Ladder approximation, scalar-exchange mesons, and Bethe-Salpeter scattering states

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We investigate the conditions under which the Bethe-Salpeter equation with a scalar-exchange interaction (in the ladder approximation) will in the small-relative-time limit give rise to scattering states. Our results indicate that the exchange particle should be at least slightly unstable (nonzero full width Γ). The role of Γ is compared with cutoff factors occurring in nonrelativistic potential scattering.

I. INTRODUCTION

The Bethe-Salpeter (BS) equation[1] was first conceived as a relativistic analog to the nonrelativistic Schrödinger equation. The ultimate goal should then be for the BS equation to explain both bound- and scattering-state properties of relativistic systems. The primary efforts to date have been devoted to studies of the bound-state properties,[2] but increasing efforts have been recently placed on the scattering-state properties[3,4] of the equation. The work of Schwartz and Zemach[3] is of particular interest since it expresses the results explicitly in the intuitive position representation and allows for simple comparisons with the nonrelativistic Schrödinger equation.

In a recent paper[5] which we shall refer to as HH, we pointed out that conditions must be imposed in order that the scalar BS equation with causal Green's function give rise to scattering solutions. The conditions were necessary in order to eliminate certain divergent behavior of the causal Green's function in the asymptotic timelike regions of the variable x-x'. Our results indicate that one may either take the infinite-momentum limit or a small-relative-time limit. For the small-relative-time limit, suitable restrictions on the interaction are also necessary. These arise from cancellations between factors occurring in the causal Green's function and the interaction $I(x)$ in a region where the two light cones pertaining to the problem intersect. The condition on the interaction $I(x)$ takes the form of the vanishing of integrals of the general form

$$\Lambda(x)=\int_{t'\in D^*} dt'\, e^{\pm i(\omega\pm z)t'} I(x') \tag{1.1}$$

in the asymptotic region $(r\to\infty)$ where D^* refers to a space-time region. More specifically, $\Lambda(x)$ has to vanish faster than e^{-mr}, where m is the mass of the scattering scalar particles. The above condition is satisfied if $I(x)$ is short-ranged and has the form

$$I(x')=I_0(\vec{x}')e^{-a|t'|}, \tag{1.2}$$

with $a>m$. However, the latter restriction on the form of $I(x)$ is only sufficient and not necessary for $\Lambda(r\to\infty)$ to vanish exponentially. For example, the ladder approximation for a scalar-exchange particle of mass μ

$$I(x)=\frac{-i\lambda}{\pi^2}\int d^4k\, \frac{e^{ikx}}{k^2+\mu^2-i\epsilon}, \tag{1.3}$$

which plays such a prominent role in applications and investigations of the BS equation, does not seem to satisfy the above form. However, it may still satisfy the more general expression involving the exponential behavior of Eq. (1.1). We shall show in this paper the conditions under which the ladder approximation with scalar exchange will satisfy the $\Lambda(r\to\infty)$ condition for scattering. Our results indicate that this can be accomplished if the form factor for the scalar-exchange particle has a pole below the real axis in the complex mass plane.

In Sec. II we review our notation and summarize the relevant results obtained in an earlier paper.[5] The explicit form of all the necessary integrals involving the scattering condition as well as the interaction in the scalar ladder approximation are presented. Section III contains the pole approximation and the resulting scattering conditions and Sec. IV contains our summary.

II. INTEGRAL SCATTERING CONDITIONS

We shall follow the notation of our previous paper HH, with x_1 and x_2 being the space-time four-vectors which locate the particles 1 and 2. The relative space-time coordinate x is then x_1-x_2 and the magnitude of the spatial part of x is r. We let ω denote the center-of-momentum energies of our equal-mass particles and z is an energy variable which varies from 0 to m in one case and 0 to ∞ in another case. From HH, if the relative time t is small and if the interactions are short-ranged with b being the range of the force, then in the timelike region of $x-x'$, the integrals

$$\Lambda_{B1}(x)=\int_{r-b+t}^{\infty} dt'\, e^{-i(z\pm\omega)t'}I(x'), \qquad 0<z<m \tag{2.1}$$

$$\Lambda_{B2}(x)=\int_{-\infty}^{-(r-b-t)} dt'\, e^{i(z\pm\omega)t'}I(x'), \qquad 0<z<m \tag{2.2}$$

$$\Lambda_{C1}(x)=\int_{r-b+t}^{\infty} dt'\, e^{-(z\pm i\omega)t'}I(x'), \qquad 0<z<\infty \tag{2.3}$$

and

$$\Lambda_{C2}(x)=\int_{-\infty}^{-(r-b-t)} dt'\, e^{(z\pm i\omega)t'}I(x'), \qquad 0<z<\infty \tag{2.4}$$

must all vanish exponentially in the asymptotic region $r\to\infty$. Specifically, Λ_{B1} and Λ_{B2} must vanish faster than e^{-mr} and Λ_{C1} and Λ_{C2} must vanish faster than $\exp[-m(r^2+z^2)^{1/2}]$ in the asymptotic region. Once these asymptotic restrictions on the Λ's are established, the scattering boundary condition

$$\psi(\vec{x})=\phi(\vec{x})+f(\theta)\frac{e^{ikr}}{r} \tag{2.5}$$

is satisfied.

