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Structure of the vertex function in finite quantum electrodynamics

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We study the structure of the renormalized electromagnetic current vertex, $\tilde{\Gamma}_\mu(p,p+q,q)$, in finite quantum electrodynamics. Using conformal invariance we find that $\tilde{\Gamma}_\mu(p,p,0)$ takes the simple form of $Z_1\gamma_\mu$ when the external fermions are far off the mass shell. We interpret this result as an old theorem on the structure of the vertex function due to Gell-Mann and Zachariasen. We give the general structure of the vertex for arbitrary momentum transfer parametrically, and discuss how the Bethe-Salpeter equation and the Federbush-Johnson theorem are satisfied. We contrast the meaning of pointlike in a finite field theory with the meaning understood in the parton model. We discuss to what extent the condition $Z_1 = 0$, which may hold in conformal theories other than finite quantum electrodynamics, may be interpreted as a bootstrap condition. We show that the vanishing of Z_1 prevents there being bound states in the Migdal-Polyakov bootstrap.

I. INTRODUCTION

In recent years a serious effort has been made to try to understand the ultraviolet structure of local field theories. This requires a study of Green's functions far off the mass shell where ordinary perturbation techniques are generally unreliable, since this is the kinematic region which generates the divergences met order by order in perturbation theory. Johnson, Baker, and Willey[1] have tackled this problem vigorously and developed a finite theory of quantum electrodynamics (finite QED) in which the divergences organize themselves away nonperturbatively. Their theory may now serve as a prototype for the sort of short-distance behavior we might expect in a realistic quantum field theory, and may be characterized by the concept of dynamical dimensions.[2] In their theory an eigenvalue condition for the coupling constant is required in order to make Z_3^{-1} finite with the photon propagator being asymptotically canonical. The electron propagator acquires an anomalous dimension which can be removed in the generalized Landau gauge (the finite gauge) to make Z_2^{-1} finite, and the composite mass operator $:\bar{\psi}\psi:$ must acquire a negative anomalous dimension in order to make the bare mass vanish so that mass renormalization is then also finite. Theories of this type are also known to display asymptotic conformal invariance,[3] and hence conformal invariance may prove useful in limiting the forms of the Green's functions of the theory, and may provide nonperturbative asymptotic information about the theory.

In this work we shall use conformal invariance to construct the vertex functions of finite QED to obtain their structure. Such information is useful in itself, and it will also enable us to determine the nature of the pointlike behavior of the dressed electron in a finite renormalizable field theory. As we shall see, this behavior is very different from the meaning of pointlike understood in the parton model. Our approach will also help in studying to what extent the condition $Z_1 = 0$ is a bootstrap condition for the vertex in theories where it is expected to hold. We shall present our main results in this introduction and leave the details of the calculation to Sec. II. For completeness we shall also discuss in Sec. II how the Bethe-Salpeter equation and the Federbush-Johnson theorem are satisfied in finite QED in the vector current sector, and also make a similar analysis for the axial-vector vertex, $\tilde{\Gamma}_{\mu 5}(p,p+q,q)$.

The most convenient starting point is to introduce the canonical commutator

$$[j_0(x), \psi(y)]\delta(x_0 - y_0) = C\delta^4(x-y)\psi(x) . \qquad (1)$$

In the finite theory this commutator is not destroyed by renormalization (except that there may be a c-number anomaly), so that the renormalized current is still canonical. Moreover, in the finite gauge the electron is also canonical so that the conformal ansatz for the renormalized connected unamputated vertex function $\tilde{G}_\mu(x,z,y)$ in coordinate space gives[4]

$$\langle\Omega|T(\psi(x)j_\mu(z)\bar{\psi}(y))|\Omega\rangle = \frac{Z_2^{-1}}{4\pi^4} A\, \frac{(\hat{y}-\hat{z})}{(y-z)^4}\gamma_\mu \frac{(\hat{z}-\hat{x})}{(z-x)^4} + \frac{Z_2^{-1}}{4\pi^4} B\, \frac{(\hat{x}-\hat{y})}{(x-y)^2(y-z)^2(z-x)^2}\left[\frac{(z_\mu - x_\mu)}{(z-x)^2} + \frac{(y_\mu - z_\mu)}{(y-z)^2}\right]. \qquad (2)$$

Here A and B are arbitrary functions of the coupling constant with all of the space-time structure specified by the conformal group alone. In our notation Z_2^{-1} is the c-number anomaly of the electron anticommutator in the finite gauge, i.e., the gauge-independent part of the electron wave-function renormalization constant. [Equation (2) may not be the complete $\tilde{G}_\mu$ because of difficulties met in trying to make gauge invariance compatible with conformal invariance.[5] Extra form factors may be required,[6-8] and this point will be discussed in Appendix A.] Since the current is canonical the vacuum polarization is fixed by conformal invariance to have the form

$$\Pi_{\mu\nu}(z) = \langle\Omega|T(j_\mu(z)j_\nu(0))|\Omega\rangle = \frac{f(\alpha)}{(2\pi^2)^2}\,\frac{\mathrm{Tr}[\gamma_\mu \hat{z}\gamma_\nu(-\hat{z})]}{z^8}\ , \tag{3}$$

where we have normalized with respect to a free Fermi theory. Here $f(\alpha)$ is the function whose vanishing makes Z_3^{-1} finite.[1] (We remind the reader that since $\Pi_{\mu\nu}$ is not multiplicatively renormalizable, conformal invariance has been applied to the unsubtracted $\Pi_{\mu\nu}$.) We may now rewrite Eq. (3) in the form

