiation field, the scattering amplitudes introduced here are observable in the same way that scattering amplitudes are usually considered observable in a theory without massless particles.

The cross section (2.61) is nonperturbative, but approximate, for the recoil produced by the unobserved photon has been neglected. Suppose we wish to account for the recoil systematically. A comparison of the approximate formula (2.61) with the exact formula (2.42) shows that the approximate formula coincides with the zeroth term of the exact formula, provided the arbitrary quantity Δ is fixed at

$$\Delta=\omega_0/D\ . \tag{2.62}$$

This makes the zeroth term in the infinite series exact to all orders in α as ω_0 approaches zero. One expects the successive terms in the series, each of which corresponds to the emission of an additional photon, to decrease by $\alpha\omega_0$. This is a great improvement over the purely perturbative expansion of the cross section which is a series in $\alpha\ln\omega_0$.

D. Radiation inclusive cross section

In many cases the experimenter does not measure the momentum of every energetic particle participating in a collision. For example, in μ decay or in inelastic electron-proton scattering only the momenta of the initial particles and final electron are measured, so it is not known how the remaining energy is shared between photons and other particles. This situation may be called "radiation inclusive," for the final state may include unobserved high-energy photons. In this case radiation damping is not determined by the energy resolution of the detector, and a cross-section formula is required which manifests strong radiation damping if the unobserved photons have little energy, but not if they have a lot of energy. We surmise that the radiation-damping factor will depend on the energy of the unobserved photons.

For a first orientation let us calculate the rate $\sigma_0(\Omega_q,\omega_0)$ for emission of charged particles into a volume of charged-particle phase space Ω_q accompanied by unobserved photons, none of which individually has an energy greater than a small number ω_0. This situation is observable in principle and corresponds to surrounding the scatterer with photosensitive material which detects photons of frequency above ω_0, and rejecting events where a photon is recorded. In this case the projectors χ_n factorize

$$\chi_n(k_{f1}\cdots k_{fn},p_f)=\chi(p_f)\theta(\omega_0-\omega_{f1})\cdots\theta(\omega_0-\omega_{fn}), \tag{2.63}$$

where $\chi(p_f)$ is one for $p_f\in\Omega_q$ and zero otherwise. Again making use of infrared coherence of the S matrix below ω_0, and with neglect of recoil due to unobserved photons, we find from Eq. (2.42)

$$\sigma_0(\Omega_q,\omega_0)=\int dp_f\, Q(\omega_0)\chi(p_f)|F_0(p_f;k_i,p_i)|^2\delta^4(\textstyle\sum p_f-\sum k_i-\sum p_i), \tag{2.64}$$

$$Q(\omega_0)=e^{-J(p,\Delta)}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left[\int (dk)_\Delta\frac{1}{(2\pi)^3}\left(\sum_a\frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\right)^2\theta(\omega_0-\omega)\right]^n\right\}. \tag{2.65}$$

With $(dk)_\Delta=-\frac{1}{2}\int dk\,d\omega\ln(\omega/\Delta)\partial/\partial\omega$, one has immediately

$$Q(\omega_0)=e^{-J(p,\Delta)}\exp\left[-\int\frac{d\hat{k}}{2}\frac{1}{(2\pi)^3}\left(\sum_a\frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\right)^2\ln\left(\frac{\omega_0}{\Delta}\right)\right].$$

Comparison with Eq. (2.43) yields

$$Q(\omega_0)=e^{-J(p,\omega_0)}, \tag{2.66}$$

which gives for the cross section

$$\sigma_0(\Omega_q,\omega_0)=\int dp_f\, e^{-J(p,\omega_0)}\chi(p_f)|F_0(p_f;k_i,p_i)|^2\delta^4(\textstyle\sum p_f-\sum k_i-\sum p_i). \tag{2.67}$$

Thus the infinite sum (2.42) adds up to the zeroth term with Δ set at ω_0.

Equation (2.57), $J(p,\omega_0)=J(p,1)-B\ln\omega_0$, allows the cross section (2.67) to be expressed as

$$\sigma_0(\Omega_q,\omega_0)=\int dp_f\,\omega_0{}^B e^{-J(p,1)}\chi(p_f)|F_0(p_f;k_i,p_i)|^2\delta^4(\textstyle\sum p_f-\sum k_i-\sum p_i). \tag{2.68}$$

The usual infrared damping factor has resurfaced. The seemingly trivial identity

$$\omega_0{}^B=\int_0^{\omega_0}\frac{d\omega}{\omega}B\omega^B \tag{2.69}$$

does not possess a perturbative expansion, for B is proportional to α and the integrand is of nominal order α whereas the integral is of order unity. The first term in a perturbative expansion of the integrand would be divergent. With this identity and the definition of B, Eq. (2.38a), the cross section becomes

$$\sigma_0(\Omega_q,\omega_0)=\int dp_f\int\frac{d\hat{k}}{2}\int_0^{\omega_0}\frac{d\omega}{\omega}e^{-J(p,\omega)}\chi(p_f)\frac{(-1)}{(2\pi)^3}\left(\sum_a\frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\right)^2|F_0(p_f;k_i,p_i)|^2\delta^4(\textstyle\sum p_f-\sum k_i-\sum p_i). \tag{2.70}$$

Infrared coherence states that for sufficiently low frequency the one-photon bremsstrahlung amplitude is related to the no-photon amplitude by

$$F_1^{\mu_f}(k_f,p_f;k_i,p_i)=\sum_a\frac{-\eta_a e_a p_a^{\mu_f}}{p_a\cdot\hat{k}_f(2\pi)^{3/2}}F_0(p_f;k_i,p_i),$$

so for sufficiently small ω_0, Eq. (2.70) becomes

$$\sigma_0(\Omega_q,\omega_0)=\int dp_f\int dk_f\, e^{-J(p,\omega_f)}\chi(p_f)\theta(\omega_0-\omega_f)|F_1(k_f,p_f;k_i,p_i)|^2\delta^4(k_f+\textstyle\sum p_f-\sum k_i-\sum p_i), \tag{2.71}$$

where $dk=\frac{1}{2}d\hat{k}\,d\omega/\omega$ is the usual photon phase space. This formula expresses the cross section for emission of no photons of energy greater than ω_0 as an integral over the bremsstrahlung cross section for emission of a photon with energy less than ω_0. The infrared divergence is averted because of the radiation-damping factor ω^B. It gives instead a cross section of lower order in α than the bremsstrahlung cross section.

Another way of stating this is that the probability for a small energy loss ω per unit frequency interval is proportional to $B\omega^{B-1}$. This is singular at $\omega=0$, but is integrable and gives an integrated probability for energy loss no greater than ω_0 which is proportional to $\omega_0{}^B$. The usual zeroth or no radiation term represents a probability for energy loss per unit frequency interval of the form $\delta(\omega)$, which in a correct treatment is replaced by $B\omega^{B-1}$. This manifests the fundamentally radiative nature of the scattering of charged particles.

We now seek an exact formula which embodies the features of the approximate formula (2.71). We begin by calculating the transition rate $\sigma_1(\Omega_q,\omega_0)$ for emission of charged particles into a volume of charged-particle phase space Ω_q and emission of at least one photon with energy greater than some fixed number ω_0. Later we shall let ω_0 tend toward zero. In this case projectors χ_n factorize into a projector $\chi(p_f)$ onto Ω_q and a projector onto the n-photon phase space,

$$\chi_0(p_f)=0, \tag{2.72a}$$

and for $n\geq 1$,

$$\chi_n(k_1\cdots k_n,p)=\chi(p)[\theta(\omega_1-\omega_0)\theta(\omega_1-\omega_2)\cdots\theta(\omega_1-\omega_n)+\theta(\omega_2-\omega_0)\theta(\omega_2-\omega_1)\theta(\omega_2-\omega_3)\cdots\theta(\omega_2-\omega_n)+\cdots$$
$$+\theta(\omega_n-\omega_0)\theta(\omega_n-\omega_1)\cdots\theta(\omega_n-\omega_{n-1})]. \tag{2.72b}$$

This formula expresses the fact that some one of the n photons has an energy greater than all the others and the requirement that this be greater than ω_0. The amplitudes are symmetric in the photon variables, which allows the substitution

$$\chi_n(k_1\cdots k_n,p)\rightarrow n\chi(p)\theta(\omega_n-\omega_0)\theta(\omega_n-\omega_1)\cdots\theta(\omega_n-\omega_{n-1}).$$

The exact cross-section formula (2.42) gives

$$\sigma_1(\Omega_q,\omega_0)=\int dp_f\,\chi(p_f)\int dk_f\,\theta(\omega_f-\omega_0)\,e^{-J(p,\Delta)}$$
$$\times\Big[|F_1(k_f,p_f\,;k_i,p_i)|^2\delta^4(k_f+\textstyle\sum p_f-\sum k_i-\sum p_i)$$
$$+\sum_{n=2}^{\infty}\frac{1}{(n-1)!}\int(dk_{f1})_\Delta\theta(\omega_f-\omega_{f1})\cdots(dk_{f,n-1})_\Delta\theta(\omega_f-\omega_{f,n-1})$$
$$\times|F_n(k_f,k_{f1}\cdots k_{f,n-1},p_f;k_ip_i)|^2\delta^4(\textstyle\sum k_f+\sum p_f-\sum k_i-\sum p_i)\Big]. \tag{2.73}$$

Here we have written k_f instead of k_{fn}, and the integral

$$\int(dk_f)_\Delta\theta(\omega_f-\omega_0)\cdots=-\tfrac{1}{2}\int d\hat{k}_f\int_0^\infty d\omega_f\ln\left(\frac{\omega_f}{\Delta}\right)\frac{\partial}{\partial\omega_f}\,\theta(\omega_f-\omega_0)\cdots$$

was integrated by parts to give $\frac{1}{2}\int dk_f\int_0^\infty(d\omega_f/\omega_f)\theta(\omega_f-\omega_0)\cdots$. Observe that with the substitution $n\rightarrow n+1$, the integrand has the usual infrared-coherence properties in the variables $k_{f1}\cdots k_{fn}$, and consequently it is independent of the arbitrary parameter Δ for each value of k_f. Thus we are at liberty to set $\Delta=\omega_f$, which gives

$$\sigma_1(\Omega_q,\omega_0)=\int dp_f\,\chi(p_f)\int dk_f\,\theta(\omega_f-\omega_0)e^{-J(p,\omega_f)}$$
$$\times\Big[|F_1(k_{f1}p_f;k_i,p_i)|^2\delta^4(k_f+\textstyle\sum p_f-\sum k_i-\sum p_i)$$
$$+\sum_{n=1}^{\infty}\frac{1}{n!}\int(dk_{f1})_{\omega_f}\theta(\omega_f-\omega_{f1})\cdots(dk_{fn})_{\omega_f}\theta(\omega_f-\omega_{fn})$$
$$\times|F_n(k_{f1}\cdots k_{fn},p_f;k_i,p_i)|^2\delta^4(\textstyle\sum k_f+\sum p_f-\sum k_i-\sum p_i)\Big], \tag{2.74a}$$

$$(dk_{fr})_{\omega_f}=-\int d\hat{k}_{fr}\int_0^\infty d\omega_{fr}\ln\left(\frac{\omega_{fr}}{\omega_f}\right)\frac{\partial}{\partial\omega_{fr}}. \tag{2.74b}$$

With this choice for Δ, $\Delta=\omega_f$, the leading term in this expansion agrees with the approximate formula (2.71), except that $\theta(\omega_f-\omega_0)$ appears instead of $\theta(\omega_0-\omega_f)$. The emission of charged particles into the charged-particle phase-space volume Ω_q is accompanied by emission of either no photons of energy greater than ω_0, or at least one photon of energy greater than ω_0. Therefore, $\sigma(\Omega_q)$, the total rate for emission of charged particles into Ω_q, is given by

$$\sigma(\Omega_q)=\sigma_0(\Omega_q,\omega_0)+\sigma_1(\Omega_q,\omega_0). \tag{2.75}$$

As ω_0 approaches zero $\sigma_0(\Omega_q,\omega_0)$ vanishes like ${\omega_0}^B$, according to Eq. (2.61) and we have

$$\sigma(\Omega_q)=\sigma_1(\Omega_q,0). \tag{2.76}$$

This is a manifestation of the principle that if you look closely enough at a collision of charged particles you will see at least one photon, or for that matter, at least n photons. We thus obtain

$$\sigma(\Omega_q) = \int dp_f\, \chi(p_f) \int dk_f\, e^{-J(p,\omega_f)} \Big[|F_1(k_{f1} p_f; k_i, p_i)|^2 \delta^4(k_f + \sum p_f - \sum k_i - \sum p_i) + \sum_{n=1}^{\infty} \frac{1}{n!} \int (dk_{f1})_{\omega_f}\, \theta(\omega_f - \omega_{f1}) \cdots (dk_{fn})_{\omega_f}\, \theta(\omega_f - \omega_{fn}) \times |F_{n+1}(k_f, k_{f1} \cdots k_{fn}, p_f; k_i, p_i)|^2 \delta^4(\sum k_f + \sum p_f - \sum k_i - \sum p_i) \Big]. \quad (2.77)$$

Because $e^{-J(p,\omega_f)} = e^{-J(p,1)} \omega_f{}^B$, this formula has the desired feature that for small values of ω_f successive terms in the series decrease by $\alpha\omega_f$, and the leading term agrees with Eq. (2.71), and thus has the correct nonperturbative radiation damping. For large values of ω_f, the leading term has no particular nonperturbative virtue, but it is of order α and $e^{-J(p,\omega)}$ is of order unity, and thus it correctly gives the usual one-photon bremsstrahlung cross section. It is proposed to test this formula on the electron spectrum in μ decay and compare with known results. If the leading term is accurate, this formula may be useful in calculating radiative corrections to other processes such as inelastic electron-proton scattering or electron-positron annihilation.

