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Gravitation and positive-energy wave equations

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Relativistic equations that describe massive particles with only positive-energy eigenvalues are generalized to include gravitational interactions. The method developed, using a *vierbein* field, is applied specifically to Staunton's spin-$\frac{1}{2}$ equation. All consistency conditions are satisfied. A second-rank symmetric tensor whose covariant divergence is zero is found.

I. INTRODUCTION

Notable progress has been made in generalizing and in interpreting Dirac's[1] new relativistic equation. This equation describes a massive spin-zero positive-energy particle. It is linear in the momentum operators and involves a scalar field that depends on two internal harmonic-oscillator coordinates. The equation is interesting both because it exhibits a new method, using commutation relations, of reducing the Klein-Gordon equation and because it has the unusual property, despite the existence of a conserved current, of not permitting the minimal coupling of the particle to the electromagnetic field.

The first generalization of Dirac's equation was due to Biedenharn, Han, and van Dam.[2] They considered equations of higher order in the momentum operators that described particles of integer and half-odd-integer spin. More importantly they showed that all generalized Dirac equations could be viewed (within the framework of a quantum-front subdynamics) as describing a composite of two relativistic subparticles bound by harmonic-oscillator forces.

The theory has since been reformulated as a complete Poincaré algebra[3] and multiplet structures have been incorporated.[4] Connections between the quantum-front models and the dual-resonance model have been explored.[5] More recently, Staunton[6] has exhibited a spin-$\frac{1}{2}$ generalization of Dirac's equation that permits minimal coupling of the particle to the electromagnetic field.

The form of the interaction between particles, described by the above theories, and gravitation has not been previously elucidated. It is not apparent in what manner the consistency conditions, which often prevent a minimal electromagnetic coupling,[1,2,6] allow a gravitational generalization of the equations.

The purpose of this paper is to show how to write equations of the Dirac type using a Riemannian metric. For illustration a specific example is used, but the methods are general and readily applicable to other cases.

In Sec. II, for completeness and to fix notation, the logic and procedures of Staunton's spin-$\frac{1}{2}$ equation[6] are briefly reviewed. Section III details the procedure used to transform Staunton's equation from Minkowski to Riemannian space-time. The equations remain consistent. In Sec. IV, a second-rank symmetric tensor is found whose covariant divergence is zero. A brief summary and a short discussion comprise Sec. V.

II. STAUNTON SPIN-$\frac{1}{2}$ THEORY[6]

Consider a single-component wave function $\psi(x^a, q_1, q_2)$ depending upon Minkowski space-time coordinates x^a, and two internal dimensionless commuting harmonic-oscillator coordinates q_1 and q_2. Also consider operators[7] V_a and S_{ab}, defined on the space of functions of q_1 and q_2, that generate the Lie algebra of SO(3,2):

$$\begin{aligned}&[V_a, V_b] = iS_{ab},\\ &[V_a, S_{bc}] = i(\eta_{ac}V_b - \eta_{ab}V_c),\\ &[S_{ab}, S_{cd}] = i(\eta_{ac}S_{bd} - \eta_{ad}S_{bc} + \eta_{bd}S_{ac} - \eta_{bc}S_{ad}).\end{aligned} \tag{1}$$

Useful relations, given in Ref. 6, include

$$\begin{aligned}&3V_aV^a = S_{ab}S^{ab} = -\tfrac{3}{2},\\ &\epsilon^{abcd}S_{ab}S_{cd} = 0,\\ &V^bS_{ba} = S_{ab}V^b = -\tfrac{3}{2}iV_a,\\ &S_a{}^cS_{cb} = \tfrac{1}{2}(\eta_{ab} - V_aV_b + 3V_bV_a).\end{aligned} \tag{2}$$

The equation to be investigated is

$$T_a\psi = 0, \tag{3}$$

where T_a is taken to be

$$T_a = (-\eta_{ab} + aS_{ab})\Pi^b + mV_a. \tag{4}$$

Here $\Pi_a = P_a + eA_a = i\partial_a + eA_a$, while a and the mass m are constants. Equation (3) may not be a self-consistent set of equations.[1] In particular, Staunton[6] showed that self-consistency and the existence of a nontrivial solution imply that a could only take on the value i. We shall repeat only a portion of his demonstration below.