Let us consider the ladder approximation expressed in Eq. (1.3), with μ being the mass of the scalar-exchange particle and λ being the coupling constant. Equation (1.3) can be reduced to the form

$$I(x)=\frac{4\lambda}{r}\int_0^{\infty} dk\,\frac{k}{\omega_k}\sin(kr)e^{i\omega_k t} \tag{2.6}$$

for $t<0$ and

$$I(x)=\frac{4\lambda}{r}\int_0^{\infty} dk\,\frac{k}{\omega_k}\sin(kr)e^{-i\omega_k t} \tag{2.7}$$

for $t>0$. The above integrals can be evaluated if we add a small imaginary part to the time variable $x\to x-i\hat{\eta}$, where $\hat{\eta}=(\vec{0},\theta(t)\eta)$, and perform the limit of η going to zero after the integration. The expression

$$I(x-i\hat{\eta})=\frac{-i\lambda}{\pi^2}\int d^4k\,\frac{e^{ik(x-i\hat{\eta})}}{k^2+\mu^2-i\epsilon} \tag{2.8}$$

reduces to

$$I(x-i\hat{\eta})=\frac{4\lambda}{r}\int_0^{\infty} dk\,\frac{k\sin(kr)}{(\vec{k}^2+\mu^2)^{1/2}}\times e^{-(\eta+i|t|)(\vec{k}^2+\mu^2)^{1/2}}\,. \tag{2.9}$$

From a standard book of tables,[6]

$$I(x)=\frac{4\lambda\mu}{(r^2-t^2)^{1/2}}K_1(\mu(r^2-t^2+i\epsilon)^{1/2}) \tag{2.10}$$

for both $t<0$ and $t>0$.

Let us next work with a more general expression for the ladder approximation interaction for equal-mass scalar particles. If $\rho(\mu)$ represents the spectral distribution (we have suppressed other energy variables not pertinent to our problem), $I(x-i\hat{\eta})$ then has the form

$$I(x-i\hat{\eta})=\frac{1}{\pi}\int_{\mu_0}^{\infty} d\mu\rho(\mu)\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-i\hat{\eta})}}{k^2+\mu^2-i\epsilon}, \tag{2.11}$$

which reduces to

$$I(x)=\frac{1}{4\pi^3(t^2-r^2)^{1/2}}\int_{\mu_0}^{\infty} d\mu\,\mu\,\rho(\mu)K_1(\mu(r^2-t^2)^{1/2}). \tag{2.12}$$

III. POLE APPROXIMATION AND RESULTS

Let us begin by working with Eq. (2.1),

$$\Lambda_{B1}(x)=\frac{1}{4\pi^3}\int_{\mu_0}^{\infty} d\mu\,\mu\,\rho(\mu)\overline{\Gamma}(\mu,r,t), \tag{3.1}$$

where

$$\overline{\Gamma}(\mu,r,t)=\int_{r-b+t}^{\infty} dt'\,\frac{e^{-i\beta t'}}{(t'^2-r'^2)^{1/2}}K_1(i\mu(t'^2-r'^2)^{1/2}) \tag{3.2}$$

and $\beta=z\pm\omega$. The variable r represents the field point and r' represents the source point (region where the interaction is nonzero). For small relative time, $r-b+t$ can be replaced by $r-b$ which is much greater than 0 in the scattering region. If we let $\tau^2=t'^2-r'^2$,

$$\overline{\Gamma}(\mu,r,t)=\int_{\tau_0}^{\infty} d\tau\,\frac{e^{-i\beta(\tau^2+r'^2)^{1/2}}}{(\tau^2+r'^2)^{1/2}}K_1(i\mu\tau), \tag{3.3}$$

where $\tau_0=[(r-b)^2-r'^2]^{1/2}$. Since the range of the force b is greater or equal to r', $r\gg b$, and $\tau\gg b$,

$$\overline{\Gamma}(\mu,r,t)=\int_r^{\infty} d\tau\,\frac{e^{-i\beta\tau}}{\tau}K_1(i\mu\tau). \tag{3.4}$$