III. REDUCTION FORMULAS

A. Reduction formula for photons and superselection rules

In this section we effect Lehmann-Symanzik-Zimmermann (LSZ) projections on the asymptotic vector potential. This yields a reduction formula for photons of finite frequency and conserved zero-frequency quantities which commute with all local observables. The latter define superselection sectors.

Consider the LSZ projection on the asymptotic vector potential, Eq. (I. 2.9),

$$A_\mu^{\rm as}(x) = A_\mu^{\rm in}(x) + \frac{1}{4\pi} \int d^4p \frac{p_\mu \rho^{\rm in}(p)}{[(p\cdot x)^2 - p^2 x^2]^{1/2}} \quad (3.1)$$

and in $\to$ out,

$$c_\mu^{\rm as}(k,t) \equiv \frac{i\omega}{(2\pi)^{3/2}} \int_{x^0=t} e^{ik\cdot x} \overleftrightarrow{\partial}_0 A_\mu^{\rm as}(x) d^3x\,. \quad (3.2)$$

If the expansion (I. 4.3c) is substituted for $A^{\rm in}$ there results

$$c_\mu^{\rm as}(k,t) = c_\mu^{\rm in}(k) + \int d^4p\, g_\mu(k,p,t) \rho^{\rm in}(p)\,, \quad (3.3)$$

$$g_\mu(k,p,t) = \frac{i\omega}{(2\pi)^{3/2}} \frac{1}{4\pi} \int_{x^0=t} e^{ik\cdot x} \overleftrightarrow{\partial}_0 \frac{p_\mu}{[(p\cdot x)^2 - p^2 x^2]^{1/2}}\,. \quad (3.4)$$

The first term in Eq. (3.3) gives the expected annihilation operator and the second term represents the correction due to the Liénard-Wiechert potential of the asymptotic charged particles. Because this potential is not a solution of the free wave equation the integral (3.4) depends on the spacelike surface integrated over. In the present instance it is the plane $x^0 = t$ and the result depends on the parameter t,

$$g_\mu(k,p,t) = \frac{1}{(2\pi)^{3/2}} \frac{p_\mu}{p\cdot \hat{k}} \exp(ik\cdot pt/E)\,,$$

where $E = p^0 = (\vec{p}^2 + m^2)^{1/2}$. This gives

$$c_\mu^{\rm as}(k,t) = c_\mu^{\rm in}(k) + \int d^4p \frac{1}{(2\pi)^{3/2}} \frac{p_\mu}{p\cdot\hat{k}} \exp(ik\cdot pt/E) \rho^{\rm in}(p) \quad (3.5)$$

and in $\to$ out. Because $\rho^{\rm in}(p)$ commutes with $c_\mu^{\rm in}(k)$ and $c_\mu^{\rm in\,\dagger}(k)$, the additional term is a shift in the photon variables by a quantity which commutes with the photon variables, so $c_\mu^{\rm as}(k,t)$ is the annihilation operator for a coherent state. If this expression is contracted with a test function $f(k) \in \mathcal{S}_0$ (f vanishes like ω at $\omega = 0$) the last term gives a vanishing contribution at asymptotic times by the Riemann-Lebesgue lemma and we have

$$\lim_{t\to-\infty} \int dk f_\mu^*(k) (-g^{\mu\nu}) c_\nu^{\rm as}(k,t) = \int dk f_\mu^*(k)(-g^{\mu\nu}) c_\nu^{\rm in}(k) = c^{\rm in}(f),\quad f \in \mathcal{S}_0\,. \quad (3.6)$$

On the other hand, in the strictly zero-frequency limit of Eq. (3.5) the second term has a finite limit independent of the time,

$$\lim_{\omega\to 0} c_\mu^{\rm as}(k,t) = c_\mu^{\rm in}(\hat{k}) + \int d^4p \frac{1}{(2\pi)^{3/2}} \frac{p_\mu}{p\cdot\hat{k}} \rho^{\rm in}(p)\,, \quad (3.7)$$

where $c_\mu^{\rm in}(\hat{k}) = \lim_{\omega\to\infty} c_\mu^{\rm in}(\omega,\hat{k})$.

If these relations are expressed as projections on $A(x)$ at asymptotic times, they provide a more precise meaning to $\lim_{t\to-\infty} A(x) = A^{\rm as}(x)$. Thus for $f_\mu(k) \in \mathcal{S}_0$, with

$$f_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int dk\,\omega e^{-ik\cdot x} f_\mu(k)\,, \tag{3.8}$$

we have

$$c^{\text{in}}(f) = \lim_{t\to-\infty} i \int_{x^0=t} f_\mu^*(x)(-g^{\mu\nu})\overleftrightarrow{\partial}_0 A_\nu(x) d^3x\,, \tag{3.9}$$

and also

$$c_\mu^{\text{in}}(\hat{k}) + \int d^4p \frac{1}{(2\pi)^{3/2}} \frac{p_\mu}{p\cdot\hat{k}} \rho^{\text{in}}(p) = \lim_{t\to-\infty} \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} \int_{x^0=t} e^{ik\cdot x}\overleftrightarrow{\partial}_0 A_\mu(x) d^3x\,. \tag{3.10}$$

There are corresponding expressions for creation operators obtained by Hermitian conjugation, and also for out variables obtained as t approaches plus infinity.

The usual LSZ machinery[18] may be employed with the finite-frequency relations. It gives, for $f\in\mathcal{E}_0$ and T a time-ordered product,

$$c^{\text{out}}(f)T[\cdots] - T[\cdots]c^{\text{in}}(f) = i\int d^4x f_\mu^*(x)(-g^{\mu\nu})\partial^2 T[A_\nu(x)\cdots]\,, \tag{3.11a}$$

$$c^{\text{out}\dagger}(f)T[\cdots] - T[\cdots]c^{\text{in}\dagger}(f) = -i\int d^4x f_\mu(x)(-g^{\mu\nu})\partial^2 T[A_\nu(x)\cdots]\,. \tag{3.11b}$$

Because $c^{\text{in}\dagger}(f)$ and $c^{\text{out}\dagger}(f)$, $f\in\mathcal{E}_0$, are creation operators for retarded and advanced states, respectively, we obtain, by taking matrix elements between generic retarded and advanced states α^{ret} and β^{ad}, from Eq. (3.8),

$$\langle f, \beta^{\text{ad}} | T[\cdots] | \alpha^{\text{ret}}\rangle = \frac{-i}{(2\pi)^{3/2}} \int dk\, \omega f_\mu^*(k)(-g^{\mu\nu}) \lim_{k^2\to 0} k^2 \int d^4x\, e^{ik\cdot x} \langle \beta^{\text{ad}} | T[A_\nu(x)\cdots] | \alpha^{\text{ret}}\rangle\,, \tag{3.12a}$$

$$\langle \beta^{\text{ad}} | T[\cdots] | f, \alpha^{\text{ret}}\rangle = \frac{-i}{(2\pi)^{3/2}} \int dk\, \omega f_\mu(k)(-g^{\mu\nu}) \lim_{k^2\to 0} k^2 \int d^4x\, e^{-ik\cdot x} \langle \beta^{\text{ad}} | T[A_\nu(x)\cdots] | \alpha^{\text{ret}}\rangle\,, \tag{3.12b}$$

where photon disconnected parts have not been written explicitly. These are the desired reduction formulas for photons of finite frequency. They have the standard form, but the interpretation is somewhat different, for the retarded and advanced subspaces $\mathcal{F}^{\text{ret}} \subset \mathcal{F}^{\text{in}} = \mathcal{F}^{\text{out}}$, $\mathcal{F}^{\text{ad}} \subset \mathcal{F}^{\text{in}} = \mathcal{F}^{\text{out}}$ are distinct, $\mathcal{F}^{\text{ret}} \neq \mathcal{F}^{\text{ad}}$.

The zero-frequency radiation field appears in the expression for the asymptotic charged fields. Set

$$s_\mu^{\text{in}}(\hat{k}) \equiv c_\mu^{\text{in}}(\hat{k}) + \int d^4p \frac{1}{(2\pi)^{3/2}} \frac{p_\mu}{p\cdot\hat{k}} \rho^{\text{in}}(p) \tag{3.13}$$

and in $\to$ out. From Eq. (3.10) we have

$$\begin{aligned} s_\mu^{\text{out}}(\hat{k}) - s_\mu^{\text{in}}(\hat{k}) &= \lim_{t\to\infty} \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} \int_{-t}^{t} dx^0 d^3x\, \partial_0[e^{ik\cdot x}\overleftrightarrow{\partial}_0 A_\mu(x)] \\ &= \lim_{t\to\infty} \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} \int_{-t}^{t} dx^0 d^3x\, e^{ik\cdot x} J_\mu(x)\,. \end{aligned}$$

This expression may be estimated by its zeroth component, in which case it becomes

$$\lim_{t\to\infty} \lim_{\omega\to 0} \frac{2i\omega t}{(2\pi)^{3/2}} Q\,,$$

where Q is the charge operator. This limit vanishes and we have $s^{\mu\,\text{out}}(\hat{k}) = s^{\mu\,\text{in}}(\hat{k})$, a relation obtained earlier, Eq. (2.13), by a different method. Set

$$s^\mu(\hat{k}) \equiv s^{\mu\,\text{out}}(\hat{k}) = s^{\mu\,\text{in}}(\hat{k})\,. \tag{3.14}$$

by Eq. (3.10) we have

$$\begin{aligned} s_\mu(\hat{k}) &= \lim_{t\to\infty} \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} \int_{x^0=t} e^{ik\cdot x}\overleftrightarrow{\partial}_0 A^\mu(x) d^3x \\ &= \lim_{t\to-\infty} \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} \int_{x^0=t} e^{ik\cdot x}\overleftrightarrow{\partial}_0 A^\mu(x) d^3x\,. \end{aligned}$$

The fact that the two expressions on the right are equal for large positive and negative t suggests that the quantity

$$s^\mu(\hat{k}, t) \equiv \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} \int_{x^0=t} e^{ik\cdot x}\overleftrightarrow{\partial}_0 A^\mu(x) d^3x$$

may in fact be time-independent. Its time derivative

$$\dot{s}^\mu(\hat{k}, t) = \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} \int e^{ik\cdot x} J^\mu(x) d^3x$$

also may be estimated by its zeroth component

$$\dot{s}^\mu(\hat{k}, t) = \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} Q = 0\,,$$

so

$$s^\mu(\hat{k}) = \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} \int_{x^0=t} e^{ik\cdot x}\overleftrightarrow{\partial}_0 A^\mu(x) d^3x \tag{3.15}$$

is, in fact, time-independent.

It is tempting to set $s^\mu(\hat{k}) = 0$, which would characterize a time-symmetric subspace of $\mathcal{F}(\mathcal{E})$. However, by Eq. (3.13) we have

$$\hat{k}\cdot s(\hat{k}) = \hat{k}\cdot c^{\text{in}}(\hat{k}) + \frac{Q}{(2\pi)^{3/2}} = \hat{k}\cdot c^{\text{out}}(\hat{k}) + \frac{Q}{(2\pi)^{3/2}} , \tag{3.16}$$

so if $s_\mu(\hat{k})$ vanishes the Gupta-Bleuler condition is violated in the charged sectors. Let L be any operator with compact support in space-time. By Eq. (3.15) we have

$$[s^\mu(\hat{k}), L] = \lim_{\omega\to 0} \frac{i\omega}{(2\pi)^{3/2}} \int_{x^0=t} e^{ik\cdot x}\overleftrightarrow{\partial}_0[A^\mu(x), L]d^3x .$$

Because of locality, the integration on $\vec{x}$ extends over the intersection of the plane $x^0 = t$ with the interior of the future or past light cone subtended by the support of L. This is a finite region, so as ω approaches zero, the right-hand side is annihilated by the explicit factor of ω. Thus $s^\mu(\hat{k})$ commutes with any local operator L,

$$[s^\mu(\hat{k}), L] = 0 . \tag{3.17}$$

This result, and the explicit expressions for $s_\mu(\hat{k}) = s_\mu^{\text{in}}(\hat{k}) = s_\mu^{\text{out}}(\hat{k})$, Eq. (3.13), will be very convenient in deriving a reduction formula for the charged asymptotic field.

