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Particle spectrum in model field theories from semiclassical functional integral techniques*

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We have used a semiclassical method developed earlier to compute the particle spectrum of a field theory in two-dimensional space-time defined by the (sine-Gordon) Lagrangian $\frac{1}{2}(\partial_\mu\phi)^2 + (m^4/\lambda)\{\cos[(\sqrt{\lambda}/m)\phi]-1\}$. For weak coupling we find a heavy particle, the soliton, corresponding to a peculiar classical field configuration and an antisoliton. Below the soliton-antisoliton threshold there are a large number of further states. They can be viewed either as soliton-antisoliton bound states or as bound states of n of the usual quanta of the theory. The "elementary particle" ϕ is the lowest of these. As the coupling increases, the higher states successively unbind, decaying into soliton-antisoliton pairs. At $\lambda/m^2 = 4\pi$, the "elementary particle" unbinds leaving only solitons and antisolitons for $\lambda/m^2 > 4\pi$. Comparing our semiclassical results with recent exact results of Coleman and with perturbation theory, we find that the semiclassical calculations are *exact*. This field theory seems similar to the hydrogen atom for which the Bohr-Sommerfeld quantization rules give the energy levels exactly. We also treat a ϕ^4 theory in weak coupling and carry out a number of calculations which provide nontrivial illustrations of the semiclassical method.

I. INTRODUCTION

In an earlier paper,[1] which we will refer to as I, we developed a semiclassical WKB method which can be applied to problems in quantum field theory. The method is based on the ideas of Keller,[2] Gutzwiller,[3] and Maslov[4] and requires as input a knowledge of the solutions of the classical field equations. Since already at a classical level the field equations are difficult to handle, one generally has to resort to further approximations. For weak coupling, it was found that time-independent classical solutions are interesting. Basically what happens is that for weak coupling, a stable time-independent solution plus the small oscillations around it provide a list of classical solutions which is sufficient for application of the semiclassical quantization procedure. In another paper,[5] which we will refer to as II, we studied a two-dimensional ϕ^4 theory in this weak-coupling approximation. It was found, among other things, that a certain static, particlelike solution (the "kink") of the classical field equations turns into a heavy quantum-mechanical particle. Several other interesting static solutions are known.[6]

In the weak-coupling limit our WKB quantization of static solutions to classical field equations is equivalent to a number of other schemes.[7-9] The difference comes when one contemplates classical motions which cannot be reduced to a time-independent field. That such solutions are interesting should be obvious from the fact that the Bohr orbits of hydrogen are *not* time-independent solutions to classical equations of motion but rather are motions which are (multiply) periodic in time. The real power of the WKB method is the quantization of motions analogous to Bohr orbits.

To find an example of how the semiclassical method would work in field theory we have studied the sine-Gordon equation in one space and one time dimension.[10-13] It is defined by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+\frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\,\phi\right)-1\right], \tag{1.1}$$

and is completely solvable at the classical level: There exists an algorithm[10-12] from which *all* solutions to the Lagrange equations for ϕ can be constructed. For this system we can therefore try to apply the general methods developed in I. In particular, we look for classical solutions which will become particles when quantized. There are two types of these. First there is the soliton (and an antisoliton) which is a solution that is time-independent in its rest frame. It is analogous to the kink discussed in II. The other which we call the doublet is, loosely speaking, a soliton-antisoliton bound state. In its rest frame the doublet field oscillates periodically in time. Doublet solutions exist for a continuous range of classical energies. The WKB method will quantize the doublet energies yielding a discrete spectrum of particle masses.

For strong coupling there are some subtleties in the calculations. They are discussed in context in Secs. II and III. However, if we take our semiclassical calculations at face value, the results are quite remarkable. It makes one wonder what might happen in other more complicated field theories if only one could solve them. Actually Skyrme,[13] who worked extensively on the sine-Gordon equation, had already noted that the quantized theory could have usual properties.

The particle spectrum of the sine-Gordon Hamiltonian turns out to be the following. The soliton and antisoliton have a mass $M = 8m/\gamma'$, where $\gamma' = (\lambda/m^2)(1 - \lambda/8\pi m^2)^{-1}$. The doublet produces the remaining series of states at masses

$$M_n = \frac{16m}{\gamma'} \sin\frac{n\gamma'}{16}, \quad n = 1, 2, 3, \ldots < 8\pi/\gamma'. \tag{1.2}$$

The original "elementary particle" of the theory is the $n = 1$ state in Eq. (1.2). As $\lambda \to 0$, γ' vanishes and one easily sees that M_1 approaches the weak-coupling mass, $m + O(\lambda^2)$, of the "elementary" particle. Notice that according to Eq. (1.2) there is a finite number of doublet states. As the coupling γ' increases the states disappear one by one. What happens is that they decay into soliton-antisoliton pairs. This may be seen by observing that when the nth state disappears M_n is just $16m/\gamma'$ or twice the soliton mass. At $\gamma' = 8\pi$, the $n = 1$ or "elementary particle" state breaks up and disappears[14] from the spectrum. Only solitons and antisolitons remain.

The weak-coupling behavior of M_n is quite interesting. Expanding, one finds

$$\begin{aligned} M_n &= nM_1 - \frac{M_1}{6}\left(\frac{\lambda}{16m^2}\right)^2 (n^3 - n) + O(\lambda^3), \\ M_1 &= \frac{16m}{\gamma'} \sin\frac{\gamma'}{16} \\ &= m\left[1 - \frac{1}{6}\left(\frac{\lambda}{16m^2}\right)^2\right] + O(\lambda^3), \end{aligned} \tag{1.3}$$

which corresponds to a nonrelativistic n-body bound state made up of n particles with physical mass M_1. The binding energy is $(M_1/6)(\lambda/16m^2)^2 \times (n^3 - n)$.

This is the same as that which one finds upon solving the n-body Schrödinger equation with the δ-function potential obtained from the ϕ^4 term in the interaction Lagrangian. Thus for weak-coupling the doublet states can be thought of as bound states of n "elementary particles." Of course n cannot be too big. When $\gamma' n$ is greater than 8π, the state breaks up into a soliton-antisoliton pair. In fact for $\gamma' n$ large (but less than 8π) the states are probably best thought of as soliton-antisoliton bound states.

The semiclassical calculation suggests that all states with $\gamma' n$ less than 8π are stable. The mass ratios as given by Eq. (1.2) and the symmetry of the Lagrangian under $\phi \to -\phi$ account for the stability of the $n = 1$, 2, and 3 states. It will take further symmetry to keep the $n = 4$ state from decaying into two $n = 1$ states. At a classical level the sine-Gordon equation has an infinite number of nontrivial conserved quantities.[10-13] If these survive in the quantum theory, they could provide enough quantum numbers to stabilize the states with $n \geq 4$. We have checked that the matrix element for decay of the $n = 4$ state does, in fact, vanish to leading order in λ.

We have also extended our work on the ϕ^4 theory in two dimensions. This system is not exactly solvable. For a small coupling, however, one can find the analog of the sine-Gordon doublet states. We obtain a formula like (1.3) with a different coefficient of $n^3 - n$. The interpretation is the same except that we no longer know what happens for strong coupling. It is a reasonable speculation, however, that for large λn the states break up into a kink-antikink pair. Although our results for the ϕ^4 theory are neither as complete nor as elegant as those for the sine-Gordon case, we regard this calculation as important. It shows that the method is not restricted to special, classically solvable equations such as the sine-Gordon system. Ultimately, of course, one would like to work on realistic field theories in three space dimensions. This will require approximate methods for constructing time-dependent solutions to the classical field equations. The ϕ^4 calculation is a step in this direction.

Coleman[15] has recently obtained the remarkable result that the sine-Gordon system can be mapped into the massive Thirring model. The relationship between the sine-Gordon coupling λ and the four-fermion coupling g of the Thirring model is $\lambda/4\pi m^2 = 1/(1 + g/\pi)$ or $\gamma' = 8\pi/(1 + 2g/\pi)$. What are the fermions? They are almost certainly the solitons. To see this, we observe that at $\gamma' = 8\pi$ the Thirring model coupling g vanishes. This is just the point where the $n = 1$ state unbinds. For γ' slightly less than 8π, the four-fermion coupling is weak and attractive. There will then be one nonrelativistic fermion-antifermion bound state. Summing diagrams in the Thirring model one finds that through order g^3, the mass M_B of the bound state is given in terms of the fermion mass M_f by

$$\frac{2M_f - M_B}{M_f} = g^2 - \frac{4g^3}{\pi} + O(g^4). \tag{1.4}$$

Identifying M_B with M_1 and M_f with the soliton mass $8m/\gamma'$ we compare this to

$$\begin{aligned} \frac{2M(\text{soliton}) - M_1}{M(\text{soliton})} &= 2\left(1 - \sin\frac{\gamma'}{16}\right) \\ &= \left(\frac{\gamma' - 8\pi}{16}\right)^2 + O\big((\gamma' - 8\pi)^4\big) \\ &= g^2 - \frac{4g^3}{\pi} + O(g^4), \end{aligned} \tag{1.5}$$

where we have used Coleman's identification of the coupling constants. It is remarkable that both the g^2 and g^3 terms agree. We have not computed beyond order g^3 in the Thirring model. For $\gamma' > 8\pi$, the four-fermion coupling is repulsive and there is no bound state.

Do the solitons which appear in the classical boson field ϕ really obey Fermi statistics? The answer is presumably yes. Our formalism is capable of giving a definite answer. We would have to compute a sign associated with one of the exchange orbits discussed in Sec. V of I. For technical reasons this calculation is difficult and we have not carried it through.

Coleman also finds that the theory is singular at $\lambda/m^2 = 8\pi$. At this point γ' goes to infinity and it is evident that our semiclassical solution is also singular.

The agreement between our approximation and Coleman's precise results suggests to us that WKB may be exact for the mass spectrum of the sine-Gordon equation. This is not beyond the realm of possibility. Recall that the Bohr-Sommerfeld quantization conditions give the energy levels of hydrogen exactly. To investigate this question, we have gone to the weak-coupling regime and carried out an exact calculation of M_2/M_1 through order $(\lambda/m^2)^4$. This is done by summing Feynman diagrams in a way which is equivalent to solving the Bethe-Salpeter equation. The exact result is

$$\frac{2M_1 - M_2}{M_1} = \left(\frac{\lambda}{16m^2}\right)^2 + \frac{4}{\pi}\left(\frac{\lambda}{16m^2}\right)^3 + \left(\frac{12}{\pi^2} - \frac{1}{12}\right)\left(\frac{\lambda}{16m^2}\right)^4 + O(\lambda^5). \quad (1.6)$$

One can easily calculate the same quantity using Eq. (1.2) for M_1 and M_2. Expanding, one finds that the coefficients of λ^2, λ^3, and λ^4 are identical. This is a highly nontrivial result: To get the exact order-λ^4 term one has to keep two-loop diagrams in the kernel of the Bethe-Salpeter equation. We can show that the agreement in order λ^4 is special to the sine-Gordon equation and will not occur in the generic case.

For the sine-Gordon equation our method appears to be giving exact results for both weak and strong couplings. It may come as a surprise to some readers that a semiclassical method can give reliable, let alone exact, answers for strong coupling. In the sine-Gordon Lagrangian the dimensionless coupling parameter is really $\hbar\lambda/m^2$. A straightforward expansion in powers of $\hbar$ is therefore the same thing as the perturbation expansion in powers of λ. This is familiar from the usual loop expansion. The WKB method is something else, the nature of which is best seen in examples. First we will take a typical case where WKB is not exact. The anharmonic oscillator defined by the Lagrangian $L = \dot{x}^2 - \omega^2 x^2 - \lambda x^4$ may be thought of as a field theory in one time and no space dimension. The dimensionless coupling constant corresponding to $\hbar\lambda/m^2$ in the sine-Gordon Lagrangian is $G = \hbar\lambda/\omega^3$. On dimensional grounds we know that the energy levels can be written as $E_n = \hbar\omega f_n(G)$, $n = 0, 1, \ldots$. One sees immediately that $f_n(0) = n + \frac{1}{2}$, and a simple scaling argument shows that $f_n \sim c_n G^{1/3}$ as $G \to \infty$. The WKB approximation to f_n is easily found to be the solution of

$$\frac{2}{\pi}\int_{-y_0}^{y_0} [f_n^{\mathrm{WKB}}(G) - y^2 - Gy^4]^{1/2} dy = n + \tfrac{1}{2}, \quad (1.7)$$

where y_0 and $-y_0$ are the turning points. It is evident that WKB is not an expansion in powers of G. Actually, the WKB energies are reasonably good for any G. At $G=0$, f_n^{WKB} is equal to $n + \frac{1}{2}$ and for large G it scales in the same way as the exact f_n, giving $f_n^{\mathrm{WKB}} \sim c_n^{\mathrm{WKB}} G^{1/3}$ as $G \to \infty$. Comparing the exact[16] (numerical) and approximate c's one finds $c_0^{\mathrm{WKB}}/c_0 = 1.223$, $c_1^{\mathrm{WKB}}/c_1 = 1.013$, and $c_2^{\mathrm{WKB}}/c_2 = 1.005$. Evidently, WKB gives meaningful results even in the extreme strong-coupling limit $G \to \infty$. As another example, we could consider a Lagrangian of the form $L = \dot{x}^2 - (\omega^2/g^2) \times (e^{-gx} - 1)^2$. In this case WKB gives the bound-state energies exactly,[17] for any value of the coupling g. The sine-Gordon equation seems to be one of these special systems where WKB is exact.

It is amusing to find that WKB gives some exact results for the sine-Gordon equation. However, the real goal of our program is to develop an approximate method which can be used for physically interesting theories in four dimensions. As can be seen from the above examples, the WKB method is reasonable for strong-coupling problems in ordinary quantum mechanics. Here one does not expect WKB to yield highly accurate numbers: For a potential which is reasonably smooth one expects errors of order 20%. Still, an approximate method accurate to order 20% would be most useful in strong-coupling field theory. To go to field theory, one has to convince oneself that a semiclassical approximation continues to make sense as the number of degrees of freedom becomes infinite. In view of the fact that our sine-Gordon results are exact, it is hard to see how there could be any unforeseen trouble here.

This paper is not meant to be self-contained. The reader is expected to have some familiarity with the contents of I and II. Also we do not discuss the classical mechanics of the sine-Gordon equation in any detail. There exists an extensive literature on this subject.[10-13]

The plan of this paper is as follows. In Sec. II we review those particular features of the sine-Gordon equation which will be important to us. In Sec. III we go through the quantization of the doublet. This is done in a fairly systematic way, which we hope will clarify some of the assertions made in I. We also take care to point out those things which are peculiar to the sine-Gordon equation. This section should therefore serve as a guide to semiclassical quantization in general. Some of the longer specialized calculations have been placed in Appendixes A, B, and C. In Sec. IV we discuss the interpretation of the particle spectrum. Two Appendixes, D and E, deal with the related problems of soliton-antisoliton scattering and the derivation of Eqs. (1.4) and (1.6). Finally, the calculations for the ϕ^4 theory are done in Sec. V.

II. THE SINE-GORDON EQUATION

In this section we set down some properties of the sine-Gordon system defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right], \tag{2.1}$$

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2\phi^2}{2!} + \frac{\lambda\phi^4}{4!} + \cdots . \tag{2.2}$$

As is obvious from Eq. (2.2) the system reduces to an attractive ϕ^4 model in the limit of small coupling λ. All of the following is understood to be in two-dimensional space-time. It is extremely suggestive that any insight we obtain in dealing with the quantization of modes of the sine-Gordon equation would also be used as a strategic starting point for investigating ϕ^4, and we will do so.

