PHYSICAL REVIEW D VOLUME 11, NUMBER 12 15 JUNE 1975

Equivalence between time-dependent and time-independent formulations of time delay*

D. Bollé[†‡]
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

T. A. Osborn
Department of Physics, Brooklyn College, City University of New York, Brooklyn, New York 11210
(Received 6 November 1974; revised manuscript received 26 March 1975)

The time-dependent and time-independent treatments of time delay in two-body scattering are discussed and shown to be completely equivalent. Further, we provide a careful discussion of the definition of time delay in both approaches. We are able to establish that the customary method of using an average limiting process is unnecessary if the time delay is evaluated between normalizable wave packets.

I. INTRODUCTION

There exist in the literature two fundamental approaches to the notion of time delay for two-particle scattering. One is based upon the time-dependent description of scattering theory, the other on the time-independent formalism. It is the purpose of this paper to carefully discuss these two definitions of time delay and to establish an exact equivalence between them.

The time-independent approach to time delay was developed by Smith.[1] Using the steady-state solution of the time-independent Schrödinger equation for a single energy E, Smith calculates the excess number of particles near the scattering center after subtracting the number that would have been present in the absence of the interaction. Time delay is then determined to be this excess number divided by the incoming (or outgoing) flux at a large distance R from the scattering center.

Time-dependent approaches to aspects of time delay date back to the early work of Wigner and Eisenbud.[2] However, the time-dependent version we employ in this paper is that found in the work of Jauch and Marchand.[3] The method developed by Jauch and Marchand (JM) possesses a number of advantages not found in prior work on this problem. First, by utilizing a definition proposed by Goldberger and Watson[4] they are able to associate a Hermitian operator Q with the observable time delay. The derivation of time delay by JM uses only simple abstract operator identities, the relationship of the Møller operators to the S matrix, the intertwining property, and the singularity structure of the momentum-space representation of the Møller operators. Thus one learns that associated with time delay there are certain universal operator identities commom to all scattering problems. In fact, in other works[5,6] we have extended this operator approach to define time delay for the three-body problem. A further advantage of the abstract JM method of analysis is that it proves that the time-delay formalism is valid for all types of wave packets. No restrictions, such that the wave packet is a sharply peaked function about some central momentum, are needed.

A commom technical problem besets both approaches described above. In both one computes the delay of the wave packet for a finite region of space, generally a sphere of radius R, and then one takes $R\to\infty$. This gives one a time delay for the entire wave packet in all of the three-dimensional space. However, in both approaches the plane-wave matrix elements that arise have oscillatory terms involving $\sin 2kR$. Normally these terms do not lead to a well-defined limit as $R\to\infty$. The remedy of both Smith and JM was to first average the definition of time delay over an interval $(R, 2R)$ and then let R become infinite. This *ad hoc* averaging procedure makes the troublesome terms zero.

Because of its arbitrary character and a lack of physical foundation, this averaging process is undesirable. However, we show in Sec. II that one may define the time delay in the JM sense without this averaging process and still obtain the fundamental result. Our variation of the JM proof is based on the Riemann-Lebesgue lemma and the fact that one is always evaluating the time-delay operator between normalized square-integrable wave packets. Our results allow one to understand that the averaging process introduced by Smith permits him to compute the matrix elements of non-normalized states and still get the same answer as one obtains for normalized matrix elements. An alternate average-free approach is found in the work of Jauch, Sinha, and Misra.[7] However, they only prove that the trace of Eq. (2.9) is valid. This is a weaker statement than Eq. (2.9) itself. Specifically if one computes the time delay for an arbitrary wave packet one needs the nondiagonal matrix elements of Q.

In Sec. III we describe Smith's approach and establish the equivalence of the two different definitions of time delay. We also discuss the role of the oscillatory terms. Finally, the Appendix contains the proofs of the mathematical properties of the projection operators needed in the modified JM method presented in Sec. II.