The fact that $s_\mu(\hat{k})$ commutes with all local operators leads to a large set of superselection rules[19,20] in the retarded Hilbert space $\mathcal{H}^{\text{ret}}$ which is the completion in norm of $\mathfrak{F}^{\text{ret}\,+}$. The latter is characterized by $c^{\text{in}}(\hat{k})\mathfrak{F}^{\text{ret}\,+} = 0$ and $\hat{k}\cdot c_n^{\text{in}}(k)\mathfrak{F}^{\text{ret}\,+} = 0$. Physical observables are local quantities which leave both of these conditions invariant. Let L be one of them. By locality it commutes with $s(\hat{k})$, $[s(\hat{k}), L] = 0$. Because it is an observable it leaves $\mathfrak{F}^{\text{ret}\,+}$ invariant, $L\mathfrak{F}^{\text{ret}\,+} \subset \mathfrak{F}^{\text{ret}\,+}$, so $c^{\text{in}}(\hat{k})L\mathfrak{F}^{\text{ret}\,+} = 0$. This gives

$$\begin{aligned}[s_\mu(\hat{k}), L]\mathfrak{F}^{\text{ret}\,+} &= \left[c_\mu^{\text{in}}(\hat{k}) + \int d^4p \frac{1}{(2\pi)^{3/2}} \frac{p_\mu}{p\cdot\hat{k}} \rho^{\text{in}}(p), L\right] \times \mathfrak{F}^{\text{ret}\,+} \\ &= \left[\int d^4p \frac{1}{(2\pi)^{3/2}} \frac{p_\mu}{p\cdot\hat{k}} \rho^{\text{in}}(p), L\right]\mathfrak{F}^{\text{ret}\,+} \\ &= 0 ,\end{aligned} \tag{3.18a}$$

and hence

$$\left[\int d^4p \frac{1}{(2\pi)^{3/2}} \frac{p_\mu}{p\cdot\hat{k}} \rho^{\text{in}}(p), L\right]\mathcal{H}^{\text{ret}} = 0 . \tag{3.18b}$$

Thus all local observables commute with $\int d^4p (2\pi)^{-3/2}(p_\mu/p\cdot\hat{k})\rho^{\text{in}}(p)$, and the eigenspaces of this operator define superselection sectors. The relative phase of vectors in two different eigenspaces of this operator is without physical meaning.

From the form of this operator it follows that states with different numbers and momenta of the charged particles are in different superselection sectors, apart from exceptional momenta which are of measure zero. [For example, pairs of particles with equal velocity but opposite charge are annihilated by $\int d^4p (2\pi)^{-3/2}(p\cdot\hat{k})^{-1}p_\mu\rho^{\text{in}}(p)$ are thus in the same superselection sector as the vacuum.] Consequently, in the retarded representation, incident states are represented by density matrices which are diagonal in the numbers and momenta of the incident charged particles. A density matrix of this type obviously provides an accurate description of a large class of scattering situations. More generally such a density matrix, and the wave function to specify a state of the radiation field with finite photon number, appears in fact to offer the most complete description possible of an incident state containing charged particles.

B. Reduction formula for charged particles

To reduce out charged particles we make use of the asymptotic charged particle field found in Sec. I IV D. In particular for an electron, a particle which by convention has charge $-e$, the asymptotic charged field is given by

$$\begin{aligned}\psi^{\text{as}}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E} \sum_s [&D(p,x)b_s(p)u_s(p)e^{-ip\cdot x} \\ &+ d_s^\dagger(p)D(p,x)v_s(p)e^{ip\cdot x}] ,\end{aligned} \tag{3.19a}$$

where

$$\begin{aligned}D(p,x) = &\exp[ieQ(p)\epsilon(p\cdot x)\ln(|p\cdot x|\, l_1/m)] \\ &\times\exp[-eA^\dagger(p)\ln(\epsilon - ip\cdot x\, l_1/m)] \\ &\times\exp[eA(p)\ln(\epsilon + ip\cdot x\, l_1/m)]\end{aligned} \tag{3.19b}$$

represents the logarithmic distortion of the plane wave, and $\epsilon(s) = s/|s|$ represents the sign function. The first factor of D represents the logarithmic distortion in phase produced by the Liénard-Wichert potentials of the charged particles, with

$$Q(p) = \frac{1}{4\pi}\int d^4p' \frac{p\cdot p'}{[(p\cdot p')^2 - p^2p'^2]^{1/2}} \rho(p') , \tag{3.20}$$

and the last two factors represent the logarithmic distortion in amplitude produced by the zero-frequency radiation field

$$A(p) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\hat{k}}{2} \frac{p\cdot c(\hat{k})}{p\cdot\hat{k}} . \tag{3.21}$$

The quantities b, d, D, Q, A, ρ, and c are all understood to bear either in or out labels, and l_1 is an infrared renormalization constant.

Let p_i be the 4-momenta of a set of incoming particles with charges e_i, and correspondingly for p_f and e_f, and consider the matrix element

$$\langle\psi^{\text{as}-}(x)\rangle \equiv \langle p_f^{\text{ad}} | \psi^{\text{as}-}(x)\, T[\cdots]|p_i^{\text{ret}}\rangle\,, \tag{3.22}$$

where $T[\cdots]$ is a time-ordered product, polarization indices and photon labels are suppressed, and $\psi^{\text{as}-}$ is the annihilation part of ψ^{as},

$$\psi^{\text{as}-}(x) = \frac{1}{(2\pi)^{3/2}} \times \int \frac{d^3p}{2E}\, D^{\text{out}}(p,x)\sum_s b_s^{\text{out}}(p)u_s(p)e^{-ip\cdot x}\,. \tag{3.23}$$

We will maneuver this matrix element until an electron of momentum p has been introduced into the final state. The retarded and advanced labels mean that the incoming and outgoing states have finite photon number,

$$c_\mu^{\text{in}}(\hat{k})|p_i^{\text{ret}}\rangle = 0, \quad \langle p_f^{\text{adv}}|c_\mu^{\text{out}\dagger}(\hat{k}) = 0\,, \tag{3.24}$$

which implies in particular $\langle p_f^{\text{ad}}|A^{\text{out}\dagger}(p) = 0$. From the definition of $Q(p)$, Eq. (3.20),

$$\langle p_f^{\text{ad}}|[eQ^{\text{out}}(p)] = \gamma_f\langle p_f^{\text{ad}}|\,, \tag{3.25a}$$

$$\gamma_f = \frac{e}{4\pi}\sum_f \frac{e_f p\cdot p_f}{[(p\cdot p_f)^2 - p^2 p_f{}^2]^{1/2}} = \frac{e}{4\pi}\sum_f \frac{e_f}{\tanh\psi_f}\,, \tag{3.25b}$$

where $\psi_f > 0$ is the hyperbolic angle between p and p_f. This gives

$$\langle\psi^{\text{as}-}(x)\rangle = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E}\exp[i\gamma_f\ln(p\cdot xl_1/m)]e^{-ip\cdot x}\langle p_f, p^{\text{ad}}|\exp[eA^{\text{out}}(p)\ln(\epsilon+ip\cdot xl_1/m)]\,T[\cdots]|p_i^{\text{ret}}\rangle\,, \tag{3.26}$$

where

$$\langle p_f, p^{\text{ad}}| \equiv \sum_s u_s(p)\langle p_f; p, s^{\text{ad}}|\,. \tag{3.27}$$

To proceed further, use Eqs. (3.13) and (3.14),

$$s_\mu(\hat{k}) = c_\mu^{\text{out}}(\hat{k}) + \int d^4p'\,\frac{1}{(2\pi)^{3/2}}\frac{p'_\mu}{p'\cdot\hat{k}}\rho^{\text{out}}(p')\,,$$

and $A^{\text{out}}(p)$ may be written

$$A^{\text{out}}(p) = \frac{-1}{(2\pi)^3}\int\frac{d\hat{k}}{2}\int d^4p'\,\frac{p\cdot p'}{p\cdot\hat{k}\,p'\cdot\hat{k}}\rho^{\text{out}}(p') + \frac{1}{(2\pi)^{3/2}}\int\frac{d\hat{k}}{2}\cdot\frac{p\cdot s(\hat{k})}{p\cdot\hat{k}}\,.$$

Making use of the commutativity of $s(\hat{k})$ with any local operator, Eq. (3.17), in particular with $T[\cdots]$, we have

$$s_\mu(\hat{k})\,T[\cdots]|p_i^{\text{ret}}\rangle = T[\cdots]s_\mu(\hat{k})|p_i^{\text{ret}}\rangle\,.$$

(The fields which appear inside $T[\cdots]$ may be the fields of particles which have already been reduced out and taken off the mass shell, in which case they are integrated over all space-time. However, the values of x which contribute are of the order of the inverse distance from the mass shell, so $[s(\hat{k}), T[\cdots]] = 0$ is justified despite the infinite region of integration.) Hence we have, by Eqs. (3.13), (3.14), and (3.24)

$$s^\mu(\hat{k})\,T[\cdots]|p_i^{\text{ret}}\rangle = \frac{1}{(2\pi)^{3/2}}\sum_i\frac{e_i p_i^\mu}{p_i\cdot\hat{k}}\,, \tag{3.28}$$

which all together gives

$$\langle p_f, p^{\text{ad}}|\exp[eA^{\text{out}}(p)\ln(\epsilon+ip\cdot xl_1/m)]T[\cdots]|p_i^{\text{ret}}\rangle = \exp[\beta_f\ln(\epsilon+ip\cdot xl_1/m)]\langle p_f, p, s^{\text{ad}}|T[\cdots]|p_i^{\text{ret}}\rangle\,,$$

where

$$\beta_f = \frac{-e}{(2\pi)^3}\int\frac{d\hat{k}}{2}\frac{p}{p\cdot\hat{k}}\cdot\left(\frac{-ep}{p\cdot\hat{k}} + \sum_f\frac{e_f p_f}{p_f\cdot\hat{k}} - \sum_i\frac{e_i p_i}{p_i\cdot\hat{k}}\right). \tag{3.29a}$$

This is easily evaluated exploiting Eq. (I.3.10),

$$\beta_f = \frac{-e}{(2\pi)^2}\left[-e + \left(\sum_f\frac{e_f\psi_f}{\tanh\psi_f} - \sum_i\frac{e_i\psi_i}{\tanh\psi_i}\right)\right], \tag{3.29b}$$

where $\psi_i > 0$ is the hyperbolic angle between p and p_i and similarly for ψ_f. This gives, from Eq. (3.26),

$$\langle \psi^{\mathrm{as}\,-}(x)\rangle = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{2E}\left(\frac{l_1}{m}\right)^{\zeta_f} e^{\pi\gamma_f/2}(\epsilon + ip\cdot x)^{\zeta_f} e^{-ip\cdot x}\langle p_f, p^{\mathrm{ad}}|\, T[\circ\cdots]\|\, p_i^{\mathrm{ret}}\rangle, \tag{3.30}$$

where

$$\zeta_f = \beta_f + i\gamma_f\,. \tag{3.31}$$

Thus the x dependence of the matrix element (3.22) is simply the plane wave $e^{-ip\cdot x}$ modulated by the complex power $(\epsilon + ip\cdot x)^{\zeta_f}$.