The sine-Gordon system is remarkable on a classical level for a variety of reasons which is usually summarized by saying it is a perfect system. By that one means the following: (a) It supports isolated wave modes which have the property that in the scattering of two of them, not only is the interaction available in exact analytic form, but they emerge from the scattering region with their shape and velocity unchanged. This is usually called the soliton condition. (b) Such a dynamics must have powerful constraints built into it, and this fact is reflected in the existence of two equivalent schemes for generating all the solutions to the sine-Gordon system. The first is the reduction of the problem to an associated one which involves a linear eigenvalue problem. This goes under the name of the inverse scattering method. The second is the existence of a nonlinear superposition principle by which any two solutions induce a third one. This is called the Bäcklund transformation and is a kind of creation operation for the system. (c) The sine-Gordon equation is completely separable in the Hamiltonian sense; that is, the complete integral of the Hamilton-Jacobi equations is exactly known. We will use this fact to isolate and quantize the particlelike modes of the sine-Gordon equation.

A. The soliton mode

Classically, it is well known that

$$\phi^{\pm}(x,t) = \frac{4m}{\sqrt{\lambda}}\tan^{-1}\left[\exp\left(\pm m\,\frac{x-vt}{(1-v^2)^{1/2}}\right)\right] \tag{2.3}$$

are exact solutions to the sine-Gordon equation. The + sign refers to the soliton solution and the − sign to the antisoliton solution. They look like isolated particlelike states. For the soliton, $\phi(x=+\infty)-\phi(x=-\infty)=2\pi m/\sqrt{\lambda}$; for the antisoliton, the corresponding quantity is $-2\pi m/\sqrt{\lambda}$.

If we go to the rest frame ($v=0$) the soliton solution is time-independent and we can quantize it by using the methods of paper II. Proceeding exactly as we did for the so-called "kink" mode of paper II, we find that the soliton or antisoliton behaves like a particle with mass

$$M(\text{soliton}) = \frac{8m^3}{\lambda} - \frac{m}{\pi} . \tag{2.4}$$

The first term on the right is the classical mass. The second is the contribution of small oscillations around the static classical solution. Subtractions appropriate to the vacuum energy and mass counterterm δm^2 have been done in exactly the same way as in II. We can rewrite (2.4) in an obvious way as

$$M(\text{soliton}) = \frac{8m}{\gamma'}, \quad \gamma' = \frac{\lambda}{m^2}\left(1 - \frac{\lambda}{8\pi m^2}\right)^{-1} \tag{2.5}$$

which serves to define a dimensionless coupling parameter γ'. The significance of γ' is discussed in subsections D and E of Sec. III.

Since at the classical level all the solutions to the sine-Gordon equation are available, we can in particular generate all the time-dependent ones, as well as the static ones discussed above. We therefore look for further, nontrivial particlelike solutions. For reasons that will become clear as we progress we choose to quantize the doublet or breather mode displayed below in Eq. (3.1). It corresponds to a set of soliton-antisoliton bound states, with periodic time dependence in the rest frame. Quantizing it is a nontrivial application of the semiclassical path-integral formalism. This is done in the following section. It turns out that there are no further particlelike solutions. Finally, we remark that for solutions bounded at spatial infinity

$$\phi(t,\infty)-\phi(t,-\infty)=\left(\frac{m}{\sqrt{\lambda}}\right)2N\pi\,, \tag{2.6}$$

where the integer N is independent of time. The conserved quantity N can be thought of as the number of solitons minus the number of antisolitons. The doublet solution has N equal to zero.

B. Separation of variables[18]

In order to display the canonical structure of the sine-Gordon equation, Faddeev and Takhtajan and McLaughlin have shown[11] that if one writes the nonlinear problem as a Hamiltonian system, one can show that it is integrable. That is, the linear eigenvalue problem of the inverse scattering method is interpreted as a canonical transformation which takes the original Hamiltonian system to an "action-angle" form. The Hamiltonian is then independent of the angle coordinates q and the equations of motion trivially integrate. One finds that the q's vary linearly with time and their canonical momenta are constants of the motion.

In action-angle variables the Lagrangian has the form

$$L=\int \mathcal{L}(\dot{\phi}(\lambda))\,d\lambda+\sum_{l}L_S(\dot{q}_l)+\sum_{k}L_D(\dot{\phi}_k,\dot{\eta}_k)\,, \tag{2.7}$$

the variable q_l is related to the position of the lth soliton or antisoliton mode, and the variables ϕ_k and η_k describe the position and internal motion of the kth doublet in the model. The integral represents the continuous background motion of the field. This is the classic separated form.[18] The importance of this form for the quantization of the various modes lies in its extreme simplicity. The soliton solution is gotten by solving the Lagrange equations of motion for a particular q_l, setting all the other variables to zero. This is consistent with the equations of motion. In quantizing the soliton mode we isolate the class of periodic orbit belonging to the linearized problem around that mode, plus all the copies of that orbit as in paper I. We never compute the absolute energy of a soliton, but only the difference in energy between a soliton and the vacuum. In a separable system this is a rigorous application of the semiclassical method.[19]

If we could perform all calculations so as to s strictly maintain the separability structure of the classical sine-Gordon system the soliton mass would be exact within the WKB approximation.[19] The same comment would be true for the doublet mode considered in Sec. III. For the doublet we solve the Lagrange equations for a particular ϕ_k and η_k, setting all other coordinates to zero. Then we proceed to compute the difference between the energy of the doublet and that of the vacuum and the argument goes through as before.

However, to make a well-defined calculation we are forced to put the system in a finite box, with ultraviolet cutoff. Both the size of the box and the cutoff are sent to infinity at the end of the calculation and all measurable quantities are independent of them. Unfortunately, finite boundaries and ultraviolet cutoffs disturb the exact separability of the sine-Gordon equation in a presently unknown way. To do the problem correctly we would have to solve the inverse scattering problem for periodic boundary conditions, which is at present unavailable. So, some modification of the above line of argument has to be made. We will return to this point in Sec. III E.

III. QUANTIZING THE DOUBLET: AN EXAMPLE OF THE SEMICLASSICAL METHOD

In this section we follow through, in outline, the WKB quantization of the doublet solution to the sine-Gordon equation. We hope that the basic steps are stated reasonably clearly and that what follows will serve as a concrete example of the ideas put forward in I.

Unfortunately, there are some aspects of the problem which require long, detailed calculations. These calculations are rather special and it was felt best to put them in a set of Appendixes A–C.

When a certain result is peculiar to the sine-Gordon equation we will state this and indicate what can be expected in the general case. Thus, we hope that this section will serve as a guide to semiclassical quantization in general.

A. The periodic orbits

The semiclassical or WKB method developed in I begins with a functional integral representation for $\mathrm{tr}\,e^{-iHT}$. To make the trace well behaved we always imagine that T has a small negative imaginary part and take a finite volume with periodic boundary conditions. For one space dimension, periodic boundary conditions are equivalent to imagining that the world is a large closed loop with a perimeter of length L.

The functional integral for $\mathrm{tr}\,e^{-iHT}$ is evaluated in a stationary-phase approximation. We are integrating over a function space and the stationary-phase points are those functions which satisfy the classical equations of motion and are periodic in time with period T. They are the periodic solutions or orbits of the classical field equations. In the WKB approximation $\mathrm{tr}\,e^{-iHT}$ is a sum of terms, one from each periodic orbit.

To quantize the doublet solution to the sine-Gordon equation, we need to consider the class of solutions

$$\phi_{\tau,v}(x,t)=\frac{4m}{\sqrt{\lambda}}\tan^{-1}\left\{\left[\left(\frac{\tau m}{2\pi}\right)^2-1\right]^{1/2}\frac{\sin[(2\pi/\tau)((t-vx)/(1-v^2)^{1/2})]}{\cosh[((\tau m/2\pi)^2-1)^{1/2}(2\pi/\tau)((x-vt)/(1-v^2)^{1/2})]}\right\}\qquad(\tau>2\pi/m) \tag{3.1}$$

corresponding to the basic doublet boosted to velocity v. The parameter τ is the period of the doublet in its rest frame. In our closed-loop world, these solutions are periodic if

$$T=\frac{l\tau}{(1-v^2)^{1/2}}=\frac{nL}{v},\quad l,n=0,1,2,\ldots,\infty\,. \tag{3.2}$$

For each pair of integers l and n, Eq. (3.2) can be thought of as fixing v and τ as functions of T. The classical action for these solutions is

$$\int_0^T dt\int_{-\infty}^{\infty}dx\left\{\frac{(\partial_\mu\phi_{\tau,v}(x,t))^2}{2}+\frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi_{\tau,v}(x,t)\right)-1\right]\right\}$$
$$=l\bar{S}(\tau)=l\,\frac{32\pi m^2}{\lambda}\left\{\cos^{-1}\left(\frac{2\pi}{m\tau}\right)-\left[\left(\frac{m\tau}{2\pi}\right)^2-1\right]^{1/2}\right\}. \tag{3.3}$$

The relevant stationary-phase points of the functional integral are thus labeled by pairs of integers l and n. The WKB approximation to $\mathrm{tr}e^{-iHT}$ is explicitly

$$\mathrm{tr}e^{-iHT}=\sum_{ln}e^{il\bar{S}(\tau)}D_{ln}\,,$$

$$D_{ln}=\int\mathfrak{D}\{\psi\}\exp\left\{i\int_0^T\int_{-\infty}^{\infty}dt\,dx\left[\frac{1}{2}(\partial_\mu\psi)^2-\frac{m^2}{2}\psi^2\cos\frac{\sqrt{\lambda}}{m}\phi_{\tau,v}\right]\right\}, \tag{3.4}$$

where, for each l and n, we recognize the exponential of i times the classical action multiplied by a Gaussian functional integral around the classical orbit. The functional integral is over all functions $\psi(x,t)$ which are periodic, $\psi(t+T,x)=\psi(t,x)$, in time. One should remember that τ and v depend on l, n, and T according to Eq. (3.2).

B. The Gaussian integral

The functional integral in Eq. (3.4) is evaluated in Appendixes A, B, and C. Appendix A works out a general formalism while the latter two contain the specific calculations.

If $\phi_{\tau,v}$ were time-independent in its rest frame we could evaluate D in terms of the eigenfrequencies ω_α of a harmonic-oscillator system defined by the linearized Lagrangian in (3.4). For a periodic system, the analogs of the frequencies ω_α are the stability angles ν_α. They are defined in terms of special solutions to the linear equations for ψ following from the Lagrangian in (3.4). In the rest frame of the doublet these special solutions satisfy

$$\psi_\alpha(t+\tau,x)=e^{-i\nu_\alpha}\psi_\alpha(t,x)\,. \tag{3.5}$$

The precise analogy with harmonic-oscillator frequencies is evidently $\nu_\alpha\sim\omega_\alpha\tau$. In a finite box, the ν_α's are discrete. As the length of space L tends to infinity they form a continuum for $|\nu_\alpha|>m\tau$. There are discrete ("bound-state") angles at $\nu=0$ corresponding to the symmetries of the system. In general, there will be a further discrete spectrum of ν's in the interval $0<|\nu|<m\tau$. For the sine-Gordon doublet, however, there is no discrete spectrum other than at $\nu=0$. This leads to a considerable simplification. In Appendix A it is shown that when there is no discrete spectrum other than at $\nu=0$, then for the purpose of finding the particle spectrum of the theory a rather complicated factor can be dropped from D. Dropping this factor yields the simple result derived in Appendixes B and C,

$$D_{ln}\to\frac{1}{2\pi}\left[\frac{\tau}{l^{1/2}}\left|\frac{d^2\bar{S}}{d\tau^2}\right|^{1/2}\right]\left[L\left|\frac{-d\bar{S}/d\tau}{T(1-v^2)^{3/2}}\right|^{1/2}\right]e^{il\xi(\tau)},$$
$$\xi(\tau)=-\tfrac{1}{2}\sum_{\nu_\alpha>0}\nu_\alpha\,. \tag{3.6}$$

It is shown in Appendixes A and B that, apart from the factor $e^{il\xi(\tau)}$, the form of D is dictated by space and time translation invariance. This form already appeared in Eq. (4.12) of I and was discussed there. The factor $e^{il\xi(\tau)}$ is clearly the analog of the factor $\exp[-(i/2)\sum_\alpha\omega_\alpha T]$ involving harmonic-oscillator frequencies which shows up when one computes around a time-independent classical solution.

The sum over stability angles in ξ is divergent and must be renormalized. This will be done below. The rest of D is manifestly finite.

C. Renormalization

In Appendix C it is shown that ξ can be written as

$$\xi(\tau) = -\frac{T}{l}\Delta E - \frac{16\pi}{\lambda}\left[\left(\frac{m\tau}{2\pi}\right)^2 - 1\right]^{1/2}\delta m^2 + \bar{\xi}(\tau), \tag{3.7}$$

where ΔE is quadratically divergent constant proportional to the length of space L and is independent of τ, δm^2 is a logarithmically divergent integral

$$\delta m^2 = \frac{-\lambda}{4\pi}\int_0^{(\text{cutoff})} \frac{dk}{(k^2+m^2)^{1/2}}, \tag{3.8}$$

and $\bar{\xi}$ is finite. It is easy to dispose of the term $T\Delta E$. It is the vacuum bubble shown in Fig. 1 and can be dropped if we agree to measure energies relative to the vacuum.

The coefficient of δm^2 in (3.7) is

$$-\frac{16\pi}{\lambda}\left[\left(\frac{m\tau}{2\pi}\right)^2 - 1\right]^{1/2} = \frac{m^2}{\lambda l}\int_0^T \int_{-\infty}^{\infty}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi_{v,\tau}(x,t)\right) - 1\right] dx\, dt. \tag{3.9}$$

With the aid of this formula, we can interpret δm^2 as follows. The Lagrangian in the functional integral that we started with really contained the bare mass m_0 rather than a physical mass parameter m. Therefore, we should have actually been computing our classical periodic orbits from

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 + \frac{m^4}{\lambda}\left[\cos\frac{\sqrt{\lambda}}{m}\phi - 1\right] - \delta m^2\frac{m^2}{\lambda}\left[\cos\frac{\sqrt{\lambda}}{m}\phi - 1\right]. \tag{3.10}$$

Let $\phi(m^2 - \delta m^2)$ be a periodic orbit associated with the Lagrangian (3.10) and $\phi(m^2)$ be the corresponding orbit computed without the δm^2 counterterm.

The difference in actions is

$$S(m^2 - \delta m^2) - S(m^2) = S(\phi(m^2-\delta m^2),\ m^2 - \delta m^2) - S(\phi(m^2), m^2), \tag{3.11}$$

where we have noted that S depends on m^2 both explicitly because $\mathcal{L}$ depends on m^2 and implicitly because the classical orbit depends on m^2. However, since the action is stationary against small variations in the fields ϕ, we have to lowest order in δm^2

$$\begin{aligned} S(m^2 - \delta m^2) - S(m^2) &= S(\phi(m^2), m^2 - \delta m^2) - S(\phi(m^2), m^2) \\ &= -\delta m^2\frac{m^2}{\lambda}\int_0^T\int_{-\infty}^{\infty}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi(m^2)\right) - 1\right] dx\, dt. \end{aligned} \tag{3.12}$$

Therefore, if we had started the calculation using the bare mass $m_0^2 = m^2 - \delta m^2$ and computed classical orbits and actions $S(m_0^2)$ in terms of m_0, the δm^2 term in ξ would have just combined with $S(m_0^2)$ to give the classical action $S(m^2)$ computed with the physical parameter m^2. Therefore, we can also drop the second term in (3.7), leaving only the finite piece $\bar{\xi}$.