Contracting (3) with V_a and using (2) we obtain

$$(2+3ia)V_a\Pi^a\psi+m\psi=0. \tag{5}$$

This has the form of an interacting Majorana equation.[8] We shall restrict our attention to cases of massive particles. Accordingly the coefficient of $V_a\Pi^a$, the Majorana operator, is nonzero. We define a "Majorana mass" of the particle as $-m(2+3ia)^{-1}$. Next, contracting (3) with Π^a and using (2), we obtain

$$(\Pi^2-M^2)\psi-\tfrac{1}{2}iaeS_{ab}F^{ab}\psi=0. \tag{6}$$

Here we have introduced F_{ab} by

$$[\Pi_a,\Pi_b]=ieF_{ab}=ie(A_{b,a}-A_{a,b}), \tag{7}$$

and defined a "Klein-Gordon mass" M by

$$M^2=-m^2(2+3ia)^{-1}. \tag{8}$$

Finally, on contraction of (3) with Π_bS^{ba} we get

$$[(2+3ia)a\Pi^2-(6i-7a)M^2-\tfrac{1}{2}ie(2+3ia)^2S_{ab}F^{ab}]\psi=0. \tag{9}$$

When (6) and (9) are compared, a restriction on the constant a is obtained. It is $a=i$, or $a=2i$. The possibility $a=0$ is excluded since this implies that the particle's mass is zero. It is interesting to note that the "Majorana mass," m, and M are identical only for the case $a=i$.

The remaining consistency condition,

$$[T_a,T_b]\psi=0, \tag{10}$$

was shown by Staunton[6] to exclude the $a=2i$ possibility. This was accomplished by showing that (10) gave no new conditions on ψ for the $a=i$ case. However, in the $a=2i$ case, (10) implied an additional equation equivalent to

$$F_{ab}^{\text{dual}}S^{ab}\psi=0\quad (a=2i\text{ only}). \tag{11}$$

The spin of the particle when $a=i$ was found by Staunton to be $\frac{1}{2}$. He computed the action of the square of the Pauli-Lubanski operator W_a on ψ, when $A_a=0$, and obtained

$$W^2\psi\equiv-ms(s+1)\psi=-\tfrac{3}{4}m^2\psi\quad (a=i\text{ only}), \tag{12}$$

where

$$W^a\equiv\tfrac{1}{2}\epsilon^{abcd}S_{bc}\Pi_d. \tag{13}$$

III. EQUATION IN RIEMANNIAN SPACE

The conversion of the spin-$\frac{1}{2}$ equation of Sec. II to its analog in Riemannian space-time requires two steps. First the operators V_a and S_{ab} must be converted to their curved-space versions. This is accomplished by the well-known method[9] of introducing a tetrad relating local and general coordinate frames at each space-time point. The prescription (using a *vierbein* field $h_\mu{}^a$)[10] is, for example, $V_a\to V_\mu=h_\mu{}^aV_a$. Second, the derivative must be generalized[11] to allow the possibility of different internal transformations at each space-time point. Particular caution must be taken to ensure that execution of this program does not lead to $\psi=0$ when the consistency conditions are applied.

We proceed by considering the infinitesimal transformation properties of ψ. Under a coordinate transformation, $x^\mu\to x'^\mu=x^\mu+\xi^\mu$ (where ξ^μ are the space-time-dependent descriptors of the transformation), and simultaneously under an internal Lorentz transformation (with space-time-dependent parameters ϵ^{ab}), we have

$$\psi\to\psi'=\psi+i\epsilon^{ab}S_{ab}\psi, \tag{14}$$

and

$$\bar{\delta}\psi=-\psi_{,\mu}\xi^\mu+i\epsilon^{ab}S_{ab}\psi. \tag{15}$$

We define a generalized covariant derivative by

$$D_\mu\psi=\psi_{;\mu}-i\lambda_\mu{}^{ab}S_{ab}\psi. \tag{16}$$

The resemblance between $\lambda_\mu{}^{ab}$ and the Fock-Ivanenko coefficient is purely formal: No matrices are present. The transformation properties of $\lambda_\mu{}^{ab}$ are determined by the requirement that under a space-time transformation $D_\mu\psi$ transforms like a space-time covariant vector while under an internal transformation $D_\mu\psi$ transforms similarly to ψ. We obtain

$$\bar{\delta}\lambda_\mu{}^{ab}=\epsilon^{ab}{}_{,\mu}+i\epsilon^{cd}C_{cd}{}^{ab}{}_{ef}\lambda_\mu{}^{ef}-\lambda_\rho{}^{ab}\xi^\rho{}_{,\mu}-\lambda_\mu{}^{ab}{}_{,\rho}\xi^\rho. \tag{17}$$

The structure constants used here are defined by

$$[S_{ab},S_{cd}]=C_{ab}{}^{ef}{}_{cd}S_{ef}. \tag{18}$$

With the exception of the gauge term $\epsilon^{ab}{}_{,\mu}$, $\lambda_\mu{}^{ab}$ transforms like a space-time vector and like an internal Lorentz vector.