As desired, an electron of momentum p appears in the final state, but all values of p are integrated over, and it is necessary to produce a state with fixed momentum. The integral representation

$$(\epsilon + ip\cdot x)^{\zeta} = \frac{1}{\Gamma(-\zeta)}\int_0^\infty d\lambda\, \lambda^{-\zeta-1} e^{-ip\cdot x\lambda}$$

is helpful. It is valid for $\mathrm{Re}\zeta > 0$. If this condition does not hold, the result may be obtained by analytic continuation. Temporarily dropping the subscript on ζ_f, we have, from Eq. (3.30),

$$\langle \psi^{\mathrm{as}\,-}(x)\rangle = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{2E}\left(\frac{l_1}{m}\right)^{\zeta}\frac{e^{\pi\gamma_f/2}}{\Gamma(-\zeta)}\int_0^\infty d\lambda\, \lambda^{-\zeta-1} e^{-i(1+\lambda)p\cdot x}\langle p_f, p^{\mathrm{ad}}|\, T[\cdots]\|\, p_i^{\mathrm{ret}}\rangle\,. \tag{3.32}$$

This gives, for t an arbitrary finite number,

$$\frac{1}{(2\pi)^{3/2}}\int_t^\infty dx^0\int d^3x\, e^{iq\cdot x}\langle \psi^{\mathrm{as}\,-}(x)\rangle = \int_0^\infty d\lambda\,\frac{\lambda^{-\zeta-1}}{2E(1+\lambda)^3}\left(\frac{l_1}{m}\right)^{\zeta}\frac{e^{\pi\gamma_f/2}}{\Gamma(-\zeta)}\, i\,\frac{e^{i[q^0-E(1+\lambda)]t}}{[q^0-E(1+\lambda)]}\langle p_f, p^{\mathrm{ad}}|\, T[\cdots]\|\, p_i^{\mathrm{ret}}\rangle\,, \tag{3.33}$$

where $E = p^0 = (\vec{p}^2 + m^2)^{1/2}$ and $\vec{p} = \vec{q}(1+\lambda)^{-1}$. Let q approach the mass shell from below, $q^2 - m^2 = -\eta < 0$. Set $q^0 = (\vec{q} + m^2 - \eta)^{1/2}$, make the change of variable $\lambda = \eta\alpha$, and let η approach zero. On the right-hand side it may be replaced by zero everywhere except in the overall factor $\eta^{-\zeta}$ and in the denominator

$$q^0 - E(1+\lambda) = (\vec{q}^2 + m^2 - \eta)^{1/2} - [\vec{q}^2 + m^2(1+\eta\alpha)^2]^{1/2} \simeq -\tfrac{1}{2}\eta(\vec{q}^2 + m^2)^{-1/2}(1 + 2m^2\alpha)\,.$$

Then with

$$\int_0^\infty \frac{d\alpha}{\alpha^{\zeta+1}}\,\frac{1}{(1+2m^2\alpha)} = (2m^2)^{\zeta}\,\Gamma(-\zeta)\Gamma(1+\zeta)\,,$$

which holds for $-1 < \mathrm{Re}\zeta < 0$, and the substitution $q \to p$, one finds

$$\lim_{p^2\to m^2}\int_t^\infty dx^0\int d^3x\,\frac{e^{ip\cdot x}}{(2\pi)^{3/2}}\langle \psi^{\mathrm{as}\,-}(x)\rangle = \frac{-i\Gamma(1+\zeta)}{(m^2-p^2)^{1+\zeta}}(2ml_1)^{\zeta}e^{\pi\gamma_f/2}\langle p_f, p^{\mathrm{ad}}|\, T[\cdots]\|\, p_i^{\mathrm{ret}}\rangle\,.$$

This relation may be continued in ζ to all $\mathrm{Re}\zeta > -1$. With the definition (3.22), one has

$$\langle p_f, p^{\mathrm{ad}}|\, T[\circ\cdots]\|\, p_i^{\mathrm{ret}}\rangle = \lim_{p^2\to m^2}\frac{(m^2-p^2)^{1+\zeta}}{\Gamma(1+\zeta)}\,\frac{e^{-\pi\gamma_f/2}}{(2ml_1)^{\zeta}}\,\frac{i}{(2\pi)^{3/2}}\int_t^\infty dx^0\int d^3x\, e^{ip\cdot x}\langle p_f^{\mathrm{ad}}|\,\psi^{\mathrm{as}}(x)T[\cdots]\|\, p_i^{\mathrm{ret}}\rangle\,. \tag{3.34}$$

Here we have replaced $\psi^{\mathrm{as}\,-}$ by ψ^{as} which is justified because the creation part of ψ^{as} produces no singularity when integrated with $e^{ip\cdot x}$ and thus it gets annihilated by $(m^2-p^2)^{1+\zeta}$ as $p^2 \to m^2$. The right-hand side is independent of t because as $m^2 - p^2$ approaches zero, the contribution to the integral comes from times of order $(m^2-p^2)^{-1}$, as is clear from Eqs. (3.32) and (3.33). For this reason ψ^{as} may be replaced by $z_0^{-1/2}\psi$ by virtue of the weak asymptotic limit $\psi(x) \to z_0^{1/2}\psi^{\mathrm{as}}(x)$. Here z_0 defines the normalization of the renormalized Heisenberg field ψ according to Eqs. (I.4.47) and (I.4.75). Finally the x^0 integration may be extended to all times, provided p does not coincide with any of the initial momenta p_i, for in this case the infinite integration over early times produces no singularity and is annihilated by $(p^2-m^2)^{1+\zeta}$ as $p^2 \to m^2$,

$$\langle p_f, p^{\mathrm{ad}}|\, T[\circ\circ\cdot]\|\, p_i^{\mathrm{ret}}\rangle = \lim_{p^2\to m^2}\frac{(m^2-p^2)^{1+\zeta_f}}{z_0^{1/2}\Gamma(1+\zeta_f)}\,\frac{e^{-\pi\gamma_f/2}}{(2ml_1)^{\zeta_f}}\,\frac{i}{(2\pi)^{3/2}}\int d^4x\, e^{ip\cdot x}\langle p_f^{\mathrm{ad}}|\, T[\psi(x)\cdots]\|\, p_i^{\mathrm{ret}}\rangle\,. \tag{3.35}$$

Here we have restored the subscript to ζ_f, which, by Eqs. (3.31), is given by

$$\zeta_f = \beta_f + i\gamma_f = \frac{-e}{(2\pi)^2}\left[-e + \sum_f \frac{e_f(\psi_f - i\pi)}{\tanh\psi_f} - \sum_i \frac{e_i\psi_i}{\tanh\psi_i}\right]. \tag{3.36}$$

This is the desired formula, which reduces an electron out of a final state. To reduce an electron out of an initial state, one finds similarly

$$\langle p_f^{\mathrm{ad}} | T[\cdots] | p_i, p^{\mathrm{ret}} \rangle = \lim_{p' \to m^2} \frac{(m^2 - p^2)^{1+\zeta_i}}{z_0^{1/2} \Gamma(1+\zeta_i)} \frac{e^{-\pi \gamma_i/2}}{(2ml_1)^{\zeta_i}} \frac{i}{(2\pi)^{3/2}} \int d^4x \, e^{-ip\cdot x} \langle p_f^{\mathrm{ad}} | T[\cdots \bar{\psi}(x)] | p_i^{\mathrm{ret}} \rangle , \tag{3.37}$$

where

$$\zeta_i = \beta_i + i\gamma_i = \frac{-e}{(2\pi)^2} \left[-e + \sum_i \frac{e_i(\psi_i - i\pi)}{\tanh\psi_i} - \sum_f \frac{e_f \psi_f}{\tanh\psi_f} \right] , \tag{3.38}$$

and ψ_i (ψ_f) > 0 is the hyperbolic angle between p and p_i (p_f).

Formulas (3.35)–(3.38) which reduce out a particle of charge ($-e$) may be applied successively to particles of charge e_a. There results a formula which completely reduces out all charged particles. Let a be an index which runs over all initial and final charged particles, with $\{p_a\} = \{p_i, p_f\}$, $\{e_a\} = \{e_i, e_f\}$, $\{\eta_a\} = \{\eta_i, \eta_f\}$ with $\eta_f = +1$, $\eta_i = -1$. Let $G(p_a)$ be the renormalized Green's function in momentum space,

$$G(p_a) = G(p_f, p_i) = \int \prod_f [d^4x_f \, e^{ip_f\cdot x_f}] \prod_i [d^4x_i \, e^{-ip_i x_i}] \langle 0 | T[\prod_f \psi_f(x_f) \prod_i \bar{\psi}_i(x_i)] | 0 \rangle , \tag{3.39}$$

and let $S(p_a) = S(p_f, p_i)$ be the covariant S-matrix element

$$S(p_a) = \prod_f [u_{s_f}(p_f)] \langle p_f, s_f^{\mathrm{ad}} | p_i, s_i^{\mathrm{ret}} \rangle \prod_i [\bar{u}_{s_i}(p_i)] , \tag{3.40}$$

where a sum over each of the two-valued spin indices s_f and s_i is implicit. The complete reduction formula reads

$$S(p_a) = \prod_a^{\mathrm{O}} \left[\lim_{p_a^2 \to m_a^2} \frac{i}{(2\pi)^{3/2}} \frac{1}{z_{0a}^{1/2}} \frac{(m_a^2 - p_a^2)^{1+\zeta_a} e^{-\pi\gamma_a/2}}{\Gamma(1+\zeta_a)(2m_a l_1)^{\zeta_a}} \right] G(p_a) . \tag{3.41}$$

Here $\prod_a^{\mathrm{O}}$ means an ordered product such that the factors with lower index a are on the right of the factors with higher index. This results in an ordered limit such that the particles with lower index go on the mass shell first. With this order, by Eq. (3.36) and (3.38),

$$\zeta_a = \beta_a + i\gamma_a = \zeta_{aa} + \sum_{b<a} \zeta_{ab} \tag{3.42a}$$

$$\zeta_{aa} = \frac{e_a^2}{(2\pi)^2} , \tag{3.42b}$$

$$\zeta_{ab} = \frac{\eta_a e_a \eta_b e_b}{(2\pi)^2 \tanh\psi_{ab}} [\psi_{ab} - i\pi\theta(\eta_a\eta_b)], \quad a \neq b \tag{3.42c}$$

where β_a and γ_a are real and $\theta(1) = 1$, $\theta(-1) = 0$. The appearance of the ordered limit is characteristic of the weak asymptotic limit employed for the charged field. For each particle a, the parameter $\zeta_a = \zeta_{aa} + \sum_{b>a} \zeta_{ab}$ which determines the singularity as particle a goes on the mass shell depends on the particles b that have already gone on the mass shell. The first term ζ_{aa} leads precisely to a cancellation of the singularity of the propagator of particle a, Eqs. (I.4.74) and (I.4.75), so the reduction formula may also be expressed directly in terms of the amputated Green's function. The normalization of the Green's functions is discussed in the following section.

Our derivation gives the S matrix only if the momenta of the charged particles are distinct, $p_a \neq p_b$ for $a \neq b$. At coincident momenta it presumably has singularities which resemble those which occur in scattering by a Coulomb potential. In that case Herbst[21] showed that there is a unique unitary extension of the amplitude from nonforward directions to the forward directions which has a simple form. A similar procedure should determine the quantum-electrodynamical S matrix at coincident momenta.

Finally we state the reduction formula directly for the l-independent amplitude F, Eq. (2.40). With $z_a = z_{0a} \Gamma(1+\zeta_{aa})(2l_1/m_a)^{\zeta_{aa}}$, Eq. (I.4.75), one has

$$\delta^4(\sum_a \eta_a p_a) F(p_a) = \prod_a^{\mathrm{O}} \left[\lim_{p_a^2 \to m_a^2} \frac{i}{(2\pi)^{3/2}} \frac{\Gamma^{1/2}(1+\zeta_{aa})}{z_{0a}^{1/2} m_a^{z_a}} \frac{(m_a^2 - p_a^2)}{\Gamma(1+\zeta_a)} \left(\frac{m_a^2 - p_a^2}{m_a} \right)^{\zeta_a} e^{-\pi\gamma_a/2} \right] G(p_a) , \tag{3.43}$$

where

$$z_a = \zeta_{aa} + \tfrac{1}{2} \sum_{b \neq a} \zeta_{ab} . \tag{3.44}$$

The dimensional factor $\prod_a m_a^{-z_a}$ which appears here is invariant under relabeling of particle indices. The infrared renormalization constant l_1 has disappeared from this formula, which verifies that the original S-matrix element was proportional to l_1^{-Z}, where

$$Z=\tfrac{1}{2}\sum_{a,b}\zeta_{ab}\ . \tag{3.45}$$

Furthermore, the explicit factors $z_a^{-1/2}$ in the reduction formula (3.43) cancel the factor of $z_a^{1/2}$ which the Green's function contains for each external leg, by virtue of the Ward identities [see for example Eq. (4.22)]. Thus F is independent of both the infrared renormalization constant l_1 and the field-renormalization constants z_a.

IV. APPLICATIONS

A. On-mass-shell ultraviolet renormalization

Suppose the Green's functions of quantum electrodynamics are calculated according to a standard renormalization procedure. The resulting renormalized Green's functions are finite.[22] However, the normalization of the photon propagator, the electron propagator, and the vertex function may be varied according to the renormalization group. We show here how these normalizations may be fixed in actual practice.

