Scattering theory for quantum electrodynamics. I. Infrared renormalization and asymptotic fields

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The present article lays the theoretical foundation for a scattering theory of quantum electrodynamics, which is completed into a practical calculational scheme in the accompanying article. In order to circumvent infrared divergences, an infrared renormalization procedure is instituted whereby a Lorentz-invariant, but indefinite, inner product is defined for a class of photon test functions defined on the future light cone $k^{\mu} = \omega(1, \hat{k})$, $\omega \geq 0$. This class includes test functions whose low-frequency behavior is given by $\phi^{\mu}(k) \sim ep^{\mu}/p \cdot k$, for which the usual inner product $\int d^3k\,(2\omega)^{-1}\,\phi_{\mu}{}^*(k)(-g^{\mu\nu})\phi_{\nu}(k)$ is infrared-divergent. The Fock space of such test functions provides a representation space for the asymptotic fields of quantum electrodynamics. It contains subspaces in which the indefinite metric is non-negative which, when completed in the norm, yield physical Hilbert spaces. This Fock space of test functions thereby replaces the nonphysical Hilbert space of the usual Gupta-Bleuler method and its positive-definite but noncovariant metric. As an application the S matrix and finite transition probabilities are found for the bremsstrahlung emitted by the classical external current of a scattered charged particle. A final result is a simple weak asymptotic limit of the charged field ψ. It is used as a starting point in the accompanying article, for the derivation of reduction formulas for the quantum electrodynamical S matrix.

I. INTRODUCTION

The outcome of the present investigation is an S-matrix description of scattering processes in quantum electrodynamics. As in a theory of massive particles, the S matrix is the set of on-mass-shell momentum-space Green's functions (T products). However, because the photon is massless, the singularity at the charged particle mass shell is not a simple pole, and "going on the mass shell" requires multiplication of the Green's function by powers of $p^2 - m^2$ different from unity to get a finite S-matrix element. Before arriving at this goal, it is necessary to make the somewhat lengthy theoretical detour, which is the subject of the present article, caused by the need to construct a suitable representation space. The S-matrix and cross-section formulas follow in the accompanying article.

In any scattering process involving charged particles, infinite numbers of infrared photons are emitted coherently. Since the early work of Bloch and Nordsieck[1] it has been understood that a transformation out of the Hilbert space containing finite numbers of photons to one containing infinite numbers of coherent photons is necessary to get finite cross sections. In more recent years coherent spaces have been used particularly by Chung,[2] Kibble,[3] Kulish and Faddeev,[4] and others[5] in conjunction with renormalized Feynman amplitudes to obtain finite cross sections. Despite the important insight which these works provide, they have not so far resulted in a convenient scheme for practical calculations. These are still done, following a method introduced by Dalitz[6] and elaborated by Yennie, Frautschi, and Suura,[7] which calls for the introduction of a small but finite photon mass in intermediate stages of calculation, but which cancels out of cross sections when a sum over final photon states is formed. We hope that the calculational scheme presented here and in the accompanying article may offer some practical advantages over the massive photon method, especially in situations where radiative damping is a large effect.

The present approach grew out of an earlier attempt[8] to obtain Lehmann-Symanzik-Zimmermann (LSZ) type reduction formulas for quantum electrodynamics whereby, starting from an explicit expression for the asymptotic fields, S-matrix elements are expressed in terms of Green's functions. The asymptotic charged field presented earlier correctly accounted for the distortion of the plane wave $e^{ip\cdot x}$, occurring in the charged field, by a logarithmic operator phase $\exp[ieQ(p)\ln|p\cdot x|]$, which is produced by the long-range Liénard-Wiechert potentials. This led to a finite S matrix for Coulomb scattering. However, the earlier approach severed too violently the Liénard-Wiechert potentials from the infrared radiation, and thus led to ambiguous expressions for S-matrix elements in terms of Green's functions. The singularities of Green's functions in quantum electrodynamics at the mass shell of the charged particles have been studied by many authors,[9] with the most complete results presented by Fradkin[9] and Kibble.[3] The conclusion of these investigations is that if the charged particles go on their mass shell one at a time, the singularity for the ith particle is of the form $(p_i^2 - m_i^2)^{-1+\beta_i+i\gamma_i}$, instead of $(p_i^2 - m_i^2)^{-1}$, as in a theory without massless particles. The imaginary part of the power reflects the infinite Coulomb phase shift and corresponds to the distortion of the asymptotic charged field by $\exp(i\gamma_i \ln|p_i \cdot x|)$. The real part is an effect of the soft photons, like radiation damping. This suggests that the infrared radiation distorts the asymptotic charged field by the factor $\exp(\beta_i \ln|p_i \cdot x|)$. In the present work we derive an operator form of the asymptotic charged field which contains these distortions, Eq. (1.2). When inserted into the reduction machinery, they lead to a cancellation of the singularity of the Green's function and produce a finite S-matrix element. In this way the asymptotic field may be thought of as being reconstructed from the singularities of the Green's functions. The central problem in obtaining an asymptotic field with the requisite properties is to find a suitable vector space on which it is to be represented. This is the problem of reconstructing the asymptotic state space of quantum electrodynamics from the singularities of the Green's functions. However, we have proceeded in the opposite direction. Namely, we have obtained an asymptotic field as an ansatz solution to the equations of motion at asymptotic times. The ansatz solution contains formal operator expressions that are infrared-divergent and are thus really undefined expressions. At this point a representation space is constructed in which the undefined expressions are assigned a meaning. It is then verified that the resulting asymptotic field when used with the reduction machinery leads to finite S-matrix elements. The construction of the asymptotic representation space is the core of the present article, the reduction formula being postponed to the accompanying article.

To obtain a suitable representation space, it turned out to be necessary to modify the Gupta-Bleuler method,[10] although the basic idea of an indefinite metric, which is in fact necessary for a formulation of quantum electrodynamics in terms of local fields, is retained and even elaborated. In the usual Gupta-Bleuler method, the vector potential A and the charged field ψ are represented on a nonphysical Hilbert space H which possesses both a positive-definite, but Lorentz-noninvariant inner product $(\ ,\ )$ and also a Lorentz-invariant, but indefinite inner product $\langle\,|\,\rangle$. There is a subspace on which the indefinite inner product is non-negative which, when completed in the norm, yields a new physical, Hilbert space H_{phys}.[11] However, A and ψ are not observable fields, so the requirement that they be represented on a Hilbert space, which would be necessary for a physical interpretation, is in fact lacking. Indeed, in the usual Gupta-Bleuler method, the role of the nonphysical Hilbert space H is merely to provide a topology for the vector space on which A and ψ are represented. In the present approach, the nonphysical Hilbert space H and its positive-definite but noninvariant inner product $(\ ,\ )$ are abandoned. Instead the asymptotic fields are represented on a Fock space of test functions which is provided with an indefinite and Lorentz-invariant inner product. This inner product is nonnegative on certain subspaces which, when completed in the norm, yield physical Hilbert spaces. This approach offers the possibility of a reconstruction theorem for quantum electrodynamics. Suppose the Wightman functions of quantum electrodynamics have all the usual properties except positivity. Then the usual reconstruction theorem[12] may be followed to the point of yielding local fields which are operator-valued distributions on a Fock space of test functions that possesses an indefinite inner product. If it can be shown that the metric is nonnegative on certain subspaces which are also invariant under the action of the observable fields, then these subspaces may be completed in the norm to yield physical Hilbert spaces. From this point of view the indefinite-metric formalism does not require the additional *ad hoc* postulate of a Hilbert space with two distinct inner products. Instead, it differs from an ordinary field theory with positive-definite metric simply by a weakening of the positivity postulate.

In the present approach the nonphysical Hilbert space is replaced by an asymptotic state space which is a Fock space of test functions. In their dependence on photon variables the elements of this space are sequences of n-photon test functions, $n=0,1,2,\ldots$, which, in each photon variable, are test functions defined on the future light cone $k^\mu = \omega(1,\hat{k})$, $\omega \geq 0$. The motivation for introducing this Fock space of test functions is that it is possible to establish on it an indefinite, sesquilinear [see definition preceding Eq. (3.28)], Hermitian-symmetric, Lorentz-invariant inner product, not only for test functions $\phi_\mu(k) = \phi_\mu(\omega,\hat{k})$ which are regular at $\omega=0$, so $\int \phi_\mu^*(k)\phi^\mu(k)d^3k/2\omega$ is finite, but also for test functions whose behavior at $\omega=0$ is given by $ep_\mu/p\cdot k$, for which the usual inner product is infrared-divergent. The extension of the definition of the inner product to include such test functions proceeds by a regularization, or infrared renormalization, procedure. A subspace of the Fock space of test functions on which the extended inner product is non-negative is specified by two conditions. The first is transversality, $k\cdot a(k)|\ \rangle=0$, which is the usual Gupta-Bleuler condition. The second is a condition of infrared coherence,

$$\lim_{\omega\to 0} a^\mu(k)|\ \rangle=\frac{-1}{(2\pi)^{3/2}}\sum_a\frac{e_a p_a^\mu}{p_a\cdot k}|\ \rangle. \qquad (1.1)$$

It states that the vectors in the subspace are eigenvectors of the zero-frequency annihilation operator, with an eigenvalue, which is characteristic of the subspace, determined by a set of charges e_a and momenta p_a. Thus a subspace of non-negative metric, which may be completed in the norm to a physical photon Hilbert space, is an eigenspace of $k\cdot a(k)$ with eigenvalue zero, and an eigenspace of $\omega a^\mu(k)|_{\omega=0}$ with eigenvalue $-(2\pi)^{-3/2}\sum_a e_a p_a^\mu(E_a-\vec{p}_a\cdot\hat{k})^{-1}$, where $p_a^\mu=(E_a,\vec{p}_a)$. The quantum-electrodynamic S matrix for bremsstrahlung of low-energy photons satisfies both of these conditions. The fact that a positive-definite inner product is established for such a space means that the bremsstrahlung cross section is made finite by the infrared renormalization.

On the asymptotic space, the charged field ψ^{as} has a particularly simple weak asymptotic limit,

$$\lim_{t\to\pm\infty}\psi(x)=\psi^{as}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E}\sum_s[D(p,x)b_s(p)u_s(p)e^{-ip\cdot x}+d_s^\dagger(p)D(p,x)v_s(p)e^{ip\cdot x}]. \qquad (1.2a)$$

If the factor $D(p,x)$ were missing this would be the usual Dirac free field with electron annihilation operator $b_s(p)$ and positron creation operator $d_s^\dagger(p)$. Instead $D(p,x)$ represents a logarithmic distortion of the plane wave given by

$$D(p,x)=\exp\left[ieQ(p)\epsilon(p\cdot x)\ln\left|\frac{p\cdot xl_1}{m}\right|\right]\exp\left[-eA^\dagger(p)\ln\left(\frac{\epsilon-ip\cdot xl_1}{m}\right)\right]\exp\left[eA(p)\ln\left(\frac{\epsilon+ip\cdot xl_1}{m}\right)\right]. \qquad (1.2b)$$

The first factor, found previously,[8] represents the logarithmic distortion of the plane wave produced by the Liénard-Wiechert potentials of other particles, with

$$Q(p)=\frac{1}{4\pi}\int d^4p'\frac{p\cdot p'}{[(p\cdot p')^2-p^2p'^2]^{1/2}}\rho(p'). \qquad (1.3)$$

Here $\rho(p)$ is the charge-density operator in momentum space

$$\rho(p)=-e\delta(p^2-m^2)\theta(p^0)\sum_s[b_s^\dagger(p)b_s(p)-d_s^\dagger(p)d_s(p)] \qquad (1.4)$$

The remaining factors in Eq. (1.2b) represent a similar logarithmic distortion of the plane wave produced by the zero-frequency infrared coherent photons, but with real coefficient

$$A(p)=\frac{1}{(2\pi)^{3/2}}\int d\hat{k}\frac{p^\mu}{E-\vec{p}\cdot\hat{k}}\omega a_\mu(k)\bigg|_{\omega=0}. \qquad (1.5)$$

The constant l_1 with dimension of mass is an infrared renormalization constant which cancels out of all observable quantities. The asymptotic limit (1.2) is a weak asymptotic limit. It must be sandwiched between normalizable states; $\psi^{as}(x)$ itself is not an operator, but a bilinear form, for $A^\dagger(p)$ is defined to act only to the left. The simple spatial dependence of the asymptotic limit (1.2), namely a logarithmically distorted plane wave, allows the derivation of reduction formulas. This is taken up in the accompanying article.

The plan of the present article is as follows. In Sec. II the field equations at asymptotic times are solved by ansatz. The solution for the charged field is found to contain infrared-divergent operators that are lacking in definition. Section III is devoted to constructing a representation space on which these operators can be given a meaning. This is the space of test functions mentioned earlier. The heart of the construction is an infrared renormalization procedure which provides an inner product for wave functions that are not square integrable in the infrared. Physical Hilbert spaces are constructed by completing in the norm subspaces on which the inner product is non-negative. In Sec. IV asymptotic fields are given a meaning as operators on the Fock space of test functions. First the free vector potential is considered. Then bremsstrahlung by the classical external current of a scattered charged particle is studied in detail as a methodological introduction to the scattering problem in quantum electrodynamics: An S matrix is provided which gives finite transition probabilities. Next the normalization of the charged field is related to the normalization of the renormalized electron propagator by comparing it with the propagator of an asymptotic charged field. Finally the weak limit of the asymptotic charged field given above, Eq. (1.2), is obtained. This allows the derivation of reduction formulas for the quantum-electrodynamic S matrix, dealt with in the accompanying article.

II. EQUATIONS OF MOTION AT ASYMPTOTIC TIMES

In this section we solve the field equations at asymptotic times by ansatz. We work, in the most conventional manner, with the Fock space of free particles until divergent expressions are encountered which will force us to seek another representation space. Let the states of free incoming electrons, positrons, and photons be generated in the usual way by creation operators $b_s^{\text{in}\dagger}(p)$, $d_s^{\text{in}\dagger}(p)$, and $a_\mu^{\text{in}\dagger}(k)$, respectively, which satisfy

$$\{b_s^{\text{in}}(p), b_{s'}^{\text{in}\dagger}(p')\} = 2E\delta(\vec{p}-\vec{p}\,')\delta_{ss'}, \tag{2.1a}$$

$$\{d_s^{\text{in}}(p), d_{s'}^{\text{in}\dagger}(p')\} = 2E\delta(\vec{p}-\vec{p}\,')\delta_{ss'}, \tag{2.1b}$$

$$[a_\mu^{\text{in}}(k), a_\nu^{\text{in}\dagger}(k')] = -g_{\mu\nu}2\omega\delta(\vec{k}-\vec{k}\,'), \tag{2.1c}$$

where $g^{\mu\nu}=\text{diag}(1,-1,-1,-1)$, $p^\mu=(E,\vec{p})$, $k^\mu=(\omega,\vec{k})$, $E=(\vec{p}^2+m^2)^{1/2}$, and $\omega=|\vec{k}|$. The corresponding particles are free in the sense that they transform in the usual way under the Poincaré group. In particular the generators of space-time displacements are given by

$$P^\mu = \int \frac{d^3p}{2E}\sum_s [b_s^{\text{in}\dagger}(p)b_s^{\text{in}}(p) + d_s^{\text{in}\dagger}(p)d_s^{\text{in}}(p)]p^\mu + \int \frac{d^3k}{2\omega} a_\kappa^{\text{in}\dagger}(k)(-g^{\kappa\lambda})a_\lambda^{\text{in}}(k)k^\mu. \tag{2.2}$$

There are corresponding expressions with "in" → "out," and until further notice, the "in" label will generally be dropped.