The derivative defined by (16) is noncommutative. We find that

$$D_\nu D_\mu\psi-D_\mu D_\nu\psi=iB_{\mu\nu}{}^{ab}S_{ab}\psi, \tag{19}$$

where

$$B_{\mu\nu}{}^{ab}\equiv-\lambda_\mu{}^{ab}{}_{,\nu}+\lambda_\nu{}^{ab}{}_{,\mu}-iC_{cd}{}^{ab}{}_{ef}\lambda_\mu{}^{cd}\lambda_\nu{}^{ef}. \tag{20}$$

We relate $\lambda_\mu{}^{ab}$ and $B_{\mu\nu}{}^{ab}$ to the *vierbein* field by requiring that the generalized covariant derivative of V_μ is zero. This is important. It ensures that the consistency conditions may be satisfied nontrivially and also it uniquely fixes $\lambda_\mu{}^{ab}$:

$$D_\nu V_\mu=V_{\mu;\nu}-i\lambda_\nu{}^{ab}[S_{ab},V_\mu]=0. \tag{21}$$

Direct evaluation of (21) yields

$$\lambda_\mu{}^{ab}=\tfrac{1}{4}(h^{\nu b}h_\nu{}^a{}_{;\mu}-h^{\nu a}h_\nu{}^b{}_{;\mu}). \tag{22}$$

We also evaluate

$$(D_\nu D_\mu - D_\mu D_\nu)V_\sigma = V_{\sigma;\mu\nu} - V_{\sigma;\nu\mu} + iB_{\mu\nu}{}^{ab}[S_{ab}, V_\sigma] = 0, \tag{23}$$

and obtain a relation between $B_{\mu\nu\rho\sigma}$ and the Riemann-Christoffel tensor $R_{\mu\nu\rho\sigma}$. It is

$$B_{\mu\nu\rho\sigma} = \tfrac{1}{2}R_{\mu\nu\rho\sigma}. \tag{24}$$

Since $S_{\mu\nu}$ is constructed from the commutator of two V_μ operators,

$$D_\rho S_{\mu\nu} = 0. \tag{25}$$

The formalism above is sufficient for generalizing the material in Sec. II. We take

$$T_\mu \psi = 0, \tag{26}$$

where

$$T_\mu = (-g_{\mu\nu} + iS_{\mu\nu})\Pi^\nu + mV_\mu, \tag{27}$$

and

$$\Pi_\mu = iD_\mu + eA_\mu. \tag{28}$$

Note that the principle of equivalence implies that consistency, if it exists at all, can only apply to the case $a = i$. Equations (21) and (25) ensure that much of the analysis in Sec. II may immediately be carried over with minor changes in index labeling if a *vierbein* affinity $\lambda_\mu{}^{ab}S_{ab}$ is introduced.[12] We obtain as the analog of (5)

$$2V^\mu T_\mu \psi = (V^\mu \Pi_\mu - m)\psi = 0. \tag{29}$$

The Majorana equation holds in an arbitrary reference frame.

The saturation of (26) with either Π_μ or $S_{\mu\nu}\Pi^\nu$ gives

$$[\Pi^2 - m^2 + \tfrac{1}{2}S_{\mu\nu}(eF^{\mu\nu} + \tfrac{1}{2}R^{\mu\nu\rho\sigma}S_{\rho\sigma})]\psi = 0. \tag{30}$$

For the commutation relation we find no new condition. Using (26) to eliminate some terms, and after some straightforward algebra, we obtain

$$[T_\mu, T_\nu]\psi = -iS_{\mu\nu}[\Pi^2 - m^2 + \tfrac{1}{2}S_{\rho\sigma}(eF^{\rho\sigma} + \tfrac{1}{2}R^{\rho\sigma\tau\kappa}S_{\tau\kappa})]\psi = 0. \tag{31}$$

This is clearly only a restatement of (30).