Let $D_{\mu\nu}(k)$ be the exact renormalized photon propagator

$$D_{\mu\nu}(k)=\int e^{ik\cdot x}\langle 0|T[A_\mu(x)A_\nu(0)]|0\rangle\, d^4x\ .$$

In the Feynman gauge it is given in zeroth order by

$$D^0_{\mu\nu}(k)=(-g_{\mu\nu})\frac{i}{k^2+i\epsilon}\ . \tag{4.1}$$

No infrared divergences are encountered if $D_{\mu\nu}(k)$ is normalized to agree with (4.1) at $k^2=0$. More precisely, set

$$D_{\mu\nu}(k)=\frac{i}{k^2}\left[\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)d(k^2)-\frac{k_\mu k_\nu}{k^2}\right], \tag{4.2}$$

and normalize $d(k^2)$ according to

$$d(0)=1\ . \tag{4.3}$$

This normalization of the photon propagator is to be used with the photon reduction formula (3.12). [If one wishes to make the finite renormalization $d(k^2)\to z_3 d(k^2)$, then a factor of $z_3^{-1/2}$ would be inserted in the reduction formula for each external photon line.]

Let $G(p)$ be the renormalized electron propagator

$$G(p)=\int e^{ip\cdot x}\langle 0|T[\psi(x)\bar\psi(0)]|0\rangle\, d^4x\ .$$

In zeroth order it is given by

$$G^0(p)=\frac{i(\not p+m)}{p^2-m^2+i\epsilon}\ . \tag{4.4}$$

Set

$$G^{-1}(p)=-i[\not p-m-\Sigma(p)], \tag{4.5}$$

where $\Sigma(p)$ is the electron self-energy,

$$\Sigma(p)=A(p^2)(\not p-m)+B(p^2)\ . \tag{4.6}$$

In each order of perturbation theory there is a finite additive constant in $A(p^2)$ and $B(p^2)$, which must be fixed by some prescription. It is generally thought that the electron mass renormalization is free of infrared divergences, which means that the singularity of the propagator can be maintained at $p^2=m^2$. Thus, we require

$$\Sigma(p)|_{p^2=m^2}\,u(p)=0\ , \tag{4.7}$$

which fixes the additive constant in $B(p^2)$ according to

$$B(m^2)=0\ . \tag{4.8}$$

Set

$$B(p^2)=(p^2-m^2)\,C(p^2)\ , \tag{4.9}$$

without implying that $C(p^2)$ is regular at $p^2=m^2$, but only that its singularity there in each order of perturbation theory is annihilated by p^2-m^2. We have

$$\Sigma(p)=(\not p-m)[A(p^2)+2mC(p^2)+(\not p-m)C(p^2)]\ , \tag{4.10}$$

$$G^{-1}(p)=(-i)(\not p-m)[1-A(p^2)-2mC(p^2)-(\not p-m)C(p^2)]\ . \tag{4.11}$$

The arbitrary additive constant in $A(p^2)$ is related to the field-renormalization constant which determines the normalization of the Heisenberg field ψ. In Sec. I IV C, Eqs. (I.4.74) and (I.4.75), we have seen that if the propagator is normalized to

$$\lim_{p^2\to m^2} G(p)=iz\frac{\not p+m}{p^2-m^2}\left(\frac{m^2}{m^2-p^2}\right)^\beta,\quad \beta=\alpha/\pi$$

then the Heisenberg field is normalized such that its asymptotic limit is $\psi\to z_0\psi^{\text{as}}$, where $z=z_0\Gamma(1+\beta)(2l_1/m)^\beta$. Comparison with Eq. (4.11) gives

$$\lim_{p^2\to m^2}\left(\frac{m^2}{m^2-p^2}\right)^\beta[1-A(p^2)-2mC(p^2)]=\frac{1}{z}\ . \tag{4.12}$$

Here $z=z(\alpha)$ is a power series in α with finite coefficients. It may be fixed by any convenient normalization convention, for example, $z(\alpha)=1$, or $z(\alpha)=\Gamma(1+\beta)(2l_1/m)^{\beta}$. However it is fixed, its actual value is required for insertion into the reduction formula. The amplitude which results is then independent of the choice of $z(\alpha)$, as we have seen.

Equation (4.12) suggests the introduction of new functions $D(p^2)$ to characterize the electron propagator,

$$G(p)=iz\,\frac{\not p+m}{p^2-m^2}\left(\frac{m^2}{m^2-p^2}\right)^{\beta}\times[D(p^2)+(\not p-m)E(p^2)], \tag{4.13}$$

with

$$D(m^2)=1\,. \tag{4.14}$$

In second order perturbation theory, explicit calculation gives

$$E(p^2)=C(p^2)=\frac{\alpha}{\pi}\,\frac{m}{p^2}\left[1+\left(\frac{m^2-3p^2}{p^2}\right)\ln\left(\frac{m^2-p^2}{m^2}\right)\right], \tag{4.15}$$

$$1+A(p^2)+2mC(p^2)=(1+\alpha a)\left[1-\frac{\alpha}{\pi}\ln\left(\frac{m^2-p^2}{m^2}\right)\right]\times D(p^2), \tag{4.16}$$

$$D(p^2)=1+\frac{\alpha}{4\pi}\left(\frac{m^2-p^2}{p^2}\right)\left[1+\left(\frac{m^2-5p^2}{p^2}\right)\ln\left(\frac{m^2-p^2}{m^2}\right)\right], \tag{4.17}$$

where a is a finite additive constant in $A(p^2)$ which is fixed by a renormalization convention. The constant z, in second order, thus has the value $(1+\alpha a)$.

The three-point function is given by

$$G_\mu(p',p)=\int e^{ip'\cdot x}e^{-ip\cdot y}\langle 0|T[\psi(x)A_\mu(0)\bar\psi(y)]|0\rangle\,. \tag{4.18a}$$

It satisfies the Ward identity

$$(p'-p)^2(p'-p)^{\mu}G_\mu(p',p)=ie[G(p)-G(p')], \tag{4.18b}$$

which is an expression of the fact that the ψ field carries the conserved charge e, $[Q,\psi(x)]=e\psi(x)$, where $Q=\int J^0(x)\,d^3x$. The constant e which appears here is independent of the convention by which the propagator is normalized. In the next subsection we shall see that $-e$ is in fact the observable charge on the electron (the minus sign is a convention, obviously) as measured by the near forward cross section for scattering by a weak external potential.

Let the vertex function $\Gamma^{\nu}(p',p)$ be introduced as usual according to

$$G_\mu(p',p)=D_{\mu\nu}(p'-p)G(p')(-ie)\Gamma^{\nu}(p',p)G(p)\,. \tag{4.19}$$

It satisfies

$$(-i)(p'-p)_\nu\Gamma^{\nu}(p',p)=G^{-1}(p')-G^{-1}(p) \tag{4.20}$$

by virtue of Eqs. (4.2) and (4.18b). This fixes the normalization of the vertex in terms of the normalization of the propagator. We see that Γ^{ν} will be proportional to z^{-1}. However, we may use our knowledge of the propagator near the mass shell to obtain an explicit expression for the vertex near the forward direction. In terms of the inverse propagator (4.5), the last relation reads, with $k_\nu=(p'-p)_\nu$,

$$k_\nu\Gamma^{\nu}(p+k,p)=\not k-\Sigma(p+k)+\Sigma(p)\,. \tag{4.21}$$

Let p go on the mass shell and multiply by $u(p)$, using $\Sigma(p)u(p)=0$ at $p^2=m^2$. This gives

$$\begin{aligned}k_\nu\Gamma^{\nu}(p+k,p)|_{p^2=m^2}u(p)&=[\not k-\Sigma(p+k)]\,u(p)\\&=[\not k-\not kA((p+k)^2)-(k^2+2p\cdot k)C((p+k)^2)]\,u(p)\end{aligned}$$

by Eq. (4.10). Assume that as k_μ approaches zero with $p^2=m^2$, $A((p+k)^2)$ and $C((p+k)^2)$ develop a singularity which in each order of perturbation theory is a polynomial in $\ln(2p\cdot k)$. Then as k_μ approaches zero we may equate the coefficient of the term linear in k_μ and obtain

$$\lim_{k_\mu\to 0}\Gamma^{\nu}(p+k,p)|_{p^2=m^2}u(p)=[\gamma^{\nu}-\gamma^{\nu}A((p+k)^2)-2p^{\nu}C((p+k)^2)]u(p)$$

or

$$\lim_{p'\to p}\bar u(p)\,\Gamma^{\nu}(p',p)|_{p^2=m^2}\,u(p)=[1-A(p'^2)-2mC(p'^2)]\,\bar u(p)\gamma^{\nu}u(p)\,.$$

This gives, by Eq. (4.12),

$$\lim_{p'\to p}\left(\frac{m^2}{m^2-p'^2}\right)^{\beta}\bar u(p)\,\Gamma^{\nu}(p',p)|_{p^2=m^2}\,u(p)=\frac{1}{z}\,\bar u(p)\gamma^{\nu}u(p)\,. \tag{4.22}$$

This is a convenient explicit expression which fixes the normalization of the vertex directly in terms of the constant z. It is also a particularly simple example of the type of ordered limit which occurs in the reduction formula: p goes on the mass shell before p'.

B. Scattering of an electron by a weak potential

As a simple application, consider the scattering of an electron by a weak external potential $A^{\text{ext}}(x)$, sometimes called the Schwinger problem.[3,4] In this case the reduction formula (3.41) takes the simple form

$$S(p',p)=\frac{-1}{(2\pi)^3}\frac{1}{z_0}\frac{1}{(2ml_1)^{\beta+\beta'}}\lim_{p'^2\to m^2}\frac{(m^2-p'^2)^{1+\beta'}}{\Gamma(1+\beta')}\lim_{p^2\to m^2}\frac{(m^2-p^2)^{1+\beta}}{\Gamma(1+\beta)}G^{\text{ext}}(p',p), \tag{4.23}$$

where

$$S(p',p)=\sum_{s',s}u_{s'}(p')\langle p',s'^{\text{adv}}\,|\,p,s^{\text{ret}}\rangle\,\bar{u}_s(p), \tag{4.24}$$

$$\zeta_1=\beta_1+i\gamma_1=\beta=\alpha/\pi,\qquad \zeta_2=\beta_2+i\gamma_2=\beta'=\beta-\beta\psi\coth\psi. \tag{4.25}$$

Here, by Eq. (3.39),

$$G^{\text{ext}}(p',p)=\int e^{i(p'\cdot x-p\cdot y)}\langle 0|\,T[\psi^{\text{ext}}(x)\bar{\psi}^{\text{ext}}(y)]\,|0\rangle d^4x\,d^4y, \tag{4.26}$$

where ψ^{ext} is the Heisenberg field of the electron in the presence of an external potential. Thus

$$G^{\text{ext}}(p',p)=\int d^4x\,d^4y\,e^{i(p'\cdot x-p\cdot y)}\left\langle 0\left|T\left\{\psi(x)\bar{\psi}(y)\exp\left[-i\int J^{\text{ext}}(z)\cdot A(z)d^4z\right]\right\}\right|0\right\rangle, \tag{4.27}$$

where ψ and A are ordinary renormalized Heisenberg fields and J^{ext} is the external current $\partial^2A_\mu^{\text{ext}}(x)=J_\mu^{\text{ext}}(x)$. We are interested in the scattering to first order in the external current, so it is sufficient to retain only the term linear in $J_\mu^{\text{ext}}(x)$,

$$G^{\text{ext}}(p',p)\to(-i)G^\mu(p',p)J_\mu^{\text{ext}}(p'-p), \tag{4.28}$$

where $G(p',p)$ is the three-point function given in Eq. (4.18a), and

$$J_\mu^{\text{ext}}(k)=\int e^{ix\cdot z}J_\mu^{\text{ext}}(z)d^4z. \tag{4.29}$$

This gives

$$S(p',p)=\frac{1}{(2\pi)^3}\frac{1}{z_0}\frac{1}{(2ml_1)^{\beta_1+\beta}}\lim_{p'^2\to m^2}\frac{(m^2-p'^2)^{1+\beta'}}{\Gamma(1+\beta')}\lim_{p^2\to m^2}\frac{(m^2-p^2)^{1+\beta}}{\Gamma(1+\beta)}G^\mu(p',p)J_\mu^{\text{ext}}(p'-p). \tag{4.30}$$