The above argument is correct to first order in δm^2. Within the spirit of our WKB approximation we are justified in working only to this order. The counterterm δm^2 is of order $\hbar$ relative to m^2. (This would be explicit had we not set $\hbar = 1$.)

The expression in (3.8) for δm^2 is precisely the one-loop normal-ordering graph shown in Fig. 2. The sine-Gordon Lagrangian is super-renormalizable and all divergences are removed by setting

$$m_0^2\left(\frac{m^2}{\lambda}\right)\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right] = m^2\left(\frac{m^2}{\lambda}\right) : \left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right]:, \tag{3.13}$$

which is possible because $[\cos((\sqrt{\lambda}/m)\phi) - 1]$ is an overall factor times its normal-ordered counterpart. To lowest order in λ, $m^2 - m_0^2$ is given by (3.8). This simple renormalization structure is responsible for the simplicity of the divergence in our sum over stability angles.

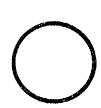

FIG. 1. The vacuum bubble corresponding to the $T\Delta E$ term in Eq. (3.7).

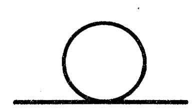

FIG. 2. The mass-renormalization graph corresponding to the δm^2 term in Eq. (3.7).

The parameter m^2 is still not the mass of a propagating particle. There are finite mass-renormalization graphs like that in Fig. 3 which have not been taken into account. Within our WKB approximation, what we have to do is to compute the mass spectrum of the theory and then see how m^2 is related to the mass of a particle. When we do this we will have made an approximate summation of all the finite mass-renormalization graphs.

D. A digression on renormalization in other models

It is appropriate at this point to leave the sine-Gordon equation for the moment and discuss renormalization in general. The above computation of $S(m^2-\delta m^2)-S(m^2)$ is a special case of the following general result. Consider a Lagrangian $\mathcal{L}(g)$ which depends explicitly on some parameter g which might be a mass or a coupling constant. The periodic orbits obviously depend on g, but the variational principle implies that for the derivative of the action

$$\frac{d}{dg}S = \int \frac{\partial}{\partial g}\mathcal{L}(g) \tag{3.14}$$

only the *explicit* dependence of $\mathcal{L}$ on g matters.

In a renormalizable theory the only divergences that can appear in the sum over stability angles are those which can be identified with replacement of a parameter g in the Lagrangian by $g-\delta g$. That this is true may be seen by noting that, except for the periodicity condition on ψ, the functional integral in (3.4) is the same as the functional integral for the usual loop functional. The periodicity condition will not affect the ultraviolet divergences for the same reason that in statistical mechanics a finite temperature does not produce new ultraviolet infinities.

Since every possible divergence will correspond to a shift $g \to g-\delta g$ in $\mathcal{L}$, one can always use (3.14) and proceed exactly as we did for the mass renormalization in the sine-Gordon equation. After removing the infinities in this way, it is consistent with the WKB approximation to assume that the classical orbits, the action, and the finite part of the determinant D in (3.4) are all computed with some set of finite parameters g. The values of these parameters are then to be fixed in terms of physical quantities (e.g., the mass spectrum), as computed in the WKB approximation.

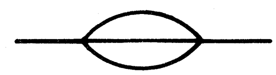

FIG. 3. A finite mass-renormalization graph of order λ^2.

E. Quantization of the doublet

The sine-Gordon Lagrangian has the remarkable property that the finite part $\bar\xi$ of ξ is just (see Appendix C)

$$\bar\xi(\tau) = -\frac{\lambda}{8\pi m^2}\bar S(\tau). \tag{3.15}$$

We can therefore write

$$\bar S(\tau)+\bar\xi(\tau) = \frac{32\pi}{\gamma'}\left\{\cos^{-1}\left(\frac{2\pi}{m\tau}\right) - \left[\left(\frac{m\tau}{2\pi}\right)^2 - 1\right]^{1/2}\right\}, \tag{3.16}$$

$$\gamma' = \frac{\lambda}{m^2}\left(1-\frac{\lambda}{8\pi m^2}\right)^{-1},$$

which clearly just amounts to a finite renormalization of the dimensionless parameter λ/m^2. This renormalized parameter γ' has already appeared in the equation (2.5) for the soliton mass. We have called it γ' to distinguish it from the unrenormalized λ/m^2, which is sometimes called γ. If we use γ' instead of λ/m^2, then the sum of stability angles will disappear from the calculation. In what follows, it will be assumed that $\bar S$ contains a factor of $(\gamma')^{-1}$ rather than m^2/λ. As with m^2, the value of γ' is to be fixed in terms of physical quantities.

The energy levels are obtained from $\mathrm{tr}\, e^{-iHT}$ through the formula

$$\begin{aligned} G(E) &= \mathrm{tr}\frac{1}{H-E} \\ &= \sum_k \frac{1}{E_k - E} \\ &= i\,\mathrm{tr}\int_0^\infty \exp[i(E-H)T]\,dT. \end{aligned} \tag{3.17}$$

We insert into this expression the WKB sum over orbits for $\mathrm{tr}\, e^{-iHT}$ in the form used in Sec. IV of I,

$$2\pi\,\mathrm{tr}\, e^{-iHT} = \sum_{nl}\int \tau\, d\tau \left| l\,\frac{d^2\bar S(\tau)}{d\tau^2}\right|^{1/2} \left[L\left|\frac{-d\bar S(\tau)/d\tau}{T(1-(nL/T)^2)^{3/2}}\right|^{1/2} \delta\left(l\tau-(T^2-n^2L^2)^{1/2}\right) e^{il\bar S(\tau)}\right], \tag{3.18}$$

where v has been explicitly set equal to nL/T and the δ function enforces the other constraint in (3.2). We now use the representation

$$\delta(l\tau-(T^2-n^2L^2)^{1/2})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{iM[l\tau-(T^2-n^2L^2)^{1/2}]\}\,dM \tag{3.19}$$

for the δ function in (3.18) and insert the whole thing into (3.17). The result can be written as

$$G(E)=\int\frac{dM}{2\pi i}\,G^0(M,E)\,\overline{G}(M), \tag{3.20}$$

with

$$\begin{aligned}G^0(M,E)&=\left(\frac{i}{2\pi}\right)^{1/2}\sum_n\int dT\,L\left[\frac{M}{T}\,\frac{1}{(1-(nL/T)^2)^{3/2}}\right]^{1/2}\exp\left\{iT\left(E-M\left[1-\left(\frac{nL}{T}\right)^2\right]^{1/2}\right)\right\},\\ \overline{G}(M)&=-\left(\frac{-i}{2\pi}\right)^{1/2}\sum_l\int_0^{\infty}\tau\,d\tau\,l^{1/2}\left|\frac{d\overline{S}/d\tau}{M}\right|^{1/2}\left|\frac{d^2\overline{S}}{d\tau^2}\right|^{1/2}\exp[i\,l(\overline{S}+M\tau)]\,.\end{aligned} \tag{3.21}$$

We evaluate the integrals in (3.20) by the stationary-phase method. The one for G^0 was evaluated in I and is equal to

$$G^0(M,E)=\left[\frac{i\partial W_0(M,E)}{\partial E}\right]\frac{e^{iW_0(M,E)}}{1-e^{iW_0(M,E)}},\quad W_0(M,E)=L(E^2-M^2)^{1/2},\quad |E|>|M|\,. \tag{3.22}$$

If $|E|<|M|$ there is no stationary-phase point for $v<1$. The integral for $\overline{G}$ is of the type computed in Sec. II of I. The stationary-phase point is at $d\overline{S}(\tau(M))/d\tau=M$. As in I the result of the stationary-phase integration and sum over l can be expressed in terms of $\overline{W}(M)=\overline{S}(\tau(M))+M\tau(M)$ as

$$\begin{aligned}\overline{G}(M)&=i\,\frac{d\overline{W}(M)}{dM}\,\frac{e^{i\overline{W}(M)}}{1-e^{i\overline{W}(M)}}\,,\\ \overline{W}(M)&=\frac{32\pi}{\gamma'}\sin^{-1}\left(\frac{\gamma' M}{16m}\right),\\ &0\le M\le 16m/\gamma'\,,\end{aligned} \tag{3.23}$$

where we have used (3.16) to work out the explicit form of $\overline{W}$. The restriction on the range of M comes from the requirement that there be a stationary-phase point corresponding to a classical orbit with a real period $\tau>2\pi/m$ as required by (3.1).

One sees immediately that $\overline{G}$ has poles when $\overline{W}(M)=2n\pi$, i.e.,

$$\overline{G}(M)\sim\frac{1}{M_n-M} \tag{3.24}$$

at

$$M_n=\frac{16m}{\gamma'}\sin\frac{n\gamma'}{16},\quad n=1,2\ldots<8\pi/\gamma'\,. \tag{3.25}$$

We now put (3.23) into (3.20) and do the M integration, picking up the poles in $\overline{G}$ to get

$$G(E)=\sum_n G^0(E,M_n)\,. \tag{3.26}$$

It was pointed out in I that $G^0(E,M)$ poles at $E=(p_k^{\,2}+M^2)^{1/2}$, $p_k=2\pi k/L$, $k=0,\pm1,\ldots,$ so that $G(E)$ has poles at

$$E_{k,n}=(p_k^{\,2}+M_n^{\,2})^{1/2}\,, \tag{3.27}$$

corresponding to a particle of mass M_n whose momenta come in units of $2\pi/L$, as it should in our periodic world.

The classical doublet solution has therefore produced a finite set of particles with masses M_n which propagate and have the correct relation between energy and momentum. We will give an interpretation of these states in Sec. IV.

F. Strong coupling and separability in sine-Gordon equation

In Sec. III B it was pointed out that if the sine-Gordon equation was truly separable then our calculation of the soliton mass and the mass spectrum of the doublet would be exact within the WKB approximation.[19] However, we must now come to terms with the fact that the sine-Gordon equation is not separable in a finite box. In order to make a sensible calculation we need to begin in a finite box and let $L\to\infty$ at the end. Also, the ultraviolet cutoffs needed to implement renormalization destroy the separation into action-angle variables.

The extent to which the sine-Gordon equation is not acting like a strictly separable system in our calculation can be seen as follows. Referring to I and Appendix A of the present paper, one can convince oneself that for a strictly separable system the sum of the stability angles $\xi(\tau)$ would have been a constant times the period T. Such a term would have been completely canceled by the vacuum-energy subtraction $T\Delta E$ and the stability angles would have played no role whatsoever. The quantization condition would then have reduced to the usual Bohr-Sommerfeld rule $\oint p\,dq = 2n\pi$ for separable systems. Of course, the vacuum-energy subtraction did not cancel all of ξ, but after renormalization the finite part $\tilde{\xi}$ was proportional to $\bar{S}$ and simply represented a renormalization of the coupling constant from λ/m^2 to γ'. The same renormalization appeared in both the calculation of the soliton mass and the mass spectrum of the doublet. Although we have not proved it, we strongly suspect that this pattern is general. That is, we suspect that for any solution of the sine-Gordon equation built out of interacting solitons, antisolitons, and doublets, the net effect of the stability angles will be to replace λ/m^2 by γ'. If this is indeed what happens, then in the WKB approximation the sine-Gordon equation will act like an exactly separable system but with a new coupling constant γ'. With this revived separability, our quantization of the soliton and doublet will be a rigorous result of the semiclassical method.

When the semiclassical method can be rigorously and fully exploited, as we believe has been done for the sine-Gordon equation, it is reasonable to expect meaningful results for both weak and strong couplings. As pointed out in the Introduction, it appears that we are actually finding exact results.

G. Remarks on quantization in more general theories

In more general theories we can look for particle-like solutions which are analogous to the sine-Gordon doublet. They will be solutions ϕ to the classical field equations which are periodic in a rest frame, i.e.,

$$\phi(\tau+t, x) = \phi(t, x), \tag{3.28}$$

and which have a behavior as $|x|\to\infty$ which is consistent with a particle interpretation. We assume the classical solution to be stable in the sense that all the stability angles are real. Given such a solution we can boost it to obtain one moving with any velocity $|v|<1$.

In Sec. V we will show how to find one of these classical periodic fields in the two-dimensional ϕ^4 theory studied in II. This solution is valid only for weak coupling, but serves to demonstrate that the phenomena which we are discussing here are not restricted to the solvable sine-Gordon equation.

If a periodic solution exists for a range of periods τ then at a classical level, there will be particles with a continuous mass spectrum. The WKB method will quantize the mass spectrum.

The quantization will proceed in the same way as for the sine-Gordon doublet. First, it is clear that the center-of-mass motion can always be taken care of just as we did above. Thus, we can concentrate on $\bar{G}(M)$. With regard to the calculation of $\bar{G}(M)$, the sine-Gordon equation is a special case for two reasons. The first is that in general the finite piece $\tilde{\xi}(\tau)$ of the sum over stability angles will not be a constant times $\bar{S}(\tau)$. Second, there will in general be a finite number of discrete positive stability angles $\nu_1, \nu_2, \ldots, \nu_N$ ($\nu_i > 0$). When these discrete stability angles are present, we have to add (See Appendix A) to the right-hand side of (3.6) a factor

$$\prod_{i=1}^{N} (1-e^{-il\nu_i})^{-1} = \sum_{\{q\}} \exp\Big[-il\sum_i q_i\nu_i\Big], \tag{3.29}$$

where the sum is over sets of positive integers. We can compute $\bar{G}(M)$ separately for each term in the sum.

For a given term in (3.29) the analog of the integral in Eq. (3.21) for $\bar{G}(M)$ is then gotten by making the replacement

$$\bar{S}(\tau)\to\bar{S}(\tau)+\tilde{\xi}(\tau)-\sum q_i\nu_i(\tau) \tag{3.30}$$

in the exponential in the integrand. In what follows we will assume a weak-coupling approximation, where $\tilde{\xi}$ and the ν_i are taken to be small relative to $\bar{S}$. With this approximation it is then straightforward to evaluate the stationary-phase integral and find the poles in $\bar{G}(M)$. The quantization condition turns out to be

$$\begin{aligned} \overline{W}(M) &= 2\pi n - \tilde{\xi}(\tau(M)) + \sum_i q_i\nu_i(\tau(M)), \\ \overline{W}(M) &= \bar{S}(\tau(M)) + M\tau(M), \end{aligned} \tag{3.31}$$

where $\tau(M)$ is the solution to

$$-\frac{d\bar{S}(\tau(M))}{d\tau} = M. \tag{3.32}$$

Taking account of the fact that $\tilde{\xi}$ and the ν_i are assumed to be small, we can solve (3.31) iteratively using $d\overline{W}/dM = \tau(M)$ to obtain

$$M_{n,\{q\}} \approx M_n + \frac{\sum_i q_i \nu_i(\tau(M_n)) - \tilde{\xi}(\tau(M_n))}{\tau(M_n)}, \tag{3.33}$$

$$\overline{W}(M_n) = 2\pi m .$$

Thus, we see that in this approximation, each basic particle $n=1, 2, \ldots$ is accompanied by excited states labeled by sets of integers $\{q\}$. This result appeared in I and has also been obtained by Voros.[20] Again, we see an analogy between the stability angles and harmonic-oscillator frequencies. In II we pointed out that the normal modes of oscillation around the static kink contained one nontrivial discrete freqnency. This produced a set of states with the spectrum $M = M_{\text{kink}} + q\omega_1$, $q = 0, 1, 2, \ldots$ in analogy with (3.33). Finally, the q-independent term $-\tilde{\xi}/\tau$ on the right-hand side of (3.33) is the analog of the quantum correction to the mass of the kink computed in II.