$$\Pi_{\mu\nu}(z) = \frac{f(\alpha)}{12\pi^4}\,(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)\,\frac{1}{z^4} \tag{4}$$

after removing a term proportional to $g_{\mu\nu}z^{-2}\delta^4(z)$ by regulating, so that $\Pi_{\mu\nu}(z)$ is now transverse. (This is a well-known feature of conformal invariance which can only be achieved for Lorentz vector operators of dimension 3.) Noting that the Fourier transform of z^{-4} is given by $i\pi^2[\Gamma(0)+\ln q^2]$ we now make a subtraction to obtain

$$\tilde{\Pi}_{\mu\nu}(q^2) = i\,\frac{f(\alpha)}{12\pi^2}\,(q_\mu q_\nu - q^2 g_{\mu\nu})\ln\left(\frac{-q^2}{\mu^2}\right) \tag{5}$$

to give the structure of the renormalized vacuum polarization.

Since we have Eq. (2) explicitly we can then in principle take the Fourier transform of $\tilde{G}_\mu$ as well, but in practice we can only do this in parametric form since the integrals do not appear to be reducible to named functions. However, for zero momentum transfer (with p^2 serving as an infrared cutoff) an analytic solution will be obtained in Sec. II which when combined with the Ward identity leads us to an amputated vertex which satisfies

$$\lim_{p^2\to-\infty} \tilde{\Gamma}_\mu(p,p,0) = Z_1\gamma_\mu\ , \tag{6}$$

a remarkably simple form. This appears to be an old result of Gell-Mann and Zachariasen[9] which says that the asymptotic behavior of the vertex function as the momentum goes to infinity is the same as that of Z_1 as the cutoff goes to infinity (Z_1^{-1} being finite in our case). Note that this is not the same as another relation which has appeared in the literature,

$$\lim_{q^2\to-\infty} \tilde{\Gamma}_\mu(p^2=m^2, (p+q)^2=m^2, q) = Z_1\gamma_\mu\ , \tag{7}$$

which would be the asymptotic behavior of the on-shell form factor. There is some confusion in the literature between Eqs. (6) and (7). Källén's famous wrong proof[10] that one of the renormalization constants in QED had to be infinite was based on the validity of Eq. (7), and it is not too clear from reading Ref. 9 as to which asymptotic limit its authors had in mind. However, we only need recall that Z_1 is a renormalization constant which multiplicatively renormalizes a product of fields at the same point and hence is related to short-distance behavior, so that it must be determined by a Green's function in which at least two legs go far off the mass shell. This is analogous to the relation for the off-shell propagator in the finite gauge.

$$\lim_{p^2\to-\infty} \tilde{S}^{-1}(p) = Z_2\not{p}\ , \tag{8}$$

and hence Eq. (6) says that Z_1 is given in a similar manner by inserting a current carrying zero momentum into the off-shell propagator and then shortening the distance between the two fermions so as to trap the insertion. Equation (7) is *a priori* unreasonable (but could still presumably hold in particular dynamical situations) since the asymptotic behavior of the on-shell form factor is not given by the short-distance limit (it is an infrared problem). It was of course realized that even if Eq. (7) were to be true in some theories it could not be expected to hold in QED since Z_1 is gauge-dependent (unless, of course, Z_1 just happened to vanish in all gauges). This in fact started the controversy which led to the development of finite QED, and in a sense we have come full circle historically by deriving Eq. (6) [instead of Eq. (7)] as a consequence of the finiteness of the theory, so that Eq. (6) is consistent with all the renormalization constants being finite.

Though we cannot determine the on-shell form factor, we can bound its asymptotic behavior by inserting the form factor into the discontinuity of the vacuum polarization [which is in fact zero since the vanishing of $f(\alpha)$ removes the discontinuity from Eq. (5)]. This then gives[11]

$$\lim_{q^2\to\pm\infty} \tilde{\Gamma}_\mu(p^2=m^2, (p+q)^2=m^2, q) = 0\ . \tag{9}$$

We mention this to contrast with the parton model. In the parton model Eq. (7) is assumed to hold (with Z_1 nonzero) rather than Eqs. (6) and (9), so that it uses a different interpretation of the word "pointlike" than that which applies to the electron. [We are not aware of any derivation of Eq. (7) in the perturbatively asymptotically free non-Abelian case[12] either where Eq. (6) will still hold since the theory is canonical (up to logarithms), since asymptotic freedom does not apply outside the light cone.] We mention this to indicate that the parton model goes a lot further than the light-cone algebra, and that the use of Eq. (7) in it (particularly when extended to timelike q^2) may be unwarranted.