It is convenient to introduce the vertex function $\Gamma^\nu(p',p)$ by means of Eq. (4.19), using the near mass shell expression for the propagator, found in Sec. IIV,

$$\lim_{p^2\to m^2}G(p)=z_0\Gamma(1+\beta)\frac{i(\not p+m)}{p^2-m^2}\left(\frac{2ml_1}{m^2-p^2}\right)^\beta, \tag{4.31a}$$

whose normalization may be expressed in terms of

$$z=z_0\Gamma(1+\beta)\left(\frac{2l_1}{m}\right)^\beta. \tag{4.31b}$$

one finds

$$S(p',p)=\frac{-ie}{(2\pi)^3}z_0\frac{\Gamma(1+\beta)}{\Gamma(1+\beta')}\left(\frac{m}{2l_1}\right)^{\beta'-\beta}\lim_{p'^2\to m^2}(\not p'+m)\left(\frac{m^2-p'^2}{m^2}\right)^{\beta'-\beta}\Gamma^\mu(p',p)\Big|_{p^2=m^2}(\not p+m)F_\gamma(k^2)A_\mu^{\text{ext}}(k). \tag{4.32}$$

Here $k=p'-p$ and we have used Eq. (4.2) for $D_{\mu\nu}$, and set $A_\mu^{\text{ext}}(k)=-(k^2)^{-1}\times J_\mu^{\text{ext}}(k)$, and written

$$F_\gamma(k^2)=d(k^2) \tag{4.33}$$

for the form factor which results from vacuum polarization. This S-matrix element depends upon the infrared renormalization constant l_1. Introduce the scattering amplitude F according to Eq. (2.40). With $\zeta_{11}=\zeta_{12}=\beta$ and $\zeta_{12}=-\beta\psi\coth\psi$ we have $F=(2l_1/m)^{\beta'}S$, where $\beta'=\beta-\beta\psi\coth\psi$. This gives

$$F(p', p) = \frac{-ie}{(2\pi)^3}\,\frac{1}{\Gamma(1+\beta')}\lim_{p'^2\to m^2}(\not{p}'+m)\left(\frac{m^2-p'^2}{m^2}\right)^{\beta'-\beta} z\Gamma^\mu(p',p)\Big|_{p^2=m^2}(\not{p}+m)F_\gamma(k^2)A_\mu^{\text{ext}}(k). \tag{4.34}$$

This expression is independent of the infrared renormalization constant. Observe that as the forward direction is approached, $\beta'-\beta=-\beta\psi\coth\psi$ approaches $-\beta$. This corresponds to the same singularity of the forward vertex function as was found using the Ward identity, Eq. (4.22). Thus we have verified that the singularity structure which we have attributed to the propagator and the forward vertex function is consistent with the Ward identity.

Let us introduce electric and magnetic form factors for the electron $F_e(k^2)$ and $F_m(k^2)$ according to

$$\bar{u}_{s'}(p')\{\gamma^\mu F_e(k^2)+(4m)^{-1}[\not{k},\gamma^\mu]F_m(u^2)\}u_s(p)=\bar{u}_{s'}(p')\left[\lim_{p'^2\to m^2}\left(\frac{m^2}{m^2-p'^2}\right)^{\beta\psi\coth\psi} z\Gamma^\mu(p',p)\Big|_{p^2=m}\right]u_s(p). \tag{4.35}$$

The left-hand side is the most general expression consistent with the symmetry properties of the right-hand side. From the normalization condition (4.22) we have

$$F_e(0)=1. \tag{4.36}$$

Hence with $F(p',s';p,s)=(2m)^2\bar{u}_{s'}(p')F(p',p)u_s(p)$, we find simply

$$F(p',s';p,s)=\frac{-ie}{(2\pi)^3}\,\frac{1}{\Gamma(1+\beta')}\,\bar{u}_{s'}(p')\{\gamma^\mu F_e(k^2)+(4m)^{-1}[\not{k},\gamma^\mu]F_m(k^2)\}u_s(p)F_\gamma(k^2). \tag{4.37}$$

Thus the scattering amplitude is expressed in terms of infrared-finite electric and magnetic form factors for the electron, F_e and F_m, and a form factor F_γ coming from vacuum polarization. The latter is a classic calculation,[3]

$$F_\gamma(k^2)=1-\frac{\alpha}{\pi}\left\{\left[1-\tfrac{1}{2}\psi\coth\left(\frac{\psi}{2}\right)\right]\left[1-\tfrac{1}{3}\coth^2\left(\frac{\psi}{2}\right)\right]-\tfrac{1}{9}\right\}, \tag{4.38}$$

with $k^2=-4m^2\sinh^2(\frac{1}{2}\psi)$. To find the electric and magnetic form factors in second order set

$$\Gamma_\mu(p',p)=\gamma_\mu+\Gamma_{1\mu}(p',p), \tag{4.39}$$

$$\Gamma_1^\mu(p',p)=\frac{-ie^2}{(2\pi)^4}\int\frac{d^4k}{k^2}\left(\frac{-\Lambda^2}{k^2-\Lambda^2}\right)\gamma_\lambda\,\frac{\not{p}'-\not{k}+m}{(p'-k)^2-m^2}\gamma^\mu\,\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\gamma^\lambda, \tag{4.40}$$

where $-\Lambda^2/(k^2-\Lambda^2)$ has been introduced to ensure ultraviolet convergence. We require this quantity for $p^2=m^2$ and for $\delta=m^2-p'^2$ small,

$$\Gamma_{1\mu}(p',p)=\frac{-ie^2}{(2\pi)^4}\int\frac{d^4k}{k^2}\left(\frac{-\Lambda^2}{k^2-\Lambda^2}\right)\gamma_\lambda\,\frac{\not{p}'-\not{k}+m}{k^2-2p'\cdot k-\delta}\gamma^\mu\,\frac{\not{p}-\not{k}+m}{k^2-2p\cdot k}\gamma^\lambda. \tag{4.41}$$

We may set $p'^2=m^2$ everywhere except in the δ which appears explicitly. For δ finite this integral is finite and one finds

$$\bar{u}_{s'}(p')\left[\lim_{p'^2\to m^2}\Gamma^\mu(p',p)|_{p^2=m^2}\right]u_s(p)=(1-\alpha a)\left[1+\frac{\alpha}{\pi}\psi\coth\psi\ln\left(\frac{m^2-p'^2}{m^2}\right)\right]$$
$$\times\,\bar{u}_{s'}(p')\{\gamma^\mu F_e(k^2)+(4m)^{-1}[\not{k},\gamma^\mu]F_m(k^2)\}u_s(p), \tag{4.42}$$

where a is an arbitrary constant arising from the ultraviolet renormalization, $k^2=2m^2(1-\cosh\psi)$, and the form factors are given by

$$F_e(k^2)=1+\frac{\alpha}{2\pi}\left[\frac{\psi(3\cosh\psi+1)}{2\sinh\psi}-\frac{\cosh\psi}{\sinh\psi}I(\psi)\right], \tag{4.43}$$

$$F_m(k^2)=\frac{\alpha}{2\pi}\,\frac{\psi}{\sinh\psi}, \tag{4.44}$$

where

$$I(\psi)=\int_0^\psi\psi'\left(\frac{3\cosh\psi'-1}{\sinh\psi'}\right)d\psi'=-\tfrac{3}{2}\psi^2-2f(e^\psi)+2f(e^{2\psi}), \tag{4.45}$$

and $f(x) = -\int_1^x dt(1-t)^{-1}\ln t$ is Spence's function. The form factors satisfy $F_e(0)=1$, $F_m(0)=\alpha/(2\pi)$.

Suppose that the external potential is time-independent, $A^{\mu\,\text{ext}}(t,\vec{x}) = V^{\mu\,\text{ext}}(\vec{x})$,

$$A^{\mu\,\text{ext}}(k) = 2\pi\delta(k^0)\,V^{\mu\,\text{ext}}(\vec{k}).$$

We seek the radiation exclusive cross section, namely only those events are counted for which the energy loss to the radiation field is no greater than a very small number ω_0, as measured in the frame in which the potential is static. In this case, the transition rate for observing electrons in a volume Ω_h of phase space is given, according to Eq. (2.61), by

$$R_{s's}(\Omega_h,\omega_0) = \int \frac{d^3p'}{2E}\exp[-J(p',p;\omega_0/D)]\chi(p')\left|\frac{A_{s's}}{(2\pi)^3}\right|^2 (2\pi)\delta(E'-E), \tag{4.46}$$

where $\chi(p')$ is the projector onto Ω_h, the usual mnemonic substitution $2\pi\delta(0)\to T$, has been made and

$$A_{s's} = \frac{-ie}{\Gamma(1+\beta')}\bar{u}_{s'}(p')\{\gamma^\mu F_e(k^2) + (4m)^{-1}[\not{k},\gamma^\mu]F_m(k^2)\}u_s(p)F_\gamma(k^2)V_\mu^{\text{ext}}(\vec{k}), \tag{4.47}$$

with $k^\mu = (0,\vec{k})$. The incident flux is $(2\pi)^{-3}2Ev = (2\pi)^{-3}2|\vec{p}|$. One finds for the differential cross section per unit solid angle, with energy loss no greater than a small amount ω_0, that

$$\frac{d\sigma_{s's}(\omega_0)}{d\Omega} = \exp[-J(p',p;\omega_0/D)]\left|\frac{A_{s's}}{4\pi}\right|^2. \tag{4.48}$$

If spins are not observed and the potential is electrostatic $V_\mu(k) = \delta_{\mu 0}V(\vec{k})$, the ratio of the radiation exclusive cross section to the uncorrected one,

$$R(\omega_0) = \frac{d\sigma(\omega_0)}{d\Omega}\Big/\frac{d\sigma_0}{d\Omega}, \tag{4.49}$$

is given by

$$R(\omega_0) = \exp[-J(p',p;\omega_0/D)]\,\frac{F_\gamma^{\,2}}{\Gamma^2(1+\beta')}\,T, \tag{4.50}$$

where

$$T = F_e^{\,2} - \frac{2\vec{p}^2\sin^2(\theta/2)}{m^2+\vec{p}^2\cos^2(\theta/2)}F_eF_m + \frac{(\vec{p}^2)^2\sin^2(\theta/2)}{m^2[m^2+\vec{p}^2\cos^2(\theta/2)]}F_m^{\,2}. \tag{4.51}$$

The form factors are functions of

$$k^2 = -\vec{k}^2 = -4m^2\sinh^2(\psi/2) = -4\vec{p}^2\sin^2(\theta/2), \tag{4.52}$$

where θ is the scattering angle.

There remains only to evaluate J, given in Eq. (2.43),

$$J(p',p;\omega_0/D) = \frac{e^2}{(2\pi)^3}\int\frac{d\hat{k}}{2}\left[\frac{p'}{p'\cdot\hat{k}}\ln\left(\frac{p'\cdot\hat{k}2\omega_1}{m^2}\right) - \frac{p}{p\cdot\hat{k}}\ln\left(\frac{p\cdot\hat{k}2\omega_1}{m^2}\right)\right]\cdot\left(\frac{p'}{p'\cdot\hat{k}} - \frac{p}{p\cdot\hat{k}}\right), \tag{4.53}$$

where $\omega_1 = e^C\omega_0/D$, which gives by Eq. (2.60)

$$\omega_1 = \frac{\omega_0}{[\Gamma(1+B)]^{1/B}}, \tag{4.54}$$

where

$$B = -2\beta' = \frac{-e^2}{(2\pi)^3}\int\frac{d\hat{k}}{2}\left(\frac{p'}{p'\cdot\hat{k}} - \frac{p}{p\cdot\hat{k}}\right)^2 = \frac{2\alpha}{\pi}(\psi\coth\psi - 1). \tag{4.55}$$

This gives

$$J(p',p;\omega_0/D) = -K(p',p) - B\ln(2\omega_1/m), \tag{4.56}$$

$$K(p',p) = \frac{-e^2}{(2\pi)^3}\int\frac{d\hat{k}}{2}\left(\frac{p'}{p'\cdot\hat{k}} - \frac{p}{p\cdot\hat{k}}\right)\cdot\left(\frac{p'}{p'\cdot\hat{k}}\ln\frac{p'\cdot\hat{k}}{m} - \frac{p}{p\cdot\hat{k}}\ln\frac{p\cdot\hat{k}}{m}\right). \tag{4.57}$$

This integral is frame-dependent, which is a reflection of the frame dependence of the condition that the energy loss be no greater than ω_0. It is to be evaluated in the frame where $p = (E,\vec{p})$, $p' = (E,\vec{p}')$; $|\vec{p}| = |\vec{p}'|$. One finds

$$K(p', p) = \frac{\alpha}{\pi}\left\{-2 + \left[\frac{1}{v} - \frac{\cosh\psi}{\sinh\psi}\ln\left(\frac{v\cosh\frac{1}{2}\psi + \sinh\frac{1}{2}\psi}{v\cosh\frac{1}{2}\psi - \sinh\frac{1}{2}\psi}\right)\right]\ln\left(\frac{1+v}{1-v}\right) + \frac{\cosh\psi}{\sinh\psi}[I + L]\right\}, \tag{4.58}$$

where $v = |\vec{p}|/E$, I is given in Eq. (4.45), and

$$L = \int_0^\psi d\psi' \frac{\psi' \sinh\psi'}{(1+v^2)(1-v^2)^{-1} - \cosh\psi'}$$
$$= -f(be^{\psi}) + f(be^{-\psi}) + \ln b \ln\left(\frac{be^{\psi}-1}{be^{-\psi}-1}\right), \tag{4.59}$$

where $b = (1+v)(1-v)^{-1}$, and $f(x)$ is Spence's function, as in Eq. (4.45). This gives from Eq. (4.49)

$$\frac{d\sigma}{d\Omega}(\omega_0) = e^{K(p',p)}\left(\frac{2\omega_0}{m}\right)^B \frac{F_\gamma^2}{\Gamma(1+B)\Gamma^2(1-B/2)} T \frac{d\sigma_0}{d\Omega}. \tag{4.60}$$

This formula is exact as the energy resolution ω_0 approaches zero. In this limit all of the frame dependence of the radiative corrections has been found explicitly to all orders in α. Only the covariant form factors F_γ, F_e, and F_m must be evaluated perturbatively. They have been calculated above to order α. The author has verified that if the cross section (4.60) is expanded to first order in α, the result agrees with perturbative calculations[23] which regularize by introducing a small photon mass λ. Finally, it is easy to see that in the forward direction $K = B = 0$, $F = T = 1$, so the cross section reduces to the cross section in the absence of radiative corrections. Thus the parameter e introduced in the Ward identity is the phenomenological electric charge, as measured in the forward scattering off a weak external potential.