The electric charge operator Q is given by

$$Q=\int d^4p\,\rho(p), \tag{2.3}$$

where $\rho(p)$ is the charge-density operator in momentum space

$$\rho(p) = -e\delta(p^2-m^2)\theta(p^0)\sum_s [b_s^\dagger(p)b_s(p) - d_s^\dagger(p)d_s(p)], \tag{2.4a}$$

or, if there are different kinds of charge-bearing particles, of masses m_a, such as electrons and muons,

$$\rho(p) = \sum_a e_a\delta(p^2-m_a{}^2)\theta(p^0) \times \sum_s [b_{sa}^\dagger(p)b_{sa}(p) - d_{sa}^\dagger(p)d_{sa}(p)]. \tag{2.4b}$$

As an ansatz the renormalized Heisenberg electric-current operator $J_\mu(x)$ is taken to have the asymptotic limit

$$\lim_{t\to\pm\infty} J_\mu(x) = J_\mu^{\text{as}}(x) = \int d^4p\,\rho(p)(p^\mu/p^0)\delta^3(\vec{x}-\vec{p}t/p^0) = \int d^4p\,\rho(p)p^\mu\int_{-\infty}^{\infty} d\tau\,\delta^4(x-p\tau)\;. \tag{2.5}$$

This corresponds to the current of classical particles moving uniformly along straight lines through the origin, as is proper because at asymptotic times any finite impact parameter is negligible compared to $\vec{p}t/E$. Because $\rho(p)$ commutes with the momentum operator, $J^{\text{as}}(x)$ does not appear to have the correct transformation law under displacement for finite values of x. However, if matrix elements of this operator between normalizable states are formed, it is found that at asymptotic times $J^{\text{as}}(x)$ decreases like t^{-3}, with a finite spatial integral. Under a finite displacement a, the change in $J^{\text{as}}(x)$ is of order at^{-4} which is negligible by comparison. Thus $J^{\text{as}}(x)$ does behave correctly under translation in the sense of a weak asymptotic limit. Other asymptotic fields will be found which behave similarly, and the results of our calculations, which make use of asymptotic properties only, will be manifestly translationally invariant.

The renormalized Heisenberg vector potential $A^\mu(x)$ in the Feynman (Gupta-Bleuler) gauge satisfies

$$\partial^2 A^\mu = J^\mu, \tag{2.6}$$

with formal solution

$$A^\mu(x) = A^{\mu\,\text{in}}(x) + \int \Delta^{\text{ret}}(x-y)J^\mu(y)d^4y, \tag{2.7a}$$

$$A^\mu(x) = A^{\mu\,\text{out}}(x) + \int \Delta^{\text{ad}}(x-y)J^\mu(y)d^4y, \tag{2.7b}$$

where $A^{\text{in}}(x)$ has the usual expansion

$$A_\mu^{\text{in}}(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2\omega}[a_\mu^{\text{in}}(k)e^{-ik\cdot x} + a_\mu^{\text{in}\dagger}(k)e^{ik\cdot x}], \tag{2.8}$$

and in → out. In Eq. (2.7a) there are contributions only for $y^0<x^0$, so for $t=x^0\to-\infty$, we insert the asymptotic limit for $\boldsymbol{J}^\mu(y)$ and obtain

$$\lim_{t\to-\infty} A_\mu(x) = A_\mu^{\text{as}}(x) = A_\mu^{\text{in}}(x) + \frac{1}{4\pi}\int d^4p\frac{\rho^{\text{in}}(p)p_\mu}{[(p\cdot x)^2-p^2x^2]^{1/2}}. \tag{2.9}$$

The last term represents the Liénard-Wiechert potential of the asymptotic charged particles. It will be estimated later in various circumstances.

The main problem is to find the asymptotic limit ψ^{as} of the charged field ψ. If the photons were massive, it would coincide with the free field

$$\psi^{in}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E}\sum_s[b_s^{in}(p)u_s(p)e^{-ip\cdot x}+d_s^{\dagger in}(p)v_s(p)e^{ip\cdot x}].$$

Instead it will be found from the equation of motion that

$$(i\partial\!\!\!/-m)\psi^{as}+e:A^{as}\psi^{as}:=0, \tag{2.10}$$

where A^{as} is the asymptotic form of A just obtained. Here normal ordering means that the creation and annihilation parts of A^{in} appear on the left and right of ψ^{as}, respectively, and the charged particle creation and annihilation operators in ψ^{as} appear respectively on the left and right of ρ^{in}. To obtain a solution of this equation at asymptotic times, consider each momentum component p of ψ^{in} as a wave packet whose center of gravity moves along

$$x_\mu=p_\mu\tau. \tag{2.11}$$

Then, in the eikonal approximation, each momentum component $b_s(p)$ becomes multiplied by

$$E(p,x)=:\exp\left[ie\int_{\mp\infty}^{\tau(x)}A_\mu^{as}(p\tau)p^\mu d\tau\right]:. \tag{2.12}$$

Here the upper or lower sign applies to early or late times, and $\tau(x)$ is obtained by contracting $x=p\tau$ with p,

$$\tau(x)=p\cdot x/m^2. \tag{2.13}$$

Upon inserting Eq. (2.9) for A^{as} one finds

$$E(p,x)=\exp[ieQ(p)\epsilon(p\cdot x)\ln|cp\cdot x|]R(p,x), \tag{2.14}$$

where ϵ is the sign function, and

$$Q(p)=\frac{1}{4\pi}\int d^4p'\frac{p\cdot p'}{[(p\cdot p')^2-p^2p'^2]^{1/2}}\rho(p'). \tag{2.15}$$

The integration constant appearing inside the logarithm, $\ln|cp\cdot x|$, may be given any desired value by a unitary transformation involving only the charged particle creation and annihilation operators. It will be assigned a convenient value in Sec. IV D. Contributions at $\tau=\pm\infty$ have been dropped. The radiation operator $R(p,x)$ is given by

$$R(p,x)=\exp[a^\dagger(\phi_{e,p}(x))]\exp[-a(\phi_{e,p}(x))], \tag{2.16}$$

where

$$a^\dagger(\phi)=\int\frac{d^3k}{2\omega}a_\mu^\dagger(k)(-g^{\mu\nu})\phi_\nu(k) \tag{2.17a}$$

(the metric on polarization indices is $-g_{\mu\nu}$),

$$\phi^\mu_{e,p}(k,x)=\frac{-e}{(2\pi)^{3/2}}\frac{p^\mu}{p\cdot k}\exp\left(\frac{ik\cdot pp\cdot x}{m^2}\right), \tag{2.17b}$$

and is recognized as the operator which, when applied to the vacuum, produces the coherent state characterized by the function $\phi_{e,p}(k,x)$.

A preliminary form of the asymptotic charged field ψ_{pre} is obtained by multiplying in normal order each momentum component of the free field by the eikonal factor $E(p,x)$,

$$\psi_{pre}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E}\sum_s[E(p,x)b_s(p)u_s(p)e^{-ip\cdot x}+d_s^\dagger(p)E(p,x)v_s(p)e^{ip\cdot x}]. \tag{2.18}$$

According to Eq. (2.14) $E(p,x)$ contains a logarithmic distortion of the plane wave caused by the Liénard-Wiechert potential of other particles, and the radiation factor $R(p,x)$, Eq. (2.16), which creates a coherent state when applied to the vacuum. However, the length square of the n-photon component of this state is given by

$$\frac{(-1)^n}{n!}\left[\frac{e^2}{(2\pi)^3}\int\frac{d^3k}{2\omega}\frac{p^2}{(p\cdot k)^2}\right]^n, \tag{2.19}$$

which diverges in both the ultraviolet and the infrared. The ultraviolet divergence is presumably controlled when $\psi_{pre}(x)$ is smeared with a test function $g(x)$ because the ω integration in Eq. (2.16) converges weakly at large ω and, in fact, no ultraviolet divergences are encountered in the present work. However, the divergence at $\omega=0$ is not eliminated when $\psi_{pre}(x)$ is smeared, because in $R(p,x)$, x always appears multiplied by ω. This means that when (smeared) ψ_{pre} is applied to any vector in the Fock space, the result is either meaningless, or zero, or lies outside the Fock space. Thus the operator expression for ψ_{pre} is really undefined. In the next section a representation space will be constructed by an infrared renormalization procedure in which ψ_{pre} will have a finite operator meaning.

III. INFRARED RENORMALIZATION

A. One-photon space

In the preliminary asymptotic field just found, one encounters the formal expressions

$$\int\frac{d^3k}{2\omega}a_\mu^\dagger(k)(-g^{\mu\nu})\phi_\nu(k,x) \tag{3.1a}$$

$$\int \frac{d^3k}{2\omega} a_\mu(k)(-g^{\mu\nu})\phi_\nu^*(k,x)\,, \tag{3.1b}$$

with

$$\phi^\mu(k,x) = \frac{-e}{(2\pi)^{3/2}} \frac{p^\mu}{p\cdot k} \exp\left(\frac{ik\cdot pp\cdot x}{m^2}\right). \tag{3.2}$$

This function has the low-frequency limit

$$\lim_{\omega\to 0} \phi^\mu(k,x) = \frac{-e}{(2\pi)^{3/2}} \frac{p^\mu}{p\cdot k}, \tag{3.3}$$

which in independent of x. For such a wave function the usual inner product

$$\int \phi_\mu^*(k)(-g^{\mu\nu})\phi_\nu(k)\frac{d^3k}{2\omega} \tag{3.4}$$

diverges at $\omega = 0$, which is the crudest form of the infrared divergence. Consequently the operators (3.1) are without precise meaning and calculations with them lead to infrared-divergent expressions.

These divergences will be circumvented by a modification and elaboration of the Gupta-Bleuler method, described in outline as follows. It will be shown that a Lorentz-invariant, Hermitian-symmetric, sesquilinear but indefinite form may be established on a space of test functions which includes functions whose low-frequency behavior is given by

$$\lim_{\omega\to 0} \phi^\mu(k) = \sum_{a=1}^{A} \frac{-c_a}{(2\pi)^{3/2}} \frac{p_a^\mu}{p_a\cdot k}, \tag{3.5}$$

where the c_a are complex constants. This induces an indefinite metric on the Fock space constructed with such test functions, namely the space of sequences of symmetric n-photon wave functions, $n = 0, 1, \ldots, \infty$, which are test functions of type (3.5) in each variable. It will be shown that the indefinite metric is non-negative on certain subspaces of this Fock space of test functions. A class of physically equivalent Hilbert spaces will be obtained by completing each of the subspaces in the norm.

Before embarking on the formal construction it is convenient to establish notation and some elementary Lorentz transformation properties. Photon wave functions are defined on the future light cone $k^\mu = \omega(1,\hat{k})$, where $\hat{k}$ is a point on the unit sphere that represents a light ray, and the frequency ω measures distance along a light ray, $\phi^\mu(k) = \phi^\mu(\omega,\hat{k})$. Under Lorentz transformation $k'^\mu = \Lambda^\mu{}_\nu k^\nu$, ω and $\hat{k} = (\hat{k}^1, \hat{k}^2, \hat{k}^3) = \hat{k}^i$ transform according to

$$\omega' = (\Lambda^0{}_0 + \Lambda^0{}_j \hat{k}^j)\omega\,, \tag{3.6a}$$

$$\hat{k}'^i = \frac{\Lambda^i{}_0 + \Lambda^i{}_j \hat{k}^j}{\Lambda^0{}_0 + \Lambda^0{}_j \hat{k}^j}. \tag{3.6b}$$

Observe that light rays transform independently of frequencies. Observe also that frequencies transform multiplicatively so the vertex of the light cone $\omega = 0$ is never reached under Lorentz transformations. The orbit of k under Λ is thus the product of the unit sphere $\hat{k}^2 = 1$ and the open half line $\omega > 0$. Because $d^3k/2\omega = d\hat{k}\omega^2 d\omega/2\omega$ is known to be invariant under Lorentz transformation, and because $d\omega/\omega$ is invariant under the transformation (3.6a), we obtain for the Jacobian of the transformation (3.6b)

$$d\hat{k}' = d\hat{k}(\Lambda^0{}_0 + \Lambda^0{}_j \hat{k}^j)^{-2}\,. \tag{3.7}$$

It is convenient to introduce for each light ray $\hat{k}$ the 4-vector quantity

$$\hat{k}^\mu \equiv (1, \hat{k}^i) = k^\mu/\omega\,, \tag{3.8a}$$

which, however, under Lorentz transformation does not quite transform vectorially

$$\hat{k}'^\mu = \frac{\Lambda^\mu{}_\nu \hat{k}^\nu}{\Lambda^0{}_\nu \hat{k}^\nu}. \tag{3.8b}$$

Its contraction with an ordinary 4-vector $p\cdot\hat{k} = p^0 - p^i\hat{k}^i = E - p^i\hat{k}^i$ is not quite a scalar, but instead under the change of variables (3.6b) it satisfies

$$p\cdot\hat{k} = \frac{\Lambda p\cdot\hat{k}'}{\Lambda_\mu{}^0 \hat{k}'^\mu}. \tag{3.9}$$

This yields a result which will be useful later. Let $F(p_a\cdot\hat{k})$ be a function of $p_a\cdot\hat{k}$, $a = 1, \ldots, A$, which is homogeneous of degree -2 in the $p_a\cdot\hat{k}$. Then the integral $J(p_a) = \int d\hat{k}\, F(p_a\cdot\hat{k})$ is an invariant function of the p_a,

$$J(p_a) = J(\Lambda p_a)\,. \tag{3.10}$$

Because the wave functions of interest, $\phi^\mu(\omega,\hat{k})$ diverge as ω approaches zero according to Eq. (3.5), it is more convenient to work with wave functions

$$f^\mu(\omega,\hat{k}) \equiv \omega\phi^\mu(\omega,\hat{k}) \tag{3.11}$$

with finite infrared limit

$$\lim_{\omega\to 0} f^\mu(\omega,\hat{k}) = \sum_{a=1}^{A} \frac{-c_a}{(2\pi)^{3/2}} \frac{p_a^\mu}{p_a\cdot\hat{k}}. \tag{3.12}$$

Under Lorentz transformation the f^μ transform according to

$$f^\mu(k) \to (U(\Lambda)f)^\mu(k) = (\Lambda^0{}_\mu \hat{k}^\mu)^{-1}\Lambda^\mu{}_\nu f^\nu(\Lambda^{-1}k)\,. \tag{3.13}$$

The usual inner product takes the form

$$\langle f_1|f_2\rangle = \int \frac{d\hat{k}}{2}\int_0^\infty \frac{d\omega}{\omega} f_1^{\mu *}(\omega,\hat{k})(-g_{\mu\nu}) f_2^\nu(\omega,\hat{k})\,, \tag{3.14}$$

which makes the infrared divergences painfully apparent when the f have a finite limit at $\omega=0$. This completes the preliminaries and we proceed to the formal construction.

Consider a space of complex test functions $f^\mu(\omega,\hat{k})$, defined on the product of the closed half line $\omega \geq 0$ and the unit sphere $\hat{k}^2=1$, which are of fast decrease as ω approaches infinity. For our purposes, the only essential feature of this space is that $f^\mu(\omega,\hat{k})$ and $\partial f^\mu(\omega,\hat{k})/\partial\omega$ be continuous as ω approaches zero and have finite limits at $\omega=0$ which may depend on $\hat{k}$,

$$f^\mu(0,\hat{k})=\lim_{\omega\to 0} f^\mu(\omega,\hat{k}), \quad \frac{\partial f^\mu}{\partial\omega}(0,\hat{k})=\lim_{\omega\to 0}\frac{\partial f^\mu}{\partial\omega}(\omega,\hat{k}). \tag{3.15}$$

Let $\mathcal{E}_0$ be the subspace of such test functions which vanish at $\omega=0$,

$$f^\mu(\omega,\hat{k})\in\mathcal{E}_0:\ f^\mu(0,\hat{k})=0. \tag{3.16}$$

Since, in addition, their first derivative is finite, they vanish like ω to the first power, and the usual inner product (3.14) is finite. Let $\mathcal{E}_p$ be the space of test functions with behavior at $\omega=0$ specified by

$$f^\mu(\omega,\hat{k})\in\mathcal{E}_p:\ f^\mu(0,\hat{k})=cf_p^\mu(\hat{k}), \tag{3.17}$$

where c is a complex constant and

$$f_p^\mu(\hat{k})\equiv\frac{-1}{(2\pi)^{3/2}}\frac{p^\mu}{p\cdot\hat{k}}. \tag{3.18}$$

This space may be loosely thought of as the space of one-photon wave functions associated with a charged particle of momentum p. For these, the inner product (3.14) is obviously divergent. Finally let $\mathcal{E}$ be the space of test functions $f^\mu(\omega,\hat{k})$ made up of finite linear combinations of functions of $\mathcal{E}_p$ for various p,

$$f^\mu(\omega,\hat{k})\in\mathcal{E}:\ f^\mu(0,\hat{k})=\sum_{a=1}^{A}c_a f_{p_a}^\mu(\hat{k}). \tag{3.19}$$

The space $\mathcal{E}$ appears to be the natural space for the study of the infrared problem. It may be thought of as the space of possible one-photon test functions in the presence of several charged particles. We have obviously $\mathcal{E}\supset\mathcal{E}_p\supset\mathcal{E}_0$.