Though this paper is concerned with finite QED we would like to discuss here the closely related conformal bootstrap developed by Migdal and Polyakov. Our discussion will give us some new information about the bootstrap nature of the ansatz of conformal invariance with anomalous dimensions, and we will also see (in Sec. II) why finite QED is not a bootstrap of this type. We shall analyze the discussion given by Migdal[3] of the $:i\lambda\phi\bar{\psi}\gamma_5\psi:$ theory under the ansatz of conformal invariance with anomalous dimensions. Let d_M and d_F be the anomalous dimensions of the meson and the fermion, respectively. (In this theory there is no gauge, so that the usual positivity requirement leads to $d_F > \frac{3}{2}$, $d_M > 1$.) If we renormalize the theory at some mass point μ, then the wave-function renormalization constants are given by

$$Z_F \sim \left(\frac{\Lambda^2}{\mu^2}\right)^{3/2-d_F}, \quad Z_M \sim \left(\frac{\Lambda^2}{\mu^2}\right)^{1-d_M}, \tag{10}$$

so that both Z_F and Z_M vanish. Let us introduce $Z_\theta^{-1/2}$, which renormalizes $:i\bar{\psi}\gamma_5\psi:$ so that

$$Z_\theta \sim \left(\frac{\Lambda^2}{\mu^2}\right)^{3-d_\theta}, \tag{11}$$

where d_θ is the anomalous dimension of the pseudoscalar composite. We introduce Z_P which renormalizes Γ_P, the vertex made by inserting $:i\bar{\psi}\gamma_5\psi:$ into the inverse fermion propagator, which then satisfies

$$\begin{aligned} Z_P &= Z_F Z_\theta^{-1/2} \\ &\sim \left(\frac{\Lambda^2}{\mu^2}\right)^{d_\theta/2-d_F}. \end{aligned} \tag{12}$$

Now since we are in an infrared-stable theory we require an eigenvalue condition for the bare coupling constant, so that there will in fact be conformal invariance with anomalous dimensions. Thus

$$\begin{aligned} \frac{\lambda}{\lambda_0} &= Z_M^{1/2} Z_\theta^{1/2} \\ &\sim \left(\frac{\Lambda^2}{\mu^2}\right)^{(4-d_M-d_\theta)/2} \end{aligned} \tag{13}$$

is finite (with $\lambda\bar{\psi}\gamma_5\psi\phi$ being a renormalization invariant), so that we obtain a consistency condition

$$d_M + d_\theta = 4\ . \tag{14}$$

But $d_M > 1$, so we see that $d_\theta < 3$, so that $:i\bar{\psi}\gamma_5\psi:$ has to have a dimension less than canonical. (Exactly the same situation is met in discussing mass renormalization in finite QED where the dimension of $:\bar{\psi}\psi:$ is required to be less than canonical.[13]) Now $d_F > \frac{3}{2}$, so that from Eq. (12) we conclude that $Z_P = 0$.

Now $\tilde{\Gamma}_P$ satisfies the Bethe-Salpeter equation

$$\tilde{\Gamma}_P(p, p+q, q) = Z_P\gamma_5 + \int d^4k\, \tilde{K}(p, k, q)\tilde{S}(k)\tilde{\Gamma}_P(k, k+q, q)\tilde{S}(k+q)\ , \tag{15}$$

where the renormalized kernel is built out of the dressed propagators and the dressed three-point function $\langle\Omega|T(\psi(x)\phi(z)\bar{\psi}(y))|\Omega\rangle$ $(=\tilde{G}_\phi)$. The conformal ansatz for $\tilde{\Gamma}_P$ (unamputated) gives[3]

$$\tilde{\Gamma}_P(x, z, y) = C\, \frac{(\hat{y} - \hat{z})}{[(y-z)^2]^{(1+d_\theta)/2}}\, \gamma_5\, \frac{(\hat{z} - \hat{x})}{[(z-x)^2]^{(1+d_\theta)/2}}\, \frac{1}{[(x-y)^2]^{(2d_F - d_\theta)/2}}\ . \tag{16}$$

This $\tilde{\Gamma}_P$ (together with a similar structure for $\tilde{G}_\phi$) then reproduces itself self-consistently in Eq. (15) when $Z_P = 0$. Thus the bare vertex which has canonical dimension is eliminated, and the ansatz of conformal invariance with anomalous dimensions reproduces itself with the convergence of the integration in Eq. (15) achieved since $d_\theta < 3$. Hence we recognize the condition $d_\theta < 3$ as the consistency condition that the conformal bootstrap be implementable.

Migdal in his paper did not actually discuss the equation for $\tilde{\Gamma}_P$ but rather that for $\tilde{G}_\phi$ which is also homogeneous when $Z_P = 0$. This equation is then nonlinear in $\tilde{G}_\phi$ and is thus a bootstrap equation for the dressed $\tilde{G}_\phi$ three-point function. This equation also looks like a typical homogeneous bound-state equation in which the meson could have been a pole in the off-shell fermion-antifermion scattering amplitude. Since the kernel is itself made up by exchanging the same meson the situation looks like a bound-state bootstrap. Hence it is often stated that the vanishing of the renormalization constants Z_P, Z_F, and Z_M gives a bound-state bootstrap. Moreover, this approach is usually coupled with an analog of Eq. (7) (see, e.g., Ref. 14) where the implied fast-falling on-shell form factor is interpreted as a compositeness condition. Now of course if there are bound states in the fermion-antifermion scattering amplitude their vertices will satisfy homogeneous equations at the pole. So let us suppose there is a pseudoscalar bound state in the conformal theory. Such a bound state would have to appear as a pole in $\tilde{\Gamma}_P$, which symbolically behaves as

$$\tilde{\Gamma}_P \sim \frac{Z_P \gamma_5}{1 - \int \tilde{K}\tilde{S}\gamma_5\tilde{S}\gamma_5} \quad . \tag{17}$$

However, we are in a situation in which $Z_P = 0$, and thus $\tilde{\Gamma}_P$ cannot contain a pole. Hence the existence of homogeneous equations is a necessary but not sufficient condition that there be bound states. (The sufficient condition is that the off-shell bound-state vertex function satisfy an inhomogeneous equation.[15]) Thus in the example discussed by Migdal we see that the condition $Z_P = 0$ actually prevents the meson from being a bootstrapped bound state, and in his model $:i\bar{\psi}\gamma_5\psi:$ is the source of the meson field instead of the meson being a pole in $\tilde{\Gamma}_P$.[16] Thus though the conformal bootstrap leads to self-consistent equations for the dressed vertices and to an eigenvalue condition for the (bare) coupling constant, it contains no bound states, and hence it appears to us that the conformal bootstrap cannot provide a field-theoretic realization of the S-matrix bootstrap.