V. CONCLUSION

A. Rules for practical calculations

The Green's functions of quantum electrodynamics are calculated according to perturbative renormalization theory, with the normalization conventions as follows. Let the photon propagator

$$D_{\mu\nu}(k) = \frac{i}{k^2}\left[\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)d(k^2) - \frac{k_\mu k_\nu}{k^2}\right] \tag{5.1}$$

be normalized such that

$$d(0) = 1. \tag{5.2}$$

The normalization of the longitudinal part is characteristic of the Feynman gauge, which we have used throughout. In this gauge the free vector potential satisfies the wave equation, which simplifies the construction of the asymptotic charged field and the asymptotic-state space. To work in other gauges, one would first have to effect a similar construction and derive a new reduction formula, or else find the near mass shell gauge transformation properties of the renormalized Green's functions[24] and convert back to the Feynman gauge before using the reduction formula given here.

Write the inverse of the electron propagator in the form

$$G^{-1}(p) = -i[\not p - m - \Sigma(p)]. \tag{5.3}$$

The electron mass renormalization is effected by requiring the self-energy to vanish at its physical value,

$$\Sigma(p)|_{p^2=m^2} u(p) = 0. \tag{5.4}$$

Electron field normalization is defined by the condition

$$\lim_{p^2\to m^2}\left(\frac{m^2}{m^2-p^2}\right)^\beta G^{-1}(p) = \frac{-i}{z}(\not p - m), \quad \beta = \frac{\alpha}{\pi}. \tag{5.5}$$

Here $z = z(\alpha)$ may be fixed by any convenient convention. However, once this convention is established, for example, $z = 1$, the actual value of z is required for the reduction formula. The Ward identity fixes the normalization of the vertex function in terms of the normalization of the electron propagator according to

$$\lim_{p'\to p}\left(\frac{m^2}{m^2-p'^2}\right)^\beta \bar u(p)\Gamma^\nu(p',p)\Big|_{p^2=m^2} u(p) = \frac{1}{z}\bar u(p)\gamma^\nu u(p). \tag{5.6}$$

The generalization to different types of particle a with propagators normalized to z_a is obvious.

We now turn to the reduction formula, whose validity presupposes the normalization conventions described above. Let $F(p_f, s_f, k_f, {}_f; p_i, s_i, k_i, \mu_i)$ be a connected scattering amplitude, where $\{p_i, s_i, k_i, \mu_i\}$ are a set of charged particle momenta and spins, and photon momenta and 4-vector polarization indices, and correspondingly for final particles. Introduce indices a and b which run over initial and final charged particles and photons, $\{p_a\} = \{p_f, p_i\}$, $\{k_b\} = \{k_f, k_i\}$, $\{\mu_b\} = \{\mu_f, \mu_i\}$, and set

$$F(p_a, k_b)_{\mu_b} = \sum_{s_f}\sum_{s_i} u_{s_f}(p_f)F(p_f, s_f, k_f, \mu_f; p_i, s_i, k_i, \mu_i)\bar{u}_{s_i}(p_i), \tag{5.7}$$

where $\sum_s u_s(p)\bar{u}_s(p) = \not p + m$. The reduction formula reads

$$\delta^4(\sum n_a p_a - \sum n_b p_b)F(p_a, k_b)_{\mu_b} = \prod_a{}^{O}\left\{\lim_{p_a^2\to m_a^2}\frac{i}{(2\pi)^{3/2}}\left[\frac{\Gamma(1+\zeta_a)}{z_a}\right]^{1/2}\frac{e^{-\pi\gamma_a/2}}{\Gamma(1+\zeta_a)}\frac{(m_a^2-p_a^2)^{\zeta_a}}{m_a^{Z_a+\zeta_a}}\right\}$$
$$\times\prod_b\left\{\frac{-i\,\omega_b}{(2\pi)^{3/2}}\lim_{k_b^2\to 0}k_b^2\right\}G(p_a, k_b)_{\mu_b}, \tag{5.8}$$

with quantities defined as follows. The η_a and η_b are sign functions with $\eta_f = +1$, $\eta_i = -1$. $\prod_a^O$ is an ordered product such that the factors with lower index a are on the right of the factors with higher index a. This results in an ordered limit such that the charged particles with lower index go on the mass shell first. The complex constants ζ_a are defined by

$$\zeta_a = \beta_a + i\gamma_a = \zeta_{aa} + \sum_{b<a}\zeta_{ab}, \tag{5.9a}$$

where β_a and γ_a are real, and

$$\zeta_{aa} = \frac{e_a^2}{(2\pi)^2}, \tag{5.9b}$$

$$\zeta_{ab} = \frac{\eta_a e_a \eta_b e_b}{(2\pi)^2\tanh\psi_{ab}}[\psi_{ab} - i\pi\theta(\eta_a\eta_b)], \quad a\neq b. \tag{5.9c}$$

Here $\psi_{ab} > 0$ is the hyperbolic angle between p_a and p_b, and $\theta(x)$ is the step function $\theta(x) = \frac{1}{2}(1 + x/|x|)$. In the reduction formula the factor $(m_a^2 - p_a^2)^{1+\zeta_{aa}}$ cancels the singularity of the external leg of the Green's function. The factor $(m_a^2 - p_a^2)^{\zeta_{ab}}$ cancels the infrared singularity of the Green's function associated with photons exchanged between external leg a and external leg b that has already gone on mass shell. The imaginary part of this power, proportional to $\theta(\eta_a\eta_b)$, is the famous infinite Coulomb phase. Also

$$Z_a = \zeta_{aa} + \tfrac{1}{2}\sum_{b\neq a}\zeta_{ab}, \tag{5.10}$$

$$G(p_a, k_b)_{\mu_b} = \int\prod_f(d^4x_f e^{ip_f\cdot x_f})\prod_i(d^4x_i e^{-ip_i\cdot x_i})\prod_b(d^4y_b e^{i\eta_b k_b\cdot y_b})\langle 0|T[\psi(x_f)A_{\mu_b}(y_b)\bar\psi(x_i)]|0\rangle. \tag{5.11}$$

Because of the Ward-Takahashi identities and the scaling law of the Green's functions under the renormalization group, $G(p_a, k_b)$ contains a factor $z_a^{1/2}$ for each external charged particle leg. This is canceled in the reduction formula by the explicit appearance of $z_a^{-1/2}$, so the amplitude F is independent of z_a.

In actual calculations of Feynman integrals the parameters $\delta_a = m_a^2 - p_a^2$ provide an infrared regularization and thereby replace the photon mass λ which is traditionally used to eliminate virtual infrared divergences. As $\delta_a \to 0$ the Green's function to order N in α is a polynomial in $\ln\delta_a$ of degree N. When the factors $(m^2 - p_a^2)^{\zeta_a}$ are expanded in powers of α, with ζ_a proportional to α by Eqs. (5.9),

$$(m_a^2 - p_a^2)^{\zeta_a} = \sum_{n=0}^{\infty}\frac{1}{n!}\zeta_a^n(\ln\delta_a)^n,$$

the dependence on $\ln\delta_a$ cancels out to order N, giving a finite perturbative expression for the on-shell amplitude F. The mass factor $\prod_a m_a^{-Z_a}$ ensures that F has the usual engineering dimension.

The amplitude is normalized corresponding to a charged particle density in configuration space of $(2\pi)^{-3}(2E)$, so the sum over charged particle states is effected with volume element $(2E)^{-1}d^3p$. As a matter of convenience, a factor of ω has been included in the amplitude F for each photon leg, so individual photons have a density in configuration space of $(2\pi)^{-3}2\omega^3$. We describe the radiation associated with the scattering in the retarded representation according to which the incident radiation is described by a finite number of photons. In practice, for two-body scattering, zero, one, or two photons are assumed incident. (Although from another point of view an infinite number of infrared photons may be incident, the error made in assuming a finite number may be made arbitrarily small, for the effect on measurements of incident photons which are neglected is proportional to their energy, not their number.) However, it is a fact of theoretical physics that with this description of the initial state, the final state must be described as having an infinite number of photons. This is true even though the most sensitive photon counter imaginable will count only the finite number of final photons which have a frequency above some minimum positive frequency. The counting rate of an actual detector of final particles is obtained from the amplitude F, defined above, as follows.

Let the detector be such that it registers a count if and only if the system is in some volume of final-state phase space Ω. For example, Ω may be defined by the condition that an electron be emitted into a given solid angle and energy interval. In terms of states of the system, Ω will be a set of volumes $\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_n \ldots$ in the $0, 1, 2, \ldots, n, \ldots$ photon phase space. In the example, the electron may be accompanied by $0, 1, 2, \ldots, n, \ldots$ photons. Let $\chi_0(p_f)$, $\chi_1(k_{f1}, p_f), \ldots, \chi_n(k_{f1}\cdots k_{fn}, p_f)$ be the characteristic function[15] on $\Omega_0, \Omega_1, \ldots, \Omega_n, \ldots$:

$$\chi_n(k_{f1}\cdots k_{fn}, p_f) = 1 \text{ for } k_{f1}\cdots k_{fn}, p_f \in \Omega_n \tag{5.12a}$$

$$= 0 \text{ otherwise}. \tag{5.12b}$$

Here p_f represents a set of final charged particle momenta, and $\chi_n(k_{f1}\cdots k_{fn}, p_f)$ is symmetric in $k_{f1}\cdots k_{fn}$. The important point is that the nonobservability of zero-frequency photons imposes the restriction on the χ_n given by

$$\lim_{\omega_1\to 0} \chi_n(k_1\cdots k_n, p) = \chi_{n-1}(k_2\cdots k_n, p), \quad n \geq 1. \tag{5.13}$$

This restriction has the consequence that if χ_m is not identically zero, then χ_n is not identically zero for all $n > m$. Thus, corresponding to any yes-no detector there is an infinite series of nonvanishing χ_n.