Let us describe briefly, to establish notation and as a model for future constructions, how the usual one-photon physical Hilbert space may be constructed. On $\mathcal{E}_0$ the inner product (3.14) is finite, but indefinite. However, on $\mathcal{E}_0^+$, the subspace of $\mathcal{E}_0$ consisting of transverse wave functions,

$$f^\mu\in\mathcal{E}_0^+:\ k_\mu f^\mu(k)=0 \text{ and } f^\mu\in\mathcal{E}_0, \tag{3.20}$$

the inner product is non-negative, $\langle f|f\rangle\geq 0$. The transversality condition also reduces the number of polarization components from four to three. However, the space $\mathcal{E}_0^+$ also contains test functions of vanishing norm which form the subspace $\mathcal{E}_0^0$ of functions f^μ parallel to k^μ:

$$f^\mu\in\mathcal{E}_0^0:\ f^\mu(k)=k^\mu f(k) \text{ and } f^\mu\in\mathcal{E}_0^+. \tag{3.21}$$

Let $[\mathcal{E}_0^+]$ be the space of equivalence classes $[f]$ of test functions in $\mathcal{E}_0^+$ (modulo test functions in $\mathcal{E}_0^0$): $[\mathcal{E}_0^+]=\mathcal{E}_0^+/\mathcal{E}_0^0$ or

$$f,f_1\in[f]:\ f_1-f\in\mathcal{E}_0^0. \tag{3.22}$$

In $[\mathcal{E}_0^+]$ the inner product $\langle[f]|[g]\rangle$, for $[f],[g]\in[\mathcal{E}_0^+]$, calculated by taking any representative elements, $\langle[f]|[g]\rangle=\langle f|g\rangle$, is positive-definite. The formation of equivalence classes of transverse test functions reduces the number of polarization components from three to two, as desired. Finally the physical one-photon Hilbert space $\mathcal{H}^{(1)}$ is the closure of $[\mathcal{E}_0^+]$. Namely, it is the vector space whose elements are equivalence classes of Cauchy sequences of vectors in $[\mathcal{E}_0^+]$ (modulo Cauchy sequences which converge to zero) with an inner product defined to be the limit of the inner product in $[\mathcal{E}_0^+]$. The construction of the Hilbert space $\mathcal{H}^{(1)}$ from the space of non-negative norm $\mathcal{E}_0^+$ is a standard construction, called completion in the norm, which in no way depends on the nature of the space $\mathcal{E}_0^+$. It has been described many times[12] and will be used here repeatedly. Note that, in contrast to the usual Gupta-Bleuler method, we have not introduced a nonphysical Hilbert space with noncovariant but positive-definite norm defined on the 4-component wave-function space. Nor shall we. By establishing an indefinite form on $\mathcal{E}$ instead of just on $\mathcal{E}_0$ it will be possible to effect a similar construction for photons associated with charged particles. However, in this case it will be necessary to go to the Fock space $\mathfrak{F}(\mathcal{E})$ before a suitable subspace of non-negative metric may be found.

Any vector f_p in $\mathcal{E}_p$ may be written as

$$f_p^\mu(k)=cf_p^\mu(\hat{k})f_1(k)+f_0^\mu(k), \tag{3.23}$$

where $f_1(\omega,\hat{k})$ satisfies $f_1(0,\hat{k})=1$, $f_0\in\mathcal{E}_0$, and $f_p^\mu(\hat{k})$ is given by Eq. (3.18). The generic element of $\mathcal{E}$ may thus be written

$$f^\mu(k)=\sum_{a=1}^{A}c_a f_{p_a}^\mu(\hat{k})f_1(k)+f_0^\mu(k). \tag{3.24}$$

The linear form on $\mathcal{E}$ will be obtained by (anti) linear extension after it is defined for the separate terms in this expansion. For $f_0,f_0'\in\mathcal{E}_0$ the inner product is the usual one,

$$\langle f_0|f_0'\rangle=\int\frac{d\hat{k}}{2}\int_0^\infty\frac{d\omega}{\omega}f_0^{\mu *}(k)(-g_{\mu\nu})f_0'^{\nu}(k), \tag{3.25}$$

and similarly for the inner product between $f_p^\mu(k) = f_p^\mu(\hat{k}) f_1(k)$ and $f_0 \in \mathcal{E}_0$

$$\langle f_p | f_0 \rangle = \langle f_0 | f_p \rangle^* = \int \frac{d\hat{k}}{2} \int_0^\infty \frac{d\omega}{\omega} f_p^{\mu *}(k)(-g_{\mu\nu}) f_0^\nu(k) . \tag{3.26}$$

These inner products are convergent because f_p is finite at $\omega = 0$ while f_0 and f_0', which are zero at $\omega = 0$ with finite derivative, vanish like ω. [This is not true for vectors in $\mathcal{H}^{(1)}$, and (3.26) would diverge for some f_0 in $\mathcal{H}^{(1)}$. However, an inner product between f_p and the generic element of $\mathcal{H}^{(1)}$ is not defined.] By (anti) linear extension from Eq. (3.23), the inner product for any $f \in \mathcal{E}$ and $f_0 \in \mathcal{E}_0$ is given by

$$\langle f | f_0 \rangle = \langle f_0 | f \rangle^* = \int \frac{d\hat{k}}{2} \int \frac{d\omega}{\omega} f^{\mu *}(k)(-g_{\mu\nu}) f_0^\nu(k) . \tag{3.27}$$

There remains only to define an inner product between $f_p(k) = f_p(\hat{k}) f_1(k)$ and $f_{p'}(k) = f_{p'}(\hat{k}) f_1(k)$. It would diverge under the usual definition. Consider instead the usual inner product in which the left factor is regularized by the substitution $f_p^\mu(k) \to f_p^\mu(k)\theta(p\cdot k - ml)$, where $p^2 = m^2$ and l is a positive number with dimensions of mass,

$$\int \frac{d\hat{k}}{2} f_p^{\mu *}(\hat{k})(-g_{\mu\nu}) f_{p'}^\nu(\hat{k}) \int_{\omega(l)}^\infty \frac{d\omega}{\omega} |f_1(k)|^2 = \int \frac{d\hat{k}}{2} f_p^{\mu *}(\hat{k}) f_{p'\mu}(\hat{k}) \int_{\omega(l)}^\infty d\omega \ln\left(\frac{\omega}{\omega(l)}\right) \frac{\partial}{\partial\omega} |f_1(k)|^2 .$$

Here $\omega(l) = ml(p\cdot\hat{k})^{-1}$, and the contribution at $\omega = \infty$ vanishes because all test functions are of fast decrease there. The last integral remains convergent if the lower limit of integration is replaced by zero, which involves neglect of terms which vanish with l. Let the result be the definition of the inner product for $f_p^\mu(k) = f_p^\mu(\hat{k}) f_1(k)$, $f_{p'}^\mu(k) = f_{p'}^\mu(\hat{k}) f_1(k)$,

$$\langle f_p | f_{p'} \rangle \equiv \int \frac{d\hat{k}}{2} f_p^{\mu *}(\hat{k}) f_{p'\mu}(\hat{k}) \int_0^\infty d\omega \ln\left(\frac{p\cdot k}{ml}\right) \frac{\partial}{\partial\omega} |f_1(k)|^2$$

$$= \int \frac{d\hat{k}}{2} \int_0^\infty d\omega \ln\left(\frac{p\cdot k}{ml}\right) \frac{\partial}{\partial\omega} [f_p^{\mu *}(k) f_{p'u}(k)] .$$

With this definition the divergence of $\int_0 d\omega/\omega$ is replaced by the convergence of $-\int_0 d\omega \ln\omega \partial/\partial\omega$. As defined, the inner product depends on an arbitrary constant l, which is a typical renormalization phenomenon. Observe that the inner products (3.25) and (3.26) may be integrated by parts with vanishing contributions at $\omega = 0$,

$$\langle f_0 | f_0' \rangle = \int \frac{d\hat{k}}{2} \int_0^\infty d\omega \ln\left(\frac{p\cdot k}{ml}\right) \frac{\partial}{\partial\omega} [f_0^{\mu *}(k) f_{0\mu}'(k)] ,$$

$$\langle f_p | f_0' \rangle = \int \frac{d\hat{k}}{2} \int_0^\infty d\omega \ln\left(\frac{p\cdot k}{ml}\right) \frac{\partial}{\partial\omega} [f_p^{\mu *}(k) f_{0\mu}'(k)] .$$

Therefore, by requiring that the inner product be sesquilinear, i.e., linear in the second factor and antilinear in the first factor, the inner product has a unique extension to all $f_p \in \mathcal{E}_p$ and all $f' \in \mathcal{E}$ given by

$$\langle f_p | f' \rangle = \int \frac{d\hat{k}}{2} \int_0^\infty d\omega \ln\left(\frac{p\cdot k}{ml}\right) \frac{\partial}{\partial\omega} [f_p^{\mu *}(k) f_\mu'(k)] . \tag{3.28}$$

Finally because by definition every vector f in $\mathcal{E}$ is the sum of vectors in $\mathcal{E}_p$, $f = \sum_{a=1}^A f_{p_a}$, the inner product for any $f, f' \in \mathcal{E}$ becomes by antilinearity in the first factor

$$\langle f | f' \rangle = \int \frac{d\hat{k}}{2} \int_0^\infty d\omega \sum_{a=1}^A \left\{ \ln\left(\frac{p_a\cdot k}{m_a l}\right) \frac{\partial}{\partial\omega} [f_{p_a}^{\mu *}(k) f_\mu'(k)] \right\} . \tag{3.29}$$

Because $d\omega \cdots \partial/\partial\omega \cdots$ is invariant under the Lorentz transformation (3.6a), this inner product is Lorentz-invariant. Thus a covariant sesquilinear form has been established on $\mathcal{E}$.

We shall prove that despite its unsymmetrical appearance and derivation, the inner product is in fact Hermitian symmetric, $\langle f | f' \rangle = \langle f' | f \rangle^*$. Let $\Delta(\hat{k})$ be an arbitrary function of $\hat{k}$, and possibly depending also on the p_a or p_a'. We have

$$\langle f | f' \rangle = \int \frac{d\hat{k}}{2} \int_0^\infty d\omega \sum_{a=1}^A \left\{ \left(\ln\frac{p_a\cdot\hat{k}\Delta}{m_a l} + \ln\frac{\omega}{\Delta} \right) \frac{\partial}{\partial\omega} [f_{p_a}^{\mu *}(k) f_\mu'(k)] \right\} .$$

The first term is the integral of an exact derivative with respect to ω. Therefore, with $f^{\mu}_{p_a}(0,\hat{k}) = c_a f^{\mu}_{p_a}(\hat{k})$, $f^{\mu}(k) = \sum_a f^{\mu}_{p_a}(k)$, $f'^{\mu}(k) = \sum_b f'_{p'_b}(k)$, and $f'^{\mu}_{p'_b}(0,\hat{k}) = c'_b f^{\mu}_{p'_b}(\hat{k})$, where $f^{\mu}_{p}(\hat{k}) = -(2\pi)^{-3/2} p^{\mu}/p\cdot\hat{k}$ we have

$$\langle f|f'\rangle = \sum_{a,b} c_a^* c'_b \langle p_a | p'_b\rangle_\Delta + \langle f|f'\rangle_\Delta , \tag{3.30a}$$

where

$$\langle p_a | p'_b\rangle_\Delta \equiv \int \frac{d\hat{k}}{2} \ln\left(\frac{p_a\cdot\hat{k}\Delta(\hat{k})}{m_a l}\right) f^{\mu}_{p_a}(\hat{k})(-g_{\mu\nu}) f^{\nu}_{p'_b}(\hat{k}) , \tag{3.30b}$$

$$\langle f|f'\rangle_\Delta \equiv \int \frac{d\hat{k}}{2}\int_0^\infty d\omega \ln\left(\frac{\omega}{\Delta(\hat{k})}\right)\frac{\partial}{\partial\omega}[f^{\mu *}(k) f'_{\mu}(k)] . \tag{3.30c}$$

Thus the inner product has been written as the sum of two terms. The second is obviously Hermitian symmetric and makes no explicit reference to the zero-frequency limit of the test functions. The first involves an integral over light rays and depends only on the constants c_a (c'_b) and momenta p_a (p'_b) which characterize the zero-frequency limit of the test functions. Because $\langle p|p'\rangle_\Delta$ is real, to establish Hermiticity it is sufficient to show that $I \equiv \langle p|p'\rangle_\Delta - \langle p'|p\rangle_\Delta$ vanishes. With $f^{\mu}_{p}(\hat{k}) = -(2\pi)^{-3/2} p^{\mu}/p\circ\hat{k}$ we have

$$I = \frac{1}{(2\pi)^3}\int \frac{d\hat{k}}{2}\ln\left(\frac{p\circ\hat{k}m'}{p'\cdot\hat{k}m}\right)\frac{p\cdot p'}{p\cdot\hat{k}p'\circ\hat{k}} .$$

The integrand is homogeneous of degree -2 in $p\cdot\hat{k}$ and $p'\cdot\hat{k}$, so Eq. (3.10) applies and I is invariant, $I(\Lambda p, \Lambda p') = I(p,p')$. Moreover, I depends only on the unit 4-vectors p/m and p'/m', $I(\Lambda p/m, \Lambda p'/m') = I(p/m, p'/m')$. It is always possible to choose a Lorentz transformation Λ such that $\Lambda p/m = p'/m'$ and $\Lambda p'/m' = p/m$ (Λ is a 180° rotation in the center of mass of p/m and p'/m'), so $I(p/m, p'/m') = I(p'/m', p/m)$. However, by inspection of the integral, I is odd under the interchange $p/m \leftrightarrow p'/m'$, so $I=0$. Q.E.D.

Note that in dealing with particles of different mass, the appearance of m in the definition of the inner product is essential in establishing Hermiticity. The constant l represents an infrared renormalization constant which is common to all charged particles. It will be shown that all physical quantities are independent of l, which may therefore be assigned any value, for example, $l=1$. However, it is convenient to keep it as an arbitrary parameter for bookkeeping purposes.

Thus a covariant, sesquilinear, Hermitian inner product has been established on $\mathcal{E}$, for which Eqs. (3.29) and (3.30) are convenient representations. Since we will shortly deal with wave functions of several variables it is convenient to write the inner product as

$$\langle f|f'\rangle = \int (dk)_l f^{\mu *}(k)(-g_{\mu\nu}) f'^{\nu}(k) , \tag{3.31}$$

which represents its formal properties and indicates which variable is eliminated when the inner product is formed. However, $(dk)_l$ does not satisfy the usual definition of a measure, although, if either f or f' is in $\mathcal{E}_0$ then $(dk)_l = dk \equiv d\hat{k}d\omega(2\omega)^{-1}$, which is a measure.[13]

B. Photon Fock space

The extension of the inner product from $\mathcal{E}_0$ to $\mathcal{E}$ involves what is essentially a linear subtraction. Therefore, one does not expect to find a subspace in $\mathcal{E}$ where the inner product is non-negative, beyond the original one, $\mathcal{E}_0^+$. However, the situation is quite different in the Fock space $\mathfrak{F}(\mathcal{E})$ constructed from $\mathcal{E}$ where, on certain subspaces, the linear subtraction is converted to a multiplicative renormalization. Let $\mathfrak{F}(\mathcal{E})$ be the direct sum

$$\mathfrak{F}(\mathcal{E}) = \bigoplus_{n=0}^{\infty} \mathcal{E}^{(n)} , \tag{3.32a}$$

where $\mathcal{E}^{(0)}$ is a one dimensional vector space, $\mathcal{E}^{(1)} = \mathcal{E}$, and $\mathcal{E}^{(n)}$ is the symmetric tensor product of $\mathcal{E}^{(1)}$ with itself n times

$$\mathcal{E}^{(n)} = (\mathcal{E}^{\otimes n})_s . \tag{3.32b}$$

The elements F, G of $\mathfrak{F}(\mathcal{E})$ are thus infinite sequences of symmetric n-photon tests functions which are test functions in $\mathcal{E}$ in each variable:

$$F = \{F_{(n)}\} = \{F_{(0)}, F^{\mu}_{(1)}(k), \ldots, F^{\mu_1\cdots\mu_n}_{(n)}(k_1\cdots k_n), \ldots\} . \tag{3.32c}$$

Addition and multiplication by complex numbers have the usual definition. We shall call F a Fock test function and $\mathfrak{F}(\mathcal{E})$ a test-function Fock space. An inner product between $F=\{F_{(n)}\}$ and $G=\{G_{(n)}\}$ is provided by the definition

$$\langle F|G\rangle = F_0^* G_0 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int\prod_{r=1}^{n}(dk_r)_l F^{\mu_1\cdots\mu_n *}_{(n)}(k_1\cdots k_n) G_{(n)\mu_1\cdots\mu_n}(k_1\cdots k_n) . \tag{3.33}$$

The inner product may diverge because of the infinite sum. However, it is finite on many subspaces, for example, on the sequences with only a finite number of terms different from zero.