II. CALCULATION OF THE VERTEX FUNCTION

We shall proceed directly to take the Fourier transform of Eq. (2). The A term is simply done and gives

$$\tilde{G}_{\mu A} = Z_2^{-1} \frac{A}{\not{p}} \gamma_\mu \frac{1}{(\not{p} + \not{q})} \quad . \tag{18}$$

The second term can be handled by noting that the Fourier transform (FT) of $f(z-x)g(x-y)h(y-z)$ is given as a Feynman diagram by

$$\mathrm{FT} = \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k)\tilde{g}(k+p)\tilde{h}(k-q) \ , \tag{19}$$

so that we obtain

$$\tilde{G}_{\mu B} = iZ_2^{-1} \frac{B}{\pi^2} \int d^4k \frac{(\not{k}+\not{p})(2k_\mu - q_\mu)}{(k+p)^4(k-q)^2k^2} \ . \tag{20}$$

We evaluate this integral using Feynman parameters. If we introduce

$$\begin{aligned} D &= \alpha(1-\alpha)p^2 + \beta(1-\beta)q^2 + 2\alpha\beta p\cdot q \ , \\ C_{\sigma\mu} &= -4\alpha(1-\alpha)p_\sigma p_\mu - 2\beta(1-2\beta)q_\sigma q_\mu \\ &\quad -2p_\sigma q_\mu(1-\alpha)(1-2\beta) - 4p_\mu q_\sigma \alpha\beta \ , \end{aligned} \tag{21}$$

then

$$\tilde{G}_{\mu B} = Z_2^{-1} B\gamma^\sigma \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left(\frac{\alpha g_{\sigma\mu}}{D} + \frac{\alpha C_{\sigma\mu}}{2D^2} \right) . \tag{22}$$

It does not appear possible to integrate Eq. (22) right out analytically so we keep it in this form. Further simplification is possible at $q_\mu = 0$, however, and yields

$$\tilde{G}_{\mu B}(p,p,0) = Z_2^{-1} B\gamma^\sigma \left(\frac{g_{\sigma\mu}}{p^2} - \frac{2p_\sigma p_\mu}{p^4} \right) \ . \tag{23}$$

We now amputate the fermion legs using Eq. (8). (This definition of Z_2 may not necessarily coincide with the definition that Z_2 is the residue at the pole of the unrenormalized propagator, since a conformal theory has infrared problems, but will suffice for our purposes.) Thus we obtain

$$\tilde{\Gamma}_\mu(p,p,0) = Z_2(A-B)\gamma_\mu \ , \tag{24}$$

whose nontrivial content is that there is no $\not{p}\gamma_\mu\not{p}$ term at $q_\mu = 0$. However, the renormalized Ward identity in differential form gives

$$\frac{Z_2}{Z_1} \gamma^\mu \tilde{\Gamma}_\mu(p,p,0) = \gamma^\mu \frac{\partial}{\partial p_\mu} \tilde{S}^{-1}(p) \ , \tag{25}$$

so that

$$Z_1 = (A-B)Z_2 \ . \tag{26}$$

Thus finally we obtain

$$\tilde{\Gamma}_\mu(p,p,0) = Z_1\gamma_\mu \ , \tag{27}$$

the advertised result. Also we note that in general $\tilde{\Gamma}_\mu(p,p+q,q)$ will not have the simple form of Eq. (27) since the other form factors supplied by $C_{\sigma\mu}$ will appear as well so that there will be terms of the form $\not{p}\gamma_\mu(\not{p}+\not{q})$, etc. In passing we also remark that $\tilde{\Gamma}_\mu(p^2=0, (p+q)^2=0, q)$ is found to be infrared-divergent, as is expected in a massless theory. For completeness we have also calculated $\tilde{\Gamma}_\mu(p,p,0)$ in an arbitrary covariant gauge, and we leave the details to Appendix B.

It is instructive at this point to consider the Schwinger-Dyson equation satisfied by the vacuum polarization,

$$\Pi_{\mu\nu}(x-z) = \lim_{x\to y} \frac{Z_2^2}{Z_1} \mathrm{Tr}\gamma_\mu \tilde{G}_\nu(x,z,y) \ , \tag{28}$$

where the limit is taken symmetrically. Thus from Eqs. (2) and (3) we obtain the relation

$$\frac{Z_2}{Z_1}\left(A-\frac{B}{4}\right)=f(\alpha)\,, \tag{29}$$

so that there is effectively only one form factor at the eigenvalue. Combining Eqs. (26) and (29) we then obtain the relations

$$\begin{aligned}\frac{Z_2}{Z_1}A&=\tfrac{1}{3}[4f(\alpha)-1]\,,\\ \frac{Z_2}{Z_1}B&=\tfrac{4}{3}[f(\alpha)-1]\,.\end{aligned} \tag{30}$$

Thus if we can go to the mass shell in Eq. (25) to recover the familiar $Z_1=Z_2$, we can then infer the interesting fact that both A and B are necessarily nonvanishing at the eigenvalue.