We close this introduction with the observation that the time delay discussed in this paper is a global property of the entire scattering process. This "global time delay" is not the only concept of time delay found in the literature. A distinct and contrasting form of time delay is found in the work of Brenig and Haag,[8] and its extensions are studied by Froissart, Goldberger, and Watson.[9] In these later works the time delay is an angle-dependent quantity. It seems obvious that these different kinds of time delay must be interrelated. In fact, an explicit connection between the two may be found in a paper by Nussenzveig.[10]

II. THE MODIFIED JAUCH-MARCHAND METHOD

On the basis of the definition given in the Introduction, we can write down the following formula for the matrix elements of the time delay Q of an incident wave packet ϕ:

$$(\phi, Q\phi) \equiv \lim_{R\to\infty} \frac{1}{R}\int_R^{2R} dR \int_{-\infty}^{+\infty} dt\,[(\psi_t, P_R\psi_t) - (\phi_t, P_R\phi_t)]. \tag{2.1}$$

The operator P_R is defined as the projection operator associated with the particle being inside a sphere of radius R around the scattering center, viz.,

$$(P_R\phi)(\vec{x}) = \begin{cases} \phi(\vec{x}) & \text{for } |\vec{x}| \le R \\ 0 & \text{for } |\vec{x}| > R. \end{cases} \tag{2.2}$$

The vector coordinate $\vec{x}$ denotes the separation between the incident and target particles. We see that the first scalar product in Eq. (2.1) represents the probability of finding the particle described by the wave function ψ_t inside the above-mentioned sphere at time t. This gives, integrated over all times, the average total time spent by the particle in that sphere during the scattering process described by ψ. The second integral in Eq. (2.1) represents the same quantity for the free particle described by the associated asymptotic wave function ϕ.

To find the time delay itself, we then have to take the limit $R\to\infty$ of this time difference. That limit in Eq. (2.1) is clearly taken in the average sense in order to get rid of oscillatory terms[3] in R. In the following we shall denote this average limit by $\langle\lim\rangle_{R\to\infty}$.

Introducing now the well-known time evolution properties of ψ_t and ϕ_t and using the properties of the Møller wave operators $\Omega^{(+)}$ involved,[3] we can define a time-delay operator from Eq. (2.1) in a straightforward way:

$$Q \equiv \langle\lim_{R\to\infty}\rangle \int_{-\infty}^{+\infty} dt\, e^{iH_0t}[\Omega^{(+)\dagger}P_R\Omega^{(+)} - P_R]e^{-iH_0t}. \tag{2.3}$$

We next recall that the S operator is an on-shell operator so that we can write

$$\langle\vec{k}|S|\vec{k}'\rangle = \frac{\delta(E-E')}{\mu k}\langle\hat{k}|s(E)|\hat{k}'\rangle, \tag{2.4}$$

where the reduced $s(E)$ matrix is given in terms of the two-particle t matrix by

$$\langle\hat{k}|s(E)|\hat{k}'\rangle = \delta(\hat{k}-\hat{k}') - 2\pi i\mu k\langle k\hat{k}|t(E)|k\hat{k}'\rangle \tag{2.5}$$

and where μ is the reduced mass. This definition of $s(E)$ is constructed such that the important algebraic properties of S (e.g., unitarity) are preserved.

Looking then at expression (2.3) we can easily derive that the time-delay operator Q is also an on-shell operator, viz.,

$$\langle\vec{k}|Q|\vec{k}'\rangle = \frac{\delta(E-E')}{\mu k}\langle\hat{k}|q(E)|\hat{k}'\rangle, \tag{2.6}$$

where

$$\langle\hat{k}|q(E)|\hat{k}'\rangle = \langle\lim_{R\to\infty}\rangle\, 2\pi\mu k\langle k\hat{k}|\Omega^{(+)\dagger}P_R\Omega^{(+)} - P_R|k\hat{k}'\rangle. \tag{2.7}$$

Using these reduced expressions we can quote the JM result, which they get by calculating $\langle\hat{k}|s(E)q(E)|\hat{k}'\rangle$ starting from Eq. (2.3), in the following way. Let the kernels $\langle\vec{k}|P_R|\vec{k}'\rangle$ be distributions on a test-function space containing at least the integral kernels $\langle\vec{k}|t^{(+)}|\vec{k}'\rangle$ and $\langle\vec{k}|t^{(-)\dagger}|\vec{k}'\rangle$ considered as functions of $\vec{k}'$ (for fixed $\vec{k}$) or as functions of $\vec{k}$ (for fixed $\vec{k}'$) and suppose that the following assumptions hold for the distributions:

$$\text{(a) } \langle\vec{k}|P_R|\vec{k}'\rangle \xrightarrow[R\to\infty]{} \delta(\vec{k}-\vec{k}'),$$

$$\text{(b) } P\left(\frac{1}{k'-k}\right)\langle\vec{k}|P_R|\vec{k}'\rangle \xrightarrow[R\to\infty]{} -\delta'(\vec{k}-\vec{k}'), \tag{2.8}$$

$$\text{(c) } \langle\lim_{R\to\infty}\rangle\langle k\hat{k}|P_R|k\hat{k}'\rangle = 0.$$