A creation operator denoted by $c^\dagger(f)$, $f\in\mathcal{S}$ which maps $\mathfrak{F}(\mathcal{S})$ into $\mathfrak{F}(\mathcal{S})$ may be defined in the usual way. With $F=\{F_{(n)}\}$ and $F_{(-1)}\equiv 0$, a new sequence $c^\dagger(f)F\in\mathfrak{F}(\mathcal{S})$ is provided by the sequence

$$[c^\dagger(f)F]_{(n)}^{\mu_1\cdots\mu_n}(k_1\cdots k_n)=f^{\mu_1}(k_1)F_{(n-1)}^{\mu_2\cdots\mu_n}(k_2\cdots k_n)+\cdots+f^{\mu_n}(k_n)F_{(n-1)}^{\mu_1\cdots\mu_{n-1}}(k_1\cdots k_{n-1}). \tag{3.34}$$

Comparison with Eq. (3.33) shows that the Hermitian conjugate operator to $c^\dagger(f)$, satisfying, for $F,G\in\mathfrak{F}(\mathcal{S})$,

$$\langle c(f)G\,|\,F\rangle=\langle G|\,c^\dagger(f)F\rangle \tag{3.35}$$

is given by

$$[c(f)F]_{(n)}^{\mu_1\cdots\mu_n}(k_1\cdots k_n)=-\int(dk)_l f_\mu^*(k)F_{(n+1)}^{\mu,\mu_1\cdots\mu_n}(k,k_1\cdots k_n). \tag{3.36}$$

Note that $c^\dagger(f)$ depends linearly on f, whereas $c(f)$ is antilinear. These operators satisfy the commutation relations

$$[c(f),c^\dagger(g)]=\langle f\,|g\rangle . \tag{3.37}$$

The vacuum state $|0\rangle\equiv(1,0,0,\ldots)$ is annihilated by all the $c(f)$, $c(f)|0\rangle=0$. It is now possible to assign a meaning to the formal expression (3.1) introduced at the beginning of this section, namely by the replacements

$$\int\frac{d^3k}{2\omega}\,a_\mu^\dagger(k)(-g^{\mu\nu})\phi_\nu(k)\rightarrow c^\dagger(\omega\phi), \tag{3.38a}$$

$$\int\frac{d^3k}{2\omega}\,a_\mu(k)(-g^{\mu\nu})\phi_\nu^*(k)\rightarrow c(\omega\phi). \tag{3.38b}$$

This gives a meaning to $\psi_{\text{pre}}(x)$, the preliminary form of the asymptotic charged field. (The reader who prefers to see an application before continuing the present formal development may turn to Sec. IV C where the propagator $\langle 0|\,T[\psi_{\text{pre}}(x)\,\bar\psi_{\text{pre}}(y)]|\,0\rangle$ is calculated.)

The annihilation operator $c^\mu(k)$ depending on a single momentum vector k, defined by

$$[c^\mu(k)F]_{(n)}^{\mu_1\cdots\mu_n}(k_1\cdots k_n)=F_{(n+1)}^{\mu,\mu_1\cdots\mu_n}(k,k_1\cdots k_n) \tag{3.39a}$$

for $F=\{F_{(n)}^{\mu_1\cdots\mu_n}(k_1\cdots k_n)\}$, satisfies

$$[c^\mu(k),c^\dagger(f)]=f^\mu(k) \tag{3.39b}$$

and maps $\mathfrak{F}(\mathcal{S})$ into $\mathfrak{F}(\mathcal{S})$. (However, the corresponding creation operator $c^{\mu\dagger}(k)$, with $[c_\mu(k),c_\nu^\dagger(k')]=-g_{\mu\nu}\delta_{(l)}(k-k')$ produces a sequence of distributions and not another Fock test function.) Comparison with Eq. (3.37) yields

$$c(f)=\int(dk)_l f^{\mu*}(k)(-g_{\mu\nu})c^\nu(k), \tag{3.40a}$$

or explicitly, by Eq. (3.29), with $f=\sum_a f_{p_a}, f_{p_a}\in\mathcal{S}_{p_a}$,

$$c(f)=\int\frac{d\hat k}{2}\int_0^\infty d\omega\sum_{a=1}^{A}\left\{\ln\left(\frac{p_a\cdot k}{m_a l}\right)\frac{\partial}{\partial\omega}[f_{p_a}^{\mu*}(k)c_\mu(k)]\right\}. \tag{3.40b}$$

The operator $c^\mu(k)$ is useful for characterizing various subspaces. In particular the subspace on which, as will be seen, Maxwell's equations are satisfied in expectation value is provided by

$$k_\mu c^\mu(k)F=0, \tag{3.41a}$$

which implies transversality of all test functions in the sequence $F=\{F_n\}$, $n=1,2,\ldots$

$$k_\mu F_{(n)}^{\mu,\mu_2\cdots\mu_n}(k,k_2\cdots k_n)=0,\quad n=1,2,\ldots. \tag{3.41b}$$

Because $F_{(n)}$ is symmetric, this is the same as transversality in each variable. With $c^\mu(k)=c^\mu(\omega,\hat k)$, the zero-frequency annihilation operator depending on the light ray $\hat k$ is defined by

$$c^\mu(\hat{k}) \equiv c^\mu(0, \hat{k}) . \tag{3.42}$$

The subspace $\mathfrak{F}(\mathcal{S}_0)$ of sequences of test function which vanish at the origin in each argument is specified by

$$c^\mu(\hat{k})F = 0 . \tag{3.43}$$

Let us construct the physical Hilbert space for photons in the absence of charged particles. The method is the same as for the one-photon space. For $F, G \in \mathfrak{F}(\mathcal{S}_0)$ the inner product is given by ordinary integration

$$\langle F|G\rangle = F_0^* G_0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int dk_1 \cdots dk_n\, F_{(n)}^{\mu_1 \cdots \mu_n *}(k_1 \cdots k_n) G_{(n)\mu_1 \cdots \mu_n}(k_1 \cdots k_n) , \tag{3.44}$$

where $dk \equiv d\hat{k}\, d\omega (2\omega)^{-1}$. Let $\mathfrak{F}(\mathcal{S}_0^+)$ be the subspace of $\mathfrak{F}(\mathcal{S}_0)$ consisting of transverse Fock test functions, i.e., they satisfy Eqs. (3.41). For these, the norm $\langle F|F\rangle$ corresponding to Eq. (3.44) is a sum, possibly divergent, of non-negative numbers. Let $\mathfrak{F}^0(\mathcal{S}_0^+)$ be the subspace of $\mathfrak{F}(\mathcal{S}_0^+)$ consisting of vectors of vanishing norm. They are characterized by $F_{(0)} = 0$ and every $F_{(n)}^{\mu_1 \cdots \mu_n}(k_1 \cdots k_n)$ has at least one component parallel to k. Let $[\mathfrak{F}(\mathcal{S}_0^+)]$ be the space of equivalence classes of vectors in $\mathfrak{F}(\mathcal{S}_0^+)$ [modulo vectors in $\mathfrak{F}^0(\mathcal{S}_0^+)$] that have finite norm. On $[\mathfrak{F}(\mathcal{S}_0^+)]$ the norm corresponding to Eq. (3.44) is positive-definite. The physical Hilbert space of photons in the absence of charged particles, which we shall denote by $\mathcal{H}(0)$, is the closure of $[\mathfrak{F}(\mathcal{S}_0^+)]$, namely the vector space whose elements are equivalence classes of Cauchy sequences in $[\mathfrak{F}(\mathcal{S}_0^+)]$ (modulo Cauchy sequences that converge to zero).

Note again, as in the one-photon case, that we have not introduced the traditional device of a nonphysical Hilbert space with noncovariant but positive-definite norm defined on the 4-component wave-function space.

The essential point in the construction is the existence of the subspace $\mathfrak{F}(\mathcal{S}_0^+) \subset \mathfrak{F}(\mathcal{S})$ with non-negative metric. Having found one such subspace, we obtain a large class of new subspaces with non-negative metric simply by applying any operator of $\mathfrak{F}(\mathcal{S})$ which preserves the inner product. For our purposes operators of the form

$$U(f) = \exp[a^\dagger(f) - a(f)] = \exp[-\tfrac{1}{2}\langle f|f\rangle] \exp[a^\dagger(f)] \exp[-a(f)] \tag{3.45}$$

are more than sufficient.

These operators satisfy the Weyl relation

$$U(f)U(g) = \exp[-\tfrac{1}{2}\langle f|g\rangle + \tfrac{1}{2}\langle g|f\rangle] U(f+g) . \tag{3.46}$$

Vectors of the form $F(f) = U(f)|0\rangle$ with $f \in \mathcal{S}$ are coherent states, with explicit form given by

$$F(f) = \exp[-\tfrac{1}{2}\langle f|f\rangle] \exp[a^\dagger(f)]|0\rangle , \tag{3.47a}$$

or, in terms of components $F_{(0)}(f) = \exp[-\frac{1}{2}\langle f|f\rangle]$, and for $n \geq 1$,

$$F_{(n)}(f)^{\mu_1 \cdots \mu_n}(k_1 \cdots k_n) = \exp[-\tfrac{1}{2}\langle f|f\rangle] f^{\mu_1}(k_1) \cdots f^{\mu_n}(k_n) . \tag{3.47b}$$

They have unit norm $\langle F(f)|F(f)\rangle = 1$ and are eigenstates of the annihilation operator

$$c^\mu(k)F(f) = f^\mu(k)F(f) . \tag{3.48}$$

Let $\mathcal{W}(\mathcal{S})$ designate the subspace of $\mathfrak{F}(\mathcal{S})$ formed of finite linear combinations of coherent states,

$$F \in \mathcal{W}(\mathcal{S}):\quad F = \sum_{i=1}^{N} c_i F(f_i) . \tag{3.49}$$

From Eqs. (3.45) and (3.46) we see immediately that the inner product $\langle F|G\rangle$ is finite for all $F, G \in \mathcal{W}(\mathcal{S})$. Moreover, $U(f)$ is well defined on $\mathcal{W}(\mathcal{S})$ and in fact it maps $\mathcal{W}(\mathcal{S})$ isometrically one-to-one onto itself. Let $\mathcal{W}(\mathcal{S}_0^+)$ be the subspace of $\mathcal{W}(\mathcal{S})$ consisting of vectors of the form $\sum_{i=1}^{N} c_i F(f_i)$, $f_i \in \mathcal{S}_0^+$. It is also a subspace of $\mathfrak{F}(\mathcal{S}_0^+)$, so the metric is non-negative in $\mathcal{W}(\mathcal{S}_0^+)$. In fact, completion of $\mathcal{W}(\mathcal{S}_0^+)$ in the norm again produces $\mathcal{H}(0)$, because finite linear combinations of coherent states are dense. Furthermore, $U(f)$ is well defined on every vector of $\mathcal{W}(\mathcal{S}_0^+)$ for every $f \in \mathcal{S}$, and therefore produces a subspace of non-negative metric which is different from $\mathcal{W}(\mathcal{S}_0^+)$ if $f \in \mathcal{S}$ is not in $\mathcal{S}_0^+$. When completed in the norm it produces a new Hilbert space different from $\mathcal{H}(0)$.

We consider only the positive metric subspaces produced by $U(f_{\{e,p\}})$, with $f_{\{e,p\}}$ specified as follows. Let $f_{\{e,p\}}^\mu(k)$ be a transverse test function $k_\mu f_{\{e,p\}}^\mu(k) = 0$, satisfying, with $k = (\omega, \hat{k})$,

$$f^{\mu}_{\{e,p\}}(0,\hat{k}) = \sum_{a=1}^{A} e_a f^{\mu}_{p_a}(\hat{k}) = \sum_{a=1}^{A} \frac{-e_a}{(2\pi)^{3/2}} \frac{p_a^{\mu}}{p_a \cdot \hat{k}} . \tag{3.50a}$$

The e_a are the electric charges of the particles with momenta p_a. Transversality implies

$$\sum_a e_a = 0 . \tag{3.50b}$$

Let $\mathcal{W}^+\{e,p\}$ be the image of $\mathcal{W}(\mathcal{S}_0^+)$ under $U(f_{\{e,p\}})$

$$\mathcal{W}^+\{e,p\} \equiv U(f_{\{e,p\}})\mathcal{W}(\mathcal{S}_0^+) , \tag{3.51a}$$

$$\mathcal{W}^+\{0\} \equiv \mathcal{W}(\mathcal{S}_0^+) . \tag{3.51b}$$

This space is independent of which transverse f satisfying Eq. (3.50) is chosen, for we have

$$U(f') = U(f)[U^{-1}(f)U(f')] = U(f)\exp[\tfrac{1}{2}\langle f|f'\rangle - \tfrac{1}{2}\langle f'|f\rangle]U(-f+f') ,$$

and if f and f' are both transverse and both satisfy Eq. (3.50) then $-f+f' \in \mathcal{S}_0^+$ and $U(-f+f')$ maps $\mathcal{W}(\mathcal{S}_0^+)$ into itself. The space $\mathcal{W}^+\{e,p\}$ enjoys the following interesting properties:

(1) The metric is finite and non-negative on $\mathcal{W}^+\{e,p\}$, so $\mathcal{W}^+\{e,p\}$ may be completed to a Hilbert space $\mathcal{H}\{e,p\}$.

(2) $\mathcal{W}^+\{e,p\}$ is transverse, which means, as will be seen in the next section, that Maxwell's equations hold in $\mathcal{H}\{e,p\}$. To show transversality observe that for $F \in \mathcal{W}\{e,p\}$, we have $F = U(f_{\{e,p\}})F_0$ where $F_0 \in \mathcal{W}^+\{0\} \subset \mathcal{F}(\mathcal{S}_0^+)$. Because $[k_\mu c^\mu(k), U(f_{\{e,p\}})] = k_\mu f^{\mu}_{\{e,p\}}(k)U(f_{\{e,p\}}) = 0$ and $k_\mu c^\mu(k)F_0 = 0$, it follows that $k_\mu c^\mu(k)F = 0$.

(3) $\mathcal{W}^+\{e,p\}$ is an eigenspace of the zero-frequency annihilation operator $c^\mu(\hat{k})$ with eigenvalue $\sum_a e_a f^{\mu}_{p_a}(\hat{k}) = -(2\pi)^{-3/2}\sum_a e_a p_a^\mu/p_a \cdot \hat{k}$. In terms of the component test functions of $F = \{F_{(n)}\} \in \mathcal{W}^+\{e,p\}$, this condition means

$$\lim_{\omega_1 \to 0} F_{(n)}^{\mu_1 \cdots \mu_n}(k_1 \cdots k_n) = \sum_{a=1}^{A} e_a f^{\mu_1}_{p_a}(\hat{k}_1) F_{(n-1)}^{\mu_2 \cdots \mu_n}(k_2 \cdots k_n) . \tag{3.52}$$

(It follows from $[c^\mu(\hat{k}), U(f)] = f^\mu(0,\hat{k})U(f)$ and $c^\mu(\hat{k})F_0 = 0$ that $F_0 \in \mathcal{W}^+(0) \subset \mathcal{F}(\mathcal{S}_0)$.) Call this property infrared coherence. It is destroyed by the photon-number operator $N\{F_{(n)}\} = \{n F_{(n)}\}$, so the number operator does not exist on the infrared coherent space $\mathcal{W}^+\{e,p\}$ nor its completion $\mathcal{H}\{e,p\}$. On the other hand, the momentum operator

$$P^\mu\{F_{(n)}^{\mu_1 \cdots \mu_n}(k_1 \cdots k_n)\} = \{(k_1^\mu + \cdots + k_n^\mu)F_{(n)}^{\mu_1 \cdots \mu_n}(k_1 \cdots k_n)\}$$

preserves the infrared coherence property and transversality, so its acts within $\mathcal{H}\{e,p\}$.

The physicist immediately recognizes that properties (2) and (3) are commonly attributed to the S matrix in quantum electrodynamics, which is why $\mathcal{W}^+\{e,p\}$ has been singled out. In fact property (3) leads precisely to the usual real infrared divergence of the bremsstrahlung cross section. However, property (1) tells us that an ordinary Hilbert space has been established for precisely this case.

Each of the Hilbert spaces $\mathcal{H}\{e,p\}$ provides a representation of the canonical commutation relations of $c(f_0)$, $c^\dagger(f_0)$, $f_0 \in \mathcal{H}^{(1)}$. It may be shown that these representations are mathematically inequivalent for different sets $\{e,p\} \neq \{e',p'\}$ because they are characterized by different eigenvalues of the zero-frequency annihilation operator $c(\hat{k})$. However, the support of $f_{\{e,p\}}$ may be restricted to an arbitrarily small neighborhood of the origin. Since every measurement is made over a finite region of space-time, no measurement at strictly zero frequency is possible. Thus every Hilbert space $\mathcal{H}\{e,p\}$ is physically equivalent to $\mathcal{H}(0)$. Any one of the $\mathcal{H}\{e,p\}$ may be chosen to describe a given physical situation. It is a matter of mathematical convenience which one is chosen.