$K_1(i\mu\tau)$ can be replaced by its asymptotic form since the range of integration is only over very large values of τ. The asymptotic form is

$$K_1(i\mu\tau)=-\frac{\pi}{2}\frac{e^{-i\mu\tau}}{\tau^{1/2}}. \tag{3.5}$$

On substituting Eq. (3.5) into Eq. (3.4), we have

$$\overline{\Gamma}(\mu,r,t)=-\frac{\pi}{2}\int_r^{\infty} d\tau\,\frac{e^{-i(\beta+\mu)\tau}}{\tau^{3/2}}. \tag{3.6}$$

Equation (3.1) can now be written as

$$\Lambda_{B1}(x)=-\frac{1}{8\pi^2}\int_r^{\infty} d\tau\,\frac{e^{-i\beta\tau}}{\tau^{3/2}}Q(\tau), \tag{3.7}$$

where

$$Q(\tau)=\int_{\mu_0}^{\infty} d\mu\, \mu\, \rho(\mu) e^{-i\mu\tau} \quad (3.8)$$

The question is now: What form of $\rho(\mu)$ will lead to exponentially decreasing forms for Λ_{B1}? Many forms were tried, but the only successful simple forms found were those containing a pole in the lower complex μ plane,

$$\rho(\mu)=\frac{f(\mu)}{\mu-(\mu_p-i\bar{\mu})} \quad (3.9)$$

Giving $f(\mu)$ suitable properties, Eq. (3.8) can be evaluated giving

$$Q(\tau)=2\pi i F(\mu_p-i\bar{\mu}) e^{-i\mu_p\tau} e^{-\bar{\mu}\tau}, \quad (3.10)$$

where $F(\mu_p-i\bar{\mu})$ is the residue of $\mu f(\mu)$ evaluated at the location of the pole. On substituting into Eq. (3.7), we have

$$\Lambda_{B1}(x)=\frac{-iF(\mu_p-i\bar{\mu})}{4\pi}\int_r^{\infty} d\tau \frac{e^{-(\bar{\mu}+i\bar{\beta})\tau}}{\tau^{3/2}}, \quad (3.11)$$

where $\bar{\beta}=\beta+\mu_p$. The integral can be expressed in terms of the incomplete gamma function $\Gamma(-\frac{1}{2}, (\bar{\mu}+i\bar{\beta})r)$, with Eq. (3.11) becoming

$$\Lambda_{B1}(x)=\frac{-iF(\mu_p-i\bar{\mu})}{4\pi}(\bar{\mu}+i\bar{\beta})^{1/2}\Gamma(-\tfrac{1}{2}, (\bar{\mu}+i\bar{\beta})r). \quad (3.12)$$

For large values of r, $\Gamma(-\frac{1}{2}, (\bar{\mu}+i\bar{\beta})r)$ can be expressed in terms of an asymptotic expansion[7]

$$\Gamma(-\tfrac{1}{2}, (\bar{\mu}+i\bar{\beta})r)=\frac{e^{-(\bar{\mu}+i\bar{\beta})r}G(r)}{[(\bar{\mu}+i\bar{\beta})r]^{3/2}\Gamma(\frac{3}{2})}, \quad (3.13)$$

where

$$G(r)=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\frac{3}{2}+n)}{[(\bar{\mu}+i\bar{\beta})r]^n}. \quad (3.14)$$

The series diverges; however, it is an asymptotically convergent series, that is, the series converges only for large values of r. Since we are working in the asymptotic $r\to\infty$ region, this poses no problem.

It would be helpful if the series $G(r)$ could be replaced by a more workable function. Let us break $G(r)$ into real and imaginary parts,

$$G_{\text{Re}}=\sum_{n=0}^{\infty}(-1)^n\Gamma(\tfrac{3}{2}+n)R^{-n}\cos(n\phi) \quad (3.15)$$

and

$$G_{\text{Im}}=\sum_{n=0}^{\infty}(-1)^n\Gamma(\tfrac{3}{2}+n)R^{-n}\sin(n\phi), \quad (3.16)$$

where $R=r(\bar{\mu}^2+\bar{\beta}^2)^{1/2}$ and $\sin\phi=\bar{\beta}r/R$. It will be convenient to rewrite G_{Re} and G_{Im} as

$$G_{\text{Re}}=\frac{\sqrt{\pi}}{2}+\sum_{n_{\text{odd}}=1}^{\infty}a_n^R \quad (3.17)$$

and

$$G_{\text{Im}}=\sum_{n_{\text{odd}}=1}^{\infty}a_n^I, \quad (3.18)$$

where

$$a_n^R=\frac{\Gamma(\frac{3}{2}+n)}{R^n}\left\{(\tfrac{3}{2}+n)\frac{r}{R}[\bar{\mu}\cos(n\phi)-\bar{\beta}\sin(n\phi)] - \cos(n\phi)\right\} \quad (3.19)$$

and

$$a_n^I=\frac{\Gamma(\frac{3}{2}+n)}{R^n}\left\{(\tfrac{3}{2}+n)\frac{r}{R}[\bar{\mu}\sin(n\phi)+\bar{\beta}\cos(n\phi)] - \sin(n\phi)\right\}. \quad (3.20)$$