Then the following relation holds:

$$\langle \hat{k} | q(E) | \hat{k}' \rangle = -i \int d\hat{k}'' \langle \hat{k}'' | s(E) | \hat{k} \rangle^* \frac{d}{dE} \langle \hat{k}'' | s(E) | \hat{k}' \rangle . \tag{2.9}$$

Let us remark immediately that these three assumptions in (2.8), particularly (c), are not all physically transparent.[7] Fortunately, we can just discard them by using instead explicit calculations and the Riemann-Lebesgue lemma.[11] At the same time we will show that the average limiting process in Eq. (2.3) is not needed. To prove these last statements, we follow different steps similar to the original JM derivation.

We first calculate the product SQ starting from Eq. (2.3) with a nonaveraged limit. Using standard results of scattering theory[3] we arrive at

$$SQ = \lim_{R\to\infty} \int_{-\infty}^{+\infty} dt\, e^{iH_0 t} \Omega^{(-)\dagger} [P_R, \Omega^{(+)}] e^{-iH_0 t} . \tag{2.10}$$

Taking matrix elements of this expression (2.10) between incoming and outgoing wave packets we have

$$\langle \phi | SQ | \phi' \rangle = \lim_{R\to\infty} \int d\vec{k}\, d\vec{k}'\, \phi(\vec{k})^* 2\pi\delta(E-E') \times \langle \vec{k} | \Omega^{(-)\dagger} [P_R, \Omega^{(+)}] | \vec{k}' \rangle\, \phi'(\vec{k}') . \tag{2.11}$$

Since this expression (2.11) holds for a dense set of functions ϕ and ϕ', defined in the Appendix, we can associate it with the following equality between the kernels:

$$\langle \vec{k} | SQ | \vec{k}' \rangle = \lim_{R\to\infty} 2\pi\delta(E-E') \langle \vec{k} | \Omega^{(-)\dagger} [P_R, \Omega^{(+)}] | \vec{k}' \rangle . \tag{2.12}$$

Effectively Eq. (2.12) is a shorthand notation for Eq. (2.11) in the sense that the limiting process $R\to\infty$ is always executed after the matrix elements are sandwiched between wave packets. Introducing then the known relations

$$\langle \vec{k} | \Omega^{(\pm)} | \vec{k}' \rangle = \delta(\vec{k}-\vec{k}') - \langle \vec{k} | T^{(\pm)} | \vec{k}' \rangle \tag{2.13a}$$

where

$$\langle \vec{k} | T^{(\pm)} | \vec{k}' \rangle = \frac{t(\vec{k}, \vec{k}', E' \pm i0)}{E - E' \mp i0} , \tag{2.13b}$$

we easily get for Eq. (2.12)

$$\langle \vec{k} | SQ | \vec{k}' \rangle = \lim_{R\to\infty} 2\pi\delta(E-E') \{ \langle \vec{k} | [T^{(+)}, P_R] | \vec{k}' \rangle + \langle \vec{k} | T^{(-)\dagger} [P_R, T^{(+)}] | \vec{k}' \rangle \} . \tag{2.14}$$

We next calculate the first part on the right-hand side of this expression by working out the commutator and expressing the singular denominators in terms of their δ-function and principal-value parts. The principal value part reads then

$$\lim_{R\to\infty} 4\pi\mu\delta(E-E') \int d\vec{k}'' \left\{ t(\vec{k}, \vec{k}'', E'') \frac{\langle \vec{k}'' | P_R | \vec{k}' \rangle}{k^2 - k''^2} - \frac{\langle \vec{k} | P_R | \vec{k}'' \rangle}{k''^2 - k'^2} t(\vec{k}'', \vec{k}', E') \right\} , \tag{2.15}$$