In general, as was pointed out in I, Eq. (3.33) is valid only if the coupling of the basic orbit to the "transverse" degrees of freedom represented by the stability angles is weak. In our ϕ^4 calculations, we will work in the weak-coupling limit and this condition will automatically be satisfied. As was pointed out above, we have reason to believe that for the sine-Gordon equation, the simpler quantization rule (3.25) is valid for strong as well as weak coupling.

IV. INTERPRETATION OF THE MASS SPECTRUM

In the Introduction we have already sketched the interpretation of the mass spectrum. Here we give a few more details, operating under the assumption that our results are valid for all γ'.

It will turn out that there are two complementary ways in which the doublet states can be viewed. The first is that they are n-body bound states made out of n of the usual quanta of the theory. The second is that they are soliton-antisoliton bound states. Either view appears to be physically sensible, the former being more natural for weak coupling and the latter more natural for strong coupling.

A. Weak coupling

As has been pointed out, in the weak-coupling limit, the mass of the $n=1$ state becomes $m+O(\lambda^2)$, and it is to be identified with the "elementary" ϕ quanta of perturbation theory. It cannot be a new state since for very weak coupling the particle mass $m+O(\lambda^2)$ is surely a nondegenerate isolated point in the mass spectrum. With $\lambda \neq 0$ the mass of this state is

$$M_1 = \frac{16m}{\gamma'} \sin\frac{\gamma'}{16} . \tag{4.1}$$

For weak or moderate coupling, M_1 is a convenient "physical" scale of mass.

Expanding M_n in powers of λ gives

$$M_n = nM_1 - M_1\left(\frac{\lambda}{16m^2}\right)^2 \frac{n^3-n}{6} + O(\lambda^3), \tag{4.2}$$

which was already discussed in the Introduction. As long as $\lambda n/m^2$ is reasonably small, these states can be interpreted as weakly bound n-particle systems. For such a system only two-body interactions are important and only the ϕ^4 term in (2.2) contributes. One easily sees that in the nonrelativistic limit this gives a δ-function potential. The Schrödinger equation for the n-particle ground state is

$$\left(-\frac{1}{2m}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\,\frac{k}{m}\sum_{i<j}^{n} \delta(x_i - x_j)\right)\psi_n = -\epsilon_n \psi_n , \tag{4.3}$$

where k is a constant of order λ and ϵ_n is to be identified with $nM_1 - M_n$. In the center-of-mass system, the ground-state wave function is, up to a normalization constant,

$$\psi_n = \exp\left[-k\sum_{i<j}^{n} |x_i - x_j|\right], \tag{4.4}$$

which gives for ϵ_n

$$\epsilon_n = \tfrac{1}{6}(n^3-n)\frac{k^2}{m} . \tag{4.5}$$

The value of k is directly obtained by summing the Feynman graphs shown in Fig. 4. The sum of graphs has a pole below threshold and one can identify

$$k^2 = \frac{\lambda^2}{256m^2} . \tag{4.6}$$

The binding energy (4.5) as computed with the Schrödinger equation then agrees with the WKB formula.

The sine-Gordon equation is invariant under $\phi \to -\phi$, leading to a conserved "parity" operation R. It is easy to see that the doublet states are eigenstates of R with eigenvalue $(-1)^n$. This means, for example, that the decay of the $n=3$ state to two $n=1$ states is forbidden. Using this fact and the calculated mass ratios, one can easi-

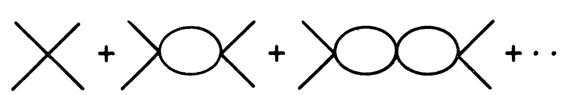

FIG. 4. The sum of bubble diagrams which corresponds to the Schrödinger equation with a δ-function potential in the nonrelativistic limit.

ly convince oneself that the $n=1$, 2, and 3 states are necessarily stable.

The first decay which could occur is the $n=4$ state going to two $n=1$ states. At a classical level, this is forbidden by special conservation laws. Classically, the sine-Gordon equation possesses an infinite number of nontrivial conserved integrals.[10-13] We do not know whether or not these conserved quantities will survive in the quantum theory. If they do, they could provide enough quantum numbers to stabilize all the doublet states. As a simple check we can calculate the matrix element for $(n=4)\to 2(n=1)$ to leading order in λ. Using the above interpretation of the $n=4$ state as a loosely bound composite of four of the usual ϕ quanta, this matrix element is easily seen to be proportional to the ordinary Feynman amplitude for $4\phi\to 2\phi$ evaluated at the 4ϕ threshold. In order λ^2 this amplitude contains a number of tree graphs with two $\lambda\phi^4$ interactions plus a single contact term from the $(\lambda^2/m^2)\phi^6$ term in the sine-Gordon Lagrangian. At threshold the graphs cancel among themselves and the amplitude vanishes. The $n=4$ state is therefore stable to leading order in λ, suggesting that the conservation laws survive quantization.

Finally, we remark that the expansion (4.2) of the exact formula $M_n=(16m/\gamma')\sin(n\gamma'/16)$ is mathematically remarkably good (10%) up to the maximum allowed value of n. Physically, this corresponds to the fact that the binding energy per particle does not exceed $\sim\frac{1}{3}$ of its rest mass, making the nonrelativistic result (4.5) quite accurate. This point will be important in Sec. V.

B. "Nuclear democracy"

To use Chew's phrase, the sine-Gordon equation is "democratic." There is no fundamental distinction between the "elementary particle" $n=1$ state and the higher-mass $n>1$ particles. They all come from one basic particlelike mode of the sine-Gordon equation. Explicitly, the classical solution corresponding to the nth quantum state is, in the rest frame,

$$\phi_{\tau_n,0}=\frac{4m}{\sqrt{\lambda}}\tan^{-1}\left[\tan\left(\frac{n\gamma'}{16}\right)\frac{\sin[mt\cos(n\gamma'/16)]}{\cosh[mx\sin(n\gamma'/16)]}\right], \tag{4.7}$$

which has no particularly distinguishing feature[21] for $n=1$. The ϕ^4 theory discussed in the next section is "democratic" in the same way. This may be a rather general phenomenon in interacting field theories.

C. Strong coupling

States with $n>8\pi/\gamma'$ do not exist. As γ' increases the nth state will disappear when $\gamma'=8\pi/n$. We have already mentioned in the Introduction that what happens is that the state breaks up into a soliton-antisoliton pair. A graphical way to see this is to note that if we keep $t/\tau_n\sim t\cos(n\gamma'/16)$ finite as $n\gamma'/16$ approaches $\pi/2$, then the argument of the inverse tangent in (4.7) will blow up and $\phi_{\tau_n,0}$ will approach $2\pi m/\sqrt{\lambda}$. Referring to Eq. (2.3) one can easily see that $|\phi|=2\pi m/\sqrt{\lambda}$ is the classical signal for decay into a soliton-antisoliton pair.

It is clear that one way to view doublet states is as soliton-antisoliton bound states. With this in mind, we study soliton-antisoliton scattering in Appendix D with the goal of checking the one-dimensional analog of Levinson's theorem[22]

$$n_{\mathrm{el}}=n_B+\frac{1}{2\pi i}[\mathrm{tr}\ln S(\infty)-\mathrm{tr}\ln S(0)] \tag{4.8}$$

where n_{el} is the number of elementary particles in the soliton-antisoliton channel, n_B is the number of bound states, and S is the S matrix. We compute the right-hand side of Eq. (4.8) using the semiclassical method, and following Jackiw and Woo[23] we find $n_{\mathrm{el}}=\frac{1}{2}n_B$. Of course this does not mean that some of our states are elementary. What it says is that on the average our states are only 50% solitons-antisolitons. The other 50% of the wave function is presumably composed of many soliton-antisoliton pairs.[22]

From Appendix D we also learn that as γ' increases and a state unbinds, it does not appear as a resonance in soliton-antisoliton scattering. It acts like an unbound S-wave state which becomes a virtual state rather than a resonance.

Finally, we remind the reader that for $\gamma'>8\pi$ only solitons and antisolitons exist. By this time the "elementary particle" has itself broken up into a soliton-antisoliton pair.

D. Comparing with some exact results

In the Introduction we compared the mass of the $n=2$ state to an exact result obtained by summing Feynman diagrams. The diagrammatic calculation has nothing to do with the semiclassical method and has been placed in Appenidx E. Specifically, we find from the diagrams

$$\Delta=\left(\frac{4M_1^2-M_2^2}{4M_1^2}\right)^{1/2}$$
$$=\frac{\lambda}{16m^2}\left[1+\frac{1}{8\pi}\frac{\lambda}{m^2}+\left(\frac{1}{8\pi}\frac{\lambda}{m^2}\right)^2-\frac{1}{6}\left(\frac{\lambda}{16m^2}\right)^2\right]+O(\lambda^4). \tag{4.9}$$

The WKB approximation to Δ is easily shown to be

$$\Delta = \sin\frac{\gamma'}{16}, \tag{4.10}$$

which agrees with (4.9) through order λ^3. Some algebra converts (4.9) into Eq. (1.6). The order-λ term in Δ was discussed above. The order-λ^2 term comes from relativistic corrections to the Schrödinger equation and an order-λ^2 potential generated by the diagrams in Fig. 5. It is not too surprising that WKB gives the order-λ^2 term exactly: WKB usually treats one-loop effects correctly. However, it is remarkable that the WKB result is still exact in the next order, λ^3. In this order two-loop diagrams enter in an essential way.

One does not generally expect WKB to be exact beyond one loop. In fact we can show that for a Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - \frac{b\lambda^2}{6!\,m^2}\phi^6, \tag{4.11}$$

the semiclassical calculation will not be exact except for the special case $b=1$ corresponding to the sine-Gordon equation.[24] In the outline, the argument goes as follows. The order-λ^4 term in $(2M_1 - M_2)/M_1$ (corresponding to the λ^3 term in Δ) contains a piece linear in b and one that is independent of b. It is easy to find the WKB approximation to the coefficient of b. The stability angles cannot produce a term of the form $b\lambda^4$. Therefore, in order λ^4 the dependence on b comes solely from the classical action $\overline{S}$. Using formulas

$$\begin{aligned}
\frac{d}{db}\overline{W}(M_n(b), b) &= \tau_n\frac{dM_n}{db} + \frac{\partial\overline{W}}{db}\\
&= \frac{d}{db}(2n\pi)\\
&= 0,\\
\frac{\partial\overline{W}}{\partial b} &= \frac{\partial\overline{S}}{\partial b}\\
&= -\frac{\lambda^2}{6!\,m^2}\int_0^\tau dt\int_{-\infty}^{\infty} dx\,\phi_{\tau_n,0}{}^6,
\end{aligned} \tag{4.12}$$

one then finds that the WKB approximation to dM_n/db is

$$\begin{aligned}
\frac{dM_n}{db} &= \frac{\lambda^2}{6!\,m^2\tau_n}\int_0^\tau dt\int_{-\infty}^{\infty}\phi_{\tau_n,0}{}^6 dx\\
&= n^5\frac{16m}{135}\left(\frac{\lambda}{m^2}\right)^4 + O(\lambda^5),
\end{aligned} \tag{4.13}$$

where we have used the expression in ~~(4.7) for~~ $\phi_{\tau_n,0}$. Using (4.13) to compute $d\Delta/db$ one finds $d\Delta/db = -\frac{16}{9}(\lambda/16m^2)^3$, whereas the diagrams (see Appendix E) give $d\Delta/db = -\frac{8}{3}(\lambda/16m^2)^3$. Therefore, the WKB result for Δ can be exact in order λ^3 only for one value of b, namely the sine-Gordon value[24] $b=1$.

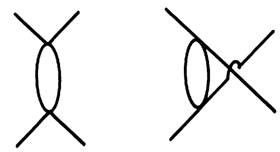

FIG. 5. Diagrams which if inserted in the chain of Fig. 4 would add an additional δ-function potential of order λ^2.

In the Introduction we also compared our results with those of Coleman.[15] He finds the remarkable fact that sine-Gordon systems can be mapped into a massive Thirring model with coupling constant g. The fermions are to be identified with solitons. As pointed out in the Introduction, we can use Coleman's correspondence between the two theories to check WKB against further exact results, this time for γ' near 8π. For γ' near 8π, g is small and we can find a bound state in the Thirring model by summing diagrams. The calculation is done in Appendix E. From the diagrams we find

$$\begin{aligned}
\Delta' &= \left(\frac{4M_f{}^2 - M_B{}^2}{4M_f{}^2}\right)^{1/2}\\
&= g - \frac{2g^2}{\pi} + O(g^3),
\end{aligned} \tag{4.14}$$

where M_f and M_B are the fermion and bound-state masses. With the identifications

$$M_B = M_1, \quad M_f = M(\text{soliton}) = \frac{8m}{\gamma'} \tag{4.15}$$

and Coleman's[15] relation

$$\gamma' = \frac{8\pi}{1 + 2g/\pi}, \tag{4.16}$$

the WKB formula for Δ' is

$$\Delta' = \cos\left(\frac{\gamma'}{16}\right) = \sin\left(\frac{g}{1+2g/\pi}\right), \tag{4.17}$$

which agrees with (4.14) through order g^2. Equation (1.4) in the Introduction follows, after some algebra, from (4.14).

Finally, we note the fact that the mass renormalization in the sine-Gordon equation is multiplicative. This means that the scale of mass is *a priori* undetermined and can change arbitrarily as the dimensionless coupling λ/m^2 changes. Consequently, one cannot give an absolute meaning to M_1, say, as a function of λ/m^2. What is meaningful is the ratio of two different masses at the same value of λ/m^2. A formal way of saying this is that the parameter m in Eqs. (1.2) and (3.25) is ambiguous owing to the possibility of a γ'-dependent rescaling $m \to Z(\gamma')m$. Of course, this does not affect any of our results for quantities such as $(2M_1 - M_2)/M_1$. On the other hand, we can compare our expression (4.1) for M_1 with the particle mass as computed in perturbation theory starting with a normal-ordered Lagrangian. The two do not agree in order λ^2. Specifically, the λ^2 term obtained from an expansion of (4.1) is exactly one half of the finite self-energy graph shown in Fig. 3. Thus our semiclassical results differ from those obtained from the standard perturbation theory by a finite but nontrivial rescaling $Z(\gamma')$.