We would like to stress that Eq. (27) has been derived from the fact that j_μ is canonical, and has not required the use of Eq. (29). In a sense we have been working with an abstracted theory in which we only look at Green's functions with external fermion lines and fermion composites such as the electromagnetic current, and have required this theory to display asymptotic conformal invariance. (We recall that the Ward identity holds before we extend the theory to a local gauge.) We can thus discuss the fermion sector in a nonperturbative and non-Lagrangian manner without an explicit photon.[17] It is only when we ask what particular Lagrangian field theory is going to provide us with these Green's functions that we introduce the dimensionless $ej_\mu A^\mu$ interaction by extending the theory to a local gauge and then make a graphical analysis. Then the equation of motion $\Box A_\mu=ej_\mu$ essentially dictates that the photon must be introduced canonically if e is to be finite. Thus finite QED may be thought of as a theory in which a global canonical current remains canonical after extension to a local gauge. Before we introduce the photon explicitly, however, $\tilde{\Pi}_{\mu\nu}$, which is defined by the fermion sector only, satisfies conformal invariance in Eq. (5) without requiring $f(\alpha)=0$. We mention this to indicate that Eq. (27) may then hold in a more general context than finite QED alone.

We turn now to the Bethe-Salpeter equation satisfied by $\tilde{\Gamma}_\mu$,

$$\tilde{\Gamma}_\mu(p,p+q,q)=Z_1\gamma_\mu+\int d^4k\,\tilde{K}(p,k,q)\tilde{S}(k)\times\tilde{\Gamma}_\mu(k,k+q,q)\tilde{S}(k+q)\,. \tag{31}$$

Ordinarily in perturbation theory the finiteness of the left-hand side is achieved by a cancellation of the infinity of Z_1 against the integration over the kernel. In finite QED we make the integration finite by fixing the gauge of the propagators of the internal photons. This is to be contrasted with the conformal bootstrap situation discussed previously in which the integration in Eq. (15) was made finite through anomalous dimensions, with the ansatz of conformal invariance with anomalous dimensions reproducing itself identically in Eq. (15). In finite QED we have strictly canonical dimensions, so that a bare vertex is compatible with conformal invariance in Eq. (31) with nonvanishing Z_1. Thus Eq. (31) is not a nonlinear integral equation for $\tilde{\Gamma}_\mu$, so that there is no vertex bootstrap in finite QED (unless perhaps Z_1 just happens to vanish at the eigenvalue). Moreover, since we have an explicit bare vertex in Eq. (31) we must ask how the conformal ansatz of Eq. (2) reproduces itself in an inhomogeneous Bethe-Salpeter equation. The answer would appear to have to be that when we build the kernel out of $\tilde{\Gamma}_\mu$ and then feed in Eq. (2), we will obtain all together two coupled equations for the different projections such as γ_μ, $\not{p}\gamma_\mu\not{q}$, etc., and these two relations will self-consistently be Eqs. (29) and (26). (Recall again the connection between canonical dimensions [used in Eq. (2)] and current conservation [used in Eq. (25)].) Unfortunately, we are unable to confirm this expected interplay between the A and B form factors since we only have Eq. (22) in parametric form.

It is instructive however, to consider the dressed ladder approximation to Eq. (31) in which we replace the kernel by a single dressed photon coupled to the dressed vertices of Eq. (2). If we only retain the A form factor, we see that Eq. (31) is not satisfied in the ladder approximation since point couplings in the integration (which is now just like the usual lowest-order radiative correction to the vector vertex) generate a nontrivial momentum dependence at arbitrary q_μ which cannot be reproduced by the remaining terms. Thus to reproduce this momentum dependence we will need the B form factor as well, so that already in the ladder approximation we will need an interplay between the form factors. [It is conceivable that the A form factor could satisfy Eq. (31) by itself when we use the exact kernel rather than just the lowest-order approximation. Since we do not know the exact kernel we cannot eliminate this possibility.] This situation should be contrasted with that found in the pseudoscalar Yukawa theory, where the dressed ladder approximation to Eq. (15) is satisfied by Eq. (16). We note that Eq. (16) does not correspond to a point coupling unless $d_F=\frac{3}{2}$, $d_\Theta=3$ (so that $d_M=1$), which was already excluded. Thus $\tilde{\Gamma}_P$ has a nontrivial momentum dependence which is reproduced identically. The only case where the dressed ladder approximation will not be satisfied is when both the meson and its source are canonical, so that the coupling is a point coupling. The only nontrivial theory in which both the meson and its source are canonical is finite QED (since Z_3 renormalizes the charge by itself), so that this is the only case where the naive pointlike A form factor will not suffice. Thus we see that finite QED is not a vertex bootstrap theory, and we see the specific role played by the additional B form factor.[18]

Another interesting formula is obtained by inserting Eq. (27) into Eq. (31), viz.

$$Z_1 \int d^4k \tilde{K}(p, k, 0)\tilde{S}(k)\gamma_\mu \tilde{S}(k) = 0 , \tag{32}$$

which must be satisfied for arbitrary external p. Thus we see that the Bethe-Salpeter kernel is compact and square-integrable in finite QED, or at least its γ_μ projection is compact, at $q_\mu = 0$. This is of course to be expected since Z_1^{-1} is finite at the finite QED eigenvalue. We shall return to this point again below in discussing the absence of Goldstone poles in the axial-vector vertex, but remark only now that a compact kernel at the eigenvalue is in fact welcome, since ultimately we would like to make contact with conventional low-energy quantum electrodynamics, which should contain a positronium bound state.