The sums are only over the odd numbers of n. In studying the convergence properties of a function, we can always replace the function with another one which is larger at every point r. We can therefore replace G_{Re} and G_{Im} by

$$\bar{G}_{\text{Re}}=\frac{\sqrt{\pi}}{2}+\sum_{n_{\text{odd}}=1}^{\infty}|a_n^R| \quad (3.21)$$

and

$$\bar{G}_{\text{Im}}=\sum_{n_{\text{odd}}=1}^{\infty}|a_n^I|. \quad (3.22)$$

Let us next consider the expansion of the function $e^{-N\bar{\mu}r}$,

$$e^{-N\bar{\mu}r}=\sum_{n=0}^{\infty}(-1)^n\frac{(N\bar{\mu}r)^n}{n!}, \quad (3.23)$$

where N is an arbitrary finite real positive number. We can rewrite this expansion as

$$e^{-N\bar{\mu}r}=1+\sum_{n_{\text{odd}}=1}^{\infty}b_n, \quad (3.24)$$

where

$$b_n=\frac{(N\bar{\mu}r)^n}{n!}\left(\frac{N\bar{\mu}r}{n+1}-1\right). \quad (3.25)$$

In the asymptotic region of $r\to\infty$, since the maximum values of $|a_n^R|$ and $|a_n^I|$ are bounded by

$$\frac{\Gamma(\frac{3}{2}+n)}{r^n}\left[\frac{(\frac{3}{2}+n)\bar{\mu}/\bar{\beta}+(\frac{5}{2}+n)}{(\bar{\mu}^2+\bar{\beta}^2)^{n/2}}\right],$$

it is clear that b_n is greater than $|a_n^R|$ and $|a_n^I|$. Since we have also $\sqrt{\pi}/2<1$, we can replace $\bar{G}_{\text{Re}}$ and $\bar{G}_{\text{Im}}$ each by $\exp(-N\bar{\mu}r)$. The asymptotic properties of $\Lambda_{B1}(x)$ are then determined from

$$\bar{\Lambda}_{B1}(x)=[(\bar{\mu}+i\bar{\beta})r]^{-3/2}e^{-i\bar{\beta}r}e^{-(N+1)\bar{\mu}r}. \quad (3.26)$$

$\Lambda_{B2}(x)$ can be treated in a similar fashion as

$\Lambda_{B1}(x)$. If we make the same assumptions of small relative time and small values of the range b, Eq. (2.2) can be written (for large r) as

$$\Lambda_{B2}(x)=\int_{-\infty}^{-r} dt' e^{i\beta t'} I(\vec{x}', t') . \quad (3.27)$$

However, on inspection of Eq. (2.12), we have

$$I(\vec{x}, t)=I(\vec{x}, -t) , \quad (3.28)$$

which will give us

$$\Lambda_{B2}(x)=\Lambda_{B1}(x) . \quad (3.29)$$

For $\Lambda_{C1}(x)$ and $\Lambda_{C2}(x)$, we can show again that $\Lambda_{C1}(x)=\Lambda_{C2}(x)$ in the asymptotic region for small relative times. Substituting Eq. (2.12) into Eq. (2.4) and making the same assumptions as before, we have

$$\Lambda_{C1}(x)=\frac{-i}{4\pi} F(\mu_p - i\bar{\mu})(\bar{\mu}+z+i\gamma)^{1/2} \times \Gamma(-\tfrac{1}{2}, (\bar{\mu}+z+i\gamma)r) , \quad (3.30)$$

where $\gamma=\mu_p \pm i\omega$. The incomplete gamma function can be represented by

$$\Gamma(-\tfrac{1}{2}, (\bar{\mu}+z+i\gamma)r)=\frac{e^{-(\bar{\mu}+z+i\gamma)r} H(r)}{[(\bar{\mu}+z+i\gamma)r]^{3/2}\Gamma(\frac{3}{2})} , \quad (3.31)$$

where

$$H(r)=\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{3}{2}+n)}{[(\bar{\mu}+z+i\gamma)r]^n} . \quad (3.32)$$