V. BOUND STATES IN $\lambda\phi^4$

From the existence and physical interpretation of the bounded classical solution (3.1) of the sine-Gordon equation, one can develop a scheme for finding similar solutions in nonexactly soluble field theories. In this section we shall consider the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4, \tag{5.1}$$

which after the usual rescaling becomes

$$\mathcal{L} = \frac{m^4}{\lambda}\left[\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}\phi^2 - \tfrac{1}{4}\phi^4\right]. \tag{5.2}$$

This Lagrangian exhibits kinklike solutions, which we discussed at length in II. We now look for time-dependent nontrivial solutions. We naturally expand around $\phi = 1$, which is one of the stable vacua of the theory. Then, setting $\phi = 1 + z$, the classical equation of motion reads

$$\ddot{z} - z'' + 2z + 3z^2 + z^3 = 0\,. \tag{5.3}$$

We look for a solution similar to Eq. (3.1) for small value of the parameter $\epsilon \equiv [(m\tau/2\pi)^2 - 1]^{1/2}$. The strategy is to expand simultaneously in harmonics of the fundamental frequency and in powers of ϵ. Using the variables

$$\tau = \frac{t\sqrt{2}}{(1+\epsilon^2)^{1/2}}, \quad \xi = \frac{\epsilon x\sqrt{2}}{(1+\epsilon^2)^{1/2}}, \tag{5.4}$$

one looks for a solution of the form

$$z = \epsilon^2 g_1(\xi) + \sum_{n=0}^{\infty} \left[\epsilon^{2n+1} f_{2n+1}(\xi)\sin(2n+1)\tau + \epsilon^{2n+2} g_{2n+2}(\xi)\cos(2n+2)\tau\right]. \tag{5.5}$$

Note that the occurrence of both odd and even powers of z in Eq. (5.3) forces the existence of both odd and even harmonics in (5.5), while the sine-Gordon equation would accommodate odd harmonics only. All functions $f_1, g_1, g_2, \ldots$ are functions of ξ only (and ϵ), to be determined. By identification in Eq. (5.3), one finds in lowest order in ϵ

$$\begin{aligned} &g_1 = -\tfrac{3}{4}f_1^2, \quad g_2 = -\tfrac{1}{4}f_1^2, \quad f_3 = -\tfrac{1}{16}f_1^3, \\ &g_4 = \tfrac{1}{64}f_1^4, \quad f_5 = \tfrac{1}{256}f_1^5, \ldots, \end{aligned} \tag{5.6}$$

and on the other hand

$$f_1'' - f_1 + \tfrac{3}{2}f_1^3 = 0\,, \tag{5.7}$$

whose solution is

$$f_1 = \frac{2}{\sqrt{3}\cosh\xi}. \tag{5.8}$$

We see that we already have achieved a bounded classical solution. For our ultimate purpose, however, it is necessary to push the calculation to the next order. This is rather lengthy, but otherwise quite straightforward. The result is

$$f_1 = \frac{2}{\sqrt{3}}(1 + \tfrac{40}{9}\epsilon^2)\frac{1}{\cosh\xi} - \frac{103}{18\sqrt{3}}\frac{\epsilon^2}{\cosh^3\xi} + O(\epsilon^4)\,, \tag{5.9}$$

$$g_1 = -\frac{1}{\cosh^2\xi}(1 + \tfrac{116}{9}\epsilon^2) + \frac{395}{36}\frac{\epsilon^2}{\cosh^4\xi} + O(\epsilon^4)\,, \tag{5.10}$$

$$g_2 = \frac{1}{\cosh\xi}(\tfrac{1}{3} + \tfrac{80}{27}\epsilon^2) + \frac{47}{27}\frac{\epsilon^2}{\cosh^4\xi} + O(\epsilon^4)\,. \tag{5.11}$$

The classical energy and action along one period for such a solution are

$$E = \frac{m^3}{\lambda}\frac{2\sqrt{2}}{3}(2\epsilon + \tfrac{37}{27}\epsilon^3) + O(\epsilon^5)\,, \tag{5.12}$$

$$\begin{aligned} W &= \int \dot{z}^2 dx\, dt \\ &= \frac{4\pi}{3}\frac{m^2}{\lambda}(2\epsilon + \tfrac{46}{27}\epsilon^3) + O(\epsilon^5)\,. \end{aligned} \tag{5.13}$$

Since the quantization condition will be in terms of W, it is convenient to eliminate ϵ:

$$E = m\sqrt{2}\,\frac{W}{2\pi} - \frac{3\sqrt{2}}{32}\left(\frac{W}{2\pi}\right)^3\frac{\lambda^2}{m^3} + O(W^5)\,. \tag{5.14}$$

Before turning to the quantization of the motion (5.5), we wish to make a few more remarks on the classical level; we have found that there exist analytic solutions to the nonlinear Eq. (5.3), which are periodic in time. It will also turn out that the solution (5.5) is stable, at least for small enough ϵ. This means that the classical ϕ^4 theory, although not completely integrable like the sine-Gordon Lagrangian, still displays interesting non-ergodic behavior.

It is also amusing to remark that in Eq. (5.6), g_2, f_3, g_4, f_5 are the first few terms of a geometric series. Although the form of solution (5.5) is specifically designed for periodic solutions and small values of ϵ, one can, by analogy with sine-Gordon, set $\epsilon = i/v$ and sum all the terms of the above-mentioned geometric series. One then obtains for $\phi = 1 + z$ and large $|t|$

$$\phi \approx \frac{\frac{1}{4}\sqrt{3}\, v \cosh[x/(1-v^2)^{1/2}] - \cosh[vt/(1-v^2)^{1/2}]}{\frac{1}{4}\sqrt{3}\, v \cosh[x/(1-v^2)^{1/2}] + \cosh[vt/(1-v^2)^{1/2}]}\,. \tag{5.15}$$

This looks like two well-separated kinks, moving with velocities $\pm v$. Whether or not Eq. (5.15) could form the starting point of an approximate analytic solution for kink-antikink scattering is unknown: There are no obvious small parameters in which to expand.

Finally, we can gain further insight from the analogy of Eq. (5.14) with the sine-Gordon result when the latter is also expanded in lowest orders in W. Since up to $\theta = \pi/2$, $\sin\theta$ is approximated within 10% by $\theta - \frac{1}{6}\theta^3$, we can conjecture a similar feature for ϕ^4. Taking (5.14) as it is up to the maximum value of E, one finds that this maximum is $\frac{16}{9}m$, which is 7% lower than twice the classical mass of the kink. This is so analogous to the sine-Gordon result that one may conjecture a very close analogy between the qualitative behavior of the two quantum-mechanical theories as a function of the coupling constant.

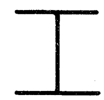

FIG. 6. A one-particle exchange graph in the ϕ^4 theory with broken symmetry.

We now turn to the quantization of our motion (5.5). Except for finer details, such as renormalization effects, everything is contained in Eq. (5.14). Up to stability angles, to be computed later, one finds the quantum-mechanical energies by setting W equal to $2n\pi$. This gives the energy levels

$$E_n = nm\sqrt{2} - \frac{3\sqrt{2}}{32}\frac{\lambda^2}{m^3}n^3 + O(\lambda^4)\,. \tag{5.16}$$

For moderate values of n, this is a nonrelativistic bound state again, and the same interpretation in terms of Feynman diagrams applies. In this case, however, the ϕ^4 interaction is repulsive, but more than compensated by the scalar exchange due to the cubic interaction, Fig. 6. The coefficient of n^3 in (5.16) naturally agrees with what one obtains by summing the Feynman diagrams.

We now turn to the calculation of the stability angles. One has to solve the linear equation

$$\delta\ddot{z} - \delta z'' + 2\delta z + 6z\delta z + 3z^2\delta z = 0\,, \tag{5.17}$$

where z is given by Eq. (5.5). Unfortunately, we have no trick analogous to that of Appendix C which would allow for an easy solution of (5.17). Instead, one must again resort to a power series in ϵ: By analogy with the corresponding problem in the sine-Gordon equation, one would write

$$\begin{aligned}\delta z = {} & \cos\frac{\delta}{2}\cos\left(\frac{t-vx}{(1-v^2)^{1/2}}\sqrt{2}+\frac{1}{2}\delta\right)+\tanh 2\epsilon x\sin\left(\frac{t-vx}{(1-v^2)^{1/2}}\sqrt{2}+\frac{\delta}{2}\right)\\ & +\frac{\epsilon}{\cosh 2\epsilon x}\left[\alpha\sin\left(\frac{t-vx}{(1-v^2)^{1/2}}\sqrt{2}+\frac{t\sqrt{2}}{(1+\epsilon^2)^{1/2}}+\frac{1}{2}\delta\right)+\beta\sin\left(\frac{t-vx}{(1-v^2)^{1/2}}\sqrt{2}-\frac{t\sqrt{2}}{(1+\epsilon^2)^{1/2}}+\frac{1}{2}\delta\right)\right]\\ & +\frac{\epsilon\sinh 2\epsilon x}{\cosh^2 2\epsilon x}\left[\gamma\cos\left(\frac{t-vx}{(1-v^2)^{1/2}}\sqrt{2}+\frac{t\sqrt{2}}{(1+\epsilon^2)^{1/2}}+\frac{1}{2}\delta\right)+\zeta\cos\left(\frac{t-vx}{(1-v^2)^{1/2}}\sqrt{2}-\frac{t\sqrt{2}}{(1+\epsilon^2)^{1/2}}+\frac{1}{2}\delta\right)\right]\\ & +\cdots\,,\end{aligned} \tag{5.18}$$

with $\alpha, \beta, \gamma, \delta, \eta$ to be determined by consistency of the expansion. We have performed the calculation of δ to leading order in ϵ only, although the results of Eqs. (5.9)–(5.11) would allow for a determination up to order ϵ^2. The result is

$$\delta(v) = \frac{2(1-v^2)^{1/2}}{v}\left[\frac{3-3v^2}{3+v^2}+1\right]\epsilon\,. \tag{5.19}$$

The rest of the calculation of the stability angles proceeds as in the sine-Gordon case. Note that the solutions $\delta z \sim \dot{z}$ and $\delta z \sim z$ contribute to order ϵ^2 only to the sum of stability angles. The renormalization is also straightforward: It exactly cancels the second term in the right-hand side of Eq. (5.19). Only the first term survives, and the new quantization condition is then found to be

$$W(\epsilon) = 2n\pi - \pi\frac{\epsilon}{\sqrt{3}} + O(\epsilon^2)\,. \tag{5.20}$$

In terms of energy levels, we thus see that to order ϵ the stability angles induce a mass renormalization of order λ. A direct calculation shows that it is identical to the finite mass renormalization induced by the diagram of Fig. 7. A more complete calculation, which would include terms of higher order in ϵ, would also renormalize the coupling constant.

Note added in proof. After this work was completed we learned of work done by L. D. Faddeev, I. Arefieva, L. A. Takhtajan, V. E. Korepin, and P. P. Kulish in which many of the above results were derived by a somewhat different path-integral method. It is unfortunate that an English translation of much of this work is not yet available, but a description of it and the original references may be found in a recent IAS report by L. D. Faddeev (unpublished).

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APPENDIX A: STABILITY ANGLES

Although we are interested in field theories we begin by studying a system with a finite number of degrees of freedom. We then have coordinates q_i, $i=1,2,\ldots,N$ and a Lagrangian $L(q,\dot{q})$.

In the semiclassical approximation one has

$$\langle q|e^{-iHT}|q'\rangle = \text{const}\times\sum_n\left|\frac{\partial^2 S_n}{\partial q\partial q'}\right|^{1/2} e^{-i[(N\pi/2)+\theta_n]}e^{iS_n}, \tag{A1}$$

where S_n is the action for the nth trajectory q^n satisfying the classical equations of motion and the end-point conditions $q^n(0)=q'$, $q^n(T)=q$. The phase $e^{i\theta_n}$ depends on the number of critical points along the trajectory.[2-4] It will not be important in what follows and will be suppressed from now on. The trace of e^{-iHT} will then be a sum of terms of the form

$$e^{-iN\pi/2}\int dq\left|\frac{\partial^2 S(q,q)}{\partial q\partial q'}\right|^{1/2} e^{iS(q,q)}. \tag{A2}$$

The point q^* is a stationary-phase point of the integral if

$$\frac{\partial S}{\partial q}(q^*,q^*)+\frac{\partial S}{\partial q'}(q^*,q^*)=P-P'=0, \tag{A3}$$

i.e., if q^* is a point lying on a periodic orbit of period T. We expand the action around $q=q^*, q'=q^*$ as

$$S(q,q')=S(q^*,q^*)+\tfrac{1}{4}(q_i+q_i'-2q_i^*)(q_j+q_j'-2q_j^*)G_{ij} + \tfrac{1}{4}(q_i-q_i')(q_j-q_j')H_{ij}+\cdots, \tag{A4}$$

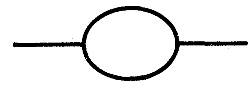

FIG. 7. A finite mass-renormalization graph in the ϕ^4 theory with broken symmetry.

where cross terms between $q+q'$ and $q-q'$ cannot occur since $S(q,q')=S(q',q)$ by time reversal. In the stationary-phase approximation the integral in (A2) is

$$e^{-iN\pi/2}\int dq\,|\tfrac{1}{2}(G-H)|^{1/2} \times\exp[iS(q^*,q^*)+i(q_i-q_i^*)(q_j-q_j^*)G_{ij}], \tag{A5}$$

where a sum over distinct stationary-phase points is implied.

Were it not for the fact that G can have zero eigenvalues the integral in (A5) would be straightforward. Assume that G has $M\le N$ vanishing eigenvalues. We can then make a local (around q^*) change of coordinates which brings G_{ij} into a block diagonal form where

$$G_{ij}=G_{ji}=0, \quad i=1,2,\ldots,M, \quad j=1,2,\ldots,N. \tag{A6}$$

In this coordinate system

$$P_i-P_i'=\frac{\partial S}{\partial q_i}+\frac{\partial S}{\partial q_i'}=\tfrac{1}{2}G_{ij}(q_j+q_j'-2q_j^*) =0, \quad i=1,2,\ldots,M. \tag{A7}$$

The M canonical momenta P_i, $i=1,2,\ldots,M$ are therefore conserved, at least locally. It is easy to see that the existence of any global conservation law implies a zero eigenvalue of G. We will assume that all zero eigenvalues of G correspond to a symmetry of the system and are therefore nonaccidental.

Further information can be extracted from the fact that q^* is a point on a periodic orbit. Let $\bar{q}_i(t)=\bar{q}_i(t+T)$ be a periodic solution to the equations of motion. We will be interested in nearby solutions of the form $q_i=\bar{q}_i+\delta q_i$, where δq_i is small. Expanding L around $q=\bar{q}$ yields an action

$$S=\int L(\bar{q},\dot{\bar{q}})dt+\int\tfrac{1}{2}[A_{ij}(t)\delta\dot{q}_i\delta\dot{q}_j+B_{ij}(t)\delta q_i\delta\dot{q}_j + \tfrac{1}{2}C_{ij}(t)\delta q_i\delta q_j]dt+O((\delta q)^3), \tag{A8}$$

where

$$A_{ij}(t)=\left.\frac{\partial^2 L}{\partial\dot{q}_i\partial\dot{q}_j}\right|_{q=\bar{q}}, \tag{A9}$$

and B and C are corresponding matrices of derivatives of L with respect to $q,\dot{q}$ and q,q. The action in (A8) leads to a set of linear equations for δq_i in which the coefficients are periodic functions of time. It is well known that from the $2N$-independent solutions to these equations one can form linear combinations with the property that

$$\xi_\alpha(t+T)=e^{-i\nu_\alpha}\xi_\alpha(t)\,. \tag{A10}$$

For a stable solution $\bar{q}_i(t)$ all the ν_α's are real. Henceforth we will assume that we are dealing with a stable orbit. The ν_α's are called stability angles. Since the equations of motion are real they come in pairs ν and $-\nu$. There are N-independent pairs making $2N$ angles in all. The presence of a symmetry makes a pair of the ν's vanish. For example, time translation invariance says that $\bar{q}(t+\Delta)=\bar{q}(t)+\dot{\bar{q}}(t)\Delta+\cdots$ is a solution to the equations of motion. Identifying δq with $\dot{\bar{q}}\Delta$ one sees that one pair of ν's must vanish.