We discuss next the axial-vector vertex. We define A_5, B_5 analogously to Eq. (2) by putting the γ_5 immediately to the right of the μ index. This leads to

$$\tilde{\Gamma}_{\mu 5}(p, p, 0) = Z_2(A_5 + B_5)\gamma_\mu \gamma_5 . \tag{33}$$

The axial-vector Ward identity then yields

$$\tilde{\Gamma}_{\mu 5}(p, p, 0) = Z_A \gamma_\mu \gamma_5 , \tag{34}$$

where Z_A renormalizes the axial-vector vertex. This time there is no relation

$$\frac{Z_2}{Z_A}\left(A_5 + \frac{B_5}{4}\right) = f(\alpha) \tag{35}$$

unless we invoke chiral symmetry to fix the "axial" vacuum polarization Schwinger term. Alternatively, Eq. (35) should also be expected to follow from the consistency of the axial-vector vertex Bethe-Salpeter equation. However, there is a relation analogous to Eq. (32),

$$Z_A \int d^4k \tilde{K}(p, k, 0)\tilde{S}(k)\gamma_\mu \gamma_5 \tilde{S}(k) = 0 . \tag{36}$$

Equation (36) is of interest since it explains why the existence of a nontrivial solution to the self-consistent equation for the fermion mass does not require the presence of a Goldstone pole in $\tilde{\Gamma}_{\mu 5}$. In finite QED the electron mass is required to arise dynamically (so that m_0 is zero to solve the mass-renormalization problem), and since QED is a closed theory there should, one hopes, be no accompanying Goldstone boson. The authors of Ref. 16 noted that the γ_5 projection of the kernel was noncompact so that the pseudoscalar bound-state problem was not of the Fredholm type. We see now that at the eigenvalue a possible pole in $\tilde{\Gamma}_{\mu 5}$ is avoided by Eq. (36) so that the $\gamma_\mu \gamma_5$ projection of the kernel can be of the Fredholm type and still not produce any Goldstone pole in $\tilde{\Gamma}_{\mu 5}$ even while it satisfies an inhomogeneous equation [similar in structure to Eq. (17) only with a nonzero Z_A and with a denominator equal to unity at $q_\mu = 0$]. These points will be explained in more detail elsewhere.[13]

We conclude this section by noting how the Federbush-Johnson theorem[19] is satisfied in finite QED. All that happens is that all matrix elements of the current j_μ are purely real (i.e., just the equal-time parts of the T products survive at the eigenvalue), while in the Wightman functions j_μ annihilates the vacuum since Z_3^{-1} is finite. In general the Federbush-Johnson theorem states that the n-point T products of a fundamental field will vanish if the imaginary part of the two-point function vanishes. For a composite operator, however, some real parts can survive even if its two-point Wightman function vanishes. This is because of the singularity at equal times that the T product possesses (i.e., the equal-time commutator of j_0 with ψ is independent of j_0). Thus in finite QED $\langle\Omega|T(j_\mu j_\nu j_\sigma j_\tau)|\Omega\rangle$ vanishes, but not $\langle\Omega|T(\psi j_\mu \bar{\psi})|\Omega\rangle$, so that the Federbush-Johnson theorem does not entail the separate vanishing of A and B. We mention this because the discussion of the Federbush-Johnson theorem is obscured by the lack of positivity in certain gauges in QED.[20] Our point is that the real part of $\tilde{\Gamma}_\mu$ is not obliged to vanish even if the Federbush-Johnson theorem can be extended to the charge sector in the finite gauge. Also $\tilde{\Pi}_{\mu\nu}$ is purely real at the eigenvalue. However, the subtlety is that even a real $\tilde{\Gamma}_\mu$ could contribute a logarithmic divergence to $\tilde{\Pi}_{\mu\nu}$ through the Schwinger-Dyson equation to give $\tilde{\Pi}_{\mu\nu}$ a discontinuity. Equation (29) stops this from happening by a cancellation between the form factors. Thus we need more than one form factor in order to satisfy the Federbush-Johnson theorem. Thus in conclusion we again remark that it is this interplay between the form factors of Eq. (2) which has enabled us to maintain consistency in the various situations that we have discussed throughout this work.