Using the same procedure as before and replacing both the real and imaginary parts of $H(r)$ by an exponential $e^{-N(\bar{\mu}+z)r}$ in studying its asymptotic behavior $\bar{\Lambda}_{C1}$, we have

$$\bar{\Lambda}_{C1}(x)=[(\bar{\mu}+z+i\gamma)r]^{-3/2} e^{-i\gamma r} e^{-(N+1)(\bar{\mu}+z)r} . \quad (3.33)$$

IV. SUMMARY

The conditions mentioned in Sec. II require that

$$\bar{\Lambda}_{B1}(x) \propto e^{-mr} \quad (4.1)$$

and

$$\bar{\Lambda}_{C1}(x) \propto e^{-(m^2+z^2)^{1/2} r} . \quad (4.2)$$

Our results from Eqs. (3.26) and (3.33) indicate that

$$\bar{\Lambda}_{B1}(x) \propto e^{-(N+1)\bar{\mu}r} \quad (4.3)$$

and

$$\bar{\Lambda}_{C1}(x) \propto e^{-(N+1)(\bar{\mu}+z)r} . \quad (4.4)$$

We see that the conditions (4.1) and (4.2) are satisfied if $(N+1)\bar{\mu} > m$.

Let us attempt to give a physical reason for the necessity for introducing such a procedure (imaginary mass). From standard nonrelativistic potential scattering using Schrödinger's equation, an alternate to the wave packet description is to introduce a cutoff factor in the potential, for example[8]

$$V(x)=V(\vec{x})e^{-\alpha|t|} , \quad (4.5)$$

to avoid divergent expressions of the form

$$\int_{-\infty}^{t} dt' e^{i\omega t'} . \quad (4.6)$$

This cutoff factor can be transferred over to the Green's function in the form of an imaginary part added to the energy. One then takes the limit $\alpha \to 0$ after the integration operation. We interpret the above in terms of assuring ourselves that the potential is gradually turned on in the interaction region and is not present at asymptotic times $|t| \to \infty$. If we consider the $\hat{\eta}$ procedure of Sec. II, $I(x)$ of Eq. (2.8) has the form

$$\lim_{\tau\to\infty} I(x-i\hat{\eta})=\frac{2\pi\lambda\mu i}{\tau^{3/2}} e^{-i\mu\tau} e^{-\eta\mu} \quad (4.7)$$

if we allow η to remain small but not zero after the integration. The interaction clearly does not cut off at large time values and therefore does not play the same role as Eq. (4.5) in nonrelativistic scattering. However, if we let $\mu=\mu_p - i\bar{\mu}$,

$$\lim_{\tau\to\infty} I(x)=\frac{2\pi\lambda\mu i}{\tau^{3/2}} e^{-i\mu_p\tau} e^{-\bar{\mu}\tau} . \quad (4.8)$$

The interaction is damped out at asymptotic values of τ, essentially the same behavior for the nonrelativistic potential scattering. The main difference is that we can let $\alpha \to 0$ after the integration in nonrelativistic potential scattering, while in the relativistic Bethe-Salpeter equation we must not set $\bar{\mu}=0$ after the integration, as it is needed to damp out an e^{mr} factor appearing elsewhere in the timelike region.

Since $\bar{\mu}$ is just the half width of the unstable particle, we must have $(N+1)\Gamma/2$, where Γ is the full width larger than the mass m of the scattering particles. Let us compare again with nonrelativistic scattering theory. From Eq. (4.5), $1/\alpha$ gives an order-of-magnitude estimate for the time interval of the interaction. In the relativistic BS equation scattering from Eq. (4.8), $1/\Gamma$ is an estimate for the time the interaction is on. That this also corresponds to the lifetime of the exchange particle is somewhat interesting, but not surprising since we would not expect the time interval for the interaction to exceed the lifetime of the exchange particle. Since N is an arbitrary real number, we can satisfy the condition $(N+1)\Gamma/2 > m$ with a sufficiently large N and any nonzero Γ. Therefore, we need the exchange particle to be just slightly unstable. This is a feature that never shows up in nonrelativistic formalisms, but enters into the BS equation via the mass terms in the timelike regions of $x-x'$.

All known scalar mesons have nonzero values[9] of Γ and thus can contribute to BS scattering as exchange particles at small relative times. For perfectly stable ($\Gamma = 0$) scalar-exchange mesons, the scattering conditions at small relative times can still be satisfied if they are also tachyons, with $i\bar{\mu}$ being the imaginary rest mass of the particles.

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