More generally, let $\pi(q(t),\dot{q}(t))$ be a time-independent quantity. To first order in δq

$$\pi=\pi(\bar{q},\dot{\bar{q}})+\frac{\partial\pi}{\partial q_i}(\bar{q},\dot{\bar{q}})\delta q_i+\frac{\partial\pi}{\partial\dot{q}_i}(\bar{q},\dot{\bar{q}})\delta\dot{q}_i\,. \tag{A11}$$

The coefficients of δq and $\delta\dot{q}$ in (A11) are periodic. Any solution ξ_α with $\nu_\alpha\neq 0$ will contribute a nonperiodic term which must vanish if π is to be time-independent. Therefore, only $\nu=0$ modes contribute to the right-hand side of (A11). One can convince himself that if there are M-independent (in the sense that the Poisson bracket of any pair vanishes) conserved quantities then there will be M pairs of zero stability angles.

We can identify the zero eigenvalues of G with pairs of zero stability angles. Let $\delta q_i(0)=q_i'-q_i^*$ and $\delta q_i(T)=q_i-q_i^*$. In our coordinate system where $q_i+q_i'-2q_i^*$, $i=1,\ldots,M$ span the null space of G, some thought shows that $\delta q(t)$ satisfying $\delta q_i(0)=\delta q_i(T)=0$, $i=M+1,\ldots,N$ span the $\nu=0$ manifold of solutions. This allows us to show easily that H is also block diagonal in this coordinate system. The conserved momenta P_i are

$$P_i=\frac{1}{2}\left[\frac{\partial S}{\partial q_i}-\frac{\partial S}{\partial q_i'}\right]=\tfrac{1}{2}H_{ij}(q_j-q_j') \tag{A12}$$

for $i=1,\ldots,M$. For $j>M$ the q_j correspond to solutions with $\nu\neq 0$, which we know cannot contribute in first order to a conserved quantity. Therefore,

$$H_{ij}=H_{ji}=0\,,\quad i=1,2,\ldots,M\,,\quad j=1,2,\ldots,N\,. \tag{A13}$$

Having concluded that both G and H can be block diagonalized, let us define $\tilde{G}$ and $\tilde{H}$ as the $(N-M)$-dimensional matrices obtained by ignoring the first M rows and columns of G and H. Then changing variables to $q_i-q_i^*=X_i$ for $i=M+1,\ldots,N$, (A5) reduces to

$$e^{iS(q^*,q^*)}e^{-iN\pi/2}\int d^Mq\left|\frac{\partial^2 S}{\partial q\partial q'}\right|^{1/2}\times\int d^{N-M}X\left|\tfrac{1}{2}(\tilde{G}-\tilde{H})\right|^{1/2}e^{i\tilde{G}_{ij}X_iX_j}\,, \tag{A14}$$

where $\partial^2S/\partial q\partial q'$ is now an M by M matrix. We have thus factored (A5) into a piece involving nonzero stability angles and a piece which depends on the symmetries of the system.

Our next task is to compute the last factor in (A14) in terms of the nonzero stability angles. One can do this by a straightforward manipulation of classical quantities.[3] We will use a heuristic functional integral approach which relates more directly to quantum field theory. In I we wrote (A5) as a known factor times

$$\int\mathfrak{D}\{X(t)\}\exp\left[i\tfrac{1}{2}\int_0^T(\dot{X}_i(t)\dot{X}_j(t)A_{ij}(t)+X_i(t)\dot{X}_j(t)B_{ij}(t)+X_i(t)X_j(t)C_{ij}(t))dt\right], \tag{A15}$$

where the functional integral is over all paths satisfying $X(0)=X(T)$. This is equivalent to the integral in (A5). A Gaussian integral like that in (A15) is proportional to the inverse square root of the determinant of the operator in the exponential. It is a differential operator with periodic boundary conditions. A zero stability angle implies a zero eigenvalue and consequently makes the inverse determinant singular. However, we already know how to extract these singular pieces. To compute (A15) it is easiest to pass to the equivalent Hamiltonian functional integral

$$\int\mathfrak{D}\{P(t)\}\mathfrak{D}\{X(t)\}\exp\left[i\int_0^T\left(\sum_i P_i(t)\dot{X}_i(t)-H(P(t),X(t))\right)dt\right], \tag{A16}$$

which is equal to (A15) if H is the quadratic Hamiltonian corresponding to the quadratic Lagrangian in (A15). It is easy to find the eigenvalues of the operator in the exponential of (A16). There are solutions to Hamilton's equations of the form $X(t)=\xi_\alpha(t)$, $P(t)=\eta_\alpha(t)$, where η_α like ξ_α satisfies $\eta_\alpha(t+T)=e^{-i\nu_\alpha}\eta_\alpha(t)$. These are not eigenfunctions since X is not periodic. However, since Hamilton's equations are first order the periodic functions $X_{n,\alpha}(t)=\xi_\alpha(t)\exp[i(t/T)(\nu_\alpha+2n\pi)]$ are eigenfunctions with eigenvalue $i(\nu_\alpha+2n\pi)T^{-1}$. Taking account of the fact that the ν_α's come in pairs with equal magnitude and opposite sign and that n can range from $-\infty$ to $+\infty$, one finds that the *square root* of the product of the eigenvalues is a constant times

$$\exp\left(-i\frac{(N-M)}{2}\right)\prod_{\nu_\alpha>0}\prod_{n=1}\nu_\alpha\left[1-\left(\frac{\nu_\alpha}{2n\pi}\right)^2\right]$$
$$=\prod_{\nu_\alpha>0}\left(e^{-i\pi/2}2\sin\frac{\nu_\alpha}{2}\right), \quad \text{(A17)}$$

where the subscript $\nu_\alpha > 0$ means the product over the $N-M$ positive ν_α.

The last factor in (A14) is just the inverse of the product in (A17) with any vanishing ν_α's eliminated. For each factor on the right in (A17) we can write

$$\left[e^{-i\pi/2}2\sin\frac{\nu_\alpha}{2}\right]^{-1}=\frac{e^{i\pi/2}e^{-i\nu_\alpha/2}}{1-e^{-i\nu_\alpha}}.$$

Then collecting everything and supplying suppressed constants yields

$$\operatorname{tr}e^{-iHT}=\sum_{\text{periodic orbits}}e^{i(S+\xi)}\Delta_1\Delta_2, \quad \text{(A18)}$$

$$\xi=-\tfrac{1}{2}\sum_{\nu_\alpha>0}\nu_\alpha,$$

$$\Delta_1=\left(\frac{1}{2\pi i}\right)^{M/2}\int\left|\frac{\partial^2 S}{\partial q\partial q'}\right|^{1/2}d^Mq, \quad \text{(A19)}$$

$$\Delta_2=\prod_{\nu_\alpha>0}(1-e^{-i\nu_\alpha})^{-1}. \quad \text{(A20)}$$

This is our desired result. The factor Δ_1 depends only on the symmetries of the system. Note that the integrand in (A19) is invariant under a transformation of coordinates $q_i \to \tilde{q}_i(q)$, $i=1,\ldots,M$. Conversely, Δ_2 and ξ contain only nonzero stability angles about which symmetry has little to say. We have not derived the normalization of (A18), which would be a factor independent of the dynamics. It can be checked by comparing to harmonic oscillators and separable systems.

As mentioned in Sec. III when we quantize, each factor in Δ_2 is expanded as

$$\frac{1}{1-e^{-i\nu_\alpha}}=\sum_{q=0}^{\infty}e^{-i\nu_\alpha q},$$

and each term in the resulting series produces a distinct energy level. Thus for the purpose of finding a given energy level, Δ_2 can be thought of as $\exp(-i\sum_\alpha q_\alpha\nu_\alpha)$, where the q_α's are some finite set of integers.

Letting the number of degrees of freedom N go to infinity to make contact with field theory, one sees that in practice all ultraviolet divergences will be contained in ξ. The renormalization of ξ was discussed in Sec. III.

In a finite box the ν_α's are a discrete series. As the length of the box goes to infinity there will be in general a discrete spectrum ν_k, $k=1,\ldots,n$ and a continuous spectrum. The discrete spectrum was discussed in the last subsection of Sec. III. The continuous spectrum corresponds to scattering states of the theory. To correctly interpret these states one must keep the volume large but finite and proceed along the lines of Sec. V of I. If we are interested only in the discrete part of the mass spectrum, these scattering states can be ignored. Therefore, for the purposes of this paper the factors of $(1-e^{i\nu_\alpha})^{-1}$ in Δ_2 corresponding to ν_α's in the continuum can be set equal to unity. This is why Δ_2 does not appear in (3.6).

Finally, for a periodic motion which corresponds to traversing a basic orbit l times the stability angles are just l times the stability angles for the basic orbit. This is why (3.6) contains $e^{il\xi}$ and (3.29) contains $e^{-il\nu_i}$.

APPENDIX B: CALCULATION OF Δ_1

We wish to compute Δ_1 defined in (A19) for a particlelike solution (3.28) in one space dimension. As in Sec. III we consider motions where the internal and translational periods are related by

$$T=\frac{l\tau}{(1-v^2)^{1/2}}=\frac{nL}{v}, \quad n, l=0,1,2,\ldots. \quad \text{(B1)}$$

There are two symmetries corresponding to space and time translation invariance so that $|\partial^2S/\partial q\partial q'|$ is a two-by-two determinant. We can use the fact that (A19) is invariant under coordinate transformations to simplify the calculation. One can choose any two coordinates which have the effect of changing the period and spatial extent of the classical motion. The simplest thing to do is to let one of the coordinates be the length of space L and the other be the period T. The integration volume is

$$\int d^2q=\frac{TL}{nl}, \quad \text{(B2)}$$

which is clearly the volume per period for spatial and temporal translations.

The total action is

$$S=l\bar{S}(\tau), \quad \text{(B3)}$$

where $\bar{S}$ is the action for one period in the rest frame. We then have

$$2\pi i\Delta_1=\frac{TL}{n}\begin{vmatrix}\dfrac{\partial^2\bar{S}(\tau)}{\partial T^2} & \dfrac{\partial^2\bar{S}(\tau)}{\partial T\partial L}\\[2ex] \dfrac{\partial^2\bar{S}(\tau)}{\partial T\partial L} & \dfrac{\partial^2\bar{S}(\tau)}{\partial L^2}\end{vmatrix}^{1/2}, \quad \text{(B4)}$$

where according to (B1)

$$\tau(T,L)=\frac{T}{l}\left[1-\left(\frac{nL}{T}\right)^2\right]^{1/2}. \quad \text{(B5)}$$

It is easy enough to compute the determinant and one finds

$$2\pi\Delta_1=\left[\frac{\tau}{l^{1/2}}\left|\frac{d^2\overline{S}(\tau)}{d\tau^2}\right|^{1/2}\right]\left[L\left|\frac{-d\overline{S}(\tau)/d\tau}{T(1-v^2)^{3/2}}\right|^{1/2}\right]. \tag{B6}$$

The second factor appeared in Eq. (4.12) of I. Its interpretation is given there. The determinant in (B4) is negative and a factor of i has been canceled on both sides of (B6). The equations in Sec. III are understood to contain the absolute values of $d^2\overline{S}/d\tau^2$ and $d\overline{S}/d\tau$.

APPENDIX C: TIME-DEPENDENT CASE

1. Stability angles for the sine-Gordon equation

Essentially, one wants to find the eigenfrequencies of the linearized sine-Gordon problem, where one expands about a particular mode, namely the doublet solution. This is the analog of finding the eigenfrequencies of the linearized problem for the static case, which are those induced by the shift $y=y_0+\alpha y_1$, where y_0 is a particular solution to the static equation and α is some expansion parameter.

Since the solutions we are interested in here are periodic in their rest frame, and one wants to use only quantities characterizing the orbit as a whole, then the appropriate parameter to compute is the angle by which the eigenfrequencies change over one traverse of the orbit, for each frequency mode. These are the stability angles.

First we will solve the linearized problem by an indirect method which takes advantage of the fact that the sine-Gordon equation is a perfect system. We will use the Bäcklund recursion algorithm which generates all of the solutions to the equation to solve the linear problem in a particularly convenient form. The system is always taken to be in a very large box of length L, with periodic boundary conditions. This makes the spectrum discrete so one can explicitly count modes. Additionally, we put the system on a lattice which regulates the ultraviolet behavior.

After doing this we subtract out the effect of vacuum excitations and show that in the limit $L\to\infty$, the sum of the stability angles is proportional to an integral over the phase shift δ of the linearized problem.

This integral contains a logarithmically divergent piece, absorbable into a mass renormalization. The rest of the stability-angle sum is finite and directly proportional to the classical action computed around one period of the doublet orbit, and so defines a renormalized action for the system. This amounts to an overall rescaling of the coupling constant.

2. The linearized problem

We set $m=1$ and rescale the fields according to $(\lambda/m^2)^{1/2}\phi\to\phi$ so that the linearized problem takes the form

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\delta\phi=\cos\phi\,\delta\phi\,. \tag{C1}$$

To find $\delta\phi$ we examine a particular four-soliton solution to the sine-Gordon equation, which consists of our usual doublet with internal period $\tau=2\pi(1+\epsilon^2)^{1/2}$ and another bound doublet, which we will use as a probe, having internal period $2\pi(1+\eta^2)^{1/2}$ and which has been boosted by a velocity v. We will examine the behavior of this system as $\eta\to 0$. This corresponds to the η doublet having vanishingly small amplitude. Working to first order in η is precisely the linearized problem and will give $\delta\phi$ in a very convenient form.

To define notation and show how to construct such a four-soliton solution algebraically, we display a form of the Bäcklund transformation due to Hirota,[10] which is an algebraic algorithm for creating the N-soliton mode of the sine-Gordon equation. We emphasize that this is essentially a trick available only because sine-Gordon is a perfect system, which means that there are algebraic algorithms available for building new solutions from old ones. In a nonperfect system one would have to resort to solving the linearized problem in a less elegant and nonclosed form, but the principle of finding stability angles is exactly the same, as is their role in the quantization of periodic orbits.