Let us compute the explicit form of the inner product in $\mathcal{W}^+\{e,p\}$. Without greater effort, we may in fact obtain the explicit form of the inner product between a vector in $\mathcal{W}^+\{e,p\}$ and a vector in $\mathcal{W}^+\{e',p'\}$. Let F and F' be generic elements of $\mathcal{W}^+\{e,p\}$ and $\mathcal{W}^+\{e',p'\}$, respectively:

$$F = \sum_{i=1}^{I} c_i F(f_i), \quad F' = \sum_{j=1}^{J} c_j' F(f_j') , \tag{3.53}$$

where $F(f_i)$ is the coherent sequence $F(f_i) = \{F_{(n)}(f_i)\}$

$$F_{(n)}(f_i)^{\mu_1 \cdots \mu_n}(k_1 \cdots k_n) = \exp[-\tfrac{1}{2}\langle f_i|f_i\rangle] f_i^{\mu_1}(k_1) \cdots f_i^{\mu_n}(k_n) , \tag{3.54}$$

and similarly for $F(f_j')$. The test function $f_i^\mu(k)$ is transverse and its zero-frequency limit is independent of the index i,

$$\lim_{\omega \to 0} f_i^\mu(\omega,\hat{k}) = \sum_{a=1}^{A} \frac{-e_a}{(2\pi)^{3/2}} \frac{p_a^\mu}{p_a \cdot \hat{k}} , \tag{3.55a}$$

and similarly for f_j'

$$\lim_{\omega \to 0} f_j'^{\mu}(\omega,\hat{k}) = \sum_{b=1}^{B} \frac{-e_b'}{(2\pi)^{3/2}} \frac{p_b'^{\mu}}{p_b' \cdot \hat{k}} . \tag{3.55b}$$

We have

$$\langle F|F'\rangle=\sum_{i=1}^{I}\sum_{j=1}^{J} c_i^* c_j \langle F(f_i)|F(f_j)\rangle \tag{3.56}$$

and

$$\begin{aligned}\langle F(f_i)|F(f_j')\rangle &= \langle 0|U^\dagger(f_i)U(f_j')|0\rangle \\ &= \exp[-\tfrac{1}{2}\langle f_i|f_i\rangle-\tfrac{1}{2}\langle f_j'|f_j'\rangle+\langle f_i|f_j'\rangle]\,.\end{aligned} \tag{3.57}$$

From Eq. (3.30) we find

$$\langle f_i|f_j'\rangle=\sum_{a,b} e_a e_b'\langle p_a|p_b'\rangle_\Delta + \int\frac{d\hat{k}}{2}\int_0^\infty d\omega\,\ln\left(\frac{\omega}{\Delta}\right)\frac{\partial}{\partial\omega}[f_i^{\mu *}(k)f_{j\mu}'(k)]. \tag{3.58}$$

It is convenient to define an inner product $\langle F|G\rangle_\Delta$ depending on the arbitrary function of $\hat{k}$, $\Delta(\hat{k})$, for any $F,G\in\mathfrak{F}(\mathcal{S})$. With $F=\{F_{(n)}\}$, $G=\{G_{(n)}\}$, set

$$\langle F|G\rangle_\Delta \equiv F_0^* G_0+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int(dk_1)_\Delta\cdots(dk_n)_\Delta F_{(n)}^{\mu_1\cdots\mu_n *}(k_1\cdots k_n)\,G_{(n)\mu_1\cdots\mu_n}(k_1\cdots k_n)\,, \tag{3.59a}$$

$$(dk_r)_\Delta\equiv -d\hat{k}_r\,d\omega_r\ln\left(\frac{\omega_r}{\Delta(\hat{k})}\right)\frac{\partial}{\partial\omega_r}\,. \tag{3.59b}$$

The derivatives act on $F_{(n)}G_{(n)}$. This inner product is obviously sesquilinear and Hermitian symmetric. In view of Eqs. (3.54) and (3.58), the inner product (3.57) may be rewritten as

$$\langle F(f_i)|F(f_j')\rangle=\exp\left(\sum_{a,b}e_a e_b'\langle p_a|p_b'\rangle_\Delta\right)\times\langle F(f_i)|F(f_j')\rangle_\Delta\,. \tag{3.60}$$

Because the inner product $\langle F|G\rangle_\Delta$ is sesquilinear and because the exponential coefficient is independent of the indices i and j, we obtain from Eq. (3.56) for the inner product between generic elements F and F' of $\mathcal{W}\{e,p\}$ and $\mathcal{W}\{e',p'\}$ the explicit expression

$$\langle F|F'\rangle=\exp\left(\sum_{a,b}e_a e_b'\langle p_a|p_b'\rangle_\Delta\right)\langle F|F'\rangle_\Delta. \tag{3.61}$$

For $F,G\in\mathcal{W}^+\{e,p\}$ this reduces to

$$\langle F|G\rangle=\exp\left(\sum_{a,b}e_a e_b\langle p_a|p_b\rangle_\Delta\right)\langle F|G\rangle_\Delta\,. \tag{3.62}$$

The inner product $\langle F|G\rangle_\Delta$ makes no references to which infrared-coherent subspace the vectors may be in, and the exponential coefficient depends only on the infrared-coherent subspace $\mathcal{W}^+\{e,p\}$ and not on the particular vector.

The topology of $\mathfrak{F}(\mathcal{S})$ is such that $\mathcal{W}(\mathcal{S})$ is a dense set and the conditions of transversality and infrared coherence define closed subspaces. It follows by continuity that the inner product is given by Eq. (3.62) throughout each subspace $F^+\{e,p\}$ defined by these two conditions. For the same reason it is also non-negative throughout $\mathfrak{F}^+\{e,p\}$, the subspace consisting of those vectors[14] that are annihilated by $k_\mu c^\mu(k)$ and that are eigenfunctions of the zero-frequency annihilation operator $c_\mu(\hat{k})$ with eigenvalue $-(2\pi)^{-3/2}\sum_a e_a p_a^\mu/p_a\cdot\hat{k}$. If the subspace $\mathcal{W}^+\{e,p\}$ or $\mathfrak{F}^+\{e,p\}$ is completed in the norm (3.62) the physical Hilbert space $\mathcal{H}\{e,p\}$ results. The reader may wonder why the space $\mathfrak{F}(\mathcal{S})$ is introduced at all since $\mathcal{W}(\mathcal{S})$ suffices for the construction of the physical Hilbert spaces. Later we will want to consider states with definite numbers of photons. They are in $\mathfrak{F}(\mathcal{S}_0)$ but not in $\mathcal{W}(\mathcal{S}_0)$.

Although the formal construction of $\mathcal{H}\{e,p\}$ is the same as for $\mathcal{H}(0)$, there is a difference between the spaces which is worth mentioning. The two Hilbert spaces are obtained from the closure of the space of equivalence classes of Fock test functions $[\mathcal{W}^+\{e,p\}]=\mathcal{W}^+\{e,p\}/\mathcal{W}^0\{e,p\}$ and $[\mathcal{W}^+\{0\}]=\mathcal{W}^+\{0\}/\mathcal{W}^0\{0\}$, respectively. The space of equivalence classes $[\mathcal{W}^+\{\cdots\}]$ may in each case be made into a space of test functions by choosing a representative, for example, the unique Fock test function F in each equivalence class $[F]$, all of whose component test functions $F_{(n)}^{\mu_1\cdots\mu_n}(k_1\cdots k_n)$, $n=1,2,\ldots$, vanish when any polarization index μ_r is zero. With this choice it may be shown that when Cauchy sequences are formed (modulo Cauchy sequences that converge to zero in norm) the corresponding sequences of n-photon test functions, in the case of $\mathcal{H}\{0\}$, converge almost everywhere to n-photon wave functions. However, due to a resurgence of the infrared divergence, they do not necessarily converge to functions at all, in the case of $\mathcal{H}\{e,p\}$. This does not invalidate our construction; $\mathcal{H}\{e,p\}$ is simply a more abstract space. Inner products must be calculated in the dense subset of $\mathcal{H}\{e,p\}$ whose n-photon components are wave functions, before taking limits.

For applications, the inner product (3.62) is the principal result of this section. The appearance of the arbitrary Δ which cancels between the exponential coefficient and the inner product $\langle F|G\rangle_\Delta$ is very convenient in practice, for Δ may be chosen to make the infinite series (3.59), which defines the inner product $\langle F|G\rangle_\Delta$, converge as rapidly as possible. In particular in a scattering experiment with good resolution it will be seen later that if Δ is appropriately chosen (appropriate means that Δ is essentially set equal to the energy resolution), then the exponential coefficient contains the usual radiation damping factor, and the successive terms in the infinite series (3.59) which defines $\langle F|F\rangle_\Delta$ are of order $(\alpha\Delta)^n$ so the series is well approximated by its zeroth term. Usual perturbation theory yields a series in $(\alpha \ln\Delta)^n$. To see the radiation damping factor emerge, let $\Delta(\hat{k})$ be a constant independent of $\hat{k}$. Then by Eq. (3.30b), the Δ dependence of the exponential coefficient is $\Delta^{B\{e,p\}}$ where

$$B\{e,p\}=-\sum_{a,b} e_a e_b \int \frac{d\hat{k}}{2} f^{\mu}_{p_a}(\hat{k}) f_{\mu p_b}(\hat{k})$$

$$=\frac{-1}{(2\pi)^3}\int \frac{d\hat{k}}{2}\left(\sum_a \frac{e_a p_a}{p_a\cdot\hat{k}}\right)^2 . \qquad (3.63)$$

The 4-vector $-(2\pi)^{-3/2}\sum_a e_a p_a^\mu (p_a\cdot\hat{k})^{-1}$, which is, in fact, precisely the eigenvalue of $c^\mu(\hat{k})$ which defines the space $\mathcal{W}^+\{e,p\}$, is spacelike, because on contraction with the lightlike vector $\hat{k}^\mu=(1,\hat{k})$ it gives $\sum e_a=0$. Therefore, its square is negative and Eq. (3.63) shows that $B\{e,p\}$ is positive. So $\Delta^{B\{e,p\}}$ vanishes as Δ approaches zero. The integral is easily evaluated term by term, by exploiting the Lorentz invariance expressed in Eq. (3.10), and one finds

$$B\{e,p\}=-\sum_{a,b}\frac{e_a e_b}{(2\pi)^2}\,\psi_{ab}\coth\psi_{ab}\,, \qquad (3.64)$$

where $\psi_{ab}\geq 0$ is the hyperbolic angle between p_a and p_b: $p_a\cdot p_b=m_a m_b\cosh\psi_{ab}$. Thus $\Delta^{B\{e,p\}}$ gives the familiar dependence on the experimental resolution. It will be convenient to refer to the exponential factor $\exp(\sum e_a e_b\langle p_a|p_b\rangle_\Delta)$ generically as the damping factor.

The inner product depends on the renormalization constant l through the multiplicative factor $l^{-B\{e,p\}}$. However, if states are normalized to unity, this factor obviously cancels out of all probabilities.

IV. ASYMPTOTIC FIELDS

In the present section we will show how the asymptotic fields of quantum electrodynamics may be represented as operators on the Fock space of test functions constructed previously. We consider in Sec. IV A the free vector potential, in Sec. IV B, bremsstrahlung by the classical external current of a scattered charged particle, in Sec. IV C, the propagator of the preliminary charged asymptotic field $\psi_{\rm pre}$, and finally in Sec. IV D we obtain a simple weak asymptotic limit of the charged field.

A. Free vector potential

The free vector potential will be designated A in this subsection only. In position space $A_\mu(x)$ is an operator-valued distribution on test functions $j_\mu(x)$ which are interpreted as classical currents. Write

$$A(j)=\int A_\mu(x)(-g^{\mu\nu})j_\nu(x)d^4x \qquad (4.1)$$

and let

$$\tilde{j}_\mu(k)=\tilde{j}_\mu(\omega,\hat{k})=\int e^{ik\cdot x}j_\mu(x)d^4x \qquad (4.2)$$

be the Fourier transform of $j_\mu(x)$ with k restricted to the future light cone, $k^\mu=\omega(1,\hat{k})$, $\omega\geq 0$. The operator $A(j)$ is defined to be

$$A(j)=\frac{1}{(2\pi)^{3/2}}\left[c^\dagger(\omega\tilde{j})+c(\omega\tilde{j})\right] , \qquad (4.3a)$$

where $c^\dagger(f)$ and $c(f)$ are defined in Eqs. (3.34) and (3.36), corresponding to the symbolic representation

$$A_\mu(x)=(2\pi)^{-3/2}\int (dk)_l\,\omega[c_\mu^\dagger(k)e^{ik\cdot x}+c_\mu(k)e^{-ik\cdot x}] . \qquad (4.3b)$$

As usual, we smear only with test functions $j^\mu(x)$ which are of fast decrease at infinity. Consequently $\omega\tilde{j}_\mu(k)$ vanishes at $\omega=0$ and is an element of $\mathcal{S}_0$, so $(dk)_l$ may be replaced by $dk=\frac{1}{2}d\hat{k}d\omega/\omega$,

$$A_\mu(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d\hat{k}}{2}\int_0^\infty d\omega[c_\mu^\dagger(k)e^{ik\cdot x}+c_\mu(k)e^{-ik\cdot x}] . \qquad (4.3c)$$

To gain some insight into the space-time characteristics of the free vector potential on $\mathfrak{F}(\mathcal{S})$, consider the weak limit of this operator as $|x|\to\infty$. Namely, $c_\mu(k)$ and $c_\mu^\dagger(k)$ represent matrix elements in $\mathfrak{F}(\mathcal{S})$ and are therefore smooth in k, and we evaluate

$$A_\mu(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d\hat{k}}{2}\int_0^\infty d\omega[c_\mu^\dagger(\omega,\hat{k})e^{-(\epsilon-i\hat{k}\cdot x)\omega} + c_\mu(\omega,\hat{k})e^{-(\epsilon+i\hat{k}\cdot x)\omega}]$$

for large $|x|$. An ϵ has been introduced to make the integral on ω well defined. This gives, by the

Riemann-Lebesgue lemma,

$$\lim_{|x|\to\infty} A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\hat{k}}{2} \left[\frac{c_\mu^\dagger(\hat{k})}{\epsilon - i\hat{k}\cdot x} + \frac{c_\mu(\hat{k})}{\epsilon + i\hat{k}\cdot x}\right], \tag{4.4}$$

where $c(\hat{k}) \equiv c(0, \hat{k})$. This weak limit is a bilinear form because $c^\dagger(\hat{k})$ is defined to act only to the left. Observe that there is a leading contribution of order $|x|^{-1}$, whereas the Klein-Gordon field goes like $|x|^{-3/2}$. If the matrix element is taken on the infrared-coherent subspace $\mathcal{F}\{e_a, p_a\}$ characterized by the eigenvalue $c^\mu(\hat{k}) = -(2\pi)^{-3/2} \times \sum_a e_a p_a^\mu/(p_a\cdot\hat{k})$, we obtain

$$\lim_{|x|\to\infty} A^\mu(x)|_{\{e_a, p_a\}} = \frac{-1}{(2\pi)^3} \int \frac{d\hat{k}}{2} \sum_a \frac{e_a p_a^\mu}{p_a\cdot\hat{k}} (2\pi)\delta(\hat{k}\cdot x), \tag{4.5}$$

$$\lim_{|x|\to\infty} A^\mu(x)|_{\{e_a, p_a\}} = \frac{-\theta(-x^2)}{4\pi} \sum_a \frac{e_a p_a^\mu}{[(p_a\cdot x)^2 - m_a{}^2 x^2]^{1/2}} . \tag{4.6}$$

Thus on an infrared-coherent subspace the vector potential at large distances has a leading term of order $|x|^{-1}$ whose support lies outside the light cone, and which precisely cancels the Liénard-Wiechert potentials of a set of particles with charges e_a and momenta p_a. At large positive times this accords with the requirement of causality if the charged particles $\{e_a, p_a\}$ are produced in a collision of neutral particles, for the Liénard-Wiechert potential alone overdresses the particles, and the infrared photons remove the excess which is in the spacelike region.