With this specification, the cross section $\sigma(\Omega)$ for emission into a volume Ω of final-state phase space is given by

$$\sigma(\Omega) = \frac{1}{\phi_i}\int \prod_f \frac{d^3p_f}{2E_f}\exp[-J(p,\Delta)]\Big[\chi_0(p_f)|F_0(p_f;k_i,p_i)|^2\delta^4(\textstyle\sum p_f - \sum k_i - \sum p_i)$$
$$+\sum_{n=1}^{\infty}\frac{1}{n!}\int (dk_{f1})_\Delta\cdots(dk_{fn})_\Delta \chi_n(k_{f1}\cdots k_{fn}, p_f)$$
$$\times |F_n(k_{f1}\cdots k_{fn}, p_f; k_i, p_i)|^2\delta^4(\textstyle\sum k_f + \sum p_f - \sum k_i - \sum p_i)\Big]. \tag{5.14}$$

Here ϕ_i represents the incident flux factor. For example, if the initial state consists of a photon of frequency ω incident on a particle of mass m at rest, $\phi_i = (2\pi)^4(2\pi)^{-3}2m(2\pi)^{-3}2\omega^3$. For simplicity, polarization and spin variables have been suppressed, and also factors of $(n_a!)^{-1}$ which are present if there are n_a identical charged particles of type a in the final state. The variables k_i, p_i, and p_f represent sets of initial photon and initial or final charged-particle momenta, but the final photon momenta are represented individually by $k_{f1}\cdots k_{fn}$. The factor $\exp[-J(p,\Delta)]$ will be seen to account for radiation damping nonperturbatively,

$$J(p,\Delta) = \frac{1}{(2\pi)^3}\int\frac{d\hat{k}}{2}\Big[\sum_a \frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\ln\Big(\frac{p_a\cdot\hat{k}\,2e^C\Delta}{m_a^2}\Big)\Big]\cdot\Big(\sum_a\frac{\eta_a e_a p_a}{p_a\cdot\hat{k}}\Big) \tag{5.15a}$$

$$= J(p,1) - B\ln\Delta. \tag{5.15b}$$

Here the index a and η_a are defined as in Eq. (5.8), $C = -\int_0^\infty dt\, e^{-t}\ln t$ is Euler's constant, $p_a\cdot\hat{k} \equiv E_a - \vec{p}_a\cdot\hat{k}$, and

$$(dk)_\Delta = -\tfrac{1}{2}d\hat{k}\,d\omega\ln\Big(\frac{\omega}{\Delta}\Big)\frac{\partial}{\partial\omega}, \tag{5.16}$$

where $d\hat{k}$ is an element of solid angle. In the cross-section formula the derivatives implicit in $(dk_{f1})_\Delta\cdots(dk_{fn})_\Delta$ act on all factors to their right, namely $\chi_n F_n\delta^4$. The partially integrated photon phase space (5.16) does not possess the real infrared divergence of the bremsstrahlung cross section that the usual phase space [which would be, with our normalization, $d\hat{k}d\omega(2\omega)^{-1}$] would produce. The cross section is independent of the parameter Δ, for it cancels out between $\exp[-J(p,\Delta)]$ and the infinite sum in (5.14) due to infrared coherence of the amplitudes,

$$\lim_{\omega_{f1}\to 0} F_n(k_{f1}\cdots k_{fn}, p_f; k_i, p_i)^{\mu_{f1}\cdots\mu_{fn}} = \frac{-1}{(2\pi)^{3/2}}\sum_a\frac{\eta_a e_a p_a^{\mu_{f1}}}{p_a\cdot\hat{k}_{f1}}F_{n-1}(k_{f2}\cdots k_{fn}, p_f; k_i, p_i)^{\mu_{f2}\cdots\mu_{fn}}. \tag{5.17}$$

This cancellation occurs both in the exact cross section and in every order of the expansion of the cross section in powers of α. Consequently Δ may be assigned any value. A good value is one which makes the infinite sum (5.14) converge rapidly. Such a value is found in Sec. II C for the radiation exclusive case, namely when the energy loss to unobserved photons is less than a small amount ω_0. It is shown there that if Δ is set at the value $\Delta=\omega_0/D$, $D=e^C[\Gamma(1+B)]^{1/B}$, then the infinite sum (5.14) becomes a series in $(\alpha\omega_0)^n$ instead of $(\alpha \ln\omega_0)^n$, and the factor $\exp[-J(p,\omega_0/D)]$ correctly accounts for radiation damping. This results in a calculational scheme whereby the amplitudes F are calculated according to perturbative renormalization theory to a given order in α. They are then inserted into the nonperturbative cross-section formula with $\Delta=\omega_0/D$. This correctly accounts for radiation damping nonperturbatively. In Sec. II D a similar scheme is developed for the radiation inclusive case, namely when unobserved photons may carry a lot of energy.

B. Toward a rigorous formulation

At present a rigorous formulation must be based on the Green's functions provided by perturbative renormalization theory. A rigorous derivation would follow the reverse order of development from that of our heuristic derivation. We started with an *Ansatz* for the asymptotic field, and ended with a reduction formula for the S matrix. In a rigorous approach one could start with the reduction formula found here, and let it define the S-matrix elements. It must then be proved to every order in α that this definition yields finite on-mass-shell S-matrix elements which are independent of the order by which the charged particles go on the mass shell. It must also be proved that they possess the properties asserted in Sec. II A, namely transversality, infrared coherence, and, most critically, unitarity calculated with the infrared-renormalized photon phase space $(dk)_I$. The only results in this direction are nonrigorous investigations of the singularity structure of the Green's functions,[25] which indicate that the S matrix defined here is finite, and the fourth-order calculation of Soloviev.[26,27]

Another necessity for a rigorous theory would be to establish a topology for the asymptotic-state space discussed in Sec. I IIIB, which is a Fock space of test functions.[28] Among the *desiderata* for such a topology would be continuity of the indefinite inner product, the condition that S map the space onto inself, $\mathfrak{F}^{\text{in}}=S\mathfrak{F}^{\text{out}}=\mathfrak{F}^{\text{out}}$, and that the transverse retarded and advanced subspaces be closed subspaces on which the inner product is non-negative. They may be completed in norm to give physical Hilbert spaces.

The fields ψ and A may be constructed as operators on $\mathfrak{F}^{\text{in}}$ as follows. The reduction formulas of Sec. III not only define S-matrix elements, but also matrix elements of any T product of fields between in and out states,

$$\langle p', k'^{\text{out}} | T[\psi(x_1)\cdots A_\mu(y_1)\cdots\bar\psi(z_1)\cdots] | p, k^{\text{in}}\rangle, \tag{5.18}$$

where p and k represent sets of momenta. The S matrix may be used to give the corresponding matrix element between in-states,

$$\langle p', k'^{\text{in}} | T[\psi(x_1)\cdots A_\mu(y_1)\cdots\bar\psi(z_1)\cdots] | p, k^{\text{in}}\rangle = \int S^*(p'',k'';p',k')\langle p'', k''^{\text{out}} | T[\psi(x_1)\cdots A_\mu(y_1)\cdots\bar\psi(z_1)\cdots] | p, k^{\text{in}}\rangle\, dp''(dk'')_I\,, \tag{5.19}$$

where summation over particle number is implicit. This gives, in particular, the matrix elements for single fields

$$\langle p', k'^{\text{in}} | \psi(x) | p, k^{\text{in}}\rangle,\quad \langle p', k'^{\text{in}} | A_\mu(x) | p, k^{\text{in}}\rangle\,. \tag{5.20}$$

These matrix elements define the (smeared) fields as integral operators on the sequences of test functions $\langle p, k^{\text{in}} | F\rangle$ which are the elements of $\mathfrak{F}^{\text{in}}$,

$$\langle p, k^{\text{in}} | \psi(x) | F\rangle = \int \langle p, k^{\text{in}} | \psi(x) | p', k'^{\text{in}}\rangle\langle p', k'^{\text{in}} | F\rangle \times dp'(dk')_I\,. \tag{5.21}$$

Asymptotic completeness and the consistency of this definition of the field operators may be verified by comparing the product of several fields calculated as the product of these integral operators, with the information on ordinary products contained in the T products (5.18) or (5.19). This defines the unobservable fields ψ and A as operators on a Fock space of test functions with indefinite metric. Finally it must be verified that the observable fields $F_{\mu\nu}$ and J_μ act properly within the physical subspaces. All of these constructions and verifications may be effected to any finite order in α. It may require a lot of work to prove our assertions to every order in α. However, if they are wrong they may be disproved by a single perturbative calculation.

ACKNOWLEDGMENTS

It is a pleasure to thank many friends and colleagues for enjoyable discussions, especially Gianfausto Dell'Antonio, Jürg Fröhlich, Francis Low, John Lowenstein, Dieter Maison, Nicolas Papanicolaou, Alberto Sirlin, and Arthur Wightman. I am also grateful for the hospitality extended to me at the Max-Planck-Institut für Physik und Astrophysik, in Munich, where part of this work was done.

---

*Work supported in part by funds from the National Science Foundation under Grant No. GP-36481X.

[1]The accompanying article [D. Zwanziger, Phys. Rev. D 11, 3481 (1975)] may serve as an extended introduction to the present one. It establishes a theoretical foundation for the practical computational method described herein. Equations, sections, and references of the accompanying article are designated by I preceding their number.

[2]F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).

[3]J. Schwinger, Phys. Rev. 75, 898 (1949); 76, 790 (1949).

[4]R. P. Feynman, Phys. Rev. 76, 769 (1949). Note, however, footnote 13 of this reference, which indicates how early difficulties were surmounted.

[5]R. Dalitz, Proc. R. Soc. A206, 509 (1951).

[6]D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961).

[7]K. E. Ericksson, Nuovo Cimento 19, 1010 (1961).

[8]K. T. Mahanthappa, Phys. Rev. 126, 329 (1961).

[9]T. D. Lee and M. Nauenberg, Phys. Rev. 133, B1549 (1964).

[10]G. Q. Hassoun and D. R. Yennie, Phys. Rev. 134, B436 (1964).

[11]A. C. Hearn, P. K. Kuo, and D. R. Yennie, Phys. Rev. 187, 1950 (1969).

[12]G. Grammer and D. R. Yennie, Phys. Rev. D 8, 4332 (1973).

[13]Kwang-Je Kim has suggested the possibility of expressing the total cross section as an integral over the n-photon emission cross section for any n (private communication).

[14]As in I, we use the notations $\hat{k}^\mu = k^\mu/\omega = (1, \hat{k})$ and $c(k) = \omega a(k)$, where $a(k)$ is the usual annihilation operator.

[15]The χ_n, with $\chi_n = 0$ or $\chi_n = 1$, define a projector onto the volume $\Omega = \{\Omega_n\}$. More generally they may be replaced by functions $\rho_n(k_{f1} \cdots k_{fn}, p_f)$, $0 \leq \rho_n \leq 1$ which describe the efficiency of the final-state counter. The ρ_n define a density matrix which is diagonal in momentum space. They satisfy Eq. (2.31) and are right differentiable in $\omega_{f1} \cdots \omega_{fn}$ at the origin.

[16]If the final-state detector counts photons, it can only be sensitive to photons of frequency greater than some positive number. For these photons, a partial integration in Eq. (2.42) is permissible and $-\frac{1}{2}\int d\hat{k}\, d\omega \ln(\omega/\Delta)\partial/\partial\omega$ gets replaced by $\int d\hat{k}\, d\omega (2\omega)^{-1}$.

[17]Ref. 6 and other works referred to therein.

[18]See, for example, J. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), pp. 146–165.

[19]The argument follows that of F. Strocchi and A. S. Wightman, J. Math. Phys. 15, 2198 (1974). I am grateful to Professor Wightman for suggesting to me the possibility that additional superselection rules may emerge in the asymptotic-state space of quantum electrodynamics.

[20]A mathematically precise result which states that the scattering states in quantum electrodynamics are incoherent superpositions of pure states labeled by the momenta of the asymptotic charged particles with respect to the electromagnetic field has been obtained by Jürg Fröhlich, working within the Hilbert-space structure offered by the Coulomb gauge (private communication). I am grateful to Dr. Fröhlich for making his manuscript available to me.

[21]I. Herbst, Commun. Math. Phys. 35, 181 (1974).

[22]P. Blanchard and R. Sénéor, CERN Report No. TH1420 (unpublished).

[23]A. Akhiezer and V. Berestetsky, *Quantum Electrodynamics* (State Technico-Theoretical Literature Press, Moscow, 1953), Eq. (45.12″).

[24]The gauge-transformation law for unrenormalized Green's functions which contain infinite constants may be found in B. Zumino, J. Math. Phys. 1, 1 (1960). See N. Papanicolaou, Ann. Phys. (N.Y.) 89, 423 (1975), for a recent discussion of mass-shell singularities of Green's functions. I am grateful to Mr. Papanicolaou for many discussions on this subject.

[25]See Refs. 3 and 9 of I.

[26]Ref. 9 of I.

[27]The reduction formula for scattering by an external Coulomb potential in the absence of radiation has been verified in the nonrelativistic case by N. Papanicolaou, Nucl. Phys. B75, 483 (1974), and in the relativistic case by G. Marques and N. Papanicolaou, Nucl. Phys. B80, 247 (1974).

[28]There is considerable similarity between the Fock space of test functions with indefinite metric which is used here and the representation of the massless scalar field in two-dimensional space-time as an operator on a Fock space of test functions with indefinite metric; see A. S. Wightman, in *Cargèse Lectures in Theoretical Physics, High Energy Electromagnetic Interactions and Field Theory*, edited by M. Lévy (Gordon and Breach, New York, 1967).