where we have systematically suppressed the $+i0$ in the energy argument of the t matrix. Now, for nonpathological potentials, we may assume that the t matrix is a smooth function of k'' and is differentiable with respect to k'' so that we can apply the first property of the projection operator P_R proved in the Appendix, viz.,

$$\lim_{R\to\infty} \int d\vec{k}'' t(\vec{k}, \vec{k}'', E'') \frac{\langle \vec{k}'' | P_R | \vec{k}' \rangle}{k'^2 - k''^2} = -\frac{d}{dk''} \left[\frac{k''}{k'} \frac{t(\vec{k}, k''\hat{k}', E'')}{k'' + k'} \right] \Bigg|_{k''=k'} . \tag{2.16}$$

We get then for the expression (2.15), after some algebra,

$$4\pi\mu\delta(E-E') \left[-\frac{1}{2k^2} t(\vec{k}, \vec{k}', E') - \frac{1}{2k} \frac{d}{dk''} t(\vec{k}, k''\hat{k}', E') \Bigg|_{k''=k'} - \frac{1}{2k} \frac{d}{dk''} t(k''\hat{k}, \vec{k}', E') \Bigg|_{k''=k} \right] . \tag{2.17}$$

To finish the calculation of the first part of expression (2.14) we still have to consider the terms containing the δ-function parts, namely,

$$\lim_{R\to\infty} 2\pi\delta(E-E') \int d\vec{k}'' [t(\vec{k}, \vec{k}'', E'') i\pi\delta(E-E'') \langle \vec{k}'' | P_R | \vec{k}' \rangle - \langle \vec{k} | P_R | \vec{k}'' \rangle t(\vec{k}'', \vec{k}', E') i\pi\delta(E''-E')] . \tag{2.18}$$

We then apply the second property of P_R proved in the Appendix, viz.,

$$\int d\vec{k}''\, t(k'\hat{k},\vec{k}'',E'')\delta\left(\frac{k'^2}{2\mu}-\frac{k''^2}{2\mu}\right)\langle\vec{k}''|P_R|\vec{k}'\rangle \underset{R\to\infty}{\sim} \frac{R\mu}{\pi k'}t(k'\hat{k},\vec{k}',E')-\frac{\sin 2k'R}{2k'}\frac{\mu}{\pi k'}t(k'\hat{k},-\vec{k}',E'). \quad (2.19)$$

We get then for expression (2.18) the following form:

$$-i\,2\pi^2\delta(E-E')\frac{\sin 2k'R}{2k'}\frac{\mu}{\pi k'}[t(k'\hat{k},-\vec{k}',E')-t(-k'\hat{k},\vec{k}',E')]. \quad (2.20)$$

So, if the interaction which produces the scattering conserves parity, i.e., $v(\vec{k},\vec{k}')=v(-\vec{k},-\vec{k}')$ then this term is exactly zero because of the cancellation of the t matrices. But this term disappears even in a more general scattering situation. Namely, if we recall that formula (2.20) is part of expression (2.12), which is the kernel equivalent of Eq. (2.11), we see that we have to evaluate the following:

$$\lim_{R\to\infty} -i\pi\mu^2\int k'\sin 2k'R[t(k'\hat{k},-\vec{k}',E')-t(-k'\hat{k},\vec{k}',E')]\,\phi(k'\hat{k})^*\phi(\vec{k}')dk'\,d\hat{k}\,d\hat{k}'. \quad (2.21)$$

And the integral in k' is zero in the limit $R\to\infty$, again because of the Riemann-Lebesgue lemma.

Finally, we have to examine the second part on the right-hand side of Eq. (2.14). For this term, we can easily prove that

$$\lim_{R\to\infty} 2\pi\delta(E-E')\langle\vec{k}|T^{(-)\dagger}[P_R,T^{(+)}]|\vec{k}'\rangle=0. \quad (2.22)$$

To do this we follow the argument of JM.[12] We expand this term in principal-value and δ-function parts. A straightforward calculation shows that the terms containing at least one δ function exactly cancel each other. The remaining terms containing the two principal-value parts can be shown to give zero in the limit $R\to\infty$ by using the first property of the projection operators and the symmetry features of the integrand. We also note that JM do not use the averaging procedure for this portion of the derivation. Combining now the foregoing calculation and arguments allows us to write for Eq. (2.12)

$$\langle\vec{k}|SQ|\vec{k}'\rangle=4\pi\mu\delta(E-E')\left[-\frac{1}{2k^2}t(\vec{k},\vec{k}',E)-\frac{1}{2k}\frac{d}{dk}t(k\hat{k},k\hat{k}',E)\right]. \quad (2.23)$$