Recall that the subspaces $\mathcal{F}^+\{e,p\}$ of $\mathcal{F}(\mathcal{E})$, which may be completed to physical Hilbert spaces $\mathcal{H}\{e,p\}$, are characterized by transversality $\hat{k}\cdot c(k)\mathcal{F}^+\{e,p\} = 0$, and infrared coherence

$$c^\mu(\hat{k})\mathcal{F}^+\{e,p\} = \left[-(2\pi)^{-3/2}\sum_a e_a p_a^\mu/p_a\cdot\hat{k}\right]\mathcal{F}^+\{e,p\},$$

where $c^\mu(\hat{k}) = c^\mu(k)|_{\omega=0}$. Each of these subspaces must be left invariant by physical observables. From Eq. (4.3a) we have

$$[c_\mu(k), A(j)] = \frac{\omega}{(2\pi)^{3/2}} \tilde{j}_\mu(k), \tag{4.7}$$

so transversality is preserved by $A(j)$ if the current is conserved, and infrared coherence is preserved if $\omega\tilde{j}(k)|_{\omega=0} = 0$. Thus $A(j)$ is observable if and only if $\omega\tilde{j}(k) \in \mathcal{E}_0^+$.

Finally let us briefly verify that the Maxwell field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is observable and that the free Maxwell equations $\partial_\mu F^{\mu\nu} = 0$ hold as operator equations on $\mathcal{H}\{e,p\}$. Let $f_{\mu\nu}(x)$ be a classical antisymmetric tensor, and set $F(f) = \frac{1}{2}\int F_{\mu\nu} f^{\mu\nu} d^4x = -\int A_\nu \partial_\mu f^{\mu\nu} d^4x = A(j)$, with $j^\nu = \partial_\mu f^{\mu\nu}$. This current is conserved. If in addition $\omega\tilde{j}(k) \in \mathcal{E}_0$, which is true if $f^{\mu\nu}$ is of fast decrease, then $F(f)$ acts within $\mathcal{F}^+\{e,p\}$, and $F(f)$ is observable. From the equation of motion $\partial^2 A = 0$, we have $\partial_\mu F^{\mu\nu} = -\partial^\nu \partial\cdot A$, so Maxwell's equations are not satisfied on $\mathcal{F}(\mathcal{E})$. Upon smearing with a test function $a_\nu(x)$ the last term becomes

$$A(\partial\partial\cdot a) = \frac{-1}{(2\pi)^{3/2}}[c^\dagger(\omega k k\cdot\tilde{a}) + c(\omega k k\cdot\tilde{a})] . \tag{4.8}$$

When applied to a vector in $\mathcal{F}^+\{e,p\}$, the second term gives zero because $\mathcal{F}^+\{e,p\}$ is transverse, whereas the first term gives a vector of zero length. Therefore, as an operator on $[\mathcal{F}^+\{e,p\}]$, the space of equivalence classes of vectors in $\mathcal{F}^+\{e,p\}$ (modulo vectors of zero length) $A(\partial\partial\cdot a)$ is the zero operator

$$A(\partial\partial\cdot a)[\mathcal{F}^+\{e,p\}] = 0 . \tag{4.9}$$

It is therefore also the zero operator on $\mathcal{H}\{e,p\}$, the closure of $\mathcal{F}^+\{e,p\}$, so Maxwell's equations hold as operator equations on $\mathcal{H}\{e,p\}$.

B. Bremsstrahlung by the classical current of a scattered charged particle

Let the vector potential A be coupled to a given classical current j^μ,

$$\partial^2 A_\mu(x) = j_\mu(x) . \tag{4.10}$$

If $j^\mu(x)$ is a test function of fast decrease in all directions of space-time, including timelike directions, then the charge $q(t) = \int j^0(t, \vec{x}) d^3x$ approaches zero at asymptotic times. If the current is conserved the charge will consequently be zero at all times. Thus the restriction to test functions of fast decrease, however convenient mathematically, will be inadequate to deal with simple scattering situations in which there is a net charge.

Consider instead a test function of the form

$$j^\mu(x) = \int d^3s\rho(\vec{s}) \int_{-\infty}^{\infty} d\tau\, \dot{z}^\mu(\tau, \vec{s})\delta^4(x - z(\tau, \vec{s})) , \tag{4.11}$$

where $\dot{z}^\mu(\tau, \vec{s}) = \partial z^\mu(\tau, \vec{s})/\partial\tau$. This represents the current produced by a scattered classical extended charged particle whose different parts have impact parameter $\vec{s}$ and corresponding 4-trajectories $z^\mu(\tau, \vec{s})$, with charge distributed by impact parameters according to the density $\rho(\vec{s})$. Thus $\rho(\vec{s})$ may be taken to be a test function of fast decrease in R^3, and $e = \int d^3s\, \rho(\vec{s})$ is the total charge. To represent the current of a scattered extended classical particle at asymptotic times, the different parts must be at relative rest, the whole moving uniformly

$$\lim_{\tau\to\infty} \dot{z}_\mu(\tau,\vec{s})=p_{f\mu}, \quad \lim_{\tau\to-\infty} \dot{z}_\mu(\tau,\vec{s})=p_{i\mu}, \qquad (4.12)$$

the limits being independent of $\vec{s}$.

The Fourier transform of this test function

$$\tilde{j}^\mu(k)=\int d^3s\,\rho(\vec{s})\int_{-\infty}^{\infty} d\tau\,\dot{z}^\mu(\tau,\vec{s})e^{ik\cdot z(\tau,\vec{s})} \qquad (4.13)$$

is required for values of k on the future light cone, in which case it represents the classical radiation emitted by the current. The integrand is oscillatory for large values of τ. If the integral is defined by insertion of $e^{-\epsilon|\tau|}$, it is equal to the expression, obtained by partial integration,

$$\tilde{j}^\mu(k)=i\int d^3s\,\rho(\vec{s})\int_{-\infty}^{\infty} d\tau\, e^{ik\cdot z(\tau,\vec{s})}\frac{d}{d\tau}\left[\frac{\dot{z}^\mu(\tau,\vec{s})}{k\cdot\dot{z}(\tau,\vec{s})}\right]. \qquad (4.14)$$

This displays the fact that radiation occurs only if charge is accelerated. Consider now the zero-frequency limit of

$$f^\mu(k)=\frac{\omega\tilde{j}^\mu(k)}{(2\pi)^{3/2}}$$

$$=\frac{i}{(2\pi)^{3/2}}\int d^3s\,\rho(\vec{s})\int_{-\infty}^{\infty} d\tau\, e^{ik\cdot z(\tau,\vec{s})} \times\frac{d}{d\tau}\left[\frac{\dot{z}^\mu(\tau,\vec{s})}{\hat{k}\cdot\dot{z}(\tau,\vec{s})}\right] \qquad (4.15)$$

With $f^\mu(k)=f^\mu(\omega,\hat{k})$, one finds, from Eqs. (4.12),

$$f^\mu(0,\hat{k})=\left.\frac{\omega\tilde{j}^\mu(k)}{(2\pi)^{3/2}}\right|_{\omega=0}$$

$$=\frac{ie}{(2\pi)^{3/2}}\left(\frac{p_f^\mu}{p_f\cdot\hat{k}}-\frac{p_i^\mu}{p_i\cdot\hat{k}}\right), \qquad (4.16)$$

which characterizes an element of the test function space $\mathcal{E}$, described in the last section.

The equation of motion (4.10) is solved by

$$A^\mu(x)=A^{\mu\,\text{in}}(x)+\int\Delta^{\text{ret}}(x-y)j^\mu(y)d^4y, \qquad (4.17a)$$

$$A^\mu(x)=A^{\mu\,\text{out}}(x)+\int\Delta^{\text{ad}}(x-y)j^\mu(y)d^4y. \qquad (4.17b)$$

Here $A^{\mu\,\text{in}}$ and $A^{\mu\,\text{out}}$ are free vector potentials which are related by

$$A_\mu^{\text{in}}(x)=A_\mu^{\text{out}}(x)-\int\Delta(x-y)j_\mu(y)d^4y, \qquad (4.18)$$

$$\Delta(x)=\Delta^{\text{ret}}(x)-\Delta^{\text{ad}}(x)=\frac{-i}{(2\pi)^3}\int\frac{d^3k}{2\omega}(e^{ik\cdot x}-e^{-ik\cdot x}), \qquad (4.19)$$

or

$$A_\mu^{\text{in}}(x)=A_\mu^{\text{out}}(x)-\frac{1}{(2\pi)^{3/2}}\int\frac{d\hat{k}}{2}d\omega[e^{ik\cdot x}g_\mu^*(k) + e^{-ik\cdot x}g_\mu(k)], \qquad (4.20)$$

where

$$g_\mu(k)=\frac{i\omega}{(2\pi)^{3/2}}\tilde{j}_\mu(k). \qquad (4.21)$$

Let A^{out} be represented as an operator on the space of test functions $\mathfrak{F}^{\text{out}}(\mathcal{E})$ according to Eq. (4.3c),

$$A^{\mu\,\text{out}}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d\hat{k}}{2}d\omega[c_\mu^{\dagger\,\text{out}}(k)e^{ik\cdot x} + c_\mu^{\text{out}}(k)e^{-ik\cdot x}]. \qquad (4.22a)$$

Then Eq. (4.20) shows that $A^{\mu\,\text{in}}(x)$ is also an operator on the test function space $\mathfrak{F}^{\text{out}}(\mathcal{E})$, with a similar expansion

$$A^{\mu\,\text{in}}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d\hat{k}}{2}d\omega[c_\mu^{\dagger\text{in}}(k)e^{ik\cdot x} + c_\mu^{\text{in}}(k)e^{-ik\cdot x}], \qquad (4.22b)$$

where

$$c_\mu^{\text{in}}(k)=c_\mu^{\text{out}}(k)-g_\mu(k), \qquad (4.23a)$$

$$c_\mu^{\text{in}\,\dagger}(k)=c_\mu^{\text{out}\,\dagger}(k)-g_\mu^*(k), \qquad (4.23b)$$

or

$$c^{\text{in}\,\dagger}(f)=c^{\text{out}\,\dagger}(f)-\langle g|f\rangle, \qquad (4.24a)$$

$$c^{\text{in}}(f)=c^{\text{out}}(f)-\langle f|g\rangle. \qquad (4.24b)$$

Let the unitary scattering operator S on $\mathfrak{F}^{\text{out}}$ be defined by

$$S=\exp[c^{\text{out}\,\dagger}(g)-c^{\text{out}}(g)]. \qquad (4.25)$$

It satisfies

$$c^{\text{in}\,\dagger}(f)=Sc^{\text{out}\,\dagger}(f)S^\dagger, \quad c^{\text{in}}(f)=Sc^{\text{out}}(f)S^\dagger, \qquad (4.26)$$

and corresponding in-states F^{in} for any state $F^{\text{out}}\in\mathfrak{F}^{\text{out}}(\mathcal{E})$ may be introduced according to

$$F^{\text{in}}=SF^{\text{out}}. \qquad (4.27)$$

The space of in-states is given by

$$\mathfrak{F}^{\text{in}}(\mathcal{E})=S\mathfrak{F}^{\text{out}}(\mathcal{E})=\mathfrak{F}^{\text{out}}(\mathcal{E}). \qquad (4.28)$$

Although these relations have the same form as the usual quantum-mechanical scattering relations in a Hilbert space, their meaning is rather different, for $\mathfrak{F}^{\text{in}\,+}(\mathcal{E})$ contains many physical subspaces $\mathfrak{F}^{\text{in}\,+}\{e,p\}$, each of which may be completed to a Hilbert space $\mathcal{H}^{\rm in}\{e,p\}$, and correspondingly for in→out. However, S is not an operator that acts within any of the physical subspace $\mathfrak{F}^+\{e,p\}$ for we have, from Eq. (4.21),

$$g^{\mu}(k)|_{\omega=0}=\frac{-e}{(2\pi)^{3/2}}\left(\frac{p_f^\mu}{p_f\cdot\hat{k}}-\frac{p_i^\mu}{p_i\cdot\hat{k}}\right)\,, \tag{4.29}$$

which gives, with $c_\mu(\hat{k})=c_\mu(0,\hat{k})$,

$$[c^{\mu\,\rm out}(\hat{k}),S]=\frac{-e}{(2\pi)^{3/2}}\left(\frac{p_f^\mu}{p_f\cdot\hat{k}}-\frac{p_i^\mu}{p_i\cdot\hat{k}}\right)S\,. \tag{4.30}$$

Thus from a given infrared-coherent subspace, S produces another infrared-coherent subspace. In addition, because the current j_μ is conserved S preserves transversality

$$[k\cdot c(k),S]=0\,. \tag{4.31}$$

Thus we have

$$\mathfrak{F}^{\rm in\,+}\{e_a,p_a\}=S\mathfrak{F}^{\rm out\,+}\{e_a,p_a\}=\mathfrak{F}^{\rm out\,+}\{\{e_a,p_a\}\oplus\{e,p_f,-e,p_i\}\}\,. \tag{4.32a}$$

When these spaces are completed in the norm, this relation extends uniquely to

$$\mathcal{H}^{\rm in}\{e_a,p_a\}=S\mathcal{H}^{\rm out}\{e_a,p_a\}=\mathcal{H}^{\rm out}\{\{e_a,p_a\}\oplus\{e,p_f,-e,p_i\}\}\,. \tag{4.32b}$$

Thus, unless $p_f=p_i$ the S operator maps a given Hilbert space into one in which the commutation relations have a mathematically inequivalent representation, $\mathcal{H}^{\rm in}\{e_a,p_a\}\neq\mathcal{H}^{\rm out}\{e_a,p_a\}$. The S operator becomes a mapping of Hilbert spaces and of vectors within the Hilbert space. Once a choice of Hilbert space for the in states is made, the S operator specifies what Hilbert space this is in terms of out variables.

It is a matter of convenience which in-Hilbert space is chosen to represent a given incoming experimental situation, for, as remarked earlier, although they are mathematically inequivalent, they are physically equivalent in the sense that any experimental situation may be represented with arbitrary accuracy in any one of the various Hilbert spaces $\mathcal{H}^{\rm in}\{e,p\}$. Some authors[4] make use of the in space associated with the charges and momenta of the incoming particles only, $\mathcal{H}^{\rm in}\{e_i,p_i\}$. This has the advantage of symmetry $\mathcal{H}^{\rm in}\{e_i,p_i\}=S\mathcal{H}^{\rm out}\{e_i,p_i\}=\mathcal{H}^{\rm out}\{e_f,p_f\}$, and for this reason, it is the only choice if one is working within the nonphysical Hilbert space of the traditional Gupta-Bleuler method. However, unless the initial state has total charge zero, transversality cannot be maintained,[15] Eqs. (3.50). This may be dealt with by placing a positron behind the moon for every electron nearby, or by restricting the violations of transversality to arbitrarily low frequencies, so the corresponding violation of Maxwell's equations is undetected. [This can be done by defining a positive-metric subspace by the condition $\hat{k}\cdot c(k)|\,\rangle=Qf(k)|\,\rangle$, where Q is the total charge and $f(k)$ is a smooth cutoff function which is $-(2\pi)^{-3/2}$ at $\omega=0$, and whose support is restricted to arbitrarily low frequencies.] However, in the present approach it is convenient to make use of two Hilbert spaces, $\mathcal{H}^{\rm in}\{0\}$, the completion of $\mathfrak{F}^{\rm in\,+}\{0\}$, and $\mathcal{H}^{\rm out}\{0\}$, the completion of $\mathfrak{F}^{\rm out+}\{0\}$. These spaces have states in which a finite number of photons is present. They correspond closely to the retarded and advanced boundary conditions of classical physics, and we write

$$\mathfrak{F}^{\rm ret}\equiv\mathfrak{F}^{\rm in}\{0\}=\mathfrak{F}^{\rm in}(\mathcal{E}_0)\,, \tag{4.33a}$$

$$\mathfrak{F}^{\rm ret\,+}\equiv\mathfrak{F}^{\rm in\,+}\{0\}=\mathfrak{F}^{\rm in}(\mathcal{E}_0^+),\quad \mathcal{H}^{\rm ret}\equiv\mathcal{H}^{\rm in}\{0\}\,, \tag{4.33b}$$

$$\mathfrak{F}^{\rm ad}\equiv\mathfrak{F}^{\rm out}\{0\}=\mathfrak{F}^{\rm out}(\mathcal{E}_0)\,, \tag{4.33c}$$

$$\mathfrak{F}^{\rm ad\,+}\equiv\mathfrak{F}^{\rm out+}\{0\}=\mathfrak{F}^{\rm out}(\mathcal{E}_0^+),\quad \mathcal{H}^{\rm ad}=\mathcal{H}^{\rm out}\{0\}. \tag{4.33d}$$

According to Eqs. (4.32), generalized to several charged particle, these spaces are related by

$$\mathcal{H}^{\rm ret}=S\mathcal{H}^{\rm ad}=\mathcal{H}^{\rm out}\{\{e_f,p_f\}\oplus\{-e_i,p_i\}\}\,, \tag{4.34a}$$

$$\mathfrak{F}^{\rm ret}=S\mathfrak{F}^{\rm ad}=\mathfrak{F}^{\rm out}\{\{e_f,p_f\}\oplus\{-e_i,p_i\}\}\,. \tag{4.34b}$$

As in classical physics the retarded representation appears most natural for the scattering situation. This allows the incoming state to be described in terms of a finite number of particles without any infrared history. The description in terms of out variables is then fixed by the S operator, a dynamically and not historically determined entity.