APPENDIX A

In this appendix we discuss briefly the interplay of gauge invariance and conformal invariance. As we noted in the Introduction the transversality condition on $\tilde{\Pi}_{\mu\nu}$ is only achieved if j_μ has dimension 3. Thus a canonical photon of dimension 1 would not be transverse. Adler[5] and Abdellatif[6] have resolved this conflict by introducing an extra four-vector b_μ, a gauge point, so that the conformal photon propagator $\langle\Omega|T(A_\mu(x)A_\nu(y))|\Omega\rangle$ becomes a three-point function in an arbitrary conformal gauge, b_μ. Suppose we now try to construct the $\tilde{G}_\mu$ vertex in perturbation theory by considering only graphs with a single continuous electron line dressed with free conformal photons. This $\tilde{G}_\mu(x,z,y)$ will now depend on b_μ as well, so that instead of the previous two form factors of Eq. (2) there will now be six independent form factors.[6,8] Moreover, since we now have four coordinates we can build harmonic ratios, so that the coefficients of these form factors will no longer be pure constants. By using the Wilson expansion Christ[8] was able to eliminate these extra four unwanted form factors and recover Eq. (2) in the special gauge where b_μ is chosen to be z_μ, the coordinate of the external current. This then leads us again to Eq. (24) in this particular gauge but not necessarily to Eq. (27) since in the presence of b_μ the Ward identity corresponds to a different kinematic configuration than that used for the short-distance Wilson expansion analysis. Thus in perturbation theory we obtain slightly less information than deduced in this paper. However, it is a moot point as to whether Christ's discussion should actually be applied at the eigenvalue. As we have indicated, there is a nongraphical formulation of the theory which discusses the fermion sector alone without ever needing to introduce an explicit photon. In this approach we go straight to the eigenvalue and introduce Eqs. (1) and (2) without any need to introduce a gauge point at all, so that there are only two form factors which satisfy Eqs. (26) and (29). This viewpoint, which merits further study, then implies that there are two formulations of conformal finite QED, a perturbative graphical one summed loopwise with free photons, and a nonperturbative one at the eigenvalue, with perturbation theory not being a good guide to the structure of the theory at the eigenvalue itself.

There is one other open question posed by the work of Schnitzer and Christ. At the eigenvalue we have the naive Wilson expansion

$$T(\psi(x)\overline{\psi}(0)) = Z_2^{-1}\frac{\gamma_\lambda}{4} \times\left[C_1(\alpha)j^{\lambda}(0)+C_2(\alpha)\frac{x^\lambda x^\nu}{x^2}j_\nu(0)\right] \tag{A1}$$

plus additional terms which are irrelevant to this discussion. The consistency of this expansion with the structure of Eqs. (2) and (3) (the Crewther analysis) leads to the relations[7,8]

$$A = C_1(\alpha)f(\alpha)\,,\qquad B = -C_2(\alpha)f(\alpha)\,. \tag{A2}$$

Thus from Eqs. (30) (with $Z_1 = Z_2$) we deduce that the Wilson coefficients have to be infinite at the eigenvalue, which is not a desirable situation. (Anomalous infinities in Wilson coefficients will also be obtained if the axial-vector current triangle anomaly is not renormalized at the eigenvalue.[21]) We shall conclude this appendix by presenting a possibility which could remove the undesirable infinities in Eq. (A2). In obtaining Eq. (A1) it is tacitly assumed that at the eigenvalue the naive Wilson expansion still holds and has the structure suggested by perturbation theory, where j_μ is given explicitly by the fermion bilinear $:\overline{\psi}\gamma_\mu\psi:$. However, in our discussion of Eqs. (1) and (2) we never needed to make such an identification for j_μ. Now in general the conformally invariant Green's functions of fermion lines and fermion composites do correspond precisely with the naive forms that would be obtained in a free massless fermion theory, except in the special case of Eq. (2). In a free fermion theory the coefficient B vanishes identically, whereas in finite QED we need a nontrivial B form factor to maintain consistency, as we discussed in this paper. This suggests that at the eigenvalue the current j_μ introduced in Eq. (1) may be represented by a more complicated object than $:\overline{\psi}\gamma_\mu\psi:$, so that there could be possible additional terms in Eq. (A1). Unfortunately for the moment we see no way of testing this proposal, and of determining whether or not the Wilson expansion is reliable in a finite field theory.

APPENDIX B

In this appendix we discuss how Eq. (27) is modified in different gauges. In an arbitrary covariant gauge the electron propagator acquires an anomalous dimension d_F, so that

$$\tilde{S}^{-1}(p) = Z_2\frac{\Gamma(d_F+\frac{1}{2})}{\Gamma(\frac{5}{2}-d_F)}\,2^{2d_F-3}\left(\frac{-p^2}{\mu^2}\right)^{(3-2d_F)/2}\not{p} \tag{B1}$$

[i.e., in an arbitrary gauge the wave-function renormalization constant of the electron is given by $Z_2(\Lambda^2/\mu^2)^{3/2-d_F}$]. Because of this anomalous dimension Eq. (2) also needs to be modified so that

$$\tilde{G}_\mu(x,z,y)=\frac{Z_2^{-1}A}{4\pi^4}\frac{(\hat{y}-\hat{z})}{(y-z)^4}\gamma_\mu\frac{(\hat{z}-\hat{x})}{(z-x)^4}\frac{1}{[(x-y)^2]^{[d_F-3/2]}}\left(\frac{1}{\mu^2}\right)^{d_F-3/2}$$
$$+\frac{Z_2^{-1}B}{4\pi^4}\frac{(\hat{x}-\hat{y})}{[(x-y)^2]^{[d_F-1/2]}(y-z)^2(z-x)^2}\left(\frac{(z_\mu-x_\mu)}{(z-x)^2}+\frac{(y_\mu-z_\mu)}{(y-z)^2}\right)\left(\frac{1}{\mu^2}\right)^{d_F-3/2}. \tag{B2}$$

Thus with our choice of gauge-independent constants Z_2^{-1}, A, and B introduced in Eqs. (2) and (8) we see that the only modifications due to changing the gauge can be absorbed in changing the various powers that appear.