Introducing then the reduced matrix elements for S and Q [cf. Eqs. (2.4) and (2.6)], and forming the energy derivative of the reduced $s(E)$ matrix defined by Eq. (2.5), it is easy to see that Eq. (2.23) becomes

$$\int d\hat{k}''\langle\hat{k}|s(E)|\hat{k}''\rangle\langle\hat{k}''|q(E)|\hat{k}'\rangle=-i\frac{d}{dE}\langle\hat{k}|s(E)|\hat{k}'\rangle. \quad (2.24)$$

And this equation is equivalent to Eq. (2.9) because of the unitarity property of the reduced $s(E)$ matrix.

So we have proved the original JM result without making the assumptions (2.8) and without using an averaging procedure. This also means that the time-delay-operator definition (2.3) is valid without averaging the limit.

In the next section we shall compare this time-dependent treatment of time delay with the time-independent method of Smith.

III. RELATION WITH SMITH'S APPROACH

In this section we want to establish the equivalence between the time-dependent formulation of time delay discussed so far and the time-independent treatment of Smith.

Since Smith originally defined time delay in the context of elastic scattering for a spinless particle in a certain partial wave, we first have to specialize the foregoing modified JM formulation to this case. As usual, we choose the complete set of commuting observables for this scattering to be $(2\mu H_0)^{1/2}$, L^2, and L_z with spectral variables k, l, and m. So, the relevant time-delay matrix elements are [cf. Eq. (2.7)]

$$\langle lm|q(E)|l'm'\rangle=\langle\lim_{R\to\infty}\rangle\,2\pi\mu k\langle klm|\Omega^{(+)\dagger}P_R\Omega^{(+)}-P_R|kl'm'\rangle. \quad (3.1)$$

It is straightforward to see that the first term on the right-hand side of this expression (3.1) can be written as

$$\langle kl\,m|\,\Omega^{(+)\dagger}P_R\Omega^{(+)}\,|\,kl'\,m'\rangle = \int_0^R r^2 dr\, d\hat{r}\, Y^*_{lm}(\hat{r})Y_{l'm'}(\hat{r}')\langle kl\,m|\,\Omega^{(+)\dagger}\,|\,rl\,m\rangle\langle rl'\,m'\,|\Omega^{(+)}|\,kl'\,m'\rangle \tag{3.2}$$

$$= \delta_{ll'}\,\delta_{mm'}\int_0^R r^2 dr\,|\,\psi_l^{(+)}(r,k)|^2, \tag{3.3}$$

where we suppressed the index m on the radial wave function because this is independent of m for the scattering we are talking about. The second term of that expression (3.1) can be calculated using the second property of the Appendix or can be taken over from the JM paper.[13] It is equal to

$$2\pi\mu k\langle\, kl\,m|\,P_R\,|\,kl'\,m'\rangle$$

$$\underset{R\to\infty}{\sim}\ \delta_{ll'}\,\delta_{mm'}\left[2R\,\frac{\mu}{k}-(-1)^l\frac{\mu}{k}\frac{\sin 2kR}{k}\right]. \tag{3.4}$$

The first term on the right-hand side of Eq. (3.4) is then the transit time of a free particle with velocity $v=k/\mu$ (in a certain angular momentum state) through a sphere of radius R. The second oscillating term[14] will be discussed at the end of this section.