The retarded and advanced subspaces $\mathfrak{F}^{\rm ret}$ and $\mathfrak{F}^{\rm ad}$ allow a simple characterization of the S operator. Consider the generic matrix element

$$\langle G^{\rm out}|F^{\rm in}\rangle=\langle G^{\rm out}|SF^{\rm out}\rangle\,, \tag{4.35}$$

$F,G\in\mathfrak{F}(\mathcal{E})$. The explicit form of the S matrix (4.25)

$$S=\exp[-\tfrac{1}{2}\langle g|g\rangle]\exp[c^{\rm out\dagger}(g)]\exp[-c^{\rm out}(g)] \tag{4.36}$$

allows the generic S-matrix element to be written in terms of the sequences $F=\{F_{(n)}\}$, $G=\{G_{(n)}\}$,

$$\langle G^{\rm out}|F^{\rm in}\rangle=\sum_{m,n=0}^{\infty}\frac{1}{m!}\frac{1}{n!}\int(dk_1)_l\cdots(dk_m)_l(dk_1')_l\cdots(dk_n')_l\,G^*_{(n)}(k_1'\cdots k_n')\times S_{n,m}(k_1'\cdots k_n';\,k_1\cdots k_m)F_{(m)}(k_1\cdots k_m)\,, \tag{4.37}$$

where polarization indices have been suppressed. The explicit form $S^c_{n,m}$ of the connected part of the kernel $S_{n,m}$ is given by

$$S^c_{n,m}(k_1'\cdots k_n';k_1\cdots k_m)=\exp[-\tfrac{1}{2}\langle g|g\rangle]g(k_1')\cdots g(k_n')g^*(k_1)\cdots g^*(k_m)\ . \tag{4.38}$$

Of all the physical subspaces $\mathfrak{F}^+\{e,p\}$, only the subspace $\mathfrak{F}^+\{0\}=\mathfrak{F}(\mathcal{E}_0^+)$ has states which are eigenstates of photon number, and only it has the property that the inner product of any of its vectors with any vector in $\mathfrak{F}(\mathcal{E})$ is formed by ordinary integration. Let F^0_m and G^0_n be elements of $\mathfrak{F}^+\{0\}$ which lie in the m and n photon subspaces, respectively, so $F^{0\,\text{in}}_m\in\mathfrak{F}^{\text{in}+}\{0\}=\mathfrak{F}^{\text{ret}}$ and $G^{0\,\text{out}}_m\in\mathfrak{F}^{\text{out}+}\{0\}=\mathfrak{F}^{\text{ad}}$. It is convenient to write $F^{0\,\text{in}}_m=F^{\text{ret}}_m$, $G^{0\,\text{out}}_n=G^{\text{ret}}_n$, which gives for the S-matrix element

$$\langle G^{\text{ad}}_n|F^{\text{ret}}_m\rangle=\frac{1}{m!n!}\int dk_1\cdots dk_m dk_1'\cdots dk_n' G^*_n(k_1'\cdots k_n')S_{n,m}(k_1'\cdots k_n';k_1\cdots k_m)F_m(k_1\cdots k_m)\ , \tag{4.39a}$$

where $dk=d\hat{k}d\omega(2\omega)^{-1}$ represents ordinary integration. Taking (improper) momentum eigenstates in the usual way we rewrite this formally as

$$\langle k_1'\cdots k_n'^{\text{ad}}|k_1\cdots k_m^{\text{ret}}\rangle=S_{n,m}(k_1'\cdots k_n';k_1\cdots k_m)\ . \tag{4.39b}$$

This formula, generalized to include charged-particle variables, will be the starting point in the derivation of reduction formulas for the S matrix in quantum electrodynamics. The definition of an S-matrix element as an inner product between vectors of two disjoint non-negative subspaces $\mathfrak{F}^{\text{ret}}$ and $\mathfrak{F}^{\text{ad}}$, each of which may be completed to two disjoint Hilbert spaces $\mathcal{H}^{\text{ret}}$ and $\mathcal{H}^{\text{ad}}$, clearly distinguishes the present approach from the traditional Gupta-Bleuler method which contains one physical Hilbert space.

Use of $\mathcal{H}^{\text{ret}}$ has the advantage that the simple question, "What radiation is produced by a given classical current?" is well posed. It has a simple answer. Let the counter of outgoing photons be described by a projector $P(\Omega)$ onto a volume Ω of final-state phase space. The volume Ω is in fact a set of volumes Ω_n, for each n-photon subspace, which is symmetric under permutations. Assume for simplicity that polarizations are not measured. Then the projector $P(\Omega)$ acts on photon states $F=\{F_{(n)}\}$ according to

$$P(\Omega)\{F_{(n)}(k_1\cdots k_n)\}=\{\chi_n(k_1\cdots k_n)F_{(n)}(k_1\cdots k_n)\}\ , \tag{4.40}$$

where $\chi_n(k_1\cdots k_n)=1$ for $\{k_1\cdots k_n\}\in\Omega_n$, and $\chi_{(n)}(k_1\cdots k_n)=0$ otherwise. For example, if Ω is the volume of phase space specified by the condition that no photon is observed with energy greater than ω_0, then

$$\chi_n(k_1\cdots k_n)=\theta(\omega_0-\omega_1)\cdots\theta(\omega_0-\omega_n)\ . \tag{4.41}$$

Because zero-frequency photons are undetectable the projector must satisfy

$$\lim_{\omega_1\to 0}\chi_n(k_1\cdots k_n)=\chi_{n-1}(k_2\cdots k_n)\ . \tag{4.42}$$

This ensures that the infrared-coherence property of F is preserved by the projector $P(\Omega)$, Eq. (4.40).

For the question posed above, the initial state contains no radiation, so the system is in the state $|0^{\text{in}}\rangle=S|0^{\text{out}}\rangle$ which is an element of $\mathfrak{F}^{\text{ret}}$. The probability $p(\Omega)$ that the system be observed to lie in the volume Ω of final-state phase space is

$$p(\Omega)=\langle S0^{\text{out}}|P^{\text{out}}(\Omega)S0^{\text{out}}\rangle\ , \tag{4.43}$$

where S is given in Eq. (4.36). We may drop the out label. The state $S|0\rangle$ is the sequence

$$\{(S|0\rangle)_{(n)}\}=\exp[-\tfrac{1}{2}\langle g|g\rangle]g_{\mu_1}(k_1)\cdots g_{\mu_n}(k_n)\ . \tag{4.44}$$

Hence the inner product (4.43) is given by

$$p(\Omega)=\exp[-\langle g|g\rangle]\left\{\chi_0+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int(dk_1)_l\cdots(dk_n)_l\,\chi_n(k_1\cdots k_n)[g^*_{\mu_1}(k_1)g^{\mu_1}(k_1)]\cdots[g^*_{\mu_n}(k_n)g^{\mu_n}(k_n)]\right\}. \tag{4.45}$$

With $g^\mu(k)=(2\pi)^{-3/2}i\omega\tilde{j}^{\,\mu}(k)$, Eq. (4.21), this is the complete answer to the question "What is the radiation produced by the given classical current $j^\mu(x)$?"

As an example, suppose the projector is given by Eq (4.41). Then the probability $p_0(\omega_0)$ that no photon be observed with energy greater than ω_0 is given by

$$p_0(\omega_0)=\exp[-\langle g|g\rangle]\exp[\langle g|\chi g\rangle]\ , \tag{4.46}$$

where $\chi g(k)=\theta(\omega_0-\omega)g(k)$, or

$$p_0(\omega_0)=\exp\left[\int(dk)_t\, g^*_\mu(k)g^\mu(k)\theta(\omega-\omega_0)\right]. \tag{4.47}$$

But $\theta(\omega-\omega_0)g(k)$ vanishes for ω less than ω_0, so it is an element of $\mathcal{E}_0$, and for it the inner product is obtained by ordinary integration $(dk)_t\to dk = d\hat{k}d\omega(2\omega)^{-1}$,

$$p_0(\omega_0)=\exp\left[\int d\hat{k}\int_0^\infty \frac{d\omega}{\omega}\, g^*_\mu(k)\theta(\omega-\omega_0)g^\mu(k)\right], \tag{4.48}$$

or by Eq. (4.21)

$$p_0(\omega_0)=\exp\left[\frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega}\,\theta(\omega-\omega_0)\tilde{j}^*_\mu(k)\tilde{j}^\mu(k)\right], \tag{4.49}$$

a familiar answer. In fact with

$$|j|_{\omega_0}{}^2\equiv\frac{-1}{(2\pi)^3}\int\frac{d^3k}{2\omega}\,\theta(\omega-\omega_0)\tilde{j}^*_\mu(k)\tilde{j}^\mu(k)\geq 0 \tag{4.50}$$

the probability of observing precisely n photons with energy greater than ω_0 is similarly found to be

$$p_n(\omega_0)=\frac{(|j|_{\omega_0}{}^2)^n}{n!}\exp(-|j|_{\omega_0}{}^2), \tag{4.51}$$

the usual Poisson distribution. This is precisely what would be obtained if $j_\mu(x)$ were of fast decrease, making S an operator that acts within the Hilbert space $\mathcal{H}^{\text{in}}\{0\}$. However, with $\lim_{\omega\to 0}\tilde{j}(k) = (2\pi)^{-3/2}ie(p_f/p_f\cdot k - p_i/p_i\cdot k)$, each of these probabilities vanishes as ω_0 approaches zero like

$$\frac{(-B\ln\omega_0)^n}{n!}\,\omega_0{}^B\,; \tag{4.52}$$

$$B=-\frac{1}{2}\,\frac{1}{(2\pi)^3}\int d\hat{k}\left(\frac{ep_f}{p_f\cdot\hat{k}}-\frac{ep_i}{p_i\cdot\hat{k}}\right)^2\geq 0\,,$$
$$B=\frac{e^2}{2\pi^2}\,(\psi_{fi}\coth\psi_{fi}-1)\,, \tag{4.53}$$

which gives the expected dependence of the transition probabilities on the resolution ω_0 of the final-state detector.

C. Normalization of the charged field

The substitution (3.38) assigns a meeting to ψ_{pre}, the preliminary form of the asymptotic charged field. Even if this is done, however, ψ_{pre} does not become a satisfactory asymptotic field, for it is a mixed beast, containing elements of a strong asymptotic limit and a weak asymptotic limit. In Sec. IV D a consistent weak asymptotic limit will be obtained from ψ_{pre} and which will be used in the accompanying article to derive reduction formulas. As a strong limit it obviously has the wrong transformation properties. However, its inadequacies only show up in higher-point functions. If an asymptotic two-point function

$$G^{\text{as}}(x,y)\equiv\langle 0\,|\,T[\psi_{\text{pre}}(x)\bar{\psi}_{\text{pre}}(y)]\,|\,0\rangle \tag{4.54}$$

is calculated it has a reasonable appearance, as we shall see. In momentum space it has the same singularity at the electron mass shell as the renormalized electron propagator obtained by summing Feynman diagrams.[3,9] By comparing the propagator at the mass shell calculated in these two ways, one may relate the renormalization prescription for Feynman diagrams to the normalization of the charged Heisenberg field.[16]

So far our discussion of the asymptotic state space has been restricted to the photon degree of freedom. To deal with photons and other particles, which we call generically electrons, the appropriate space is the direct product of $\mathfrak{F}^{\text{in}}_\gamma(\mathcal{E})$ ($\mathfrak{F}^{\text{out}}_\gamma(\mathcal{E})$) the representation space for photons described previously, and a representation space for the electrons. The latter may be taken to be either the traditional Fock-Hilbert space of the electrons $\mathcal{H}^{\text{in}}_e$ ($\mathcal{H}^{\text{out}}_e$) or else, for a more symmetric treatment of photons and electrons a Fock space of test functions defined on the mass shell $\mathfrak{F}^{\text{in}}_e$ ($\mathfrak{F}^{\text{out}}_e$). On both spaces the usual inner product is positive-definite, and our notation will not distinguish between these two alternatives.

Let x^0-y^0 be positive, so the asymptotic propagator (4.54) becomes

$$G^{\text{as}}(x,y)=\langle 0|\,\psi_{\text{pre}}(x)\,\bar{\psi}_{\text{pre}}(y)\,|0\rangle,\quad x^0>y^0. \tag{4.55}$$

From Eqs. (2.14), (2.15), (2.18), and $b_s(p)|0\rangle=0$, $d_s(p)|0\rangle=0$, and $\rho(p)|0\rangle=0$, we find

$$G^{\text{as}}(x,y)=\frac{1}{(2\pi)^3}\int\frac{d^3p}{2E}(\not{p}+m)\langle 0|R(p,x)R^\dagger(p,y)|0\rangle \times e^{-ip\cdot(x-y)} \tag{4.56}$$

since $u_s(p)$ was normalized to $\sum_s u_s(p)\bar{u}_s(p)=\not{p}+m$, and by Eqs. (2.16) and (3.38)

$$R(p,x)=\exp[c^\dagger(f_{e,p}(x))]\exp[-c(f_{e,p}(x))], \tag{4.57}$$

$$f_{e,p}{}^\mu(k,x)=-(2\pi)^{-3/2}ep^\mu(p\cdot\hat{k})^{-1}\exp(ik\cdot pp\cdot x/m^2). \tag{4.58}$$

Use of the commutator (3.37) gives

$$\langle 0|R(p,x)R^\dagger(p,y)|0\rangle = \exp\left\{\frac{e^2}{(2\pi)^3}\int\frac{d\hat{k}}{2}\int_0^\infty d\omega \ln\left(\frac{p\cdot k}{ml}\right)\frac{\partial}{\partial\omega}\frac{p^2}{(p\cdot\hat{k})^2}\exp\left[-\frac{ik\cdot p\, p\cdot(x-y)}{m^2}\right]\right\},$$

$$\langle 0|R(p,x)R^\dagger(p,y)|0\rangle = \exp\left[\frac{e^2}{(2\pi)^3}\int\frac{d\hat{k}}{2}\frac{p^2}{(p\cdot\hat{k})^2}J\right], \tag{4.59}$$

where

$$J=\int_0^\infty d\omega \ln\left(\frac{p\cdot\hat{k}\omega}{ml}\right)\frac{\partial}{\partial\omega}\exp\left[-\left(\epsilon+\frac{ip\cdot z}{m}\right)\frac{p\cdot\hat{k}}{m}\omega\right], \tag{4.60}$$

and $z=x-y$. An ϵ has been introduced to define the integral. With $s=(\epsilon+ip\cdot z/m)(p\cdot\hat{k}/m)\omega$ as the integration variable, one has

$$J=\int_0^\infty ds \ln\left[\frac{sm}{(\epsilon+ip\cdot z)l}\right]\frac{d}{ds}e^{-s},$$

$$J=C+\ln[(\epsilon+ip\cdot z)l/m]. \tag{4.61}$$

Here $C=-\int_0^\infty ds \ln s\, e^{-s}$ is Euler's constant $C=0.577\ldots$. It is convenient to introduce a new infrared renormalization constant

$$l_1=l\,e^C \tag{4.62}$$

so

$$J=\ln[(\epsilon+ip\cdot z)l_1/m], \tag{4.63}$$

which gives

$$\langle 0|R(p,x)R^\dagger(p,y)|0\rangle=[\epsilon+ip\cdot(x-y)l_1/m]^\beta. \tag{4.64}$$

Here for convenience the constant β is defined by

$$\beta\equiv\frac{e^2}{(2\pi)^2}=\frac{\alpha}{\pi}. \tag{4.65}$$

This gives for the asymptotic propagator (4.56)

$$G^{as}(x,y)=\frac{1}{(2\pi)^3}\int\frac{d^3p}{2E}(\not p+m)[\epsilon+ip\cdot(x-y)l_1/m]^\beta \times e^{-ip\cdot(x-y)}. \tag{4.66a}$$

A similar calculation valid for $x^0<y^0$ gives

$$G^{as}(x,y)=\frac{1}{(2\pi)^3}\int\frac{d^3p}{2E}(-\not p+m)[\epsilon-ip\cdot(x-y)l_1/m]^\beta \times e^{ip\cdot(x-y)}. \tag{4.66b}$$

These expressions have the correct Poincaré transformation properties, so the incorrect transformation properties of ψ_{pre} only show up in the higher-point Green's functions, as asserted. Moreover, the asymptotic two-point function is even local.