We need four velocity parameters

$$v_1=\frac{i}{\epsilon},\quad v_2=\frac{-i}{\epsilon},\quad v_3=\frac{i/\eta+v}{1+(i/\eta)v}=\frac{\eta v+i}{\eta+iv},\quad v_4=\frac{\eta v-i}{\eta-iv}. \tag{C2}$$

Hirota's form[10] is in terms of the generic functional solution

$$\phi=4\tan^{-1}\left[\frac{g}{f}\right], \tag{C3}$$

where

$$f(x,t)=\sum_{\mu_i=(0,1)}^{4}{}^{(e)}\exp\left[\sum_{i<j}^{4}B_{ij}\mu_i\mu_j+\sum_{j=1}^{4}\mu_j x_j\right],$$

$$g(x,t)=\sum_{\mu_i=(0,1)}^{4}{}^{(o)}\exp\left[\sum_{i<j}^{4}B_{ij}\mu_i\mu_j+\sum_{j=1}^{4}\mu_j x_j\right], \tag{C4}$$

and

$$x_i=k_i x+\beta_i t+\gamma_i\,,\quad \beta_i^2=k_i^2-1,\quad k_i=(1-v_i^2)^{-1/2},$$

and where γ_i are arbitrary constants which we fix conveniently to

$$e^{\gamma_1}=e^{\gamma_2}=\epsilon\,,$$
$$e^{\gamma_3}=e^{\gamma_4}=\eta\,,$$

and the notation $\sum_{\mu_i=(0,1)}^{4}$ implies summation over all possible combinations of $\mu_1=0,1$, $\mu_2=0,1,\ldots$, $\mu_4=0,1$ under the condition

$$\sum_{i=1}^{4}{}^{(e),(o)}\,\mu_i=\text{(even) (odd) integer}\,.$$

The functions B_{ij} are defined through

$$\exp[B_{ij}]\equiv\frac{(k_i-k_j)^2-(\beta_i-\beta_j)^2}{(k_i+k_j)^2-(\beta_i+\beta_j)^2}\,. \tag{C5}$$

Now, considering the generic solution $\phi=4\tan^{-1}(g/f)$ and its linearized form

$$\delta\phi+\phi=4\tan^{-1}\left(\frac{g+\delta g}{f+\delta f}\right)=4\tan^{-1}\left(\frac{g}{f}\right)+4\frac{1}{f^2+g^2}[f\delta g-g\delta f] \tag{C6}$$

we see

$$\delta\phi=4\frac{1}{f^2+g^2}[f\delta g-g\delta f]\,, \tag{C7}$$

with

$$g(x,t)=2\epsilon\exp\left[\frac{\epsilon x}{(1+\epsilon^2)^{1/2}}\right]\cos\left(\frac{t}{(1+\epsilon^2)^{1/2}}\right),$$
$$f(x,t)=\exp\left[\frac{\epsilon x}{(1+\epsilon^2)^{1/2}}\right]\cosh\left(\frac{\epsilon x}{(1+\epsilon^2)^{1/2}}\right), \tag{C8}$$

$$\delta g=2\eta\left[\cos\left(\frac{vx-t}{(1-v^2)^{1/2}}\right)+\exp\left[\frac{2\epsilon x}{(1+\epsilon^2)^{1/2}}\right]\cos\left(\frac{vx-t}{(1-v^2)^{1/2}}+\delta\right)\right],$$
$$\delta f=\exp\left[\frac{\epsilon x}{(1+\epsilon^2)^{1/2}}\right](\eta\epsilon)\left[\rho\cos\left(\frac{\delta}{2}-\frac{t}{(1+\epsilon^2)^{1/2}}+\frac{vx-t}{(1-v^2)^{1/2}}\right)+\frac{1}{\rho}\cos\left(\frac{\delta}{2}+\frac{t}{(1+\epsilon^2)^{1/2}}-\frac{vx-t}{(1-v^2)^{1/2}}\right)\right], \tag{C9}$$

where the phase shift δ is given by

$$\exp(i\delta)=\left[\frac{\mathfrak{D}-i\epsilon v-1}{\mathfrak{D}+i\epsilon v+1}\right]\left[\frac{\mathfrak{D}-i\epsilon v+1}{\mathfrak{D}+i\epsilon v-1}\right] \tag{C10}$$

and $\mathfrak{D}\equiv[(1+\epsilon^2)(1-v^2)]^{1/2}$. ρ is defined through

$$\rho\exp(i\delta/2)\equiv\frac{\mathfrak{D}-i\epsilon v-1}{\mathfrak{D}+i\epsilon v+1}\,. \tag{C11}$$

One can show directly that $\delta\phi$ satisfies Eq. (C1) and is the solution to the linearized problem. The phase shift δ can be written in a more convenient form as

$$\delta=4\tan^{-1}\frac{\epsilon}{k(1+\epsilon^2)^{1/2}}\,, \tag{C12}$$

where $k=v/(1-v^2)^{1/2}$, so that as

$$k\to0,\quad \delta\to2\pi\,,$$
$$k\to\infty,\quad \delta\sim\frac{4\epsilon}{k(1+\epsilon^2)^{1/2}}\,, \tag{C13}$$

which are limits which will have immediate use.

3. Computing the sum of stability angles

If t is replaced by $t+\tau$, $\delta\phi$ undergoes a phase change of

$$\nu=\tau(1+k^2)^{1/2}=2\pi(1+\epsilon^2)^{1/2}(1+k^2)^{1/2}\,, \tag{C14}$$

which is therefore the stability angle for a given k. In a periodic box, the k's are discrete and satisfy

$$k_nL+\delta(k_n)=2n\pi\,,\quad n=\pm1,\pm2,\ldots\,, \tag{C15}$$

where we note that k_1 and k_{-1} both vanish and are therefore not distinct modes. For $n\neq1$, $k_{-n}=-k_n$ and the quantity ξ, of Appendix A and Sec. III can be written as

$$\begin{aligned}\xi&=-\tfrac{1}{2}\sum\nu\\ &=-\frac{\tau}{2}\left[1+\sum_{n=-\infty}^{-2}(k_n{}^2+1)^{1/2}+\sum_{n=2}^{\infty}(k_n{}^2+1)^{1/2}\right]\\ &=-\tau\sum_{n=1}^{\infty}(k_n{}^2+1)^{1/2}+\frac{\tau}{2}\,.\end{aligned} \tag{C16}$$

The vacuum energy is

$$\Delta E = \tfrac{1}{2} + \sum_{n=1}^{\infty} (1 + k_n'^2)^{1/2}, \tag{C17}$$

where

$$Lk_n' = 2\pi n, \tag{C18}$$

and the extra $\frac{1}{2}$ comes from the $k' = 0$ mode. In the limit of an infinite box, one finds

$$\begin{aligned}\lim_{L\to\infty} (\xi + \tau\Delta E) &= -\tau\left[\lim_{L\to\infty} \sum_{n=1}^{\infty} \big((k_n^2+1)^{1/2} - (k_n'^2+1)^{1/2}\big) - 1\right] \\ &= \tau\left[\frac{1}{2\pi}\int_0^{\infty} \frac{k\delta}{(k^2+1)^{1/2}} dk + 1\right]. \end{aligned} \tag{C19}$$

Integrating by parts and using the limits in Eq. (C13), one sees that

$$\begin{aligned}\xi + \tau\Delta E &= 4(\epsilon - \tan^{-1}\epsilon) \\ &\quad + 4\epsilon\int_0^{\Lambda} \frac{dk}{(k^2+1)^{1/2}} + 2\pi, \end{aligned} \tag{C20}$$

where Λ is a cutoff. The additive 2π in (C20) can be dropped since ξ always appears in the form $e^{i\xi}$. Defining

$$\tilde{\xi} = 4(\epsilon - \tan^{-1}\epsilon) \tag{C21}$$

we then have

$$\xi = -\tau\Delta E + 4\epsilon\int_0^{\Lambda} \frac{dk}{(k^2+1)^{1/2}} + \tilde{\xi}, \tag{C22}$$

which with $\epsilon = [(m\tau/2\pi)^2 - 1]^{1/2}$ becomes Eq. (3.7). As explained in Sec. III, the divergent terms in ξ get absorbed by renormalization counterterms. Finally, evaluating some integrals shows that

$$\begin{aligned}\tilde{\xi} &= 4(\epsilon - \tan^{-1}\epsilon) \\ &= \frac{1}{8\pi}\frac{\lambda}{m^2}\int_{-\infty}^{\infty} dx \int_0^{\tau} dt\, \mathcal{L}(\phi \text{ doublet}) \end{aligned} \tag{C23}$$

or

$$\tilde{\xi} = -\frac{1}{8\pi}\frac{\lambda}{m^2}\mathcal{S},$$

which is Eq. (3.15).

APPENDIX D: SOLITON-ANTISOLITON SCATTERING

The solution to the sine-Gordon equation which corresponds to soliton-antisoliton scattering is, in the center-of-mass system,

$$\phi_s = \frac{4m}{\sqrt{\lambda}}\tan^{-1}\left[\frac{\sinh\big(umt/(1-u^2)^{1/2}\big)}{u\cosh\big(mx/(1-u^2)^{1/2}\big)}\right]. \tag{D1}$$

As $t \to -\infty$, ϕ_s approaches

$$\begin{aligned}&-\frac{4m}{\sqrt{\lambda}}\tan^{-1}\left[\exp\left(-\frac{m(x+u(t+\Delta/2))}{(1-u^2)^{1/2}}\right)\right] \\ &\qquad -\frac{4m}{\sqrt{\lambda}}\tan^{-1}\left[\exp\left(\frac{m(x-u(t+\Delta/2))}{(1-u^2)^{1/2}}\right)\right], \end{aligned} \tag{D2}$$

while for $t \to +\infty$ one finds

$$\begin{aligned}&\frac{4m}{\sqrt{\lambda}}\tan^{-1}\left[\exp\left(\frac{m(x+u(t-\Delta/2))}{(1-u^2)^{1/2}}\right)\right] \\ &\qquad +\frac{4m}{\sqrt{\lambda}}\tan^{-1}\left[\exp\left(\frac{-m(x-u(t-\Delta/2))}{(1-u^2)^{1/2}}\right)\right], \end{aligned} \tag{D3}$$

where the time delay Δ in the collision is

$$\Delta = \frac{2}{m}\frac{(1-u^2)^{1/2}}{u}\ln u. \tag{D4}$$

From (D2) and (D3) one sees that ϕ_s can be interpreted as an elastic collision between a soliton and an antisoliton in which each "particle" suffers a time delay Δ.

One can quantize these collision states following the method of Sec. V of I. We will work in the center-of-mass system and compute the density of states dn/dM as a function of the invariant mass

$$M = \frac{2M_{\text{soliton}}}{(1-u^2)^{1/2}} = \frac{16m}{\gamma'(1-u^2)^{1/2}}. \tag{D5}$$

To do this, all we have to do is use the quantization condition

$$\overline{W}(M) = 2\pi n \tag{D6}$$

and differentiate to get

$$\frac{d\overline{W}}{dM} = \tau(M) = 2\pi\frac{dn}{dM}, \tag{D7}$$

where as before $\tau(M)$ is the period in the center-of-mass system.

In our closed-loop world, the soliton-antisoliton scattering orbits are periodic with period

$$\tau(M) = \frac{L}{u} + \Delta, \tag{D8}$$

where the first term is just the time it takes a free particle to circle the world and the second term is the extra time required because of the time delays in collisions. Following Sec. V of I we now write (D7) as

$$\frac{dn}{dM} - \frac{dn}{dM}\bigg|_{\text{free}} = \frac{dn}{dM} - \frac{L}{2\pi u} = \frac{\Delta}{2\pi}, \tag{D9}$$

where $dn/dM|_{\text{free}}$ is the density of states for two free particles.

It is interesting to compute the total number of states in the soliton-antisoliton channel less the number of states that there would be for non-interacting particles. Calling this difference ΔN we note that it has two components. One is the bound states, of which there are $8\pi/\gamma'$, provided that γ' is small enough that we do not have to distinguish between $8\pi/\gamma'$ and the largest integer less than $8\pi/\gamma'$. The other is the difference in continuum states. Their sum is

$$\begin{aligned}\Delta N &= \frac{8\pi}{\gamma'} + \int_{16m/\gamma'}^{\infty} \frac{\Delta dM}{2\pi} \\ &= \frac{8\pi}{\gamma'} + \frac{16}{\gamma'\pi}\int_{-1}^{1}\frac{\ln|u|}{(1-u^2)}du \\ &= \frac{4\pi}{\gamma'},\end{aligned} \tag{D10}$$

where we note that to integrate over all states we must let the velocity u run from -1 to $+1$.

In ordinary quantum mechanics, the time delay is equal to twice the WKB approximation to the energy derivative of the phase shift. Using this fact, we can interpret (D10) as Levinson's theorem in the familiar form

$$\Delta N = n_{\text{el}} = n_B + \frac{1}{2\pi i}[\operatorname{tr}\ln S(\infty) - \operatorname{tr}\ln S(0)], \tag{D11}$$

where n_{el} is the number of elementary particles and n_B is the number of bound states.

APPENDIX E: MASS RATIOS FROM PERTURBATION THEORY

(1) We compute the dimensionless quantity

$$\Delta = \left(\frac{4M_1^{\,2} - M_2^{\,2}}{4M_1^{\,2}}\right)^{1/2} \tag{E1}$$

by summing Feynman diagrams. We will obtain the first three terms in the expansion of Δ in powers of $\gamma = \lambda/m^2$. The expansion parameter γ is not renormalized and is related to γ' by $\gamma' = \gamma(1-\gamma/8\pi)^{-1}$.

To obtain the first approximation to Δ, we sum the chain of bubbles shown in Fig. 4. The bubbles sum to give a two-to-two amplitude,

$$i\lambda\left\{1 - \frac{\gamma}{16}\left(\frac{16m^4}{s(4m^2-s)}\right)^{1/2} \times\left[1 - \frac{2}{\pi}\tan^{-1}\left(\left(\frac{4m^2-s}{s}\right)^{1/2}\right)\right]\right\}^{-1}; \tag{E2}$$

$\sqrt{s}$ is the total energy in the center-of-mass system. For small γ there is a pole just below threshold. This pole can be interpreted as a loosely bound state made up of two nonrelativistic particles interacting through the δ-function potential. If the pole is at s_0 we can identify Δ with $(1-s_0/4m^2)^{1/2}$, and the equation for Δ is

$$\Delta = \frac{\gamma}{16}\frac{1}{(1-\Delta^2)^{1/2}}\left[1 - \frac{2}{\pi}\tan^{-1}\frac{\Delta}{(1-\Delta^2)^{1/2}}\right], \tag{E3}$$

and one finds easily that

$$\Delta = \frac{\gamma}{16} + O(\gamma^2). \tag{E4}$$

To compute the order-γ^2 term in Δ we have to do two things. First, we have to solve (E2) to order γ^2. This amounts to including relativistic corrections to the Schrödinger equation. Second, in the two-to-two amplitude we have to include the one-loop diagrams shown in Fig. 5. The latter is an order-γ^2 correction to the potential. Since the bound state is very close to threshold we may evaluate the diagrams in Fig. 5 at zero momentum transfer. Adding these diagrams to the kernel (which we then have to iterate) has the effect of replacing γ in (E3) by $\gamma(1+\gamma/4\pi)$. Doing this yields the equation

$$\Delta = \frac{\gamma}{16}\left(1+\frac{\gamma}{4\pi}\right)\frac{1}{(1-\Delta^2)^{1/2}}\left[1 - \frac{2}{\pi}\tan^{-1}\frac{\Delta}{(1-\Delta^2)^{1/2}}\right], \tag{E5}$$

and solving to order γ^2 gives

$$\Delta = \frac{\gamma}{16}\left(1+\frac{\gamma}{8\pi}\right) + O(\gamma^3). \tag{E6}$$

We now turn to the calculation of the order-γ^3 term in Δ. We will proceed in analogy with the order-γ^2 calculation. First we compute, through order γ^3, a kernel which is to be iterated. This gives a modified version of (E5) which must then be solved to order γ^3.

In order γ^3, the kernel contains three new effects.

(1) We have to include the two-loop diagrams shown in Fig. 8.

(2) The propagator modification coming from the diagram in Fig. 3 has to be taken into account.

(3) We have to take into account the momentum-

transfer dependence of the diagrams in Fig. 5.

Proceeding in analogy with the order-γ^2 calculation, we write the corrected equation for Δ as

$$\Delta=\frac{\gamma}{16}\left[1+\frac{\gamma}{4\pi}+\gamma^2(a_1+a_2+a_3)\right]\frac{1}{(1-\Delta^2)^{1/2}}\times\left[1-\frac{2}{\pi}\tan^{-1}\frac{\Delta}{(1-\Delta^2)^{1/2}}\right], \tag{E7}$$

where a_1, a_2, and a_3 come from the effects listed as (1), (2), and (3) above. We will compute the a's and then solve (E7) to order γ^3.