Combining then the foregoing calculations, we get for the time-dependent formulation of time delay

$$q_{lm}(E)=\langle\lim_{R\to\infty}\rangle\int_0^R r^2 dr\left[4\pi\frac{\mu k}{2}\,|\,\psi_l^{(+)}(r,k)|^2-\frac{2}{(k/\mu)r^2}\right]. \tag{3.5}$$

To show that this formula is the same as Smith's stationary definition, we have to keep in mind that the wave functions used by Smith are normalized to unit inward and outward flux through the surface of a sphere at large R. We can easily calculate this flux to get

$$F_{\text{inc}}=\frac{2}{k\mu}. \tag{3.6}$$

Then Eq. (3.5) becomes

$$q_{lm}(E)=\langle\lim_{R\to\infty}\rangle\int_0^R d\vec{r}\left[\frac{1}{F_{\text{inc}}}\,|\,\psi_l^{(+)}(r,k)|^2-\frac{2}{4\pi v r^2}\right]. \tag{3.7}$$

And this is indeed Smith's formula except for the factor $1/4\pi$ in the second term on the right-hand side. This factor is obviously forgotten in Smith's paper and is corrected for example in the paper by Ohmura.[15]

Starting from this definition (3.7), then, we can derive, following Smith, the same relation between time delay and the S matrix as we had before in Eq. (2.9) by using just the Schrödinger equation for ψ^* and $\partial\psi/\partial E$. In that calculation, which we do not need to repeat here, there appear some oscillating terms of the form[1]

$$-\frac{\mu}{k^2}\sin(2kR+2\delta_l).$$

The statement we want to make again is that if we calculate the time-delay operator between smooth incoming and outgoing wave packets $\phi'(\vec{k}')$ and $\phi(\vec{k})$, the oscillating terms do not contribute in the limit $R\to\infty$. To demonstrate this we write down the following straightforward steps:

$$\langle\phi|\,Q|\,\phi'\rangle=\int\phi^*(\vec{k})\langle\vec{k}|\,Q|\vec{k}'\rangle\,\phi'(\vec{k}')d\vec{k}\,d\vec{k}' \tag{3.8}$$

$$=\int k^2 dk\,d\hat{k}\,d\hat{k}'\,\phi^*(\vec{k})\times\sum_{lm}q_{lm}(E)Y_{lm}(\hat{k})Y^*_{lm}(\hat{k}')\phi'(k\hat{k}'). \tag{3.9}$$

The oscillating terms now give contributions proportional to

$$\int dk\,\phi^*(k)\phi'(k\hat{k}')\sin(2kR+2\delta_l), \tag{3.10}$$

and this integral vanishes in the limit $R\to\infty$ because of the Riemann-Lebesgue lemma.

The conclusion is then that both formulations of time delay are completely equivalent. This is very useful to know when studying different results and applications available in the literature.[10,16]

A last, important point we want to make is that these oscillating terms and their disappearance when $R\to\infty$ have a simple physical interpretation. As Nussenzveig has observed,[17] these oscillating terms arise from the fact that the wave packet has an uncertainty in position $\Delta r\sim 1/\Delta k$. For small k and fixed R, this may be an important effect. However, after one takes $R=\infty$ and one computes the time delay for a normalized wave packet in the whole space, these localization effects vanish. And this is independent of whether one has calculated the time delay in coordinate space, as Smith does, or in momentum space, as

Jauch and Marchand do.

ACKNOWLEDGMENT

One of the authors (D. B.) wants to thank the members of the Nuclear Theory group of the University of Maryland for their warm hospitality.

APPENDIX

This appendix studies some properties of the projection operators $\langle \vec{k}' | P_R | \vec{k} \rangle$ when $R \to \infty$. It is the detailed results proved here which allow us to discard the technical assumptions and the averaging in R used in the work of Jauch and Marchand.

We assume that our operators act on the following dense set of functions: the functions of compact support having at least three derivatives belonging to $L^2(R^3)$. A subset of this set is the Schwarz test-function space.

We first note, then, that

$$\lim_{R\to\infty} (f', P_R f) = (f', f), \tag{A1}$$

where f' and f are any functions in the above dense set. This is of course an immediate consequence of the strong convergence of P_R to the identity. Furthermore, the kernel for P_R in momentum space can be written as

$$\langle \vec{k}' | P_R | \vec{k} \rangle = \int_{|\vec{x}|<R} d^3x \, \frac{1}{(2\pi)^3} e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}. \tag{A2}$$