We are interested in the behavior of the asymptotic Green's function near the mass shell in momentum space. Set $G^{as}(x-y)=G^{as}(x,y)$ and use the integral representation

$$(\epsilon\pm ip\cdot x)^\beta=\frac{1}{\Gamma(-\beta)}\int_0^\infty dt\, t^{-\beta-1}e^{-(\epsilon\pm ip\cdot x)t}. \tag{4.67}$$

This representation is valid only for $\beta<0$. However, Eqs. (4.66) are entire in β, and our final expression will be continued back to positive β. One has, for $x^0>0$,

$$G^{as}(x)=\frac{1}{(2\pi)^3}\left(\frac{l_1}{m}\right)^\beta\frac{1}{\Gamma(-\beta)} \times\int_0^\infty dt\, t^{-\beta-1}\int\frac{d^3p}{2E}(\not p+m)e^{-i(1+t)p\cdot x}, \tag{4.68}$$

or, with $p'=(1+t)p$ and $M=m(1+t)$, dropping primes,

$$G^{as}(x)=\frac{1}{(2\pi)^3}\left(\frac{l_1}{m}\right)^\beta\frac{m^2}{\Gamma(-\beta)}\int_m^\infty\frac{dM}{M^3}\left(\frac{m}{M-m}\right)^{1+\beta} \times\int\frac{d^3p}{2E}(\not p+M)e^{-ip\cdot x}, \tag{4.69}$$

where $p^\mu=(E,\vec{p})$, $E=(\vec{p}^2+M^2)^{1/2}$. This has the form of a standard spectral representation, and we have, for all x,

$$G^{as}(x)=\frac{i}{(2\pi)^4}\left(\frac{l_1}{m}\right)^\beta\frac{m^2}{\Gamma(-\beta)}\int_m^\infty\frac{dM}{M^3}\left(\frac{m}{M-m}\right)^{1+\beta} \times\int d^4p\frac{(\not p+M)e^{-ip\cdot x}}{p^2-M^2+i\epsilon}. \tag{4.70}$$

With

$$G^{as}(x)=\frac{1}{(2\pi)^4}\int d^4p\, G^{as}(p)e^{-ip\cdot x} \tag{4.71}$$

we obtain for the momentum-space asymptotic propagator

$$G^{as}(p)=i\left(\frac{l_1}{m}\right)^\beta\frac{m^2}{\Gamma(-\beta)}\int_m^\infty\frac{dM}{M^3}\left(\frac{m}{M-m}\right)^{1+\beta}\frac{\not p+M}{p^2-M^2+i\epsilon}. \tag{4.72}$$

This has the correct analytic structure for a local field, so our preliminary asymptotic field is doing better than is required, which is merely to have the correct structure near the mass shell. A simple calculation yields the quantity of interest

$$\lim_{p^2\to m^2} G^{as}(p)=\Gamma(1+\beta)\frac{i(\not p+m)}{p^2-m^2+i\epsilon}\left(\frac{2ml_1}{m^2-p^2-i\epsilon}\right)^\beta, \quad (4.73)$$

$$\beta=\frac{\alpha}{\pi}.$$

The singularity is a familiar one. It has been obtained by many authors[3,9] studying renormalized Green's functions, and may be regarded as an exact result of quantum electrodynamics, although a rigorous derivation within the framework of renormalized perturbation theory is still lacking. Our derivation allows an immediate connection between the normalization of the propagator and the normalization of the corresponding Heisenberg field as an operator or the asymptotic state space. Suppose that the renormalized electron propagator $G(p)$ is calculated in perturbative renormalization theory with an arbitrary normalization prescription. Near the mass shell it will agree with an expansion in powers of α of

$$z(\alpha)\frac{i(\not p+m)}{p^2-m^2+i\epsilon}\left(\frac{m^2}{m^2-p^2-i\epsilon}\right)^\beta. \quad (4.74)$$

Here $z(\alpha)$ is a power series in α with finite coefficients. In this case the corresponding renormalized Heisenberg field ψ has the normalization

$$\lim_{x^0\to\pm\infty}\psi(x)=z_0^{1/2}\psi_{\text{pre}}(x),\quad z_0=\frac{1}{\Gamma(1+\beta)}\left(\frac{m}{2l_1}\right)^\beta z. \quad (4.75)$$

The reduction formulas derived later allow the S-matrix elements to be expressed in terms of Green's functions calculated according to the given renormalization prescription. An alternative approach is to use Eq. (4.73) to provide an on-mass-shell renormalization condition: Normalize the electron propagator so that $z_0=1$, i.e.,

$$\lim_{p^2\to m^2} G(p)=\Gamma(1+\beta)\frac{i(\not p+m)}{p^2-m^2+i\epsilon}\left(\frac{2ml_1}{m^2-p^2-i\epsilon}\right)^\beta. \quad (4.76)$$

In this case the Heisenberg field has the normalization

$$\lim_{x^0\to\pm\infty}\psi(x)=\psi_{\text{pre}}(x). \quad (4.77)$$

Finally let us make a few observations. As mentioned earlier l, and hence also $l_1=le^C$, is an infrared renormalization constant common to fields of different mass. Consequently in virtue of the infrared renormalization, the charged field has picked up an anomalous dimension (mass units)

$$[\psi]=\tfrac{3}{2}-\tfrac{1}{2}\beta,\quad \beta=\alpha/\pi. \quad (4.78)$$

As may be seen by evaluating Eq. (4.66) at large x, this anomalous dimension controls the large distance behavior of the propagator. It is gauge dependent, and Eq. (4.78) holds in the Feynman gauge. Consider next the spectral decomposition of the exact propagator $G(p)$,

$$G(p)=i\int dM^2\frac{[\not p\rho_1(M^2)+\rho_2(M^2)]}{p^2-M^2+i\epsilon}. \quad (4.79)$$

At values of M^2 sufficiently close to m^2 it must agree with Eq. (4.72),

$$\lim_{M^2\to m^2}\not p\rho_1(M^2)+\rho_2(M^2)=\frac{l_1^{\,\beta}}{2m\Gamma(-\beta)}\frac{\not p+m}{(M-m)^{1+\beta}}. \quad (4.80)$$

However, because $\beta=\alpha/\pi$ is positive, the integral (4.79) in fact does not exist, so the propagator does not possess a spectral decomposition at this point. This is a phenomenon produced by the indefinite metric, because in a positive metric Hilbert space the spectral decomposition is possible, for representations of the Poincaré group may be reduced according to mass. In support of this, observe that for intervals sufficiently close to but excluding the mass shell, $m+\epsilon<M<m+2\epsilon$, the spectral function is arbitrarily well approximated by Eq. (4.80) and is negative-definite. This is also a gauge-dependent phenomenon. However, as β approaches zero the propagator approaches the free propagator whose spectral function is a mass-shell δ function with coefficient plus one.

D. Weak asymptotic limit of the charged field

The preliminary form of the asymptotic field which was used in the last section has too complicated an x dependence to be used in reduction formulas. In the radiation operator $R(p,x)$,

$$R(p,x)=\exp[c^\dagger(f_{e,p}(x))]\exp[-c(f_{e,p}(x))], \quad (4.81a)$$

$$f_{e,p}{}^\mu(k,x)=\frac{-e}{(2\pi)^{3/2}}\frac{p^\mu}{p\cdot k}\exp\left(\frac{ik\cdot p\, p\cdot x}{m^2}\right), \quad (4.81b)$$

the x dependence involves exponentials of exponentials. However, as mentioned in the Introduction, the known singularities of the Green's functions at the mass shell suggest that the distortion of the plane wave due to radiation should be similar to that produced by the Liênard-Wiechert potential, namely logarithmic, but with a real coefficient $\exp[\beta_i\ln|p\cdot x|]$ instead on an imaginary coefficient $\exp[i\gamma_i\ln|p\cdot x|]$. Equations (4.81) in fact lead to precisely this behavior.

The annihilation operator $c(f_{e,p}(x))$ may be written using Eq. (3.40b)

$$c(f_{e,p}(x)) = \frac{-e}{(2\pi)^{3/2}}\int \frac{d\hat{k}}{2}\int d\omega \ln\left(\frac{p\cdot k}{ml}\right)\frac{\partial}{\partial\omega}\left[\frac{p^\mu}{p\cdot \hat{k}}\exp\left(\frac{-ik\cdot p p\cdot x}{m^2}\right)c_\mu(k)\right], \tag{4.82}$$

$$c(f_{e,p}(x)) = \frac{-e}{(2\pi)^{3/2}}\int \frac{d\hat{k}}{2}\frac{p^\mu}{p\cdot\hat{k}}\int_0^\infty d\omega \ln\left(\frac{p\cdot k}{ml}\right)\left\{c_\mu(k)\frac{\partial}{\partial\omega}\exp\left[\frac{-ip\cdot x\, p\cdot\hat{k}\omega}{m^2}\right]+\exp\left[\frac{-ip\cdot x\, p\cdot\hat{k}\omega}{m^2}\right]\frac{\partial c_\mu(k)}{\partial\omega}\right\}. \tag{4.83}$$

An ϵ must be introduced as usual to make the integral well defined. We are interested in the asymptotic fields at asymptotic times, so let us estimate these two terms for large times. Also, we seek a weak asymptotic limit, so the annihilation operator acts on a Fock test function F to its right, whereas the corresponding creation operator $c^\dagger(f_{e,p}(x))$ in Eq. (4.81) operates to the left. Recall that for $F=\{F_{(n)}(k_1\cdots k_n)\}$, $c(k)$ acts according to $[c(k)F]_{(n)}(k_1\cdots k_n)=F_{(n+1)}(k,k_1\cdots k_n)$. Thus in making estimates $c_n(k)$ may be replaced by a test function with argument k. By the Riemann-Lebesgue lemma, the second term in Eq. (4.83) vanishes like $t^{-1}\ln t$ for large t. The first term contains an extra power of t owing to the derivative with respect to ω. Because $c(k)=c(\omega,\hat{k})$ is differentiable in ω at $\omega=0$, and because the contribution to the integral comes from frequencies ω of order t^{-1}, $c(\omega,\hat{k})$ may be replaced by $c(\hat{k})\equiv c(0,\hat{k})$, and we have

$$\lim_{t\to\pm\infty} c(f_{e,p}(x)) = \frac{-e}{(2\pi)^{3/2}}\int\frac{d\hat{k}}{2}\frac{p^\mu c_\mu(\hat{k})}{p\cdot k}\int_0^\infty d\omega\ln\left(\frac{p\cdot\hat{k}\omega}{ml}\right)\frac{\partial}{\partial\omega}\exp\left[-\left(\epsilon+\frac{ip\cdot x}{m}\right)\frac{p\cdot\hat{k}}{m}\omega\right]. \tag{4.84}$$

The integral over ω was effected previously, Eqs. (4.60)–(4.63), with the result

$$\lim_{t\to\pm\infty} c(f_{e,p}(x)) = \frac{-e}{(2\pi)^{3/2}}\int\frac{d\hat{k}}{2}\frac{p\cdot c(\hat{k})}{p\cdot\hat{k}}\ln[\epsilon+ip\cdot x l_1/m], \tag{4.85}$$

where $l_1=l\,e^C$ and C is Euler's constant. It is convenient to introduce

$$A(p)\equiv\frac{1}{(2\pi)^{3/2}}\int\frac{d\hat{k}}{2}\frac{p\cdot c(\hat{k})}{p\cdot k}. \tag{4.86}$$

It depends only on the zero-frequency annihilation operator $c(\hat{k})$, so infrared-coherent subspaces which are eigenspaces of $c(\hat{k})$ are also eigenspaces of $A(p)$. We thus obtain the simple result

$$\lim_{t\to\pm\infty} c(f_{e,p}(x)) = -eA(p)\ln[\epsilon+ip\cdot x l_1/m]. \tag{4.87}$$

Note that the use of the test-function space was essential in obtaining this limit.

When it is substituted into Eq.(4.81) for the radiation operator, one finds, with

$$S(p,x)\equiv\lim_{t\to\pm\infty} R(p,x), \tag{4.88}$$

$$S(p,x)=\exp[-eA^\dagger(p)\ln(\epsilon-ip\cdot x l_1/m)] \times\exp[eA(p)\ln(\epsilon+ip\cdot x l_1/m)]. \tag{4.89}$$

This logarithmic distortion produced by the radiation field corresponds precisely to the x^{-1} terms in the asymptotic vector potential, Eq. (4.4). Thus the distortion of a plane-wave component of the charged field which is produced by the radiation field is in fact logarithmic, like the distortion caused by the Liénard-Wiechert potential. However, the latter is distortion by a logarithmic phase, whereas the distortion produced by the radiation field is in the magnitude, for $A(p)$ and $A^\dagger(p)$ acting to the right and left, respectively, have real eigenvalues. This may be thought of as the origin of the phenomenon of radiation damping. Observe that whereas $R(p,x)$ is an operator, its weak limit (4.89) is a bilinear form, for $A^\dagger(p)$ must act to the left. Its action to the right is undefined. Let $D(p,x)$ be the asymptotic limit of the eikonal factor $E(p,x)$, Eq. (2.14)

$$D(p,x)=\exp[ie\epsilon(p\cdot x)Q(p)\ln(|p\cdot x|l_1/m)]S(p,x). \tag{4.90}$$

For convenience, we have fixed the constant c of Eq. (2.14) at $c=l_1/m$ which involves a choice of phase in the electron space. When this is substituted into the preliminary asymptotic field, $\psi_{\rm pre}$, Eq. (2.18), we obtain the desired weak asymptotic limit of the electron field

$$\psi^{\rm as}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E}\sum_s[D(p,x)b_s(p)u_s(p)e^{-ip\cdot x} + d_s^\dagger(p)D(p,x)v_s(p)e^{ip\cdot x}]. \tag{4.91}$$

This expression will be the starting point for the derivation of reduction formulas in the accompanying article.

Note added in proof. The indefinite metric introduced in Sec. III A to deal with infrared divergences is similar to the indefinite metric which eliminates ultraviolet divergences that has been proposed by O. I. Zav'yalov (Teor. Mat. Fiz. 16, 145 (1973) [Theor. Math. Phys. 16, 735 (1974)]) and O. I. Zav'yalov and P. B. Medvedev (Teor. Mat. Fiz. 18, 27 (1974) [Theor. Math. Phys. 18, 19 (1974)]). I am grateful to Professor Zav'yalov for bringing this work to my attention.

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[13]The inner product (3.29) or (3.30) may be rewritten

$$\begin{aligned}\langle f|g\rangle &= \langle g|f\rangle^* \\ &= \int \frac{d\hat{k}}{2}\int_0^\infty d\omega \ln\left(\frac{\omega}{l}\right)\frac{\partial}{\partial\omega}[f_\mu^*(\omega,\hat{k})g^\mu(\omega,\hat{k})] \\ &\quad -\int\frac{d\hat{k}}{2}f_\mu^*(0,\hat{k})\left\{\frac{1}{4\pi}\int d\hat{k}'\,\frac{1}{1-\hat{k}\cdot\hat{k}'}[g^\mu(0,\hat{k}') - g^\mu(0,\hat{k})]\right\}.\end{aligned}$$

This provides a covariant extension of the indefinite inner product to all sufficiently regular functions f and g. The mathematically unnatural restriction to wave functions whose angular dependence at zero frequency is given by $\sum_a c_a p_a/(E_a - \vec{p}_a\cdot\hat{k})$ is thereby eliminated. An alternate characterization of the representation space $\mathcal{E}$ may be developed which is adapted to the more general form of the inner product. However, for purposes of exposition it seemed preferable to retain the less general form which was discovered first and is adapted to the particular wave functions that arise in quantum electrodynamics.

[14]There are two different causes of the indefiniteness of the metric in $\mathfrak{F}(\mathcal{E})$: (1) the contraction on polarization indices $(-g_{\mu\nu})$ is indefinite, and (2) the infrared renormalization is indefinite. That is why two conditions, transversality and infrared coherence, are required to specify a subspace of non-negative metric. In a model with scalar photons, or phonons, cause (1) is absent and with it the original motivation for the indefinite metric introduced by Gupta and Bleuler. Nevertheless the indefinite metric due to infrared renormalization may be used in such a model.

[15]See D. Maison and D. Zwanziger, Nucl. Phys. B91, 425 (1975), for a detailed discussion of this point.

[16]This result was presented earlier and was obtained by a method which hides the infrared-renormalization mechanism; see D. Zwanziger, Phys. Rev. Lett. 30, 934 (1973).