Before we calculate the Fourier transform of Eq. (B2) at zero momentum transfer we point out that this is a problem which has a general solution. Suppose we are given a three-point function which has the form of a product of powers $f(z-x)g(x-y)h(y-z)$. Then

$$\tilde{G}(p,p,0)=\int\frac{d^4k}{(2\pi)^4}\tilde{f}(k)\tilde{g}(k+p)\tilde{h}(k)\ . \tag{B3}$$

Introducing $\tilde{F}(k)=\tilde{f}(k)\tilde{h}(k)$ we can then write $\tilde{G}(p,p,0)$ as a one-parameter Fourier transform of a product of powers

$$\tilde{G}(p,p,0)=\int d^4x\,e^{ip\cdot x}g(x)F(-x)\ , \tag{B4}$$

so that the Fourier transform can be performed analytically. Thus a conformal-invariant three-point function can always be determined analytically at zero momentum transfer.

If we now apply this method to Eq. (B2) using the relation

$$\int d^4x\,\frac{e^{ip\cdot x}}{(-x^2)^\lambda}=i\pi^2 2^{4-2\lambda}\frac{\Gamma(2-\lambda)}{\Gamma(\lambda)}\frac{1}{(-p^2)^{2-\lambda}}\ , \tag{B5}$$

we obtain after some arithmetic

$$\tilde{G}_\mu(p,p,0)=Z_2^{-1}A2^{2-2d_F}\frac{\Gamma(\frac{5}{2}-d_F)}{\Gamma(d_F+\frac{1}{2})}\frac{\gamma^\sigma\gamma_\mu\gamma^\tau}{(-p^2)^{7/2-d_F}}\left[(5-2d_F)p_\sigma p_\tau+(d_F-\tfrac{3}{2})g_{\sigma\tau}p^2\right]\left(\frac{1}{\mu^2}\right)^{d_F-3/2}$$
$$+Z_2^{-1}B2^{3-2d_F}\frac{\Gamma(\frac{5}{2}-d_F)}{\Gamma(d_F+\frac{1}{2})}\frac{\gamma^\sigma}{(-p^2)^{7/2-d_F}}\left[g_{\sigma\mu}p^2-(5-2d_F)p_\sigma p_\mu\right]\left(\frac{1}{\mu^2}\right)^{d_F-3/2}. \tag{B6}$$

We now amputate, using Eq. (B1), to find

$$\tilde{\Gamma}_\mu(p,p,0)=Z_2(A-B)2^{2d_F-3}\frac{\Gamma(d_F+\frac{1}{2})}{\Gamma(\frac{5}{2}-d_F)}\left(\frac{-p^2}{\mu^2}\right)^{(3-2d_F)/2}\left[(\tfrac{5}{2}-d_F)\gamma_\mu+(\tfrac{3}{2}-d_F)\frac{\not{p}\gamma_\mu\not{p}}{p^2}\right]. \tag{B7}$$

Inserting Eqs. (B7) and (B1) into the Ward identity again leads to Eq. (26), as expected since the c-number coefficients are gauge-independent. Thus we obtain finally

$$\tilde{\Gamma}_\mu(p,p,0)=Z_1 2^{2d_F-3}\frac{\Gamma(d_F+\frac{1}{2})}{\Gamma(\frac{5}{2}-d_F)}\left(\frac{-p^2}{\mu^2}\right)^{(3-2d_F)/2}\left[(\tfrac{5}{2}-d_F)\gamma_\mu+(\tfrac{3}{2}-d_F)\frac{\not{p}\gamma_\mu\not{p}}{p^2}\right] \tag{B8}$$

to be compared with Eq. (27).

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the coupling constant. (It is not known whether this actually happens to Z_1 in finite QED.) Though both situations lead to an infinite Z_1^{-1}, the latter situation actually corresponds to a finite theory. Thus there can be theories in which the cutoff can be removed without introducing ultraviolet divergences in which some of the renormalization constants are still zero. Such theories should still be thought of as finite. Also we note that it is unlikely that Z_1 could vanish because of infrared divergences since in a massless theory ultraviolet and infrared divergences are handled simultaneously because of dimensional reasons.

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[15]If the meson were a bound state of mass m, then the coupling of the off-shell meson to an off-shell fermion-antifermion pair would satisfy a Bethe-Salpeter equation with an inhomogeneous term of the form $(q^2 - m^2)Z_P\gamma_5$.

[16]In the language of M. Baker, K. Johnson, and B. W. Lee [Phys. Rev. 133, B209 (1964)] the bound-state problem is not of the Fredholm type. Thus Z_P vanishes because there is a nontrivial infinite renormalization in the theory (the one which gave the anomalous dimensions in the first place), and consequently the γ_5 projection of the kernel is noncompact and hence non-Fredholm.

[17]Experience with the Thirring model indicates that the anomalous dimension of the fermion has no physical significance even in a theory which possesses only a global gauge invariance and has no local gauge at all. See, e.g., G. F. Dell'Antonio, Y. Frishman, and D. Zwanziger, Phys. Rev. D 6, 988 (1972). Thus we can build the whole of the fermion sector by postulating Eqs. (1) and (2) only as an ansatz without needing to refer to the photon at all.

[18]In a recent paper F. Englert, J. M. Frère, and P. Nicoletopoulos [Nuovo Cimento 19A, 395 (1974)] tried to set up a vertex bootstrap in QED to obtain the condition $Z_1 = 0$. However, it appears that they only retained the A form factor in the finite gauge, so that $\tilde{\Gamma}_\mu$ has no momentum dependence. Thus we do not share their conclusion that $Z_1 = 0$ is consistent with Eq. (31) in the finite gauge.

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