A straightforward computation of the integral gives

$$\langle \vec{k}' | P_R | \vec{k} \rangle = \left(\frac{R}{2\pi |\vec{k}' - \vec{k}|} \right)^{3/2} J_{3/2}(|\vec{k}' - \vec{k}| R), \tag{A3}$$

where $J_{3/2}$ is a Bessel function of the first kind. The first property we want to prove is then

$$\lim_{R\to\infty} \int d\vec{k}\, d\vec{k}' f'(\vec{k}')^* \frac{\langle \vec{k}' | P_R | \vec{k} \rangle}{k^2 - k'^2} f(\vec{k}) = \int d\vec{k}' f'(\vec{k}')^* \frac{d}{dk} \left[\frac{k}{k'} \frac{f(k\hat{k}')}{k + k'} \right] \Bigg|_{k=k'}, \tag{A4}$$

where this integral has a well-defined meaning as a principal-value integral. Introducing the variable $z = R(\vec{k} - \vec{k}')$ we have

$$R^2(k^2 - k'^2) = z(z + 2Rk' \cos\theta), \tag{A5}$$

where we took the z axis of the coordinate system describing the vector $\vec{z}/R$ parallel to $\vec{k}'$. Using Eq. (A3) we can write the integral on the left-hand side of Eq. (A4) as

$$\int f'(\vec{k}')^* \frac{J_{3/2}(z)}{(2\pi z)^{3/2}} F_{z/R}(\vec{k}')\, dz\, d\vec{k}', \tag{A6}$$

where $F_{z/R}$ is the following average:

$$F_{z/R} = \frac{1}{2} \int_{-1}^{+1} d\cos\theta \left[\frac{z}{R} \left(\frac{z}{R} + 2k' \cos\theta \right) \right]^{-1} \times \bar{f}_{z/R}(\vec{k}', \cos\theta), \tag{A7}$$

$$\bar{f}_{z/R}(\vec{k}', \cos\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi\, f(\vec{k}' + \vec{z}/R). \tag{A8}$$

It is clear that in domains excluding $\vec{z} = 0$, $F_{z/R}(\vec{k}')$ is integrable with respect to $d\vec{z}$ because $f(\vec{k}' + \vec{z}/R)$ is. Furthermore, we shall show that $F_{z/R}(\vec{k}')$ is continuous in z/R in the neighborhood of 0, so that we can use Eq. (A1) to conclude that

$$\lim_{R\to\infty} \int f'(\vec{k}')^* \frac{J_{3/2}(z)}{(2\pi z)^{3/2}} F_{z/R}(\vec{k}')\, d\vec{z}\, d\vec{k}' = \int f'(\vec{k}')^* F_0(\vec{k}')\, d\vec{k}'. \tag{A9}$$

Let us then examine the behavior of $F_{z/R}(\vec{k}')$ as $z/R \to 0$. If we define

$$x = \frac{z}{R} \cos\theta + \frac{z^2}{2k' R^2} \tag{A10}$$

then Eq. (A7) becomes

$$F_{z/R} = \frac{1}{2} \frac{R}{2k' z} \int_{-z/R + z^2/2k'R^2}^{z/R + z^2/2k'R^2} \bar{f}(\vec{k}', x) \frac{dx}{x}. \tag{A11}$$

This principal-value integral is of the general form

$$\frac{1}{b+a} \int_{-a}^{b} g(x) \frac{dx}{x} = \frac{1}{b+a} \int_{-a}^{b} \frac{g(x) - g(0)}{x}\, dx + \frac{g(0)}{b+a} \int_{-a}^{b} \frac{dx}{x}, \tag{A12}$$

where g is a differentiable function. Now, we may use the mean-value theorem to write (A12) as

$$\frac{1}{b+a} \int_{-a}^{b} g(x) \frac{dx}{x} = \left[\frac{g(x_1) - g(0)}{x_1} \right] \frac{g(0)}{b+a} \ln \frac{b}{a}, \tag{A13}$$

where $x_1 \in (-a, b)$. When $a, b \to 0$, the first factor on the right-hand side becomes the derivative of g at $x = 0$, while the second factor is then a constant multiplied with $g(0)$. Applying the formula (A13) to (A11) gives

$$F_0(\vec{k}') = \frac{1}{4k'^2} f(\vec{k}') + \frac{1}{2k'} \frac{d}{dk} f(k\hat{k}')\bigg|_{k=k'}. \tag{A14}$$

Finally, if we substitute this result (A14) into Eq. (A9) we obtain our first property, Eq. (A4).