The coefficient a_1 is proportional to the threshold value of the sum of two-particle irreducible diagrams shown in Fig. 8. The diagrams can be computed by standard methods. We quote the result in terms of *Wick-rotated* Feynman integrals:

$$I_1=\frac{1}{(2\pi)^2}\int d^2q\,\frac{J(q^2)}{(l+q)^2+m^2}\bigg|_{l^2=-m^2}=\frac{1}{64},$$

$$I_2=\frac{m^2}{(2\pi)^2}\int d^2q\,\frac{J(q^2)}{((l+q)^2+m^2)^2}\bigg|_{l^2=-m^2}=\frac{1}{64\pi^2}+\frac{1}{256}, \tag{E8}$$

$$J(q^2)=\frac{m^2}{(2\pi)^2}\int d^2k\,\frac{1}{(q+k)^2+m^2}\,\frac{1}{k^2+m^2},$$

$$J(0)=\frac{1}{4\pi}.$$

With this notation, a_1 is

$$a_1=\tfrac{1}{2}(J(0))^2-\tfrac{2}{3}I_1+2I_2=\frac{1}{16\pi^2}-\frac{1}{384}, \tag{E9}$$

where the three terms come from the diagrams in Figs. 8(a), 8(b), and 8(c), respectively.

The diagram in Fig. 3 modifies the propagator according to

$$p^2-m^2\to p^2-m^2+\frac{\gamma^2m^2}{6}I_1+(p^2-m^2)\gamma^2(\tfrac{1}{2}I_2-\tfrac{1}{6}I_1)+O((p^2-m^2)^2). \tag{E10}$$

Off-shell terms of order $(p^2-m^2)^2$ in the inverse propagator do not contribute to Δ in order γ^3 and may be dropped. The effect of the (finite) mass and wave-function renormalization in (E10) will be to correct the lowest-order interaction, λ, divided by the physical mass squared according to

$$\frac{\lambda}{m^2}\to\frac{\lambda}{m^2-\frac{1}{6}\gamma m^2I_1}\left[1+\gamma^2(\tfrac{1}{2}I_2-\tfrac{1}{6}I_1)\right]^{-2}=\gamma\left[1-\gamma^2(I_1-\tfrac{1}{2}I_2)\right]+O(\gamma^4), \tag{E11}$$

from which we see that

$$a_2=-I_1+\tfrac{1}{2}I_2=-\frac{1}{64\pi^2}+\frac{1}{256}. \tag{E12}$$

Next we have to take account of the momentum-transfer dependence of diagrams in Fig. 5. We do this by computing the difference between the two diagrams in Fig. 9 and the same diagrams with the exchanged bubble set equal to its value at zero momentum transfer. Since solving (E7) iterates the diagrams in Fig. 5 with the diagrams in Fig. 4 evaluated at zero momentum transfer, it is this difference which gives a_3. Evaluating the difference at threshold yields

$$a_3=\frac{1}{(2\pi)^2}\int\frac{J(x^2+y^2)-J(0)}{|(m+ix)^2-y^2-m^2|^2}\,dx\,dy=\frac{1}{64\pi^2}-\frac{1}{256}. \tag{E13}$$

Note that $a_2=-a_3$, so that the effects (2) and (3) cancel, and the final result is

$$a_1+a_2+a_3=a_1=\frac{1}{16\pi^2}-\frac{1}{384}. \tag{E14}$$

Inserting this into (E7) and solving to order γ^3 then yields

$$\Delta=\frac{\gamma}{16}\left[1+\frac{\gamma}{8\pi}+\left(\frac{\gamma}{8\pi}\right)^2-\frac{1}{6}\left(\frac{\gamma}{16}\right)^2\right], \tag{E15}$$

which agrees with the WKB result to order γ^3.

The cancellation of a_2 and a_3 will occur in any

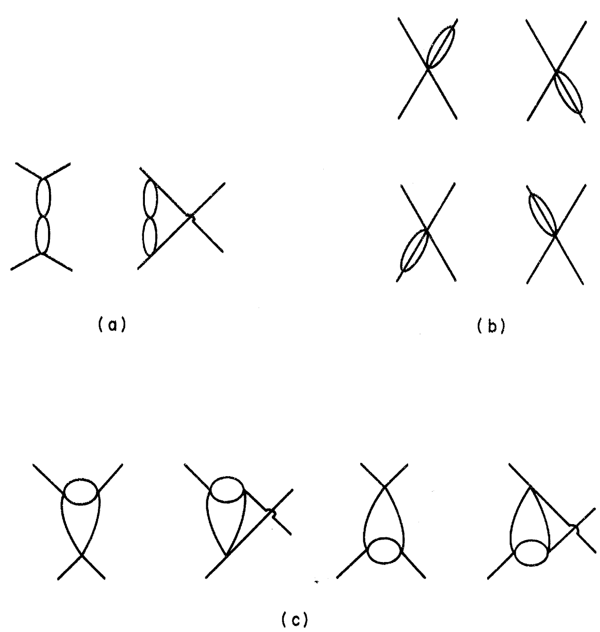

FIG. 8. Two-particle irreducible two-loop graphs.

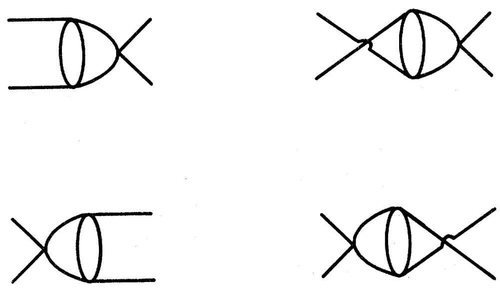

FIG. 9. An iteration of the diagrams in Fig. 4 with the four-point vertex.

Lagrangian of the form

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4-\frac{b\lambda^2}{6!m^2}\phi^6+\cdots, \tag{E16}$$

where the sine-Gordon Lagrangian corresponds to the special case $b=1$. The ϕ^6 interaction in the sine-Gordon Lagrangian yielded the term proportional to I_1 in Eq. (E9) for a_1. For the general case a_1 will be

$$a_1=\frac{1}{16\pi^2}-\frac{1}{384}-\frac{b-1}{96}, \tag{E17}$$

and Δ will be

$$\Delta=\frac{\gamma}{16}\left(1+\frac{\gamma}{8\pi}+\left(\frac{\gamma}{8\pi}\right)^2-\frac{1}{6}\left(\frac{\gamma}{16}\right)^2-\frac{(b-1)\gamma^2}{96}\right). \tag{E18}$$

As pointed out in the text, (E18) does not agree with WKB for $b\neq 1$.

(2) The Lagrangian for the massive Thirring model is

$$\mathcal{L}=\bar\psi i\partial\!\!\!/\psi-M_f\bar\psi\psi-\frac{g}{2}(\bar\psi\gamma^\mu\psi)^2. \tag{E19}$$

It is equivalent to

$$\mathcal{L}=\bar\psi i\partial\!\!\!/\psi-M_f\bar\psi\psi+\frac{\mu^2}{2}(B^\nu)^2+\mu\sqrt{g}\,B_\nu\bar\psi\gamma^\nu\psi. \tag{E20}$$

The field B^μ has a propagator $ig^{\mu\nu}/\mu^2$. We will regulate the theory by making the replacement

$$\frac{ig^{\mu\nu}}{\mu^2}\to\frac{ig^{\mu\nu}}{\mu^2-p^2} \tag{E21}$$

and letting $\mu\to\infty$ at the end of the calculation. With this regulated propagator for B^ν there is only one divergent diagram. It is the "vacuum-polarization" loop shown in Fig. 10. We use current

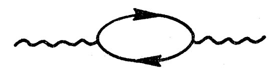

FIG. 10. "Vacuum polarization" in the massive Thirring model. The directed lines are fermions. The wiggly lines represent fictitious heavy vector mesons.

conservation to define it according to

$$\frac{-i}{(2\pi)^2}\int \mathrm{tr}\left(\gamma^\mu\frac{p\!\!\!/+k\!\!\!/+M_f}{(p+k)^2-M_f^2}\gamma^\nu\frac{k\!\!\!/+M_f}{k^2-M_f^2}\right)d^2k=\left[g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right]\left[J(-p^2)-J(0)\right], \tag{E22}$$

where J is defined as in (E8), with m^2 replaced by M_f^2.

We will have to make a mass renormalization for the fermion. After subtracting a counterterm δM_f proportional to $\ln\mu$ the diagram in Fig. 11 vanishes for large μ and the inverse fermion propagator is

$$i(p\!\!\!/-M_f)+O(g^2). \tag{E23}$$

Since there is no wave-function renormalization in order g, the Ward identity tells us that the "vertex correction" shown in Fig. 12 vanishes at zero momentum transfer. One can check that it does.

We now turn to the calculation of the bound-state mass. The "exchange" and "annihilation" diagrams shown in Fig. 13 lead to the fermion-antifermion potential

$$V=-2g\delta(x), \tag{E24}$$

which produces a nonrelativistic bound state with mass

$$M_B=2M_f-g^2M_f+O(g^3). \tag{E25}$$

To go to a higher order in g, it is convenient to define

$$\Delta'\equiv\left[\frac{4M_f^2-M_B^2}{4M_f^2}\right]^{1/2}=g+O(g^2). \tag{E26}$$

Our goal is to compute the order-g^2 term in Δ'. To accomplish this we have to do two things. First, we have to sum the diagrams shown in Fig. 14 and compute the position of the pole to order g^2. Second, we have to correct the coefficient of the δ-function potential in (E24) to include

FIG. 11. The lowest-order correction to the fermion propagator in the massive Thirring model.

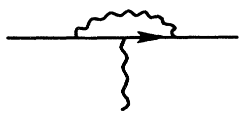

FIG. 12. The lowest-order vertex correction in the massive Thirring model.

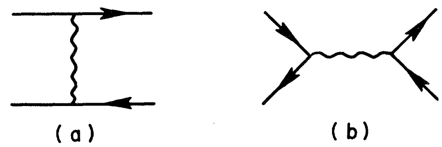

FIG. 13. The fermion-antifermion interaction in the massive Thirring model. As in Figs. 10–12 the wiggly line represents a fictitious heavy vector meson. This fictitious particle indicates the ordering of γ matrices and regulates divergent diagrams.

the diagram shown in Fig. 15. This is equivalent to solving the Bethe-Salpeter equation with the kernel correct to order g^2 and the propagator correct to order g. There is no fermion propagator modification in order g, so we can use the free fermion propagator. The complete order-g^2 Bethe-Salpeter kernel includes the diagrams shown in Fig. 16, but since they vanish at zero momentum transfer they do not affect Δ' in order g^2. We are looking for an odd-charge-conjugation state, so the diagrams shown in Fig. 17 do not appear in the kernel.

The chain of diagrams in Fig. 14 can be summed by standard means. The easiest thing to do is to make a Fierz transformation of the "exchange" diagram shown in Fig. 13(a) so that it becomes an "annihilation" diagram with γ_5 coupling in the odd-charge-conjugation channel. The position of the pole is then given by the vanishing of the determinant

$$\begin{vmatrix} 1-\frac{g}{2}\left[F(\Delta')-\frac{2}{\pi}\right] & -\frac{ig}{2}(1-\Delta'^2)^{1/2}F(\Delta') \\ \frac{ig}{2}(1-\Delta'^2)^{1/2}F(\Delta') & 1-\frac{g}{2}\left[(1-\Delta'^2)F(\Delta')+\frac{2}{\pi}\ln\left(\frac{\mu}{M_f}\right)\right] \end{vmatrix} =0\,, \tag{E27}$$

where we have taken the limit of large μ and

$$\begin{aligned} F(\Delta') &= \frac{1}{\Delta'(1-\Delta'^2)^{1/2}}\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{\Delta'}{(1-\Delta'^2)^{1/2}}\right)\right] \\ &= \frac{1}{\Delta'}-\frac{2}{\pi}+O(\Delta')\,. \end{aligned} \tag{E28}$$

From (E27) and (E28) we find that to order g^2 the equation for Δ' is

$$\begin{aligned} 1 &= g\left[\frac{1}{\Delta'}-\frac{2}{\pi}\right]+\frac{g}{\pi}\left[1-\frac{g}{2\Delta'}\right]\left[1-\ln\left(\frac{\mu}{M_f}\right)\right] \\ &=0\,, \end{aligned} \tag{E29}$$

and solving to order g^2 yields

$$\Delta' = g-\frac{2}{\pi}g^2-\frac{g^2}{2\pi}\left[1-\ln\left(\frac{\mu}{M_f}\right)\right]. \tag{E30}$$

As will be seen shortly, the $\ln\mu$ divergence in (E30) is spurious.

It is straightforward to compute the diagram in Fig. 15. One finds that it produces an attractive δ-function potential

$$V=-\frac{g^2}{\pi}\left[1-\ln\left(\frac{\mu}{M_f}\right)\right]\delta(x)\,. \tag{E31}$$

Comparing with (E24) one sees that the effect on

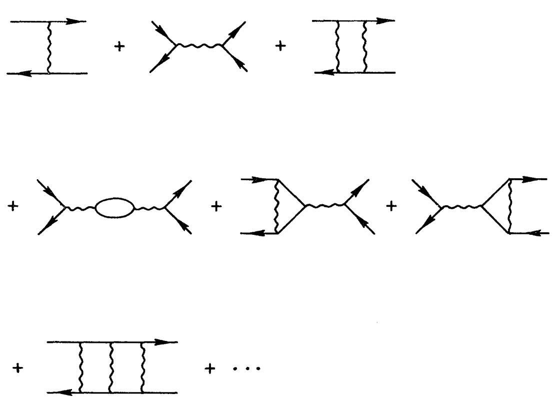

FIG. 14. The diagrams summed to give part of the order-g^2 term in Δ' as defined in the Appendix E.

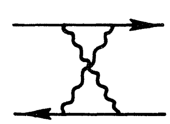

FIG. 15. A two-particle irreducible diagram appearing in the order-g^2 Bethe-Salpeter kernel.

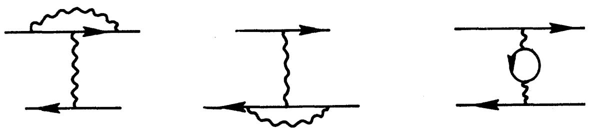

FIG. 16. Diagrams in the Bethe-Salpeter kernel which do not affect Δ' in order g^2.

FIG. 17. Diagrams which do not appear in the kernel for the C-odd channel.

Δ' will be to make the replacement

$$g \to g + \frac{g^2}{2\pi}\left[1 - \ln\left(\frac{\mu}{M_f}\right)\right] \tag{E32}$$

in (E30). The final result is then

$$\Delta' = g - \frac{2}{\pi}g^2 + O(g^3). \tag{E33}$$

As pointed out in the text, this agrees with WKB to the indicated order.

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[19]When we use phrases such as "rigorous application of the semiclassical method" or "exact within the WKB approximation," what we mean is that there is no further approximation beyond the basic WKB ansatz of evaluating certain functional integrals by the stationary-phase method. The energy levels of the anharmonic oscillator discussed in the Introduction are an example of a calculation which is a rigorous application of WKB. In paper I we stressed that the full semiclassical method is not to be confused with various weak-coupling approximations to it. We pointed out that for strong coupling, one has to find a large class of solutions to the classical field equations. In a separable system, such as the sine-Gordon equation, we can examine all the solutions and extract those which are relevant to the particle spectrum of the theory. While we would expect results which are exact within the WKB approximation to be useful for strong coupling, we would not generally expect them to be literally exact. The sine-Gordon equation seems to be a special case where WKB is literally exact.

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