IV. DIVERGENCE-FREE TENSOR

Consider the expression

$$(T_\mu\psi)^*\psi - \psi^* T_\mu \psi = 0, \tag{32}$$

which follows immediately from (26). In all equations in this section where both ψ^* and ψ appear an integration over q space is understood. Expansion of (32) using (27) yields

$$-(iD_\mu\psi)^*\psi + i\psi^* D_\mu\psi + (iS_\mu{}^\nu\Pi_\nu\psi)^*\psi - \psi^* iS_\mu{}^\nu\Pi_\nu\psi = 0. \tag{33}$$

Using the fact that the covariant derivative of the metric is zero, the first two terms in (33) may be combined. Then using (1) to replace $S_{\mu\nu}$ in the last two terms in (33) and simplifying with Majorana's equation, we get

$$i(\psi^*\delta_\mu{}^\nu\psi)_{;\nu} - (iV^\nu V_\mu D_\nu\psi)^*\psi + i\psi^*(V^\nu V_\mu D_\nu\psi) = 0. \tag{34}$$

Since D_μ commutes with V_μ we may rewrite (34) as

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{35}$$

where

$$T^{\mu\nu} \equiv \alpha\psi^*(g^{\mu\nu} + V^\mu V^\nu + V^\nu V^\mu)\psi \tag{36}$$

and where α is a constant,

$$T = T_\mu{}^\mu = 3\alpha\psi^*\psi. \tag{37}$$

In the case of Minkowski space-time Staunton[6] gives the solution of (26) for a stationary particle as

$$\psi = (Aq_1 + Bq_2)\exp[-imt - \tfrac{1}{2}(q_1{}^2 + q_2{}^2)]. \tag{38}$$

A and B are arbitrary constants. It is interesting to evaluate (36) for this case. Using the specific flat-space representation[1] of V_μ

$$\begin{aligned} V_0 &= \tfrac{1}{4}(q_1{}^2 + q_2{}^2 + \eta_1{}^2 + \eta_2{}^2),\\ V_1 &= \tfrac{1}{2}(-q_1\eta_1 + q_2\eta_2),\\ V_2 &= \tfrac{1}{2}(q_1\eta_2 + q_2\eta_1),\\ V_3 &= \tfrac{1}{4}(q_1{}^2 + q_2{}^2 - \eta_1{}^2 - \eta_2{}^2),\\ \eta_i &= -i\partial_i, \quad i = 1,2 \end{aligned} \tag{39}$$

we obtain

$$T_{00} = 3\alpha\psi^*\psi. \tag{40}$$

All other components of $T_{\mu\nu}$ are zero.

V. CONCLUSION

We have shown that it is possible to generalize Staunton's spin-$\tfrac{1}{2}$ equation to curved space-time and have explicitly found a symmetric tensor whose covariant divergence is zero. This tensor is a possible candidate for the gravitational stress-energy tensor of the particle. Many interesting questions remain that are currently being investigated.

1. What are the properties of $T_{\mu\nu}$ for moving particles both in flat and in curved space-time?
2. Can a Lagrangian density be found that upon variation uniquely yields Staunton's equation? Its variation with respect to the metric would immediately give the gravitational stress-energy tensor.
3. Do solutions of the coupled Einstein-Staunton equations exist for cases of high metric symmetry? If so, what are their properties?

[1]P. A. M. Dirac, Proc. R. Soc. A322, 435 (1971); A328, 1 (1972).

[2]L. C. Biedenharn, M. Y. Han, and H. van Dam, Phys. Rev. D 8, 1735 (1973).

[3]L. P. Staunton, Phys. Rev. D 8, 2446 (1973).

[4]L. P. Staunton and H. van Dam, Nuovo Cimento Lett. 7, 371 (1973).

[5]L. C. Biedenharn nnd H. van Dam, Phys. Rev. D 9, 471 (1974).

[6]L. P. Staunton, Phys. Rev. D 10, 1760 (1974).

[7]A specific realization is given in Ref. 6. The commutation relations and associated relations listed in Eqs. (1) and (2) are sufficient for our purposes.

[8]E. Majorana, Nuovo Cimento 9, 335 (1932). For a review in English, see D. M. Fradkin, Am. J. Phys. 34, 314 (1966).

[9]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 365.

[10]Latin lower case letters are used for local Lorentz indices as in Sec. II, while the Greek alphabet is reserved for space-time indices. Comma, semicolon, and D_μ indicate ordinary derivatives, space-time covariant derivatives, and generalized covariant derivatives, respectively. $c=\hbar=1$. $\eta_{ab}=\text{diag}\,(1,-1,-1,-1)$, $h_\mu{}^a h_{\nu a}=g_{\mu\nu}$, and $h_\mu{}^a h^{\mu b}=\eta^{ab}$. Latin and Greek indices are raised or lowered using η_{ab} and $g_{\mu\nu}$, respectively.

[11]The logic and methods of doing this are well known but have never been applied to this problem. A good treatment appears in J. L. Anderson, *Principles of Relativity Physics* (Academic, New York, 1967), pp. 35–37, 44–46.

[12]Identities (2) remain valid after the *vierbein* field has been introduced.