The second property we want to show is

$$\int d\vec{k}' f(\vec{k}') \delta\left(\frac{k'^2}{2\mu} - \frac{k^2}{2\mu}\right) \langle \vec{k}' | P_R | \vec{k}\rangle \underset{R\to\infty}{\sim} \frac{R\mu}{\pi k} f(\vec{k}) - \frac{\sin 2kR}{2k} \frac{\mu}{\pi k} f(-\vec{k}). \tag{A15}$$

We first note that the integral on the left-hand side of Eq. (A15) can be written as

$$\mu k \int d\hat{k}' f(k\hat{k}') \langle k\hat{k}' | P_R | \vec{k}\rangle . \tag{A16}$$

Introducing again a variable z,

$$z = |k\hat{k}' - \vec{k}| R = \sqrt{2}\, kR(1-\cos\theta)^{1/2}, \tag{A17}$$

where we took the z axis of the $\vec{k}'$ system parallel to $\vec{k}$, and using furthermore Eq. (A3), we get for expression (A16)

$$\mu k \int_0^{2kR} dz \frac{R}{(2\pi)^{3/2} k^2} z^{-1/2} J_{3/2}(z) 2\pi \bar{f}, \tag{A18}$$

where $\bar{f}$ is the following average:

$$\bar{f} \equiv \bar{f}\left(k, \arccos\left(1 - \frac{z^2}{2k^2R^2}\right)\right) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi f(k\hat{k}'). \tag{A19}$$

We next do a partial integration with respect to z on the integral (A18) and apply the Riemann-Lebesgue lemma to the integral term. The terms that survive in the limit $R\to\infty$ are

$$-\frac{\mu R}{(2\pi)^{1/2} k}\left[z^{-1/2} J_{1/2}(z) \bar{f}\left(k, \arccos\left(1 - \frac{z^2}{2k^2R^2}\right)\right)\right]_{z=0}^{z=2kR}. \tag{A20}$$

Finally, with our special choice of coordinate system, this result can be written as

$$\frac{R\mu}{\pi k} f(\vec{k}) - \frac{\sin 2kR}{2k} \frac{\mu}{\pi k} f(-\vec{k}), \tag{A21}$$

and this proves our second property.

We still remark that, if we multiply Eq. (A15) with the function $f^*(\vec{k})$ and integrate over $\vec{k}$, then the second oscillating term does not contribute any more because of the Riemann-Lebesgue lemma.

---

*Work supported in part by the U. S. Atomic Energy Commission.

†Onderzoeker IIKW Belgium. Work supported in part by NATO.

‡On leave from Institute for Theoretical Physics, University of Leuven, B-3030 Heverlee, Belgium.

[1]F. T. Smith, Phys. Rev. 118, 349 (1960).

[2]L. E. Eisenbud, Ph.D. thesis, Princeton University, 1948 (unpublished); E. P. Wigner, Phys. Rev. 98, 145 (1955).

[3]J. M. Jauch and J. P. Marchand, Helv. Phys. Acta 40, 217 (1967).

[4]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), p. 485.

[5]T. A. Osborn and D. Bollé, Phys. Lett. 52B, 13 (1974).

[6]T. A. Osborn and D. Bollé, J. Math. Phys. (to be published).

[7]J. M. Jauch, K. B. Sinha, and B. N. Misra, Helv. Phys. Acta 45, 398 (1972).

[8]W. Brenig and R. Haag, Fortschr. Phys. 7, 183 (1959).

[9]M. Froissart, M. L. Goldberger, and K. M. Watson, Phys. Rev. 131, 2820 (1963).

[10]H. M. Nussenzveig, Phys. Rev. D 6, 1534 (1972).

[11]G. N. Watson, *Theory of Bessel Functions* (Cambridge Univ. Press, New York, 1962), p. 457. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge Univ. Press, New York, 1958), p. 172.

[12]Ref. 3, p. 227, Eq. (34) and the argument that follows.

[13]Ref. 3, p. 228, Eq. (37).

[14]Ref. 1, p. 351, Eqs. (10), (11), and (12).

[15]T. Ohmura, Prog. Theor. Phys. Suppl. 29, 108 (1964).

[16]R. Dashen, S. Ma, and H. J. Bernstein, Phys. Rev. 187, 345 (1969).

[17]H. M. Nussenzveig, *Causality and Dispersion Relations* (Academic, New York, 1972), pp. 110–116.